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Steady Open Channel Test Problems with Analytic Solutions

by

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Numerical Analysis Report **3/95**

DEPARTMENT OF MATHEMATICS

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[1]The work reported here forms part of the research programme of the Oxford/Reading Institute of Computational Fluid Dynamics and was supported by the EPSRC and HR Wallingford Ltd.

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Abstract: In many areas of computational fluid dynamics there are benchmark test problems which have known analytic solutions. For the case of steady flow in an open channel there is a need for such test problems, so as to give a measure of the performance of numerical methods. This is especially true for problems where the flow changes between subcritical and supercritical. In this report a method is described that allows the construction of a wide range of such test problems, where in each case the exact solution to the full steady Saint Venant equation is known. In particular the report discusses how to construct problems with solutions having a hydraulic jump. Example test problems are given along with their analytic solution. These can be readily used by modellers to test their own particular software. For some of the example test problems, results from a commercial software package are compared with the exact solutions, thus demonstrating how the method is a valuable tool for validating both steady and unsteady flow solvers.

1 Introduction

The use of numerical methods to compute the water surface profile and discharge, for both unsteady and steady open channel flows, is now very common in civil engineering hydraulics. The Saint Venant equations, first formulated by De St.Venant (1871), are almost always used to model the flow. Derivations of these equations can be found, for example, in Cunge, Holly and Verwey (1981) and in Yen (1973). In this report attention is restricted to the important case of steady flow. Apart from the intrinsic interest of the steady flow problem, a reason why the computation of steady flows is so significant is that they are often required as initial data for unsteady simulations. Under steady conditions the Saint Venant equations reduce to a single Ordinary Differential Equation (ODE), which, if the flow is either wholly subcritical or wholly supercritical, can be efficiently integrated using a high accuracy numerical ODE solver. The situation is much more difficult if the flow is mixed sub-supercritical, having hydraulic jumps or critical sections. Algorithms do exist which can handle such flows, for example the method of Humpidge and Moss (1971) which combines the ODE integration with the location of control points and the fitting of hydraulic jumps. Another method is to apply an unsteady solver and proceed forward in time until all the transients in the flow have decayed and the flow has reached a steady state. In general this approach can be very time consuming for trans-critical flows, although the work of Garcia-Navarro, Priestley and Alcrudo (1994) gives a numerical method with improved efficiency.

Recently in MacDonald, Baines and Nichols (1994) a new approach has been developed which uses shock capturing methods (see Engquist and Osher (1981)) such as those that have been previously applied to the unsteady system of equations, but are there applied to the single steady equation.

With the recent emergence of these new steady solvers, some way is required to compare the performance of these methods with existing schemes. In many areas of computational fluid dynamics there are standard test problems, for which the exact solution is known. Using some measure of the difference between the numerical solution and the exact solution, the performance of a particular numerical scheme can be evaluated. Important features of the solution, for example shocks, can be compared with the exact solution. This allows the strengths and weaknesses of one scheme over another to be judged. If a particular scheme gives acceptable performance over a wide range of such test problems, then, even though there may be no theory guaranteeing good performance for a practical problem, there will be more confidence in the method. Altogether, the existence of test problems with known solutions is extremely desirable.

For the steady open channel problem, because of the non-linear nature of the differential equation, particularly the friction term, even the simplest problem of flow in a uniform rectangular channel with zero bed slope cannot be solved analytically. To obtain a solvable problem it is necessary to assume zero friction or at least to simplify the friction term significantly. Features of the problem will then be lost. For this reason, until now, the performance of methods for steady computation have mostly been judged only qualitatively.

This report presents a simple method for constructing test problems with known analytic solutions to the full steady Saint-Venant equation. The method is an "inverse method" in that some hypothetical depth profile is chosen and the bed slope that makes this profile an actual solution of the steady equation is then found. The method can be used to construct test problems with almost any desired features, including hydraulic jumps. Hence these test problems can be used to compare the numerical results, for any algorithm, with an exact solution. The method is also useful for evaluating unsteady solvers, since, if an unsteady model is given steady boundary conditions, the limiting steady solution can be compared with the analytic steady solution. The method presented in this report fits in well with the validation documentation initiative of the European hydraulics laboratories (see Dee (1993)), since it enables the creation of benchmark test problems which can be used as a standard measure for the performance of commercial software packages.

2 Background and Theory

2.1 The Equations

The Saint Venant equations are obtained from the principles of mass and momentum balance and are given by

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q, \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\beta Q^2}{A}\right) + gA\frac{\partial y}{\partial x} - gA(S_0 - S_f) = 0. \tag{2}$$

Here x is the distance along the channel, t is time, $y(x,t)$ is the depth, $Q(x,t)$ is the discharge, $T(x,y)$ is the free surface width, $A(x,y)$ is the wetted area, $S_0(x)$ is the bed slope, $S_f(x,y,Q)$ is the friction slope, q is the lateral inflow per unit length, β is the momentum coefficient and g is the acceleration due to gravity. The bed slope, S_0, is given by

$$S_0 = -\frac{dz}{dx}, \tag{3}$$

where $z(x)$ is the bed level, the elevation of the bed above some horizontal datum. The friction slope, S_f, is given by

$$S_f = \frac{Q|Q|}{K^2},$$

where $K(x,y)$ is the conveyance, which can be taken to be one of the many commonly used formulae (see Chow (1959)). Here Manning's equation, for which

$$K = \frac{A^{5/3}}{nP^{2/3}},$$

is used, where $P(x,y)$ is the wetted perimeter and n is the Manning friction coefficient. However, the method described in this work does not rely on this choice of conveyance.

In this report only the steady flow problem is considered, so it is assumed that $y = y(x)$ and $Q = Q(x)$. Under these steady conditions equations (1) and (2) reduce to

$$\frac{dQ}{dx} = q, \tag{4}$$

$$\frac{d}{dx}\left(\frac{\beta Q^2}{A}\right) + gA\frac{dy}{dx} - gA(S_0 - S_f) = 0. \tag{5}$$

Although not strictly necessary, the following simplifying assumptions are made. There is uniform velocity over each cross-section, which implies that the momentum coefficient β and the energy coefficient α are both unity. Also the lateral inflow q is assumed to be zero. With zero lateral inflow, equation (4) becomes trivial with solution $Q \equiv$constant. Since the discharge can have no jumps it must be constant throughout the entire channel reach. From now on x will be measured in the direction of this constant discharge and hence $Q > 0$.

Differentiating the momentum term in equation (5) and dividing through by gA then yields the equation

$$\left(1 - \frac{Q^2 T}{gA^3}\right)\frac{dy}{dx} - \frac{Q^2}{gA^3}\frac{\partial A}{\partial x} - S_0 + S_f = 0. \tag{6}$$

It is convenient to re-write this equation as

$$S_0(x) = f_1(x, y(x))y'(x) + f_2(x, y(x)), \tag{7}$$

where

$$f_1 = 1 - \frac{Q^2 T}{gA^3} = 1 - F_r^2 \tag{8}$$

and

$$f_2 = \frac{Q^2 n^2 P^{4/3}}{A^{10/3}} - \frac{Q^2}{gA^3}\frac{\partial A}{\partial x}. \tag{9}$$

Fr is the Froude number. The assumptions on α, β and q can be relaxed, and equation (6) can still be written in the form (7) for different functions f_1 and f_2.

2.2 Problems with Smooth Solutions

The crux of the work in this report depends on the following argument. Suppose that for some reach $0 \leq x \leq L$ the functions T and P, representing channel width and wetted perimeter respectively, are arbitrarily defined for $0 < y \leq y_{\max}$. For example for a rectangular channel define $T = B$, $P = 2y + B$ so that $A = By$, where $B(x) > 0$ gives the width. It will be required that T, P, $\partial T/\partial x$ and $\partial T/\partial y$ are continuous. These requirements are sufficient to ensure that differential equation (6) is valid and can be obtained from the integral form of the steady Saint-Venant equation. If values for the discharge Q and the Manning friction coefficient n are chosen, then the functions f_1 and f_2 given by equations (8) and (9) have been completely defined. The main part of the method is to choose some function $\hat{y}$ for

$0 \leq x \leq L$, with $0 < \hat{y}(x) \leq y_{\max}$ and having a continuous first derivative. This function will be referred to as the hypothetical depth profile. Finally, if the bed slope of the channel is given by

$$S_0(x) = f_1(x, \hat{y}(x))\hat{y}'(x) + f_2(x, \hat{y}(x)), \tag{10}$$

then it is now easy to see that $\hat{y}$ satisfies the differential equation (7) for the entire reach $0 \leq x \leq L$.

From the above, a complete test problem can be specified by the length L of the reach, the functions T, P (and hence A) which define the cross-sectional shape throughout the reach, values for the discharge Q and Manning coefficient n, and the bed slope of the channel given by equation (10). The appropriate boundary conditions are also required. These depend on the value of the hypothetical depth at the boundaries. For example, if the depth $\hat{y}(0)$ is supercritical at $x = 0$, a depth of $\hat{y}(0)$ needs to be specified at inflow. The analytic solution to this steady problem is now given by $y \equiv \hat{y}$.

For many computational models the bed level z is required rather than the bed slope. This cannot normally be found analytically from S_0, so equation (3) must be integrated with a high accuracy ODE solver. For this purpose a starting value such as $z(L) = 0$ is required.

2.3 Problems with Hydraulic Jumps

It has been shown how to construct test problems with a smooth analytic solution, i.e. for which $\hat{y}$ is differentiable. This in itself is useful, but it would be more interesting if test problems could be constructed where the known solution has a hydraulic jump. A procedure for achieving this is now given. Let the hypothetical depth profile $0 < \hat{y}(x) \leq y_{\max}$, with a hydraulic jump at some point $x = x^\star$, be defined by

$$\hat{y}(x) = \begin{cases} \hat{y}_L(x) & 0 \leq x \leq x^\star \\ \\ \hat{y}_R(x) & x^\star < x \leq L, \end{cases}$$

the functions $\hat{y}_L$ and $\hat{y}_R$ having continuous derivatives on the intervals $0 \leq x \leq x^\star$, $x^\star \leq x \leq L$ respectively, with the derivatives being one-sided at the end points. The hypothetical flow must be physically allowable, so the hydraulic jump must satisfy a jump condition and there cannot be a gain in energy across the jump (see Chow (1959)). Mathematically this requires that

$$F(x^\star, \hat{y}_L(x^\star)) = F(x^\star, \hat{y}_R(x^\star)), \tag{11}$$

$$E(x^\star, \hat{y}_L(x^\star)) \geq E(x^\star, \hat{y}_R(x^\star)),$$

where $F(x, y)$ is the specific force given by

$$F = \frac{\beta Q^2}{gA} + \int_0^y (y - \eta) T(x, \eta) d\eta$$

and $E(x, y)$ is the specific energy given by

$$E = \frac{\alpha Q^2}{2gA^2} + y.$$

For a rectangular channel the jump condition explicitly yields the required value of $\hat{y}_R(x^\star)$ for any value of $\hat{y}_L(x^\star)$ and vice versa. For other cross-sections it is necessary to solve the jump condition numerically. For many cross-sections, where there is a unique critical depth, the energy condition is satisfied if and only if the jump is from supercritical to subcritical. For cases where the situation is not clear, it is easy to check the condition for each individual jump. The bed slope of the channel can now be defined in a piecewise manner by

$$S_0(x) = \begin{cases} S_{0L}(x) & 0 \leq x \leq x^\star \\ \\ S_{0R}(x) & x^\star < x \leq L, \end{cases} \tag{12}$$

where

$$\begin{aligned} S_{0L}(x) &= f_1(x, \hat{y}_L(x))\hat{y}'_L(x) + f_2(x, \hat{y}_L(x)), \\ S_{0R}(x) &= f_1(x, \hat{y}_R(x))\hat{y}'_R(x) + f_2(x, \hat{y}_R(x)). \end{aligned}$$

For this bed slope, $\hat{y}$ satisfies the differential equation (7) everywhere except at the jump. In general the bed slope is discontinuous at $x = x^\star$, i.e.

$$S_{0L}(x^\star) \neq S_{0R}(x^\star).$$

At first sight this discontinuity might seem perfectly acceptable, since many valid test problems have such a feature. However, in general the hydraulic jump does not occur at the same position as the bed slope discontinuity and so we would like to construct problems where the jump does not coincide with a bed slope discontinuity. This can be achieved in the following way. Having chosen values for $\hat{y}_L(x^\star)$ and $\hat{y}_R(x^\star)$, choose values for $\hat{y}'_L(x^\star)$ and $\hat{y}'_L(x^\star)$ satisfying the linear relationship

$$\begin{aligned} S_{0L}(x^\star) &= f_1(x^\star, \hat{y}_L(x^\star))\hat{y}'_L(x^\star) + f_2(x^\star, \hat{y}_L(x^\star)) \\ &= f_1(x^\star, \hat{y}_R(x^\star))\hat{y}'_R(x^\star) + f_2(x^\star, \hat{y}_R(x^\star)) \quad = \quad S_{0R}(x^\star). \end{aligned} \tag{13}$$

If the functions are smooth enough, equation (12) can be differentiated to find a linear relationship between $\hat{y}_L''(x^\star)$ and $\hat{y}_R''(x^\star)$ in order to make the bed slope differentiable at the jump, i.e.

$$S_{0L}'(x^\star) = S_{0R}'(x^\star). \tag{14}$$

In the next section specific examples of test problems are given. Motivated by the theory in MacDonald, Baines and Nichols (1994), which requires the bed slope to have a continuous first derivative, these test problems satisfy (13) and (14). Further smoothness in the bed slope may be achieved if necessary.

The test problems in the next section are constructed using the following strategy. The function $\hat{y}_L$ is chosen arbitrarily. The bed slope is required to have a continuous first derivative, so equations (11), (13) and (14) determine the necessary values $\hat{y}_R$, $\hat{y}_R'$ and $\hat{y}_R''$ at $x = x^\star$. The form of the function $\hat{y}_R$ is chosen with three free parameters. Values for these parameters are chosen so as to give the required values for $\hat{y}_R$, $\hat{y}_R'$ and $\hat{y}_R''$ at $x = x^\star$. In general more than three parameters are required so that there is some control over the behaviour of $\hat{y}_R$. In particular $\hat{y}_R$ must always remain positive. The obvious choice for $\hat{y}_R$ is a cubic or higher order polynomial in $(x - x^\star)$. The examples in MacDonald (1994) use this form. The difficulty with polynomials is that they can be highly oscillatory and hence difficult to control and keep positive, a difficulty which increases the greater the distance $\hat{y}_R$ is required to cover. There are many other possible forms for $\hat{y}_R$. The examples in this report use sums of exponential functions (see Appendix III). The magnitudes of $\hat{y}_R'$ and $\hat{y}_R''$ at the jump are often required to be high. The choice of exponentials allows the restriction of these high magnitudes to the locality of the jump. Judicious choice of these exponentials allows control over the behaviour of $\hat{y}_R$ away from the jump. Even with a reasonable form for $\hat{y}_R$, it can take much trial and error to achieve a specific behaviour.

3 Examples

This section describes specific examples of test problems constructed using the method given in this report. The full details of the test problems are given in Appendix III. In Examples 1-4, a 1km long, 10m wide rectangular channel with a discharge of 20m^3/s is chosen. In Example 1 the hypothetical depth function is the subcritical hump shown in Figure 1. The resulting bed slope is also shown in the same figure. Figure 2 shows the resulting bed level and free surface level.

In Example 2 the hypothetical depth function is an inverted hump with the flow being wholly supercritical. Figure 3 shows the resulting bed level and free surface level.

In Example 3 the hypothetical depth function is chosen to be subcritical at inflow and to change smoothly to supercritical halfway along the channel, remaining supercritical for the rest of the channel. Figure 4 shows the resulting bed level and free surface level.

In Example 4 the hypothetical depth function is supercritical at inflow, changes via a hydraulic jump to subcritical halfway along the channel and remains subcritical. The hypothetical depth function is shown in Figure 5 as well as the resulting bed slope. Figure 6 shows the resulting bed level and free surface level.

In Example 5 a 5km long trapezoidal channel ($T = 10 + 4y$, $P = 10 + 2y\sqrt{5}$) with a discharge of $20\text{m}^3/\text{s}$ is chosen. The hypothetical depth function is the subcritical sine function shown in Figure 8. Figure 7 shows the resulting bed level and free surface level.

In Example 6 a 1km long trapezoidal channel ($T = 10 + 2y$, $P = 10 + 2y\sqrt{2}$) with a discharge of $20\text{m}^3/\text{s}$ is chosen. In this case the hypothetical depth function, shown in Figure 10, is subcritical at inflow, changes smoothly to supercritical and then returns to subcritical via a hydraulic jump. Figure 9 shows the resulting bed level and free surface level.

4 Comparison with Exact Solution for a Commercial Package

The FLUCOMP package, developed at HR Wallingford Ltd, simulates both steady and unsteady flow using the Saint-Venant equations. Details of FLUCOMP can be found in Price and Samuels (1980) and in Samuels and Gray (1982). The package is designed to solve for the flow in natural rivers with flood plains and has a special module for solving steady flows. This module uses finite differences, the derivatives in the dynamic equation (5) being discretised using the trapezium rule. The algorithm can only integrate in the upstream direction and is intentionally restricted to handle only subcritical flows.

Figure 8 shows the numerical results from the application of FLUCOMP to Example 5 with a grid spacing of 25m. Figure 10 shows the numerical results from the application of FLUCOMP to Example 6 with a grid spacing of 10m. These examples illustrate the value of having an exact solution with which to compare the numerical solution. It can clearly be seen that for example 5 the numerical solution follows the exact solution, whereas in example 6 the numerical solution does not follow the exact solution in the region where it is supercritical. This is a consequence of the package restricting the flow in order to avoid instability in the integration procedure. The user is warned when this situation occurs.

In MacDonald, Baines and Nichols (1994) a method is described that successfully solves for any type of flow, regardless of whether it is subcritical, supercritical or of mixed type. A comparison of this method against FLUCOMP and against exact solutions for test problems derived by the technique described in this report can be found in MacDonald, Baines, Nichols and Samuels (1995).

5 Discussion and Conclusions

In this report a method has been given for constructing steady open channel test problems to which the exact solution of the steady Saint Venant equation is known. To the authors' knowledge this is the first time that non-trivial exact solutions have been made available to the modeller. Moreover, the method can create a useful range of test problems, including almost all channel geometries and all flow types. In particular, techniques for constructing problems with hydraulic jumps have been described and it has been shown that jumps must satisfy certain conditions depending on how smooth the bed slope is required to be. For brevity, the test examples given here (see Appendix III) are restricted to rectangular and trapezoidal channels. This is not a restriction on the method, although for more complicated channel geometries, particularly those where the channel cross-section varies along the channels, the expressions for the bed slope become large and unwieldy. The symbolic computation package Mathematica (see Wolfram (1988)) helped greatly to facilitate the algebraic construction of these test problems. Examples of non-prismatic channels are to be found in MacDonald, Baines, Nichols and Samuels (1995). Details of the test examples are given in such a way that they may be used immediately by modellers without the need to understand the method of construction. In general, to apply computational methods to the test problems, the modeller will need to integrate the bed slope numerically in order to recover the bed level. This may be done by using a standard numerical software package.

For two of the examples given, numerical results from a commercial steady open-channel solver have been compared with the exact solution, demonstrating how it is possible to judge the performance of the solver.

The method described in this report is a valuable tool for developing, validating or comparing steady open-channel solvers. The method can also be used to test the performance of unsteady models as the solution tends to a steady state.

Acknowledgements

This work was supported by the Engineering and Physical Sciences Research Council, UK. and HR Wallingford Ltd, UK.

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Appendix II: Notation

x	Distance along channel (m).
t	Time (s).
y	Depth (m).
Q	Discharge (m^3/s).
A	Wetted area (m^2).
T	Channel width (m).
P	Wetted perimeter (m).
g	Acceleration due to gravity ($9.80665 m/s^2$).
L	Length of channel reach (m).
z	Level of bed above some horizontal datum (m).
S_0	Bed slope $= -dz/dx$.
S_f	Friction slope.
q	Lateral inflow (m^2/s).
K	Conveyance (m^3/s).

n	Manning friction coefficient.
F	Specific Force (m^3).
E	Specific Energy (m).
α	Energy coefficient.
β	Momentum coefficient.
$\hat{y}$	Hypothetical depth function (m).
h', h''	First and second derivatives of a function h, respectively.
f_1, f_2	Bed slope functions: $S_0 = f_1\hat{y}' + f_2$.
F_r	Froude Number.
$x^\star$	Position of hydraulic jump (m).
$\hat{y}_L$, $\hat{y}_R$	Hypothetical depth functions for left and right of jump (m).
S_{0L}, S_{0R}	Bed slope functions for left and right of jump.
$y\mathrm{max}$	Maximum depth for hypothetical depth functions (m).
B	Width for rectangular channels (m).
$a_1 ... a_3$	Coefficients in depth functions.
k	Integer index.
η	Dummy integration variable.

Appendix III: Details Of Test Problems

In this Appendix the full details for the examples described in the main text are given. Here x denotes the distance along the channel (in metres) in the direction of the flow.

Example 1

A 1km long rectangular channel of width 10m has a discharge of $20\mathrm{m}^3/\mathrm{s}$. The flow is subcritical at inflow and is subcritical at outflow with depth 0.748409m. The Manning roughness coefficient for the channel is 0.03 and the bed slope is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{y}(x)^3}\right)\hat{y}'(x) + 0.36\frac{(2\hat{y}(x)+10)^{4/3}}{(10\hat{y}(x))^{10/3}},$$

where

$$\hat{y}(x) = \left(\frac{4}{g}\right)^{1/3}\left(1 + \frac{1}{2}\exp\left(-16\left(\frac{x}{1000} - \frac{1}{2}\right)^2\right)\right),$$

and

$$\hat{y}'(x) = -\left(\frac{4}{g}\right)^{1/3} \frac{2}{125} \left(\frac{x}{1000} - \frac{1}{2}\right) \exp\left(-16\left(\frac{x}{1000} - \frac{1}{2}\right)^2\right).$$

The solution to this problem is given by $y \equiv \hat{y}$ and is shown in Figures 1 and 2.

Example 2

A 1km long rectangular channel of width 10m has a discharge of $20\text{m}^3/\text{s}$. The flow is supercritical at inflow with depth 0.741599m and is supercritical at outflow. The Manning roughness coefficient for the channel is 0.02 and the bed slope is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{y}(x)^3}\right)\hat{y}'(x) + 0.16\frac{(2\hat{y}(x) + 10)^{4/3}}{(10\hat{y}(x))^{10/3}},$$

where

$$\hat{y}(x) = \left(\frac{4}{g}\right)^{1/3} \left(1 - \frac{1}{5}\exp\left(-36\left(\frac{x}{1000} - \frac{1}{2}\right)^2\right)\right),$$

and

$$\hat{y}'(x) = \left(\frac{4}{g}\right)^{1/3} \frac{9}{625} \left(\frac{x}{1000} - \frac{1}{2}\right) \exp\left(-36\left(\frac{x}{1000} - \frac{1}{2}\right)^2\right).$$

The solution to this problem is given by $y \equiv \hat{y}$ and is shown in Figure 3.

Example 3

A 1km long rectangular channel of width 10m has a discharge of $20\text{m}^3/\text{s}$. The flow is subcritical at inflow and is supercritical at outflow. The Manning roughness coefficient for the channel is 0.02 and the bed slope is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{y}(x)^3}\right)\hat{y}'(x) + 0.16\frac{(2\hat{y}(x) + 10)^{4/3}}{(10\hat{y}(x))^{10/3}},$$

where

$$\hat{y}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(1 - \frac{1}{3}\tanh\left(3\left(\frac{x}{1000} - \frac{1}{2}\right)\right)\right) & 0 \le x \le 500 \\ \left(\frac{4}{g}\right)^{1/3}\left(1 - \frac{1}{6}\tanh\left(6\left(\frac{x}{1000} - \frac{1}{2}\right)\right)\right) & 500 < x \le 1000, \end{cases}$$

and

$$\hat{y}'(x) = \begin{cases} -\left(\frac{4}{g}\right)^{1/3} \frac{1}{1000}\mathrm{sech}^2\left(3\left(\frac{x}{1000}-\frac{1}{2}\right)\right) & 0 \le x \le 500 \\ -\left(\frac{4}{g}\right)^{1/3} \frac{1}{1000}\mathrm{sech}^2\left(6\left(\frac{x}{1000}-\frac{1}{2}\right)\right) & 500 < x \le 1000. \end{cases}$$

The solution to this problem is given by $y \equiv \hat{y}$ and is shown in Figure 4.

Example 4

A 1km long rectangular channel of width 10m has a discharge of 20m^3/s. The flow is supercritical at inflow with depth 0.543853m and is subcritical at outflow with depth 1.334899m. The Manning roughness coefficient for the channel is 0.02 and the bed slope is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{y}(x)^3}\right)\hat{y}'(x) + 0.16\frac{(2\hat{y}(x)+10)^{4/3}}{(10\hat{y}(x))^{10/3}},$$

where

$$\hat{y}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(\frac{9}{10} - \frac{1}{6}\exp\left(\frac{-x}{250}\right)\right) & 0 \le x \le 500 \\ \left(\frac{4}{g}\right)^{1/3}\left(1 + \sum_{k=1}^{3} a_k \exp\left(-20k\left(\frac{x}{1000}-\frac{1}{2}\right)\right) + \right. \\ \left. \frac{4}{5}\exp\left(\frac{x}{1000}-1\right)\right) & 500 < x \le 1000, \end{cases}$$

and

$$\hat{y}'(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\frac{1}{1500}\exp\left(\frac{-x}{250}\right) & 0 \le x \le 500 \\ \left(\frac{4}{g}\right)^{1/3}\left(-\frac{1}{50}\sum_{k=1}^{3} k a_k \exp\left(-20k\left(\frac{x}{1000}-\frac{1}{2}\right)\right) + \right. \\ \left. \frac{1}{1250}\exp\left(\frac{x}{1000}-1\right)\right) & 500 < x \le 1000, \end{cases}$$

with a_1=-0.348427, a_2=0.552264, a_3=-0.555580. The solution to this problem is given by $y \equiv \hat{y}$ and is shown in Figures 5 and 6.

Example 5

A 5km long trapezoidal channel ($T = 10+4y$, $P = 10+2y\sqrt{5}$) has a discharge of 20m^3/s. The flow is subcritical at inflow and is subcritical at outflow with depth 1.125m The Manning roughness coefficient for the channel is 0.03 and the bed slope is given by

$$S_0(x) = \left(1 - \frac{400(10 + 4\hat{y}(x))}{g(10 + 2\hat{y}(x))^3\hat{y}(x)^3}\right)\hat{y}'(x) + 0.36\frac{(10 + 2\hat{y}(x)\sqrt{5})^{4/3}}{(10 + 2\hat{y}(x))^{10/3}\hat{y}(x)^{10/3}},$$

where

$$\hat{y}(x) = \frac{9}{8} + \frac{1}{4}\sin\left(\frac{\pi x}{500}\right),$$

and

$$\hat{y}'(x) = \frac{\pi}{2000}\cos\left(\frac{\pi x}{500}\right).$$

The solution to this problem is given by $y \equiv \hat{y}$ and is shown in Figures 7 and 8.

Example 6

A 1km long trapezoidal channel ($T = 10+2y$, $P = 10+2y\sqrt{2}$) has a discharge of 20m^3/s. The flow is subcritical at inflow and is subcritical at outflow with depth 1.349963m The Manning roughness coefficient for the channel is 0.02 and the bed slope is given by

$$S_0(x) = \left(1 - \frac{400(10 + 2\hat{y}(x))}{g(10 + \hat{y}(x))^3\hat{y}(x)^3}\right)\hat{y}'(x) + 0.16\frac{(10 + 2\hat{y}(x)\sqrt{2})^{4/3}}{(10 + \hat{y}(x))^{10/3}\hat{y}(x)^{10/3}},$$

where

$$\hat{y}(x) = \begin{cases} 0.723449\left(1 - \tanh\left(\frac{x}{1000} - \frac{3}{10}\right)\right) & 0 \le x \le 300 \\ 0.723449\left(1 - \frac{1}{6}\tanh\left(6\left(\frac{x}{1000} - \frac{3}{10}\right)\right)\right) & 300 < x \le 600 \\ \frac{3}{4} + \sum_{k=1}^{3} a_k \exp\left(-20k\left(\frac{x}{1000} - \frac{3}{5}\right)\right) + \\ \frac{3}{5}\exp\left(\frac{x}{1000} - 1\right) & 600 < x \le 1000, \end{cases}$$

and

$$\hat{y}'(x) = \begin{cases} -0.723449 \times 10^{-3}\mathrm{sech}^2\left(\frac{x}{1000} - \frac{3}{10}\right) & 0 \le x \le 300 \\ -0.723449 \times 10^{-3}\mathrm{sech}^2\left(6\left(\frac{x}{1000} - \frac{3}{10}\right)\right) & 300 < x \le 600 \\ -\frac{1}{50}\sum_{k=1}^{3} k a_k \exp\left(-20k\left(\frac{x}{1000} - \frac{3}{5}\right)\right) + \\ \frac{3}{5000}\exp\left(\frac{x}{1000} - 1\right) & 600 < x \le 1000, \end{cases}$$

with a_1=-0.111051, a_2=0.026876, a_3=-0.217567. The solution to this problem is given by $y \equiv \hat{y}$ and is shown in Figures 9 and 10.

Figures

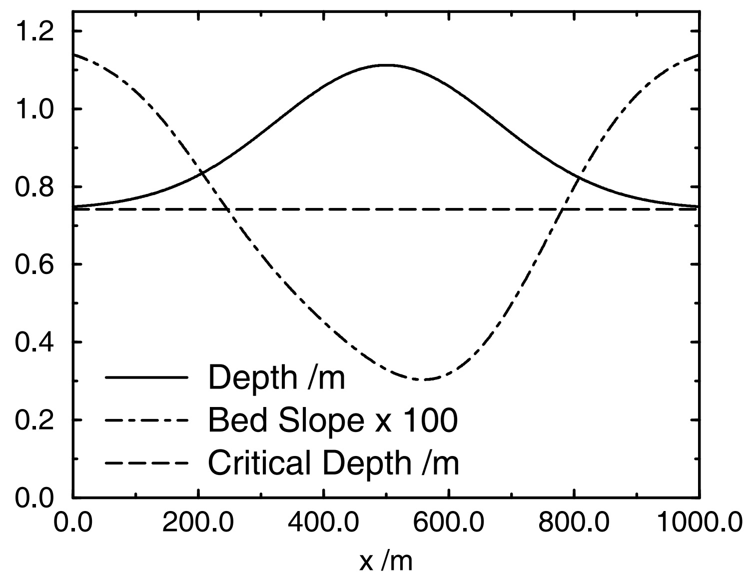

Figure 1: Depth and Bed Slope for Example 1

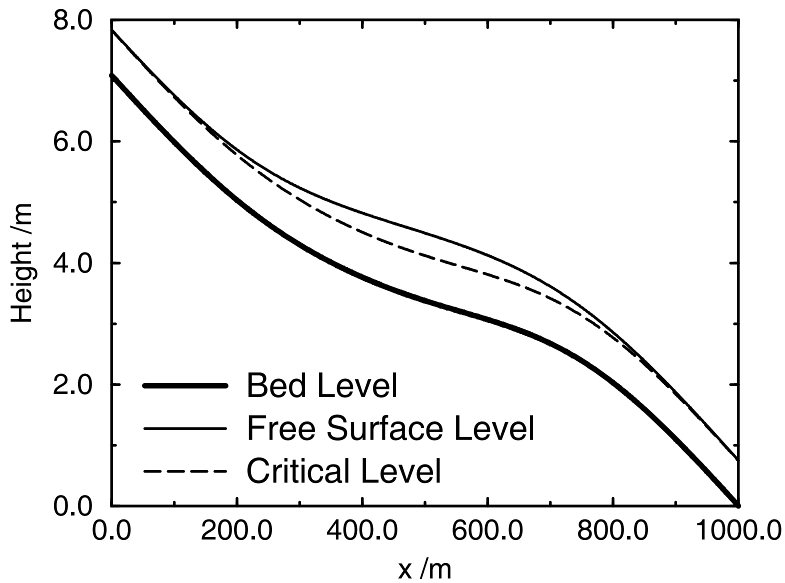

Figure 2: Surface Level and Bed Level for Example 1

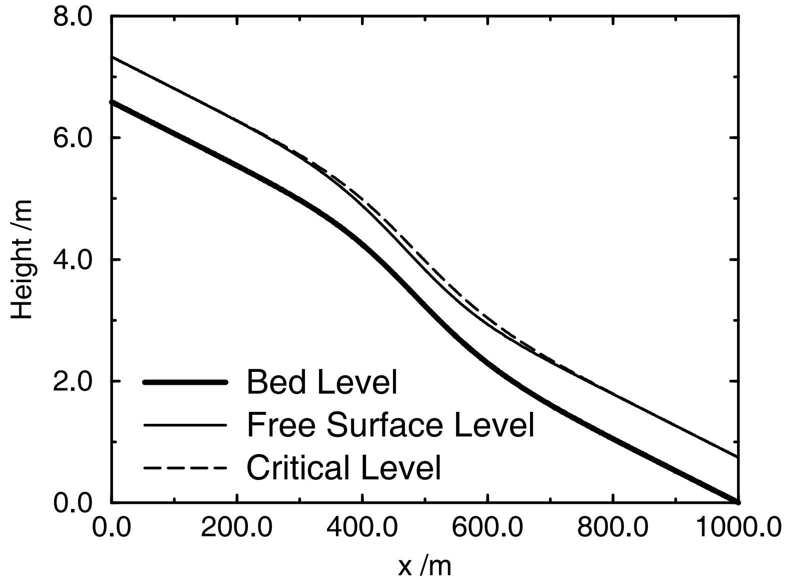

Figure 3: Surface Level and Bed Level for Example 2

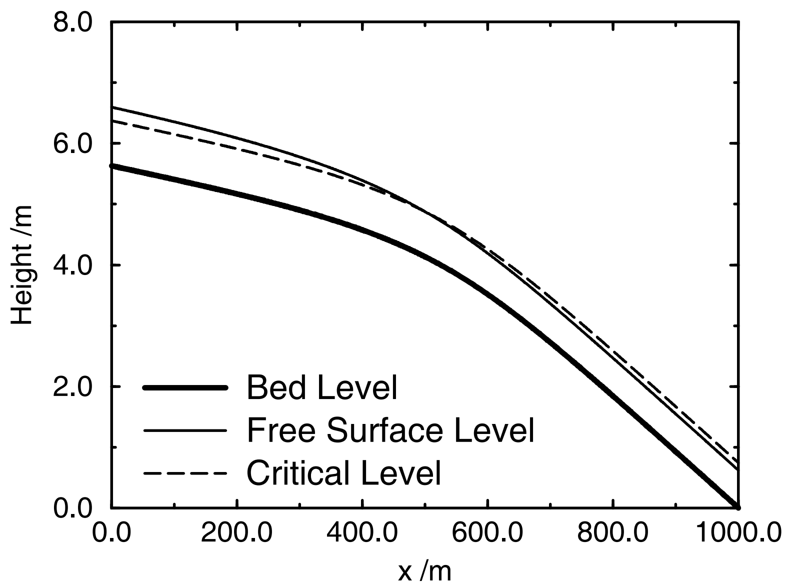

Figure 4: Surface Level and Bed Level for Example 3

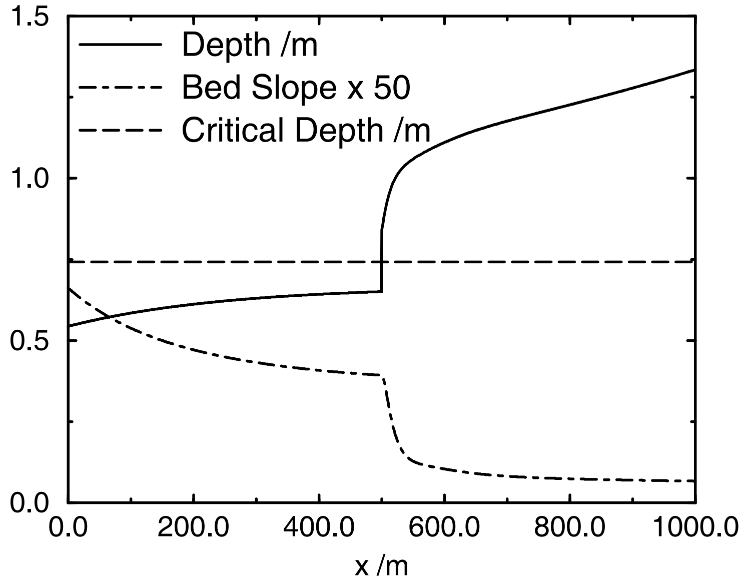

Figure 5: Depth and Bed Slope for Example 4

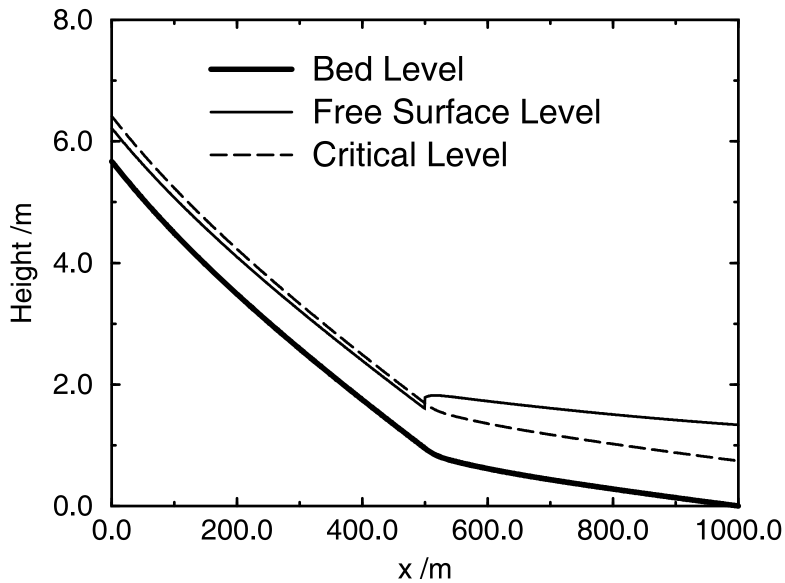

Figure 6: Surface Level and Bed Level for Example 4

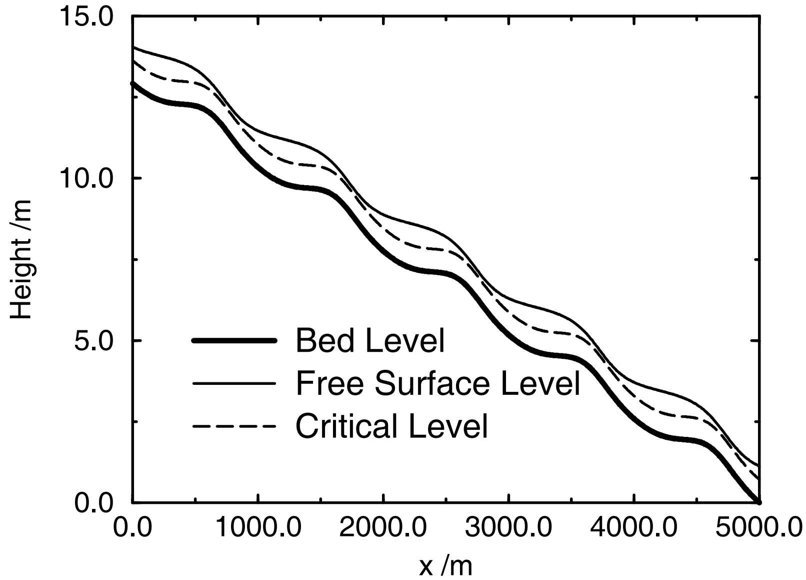

Figure 7: Surface Level and Bed Level for Example 5

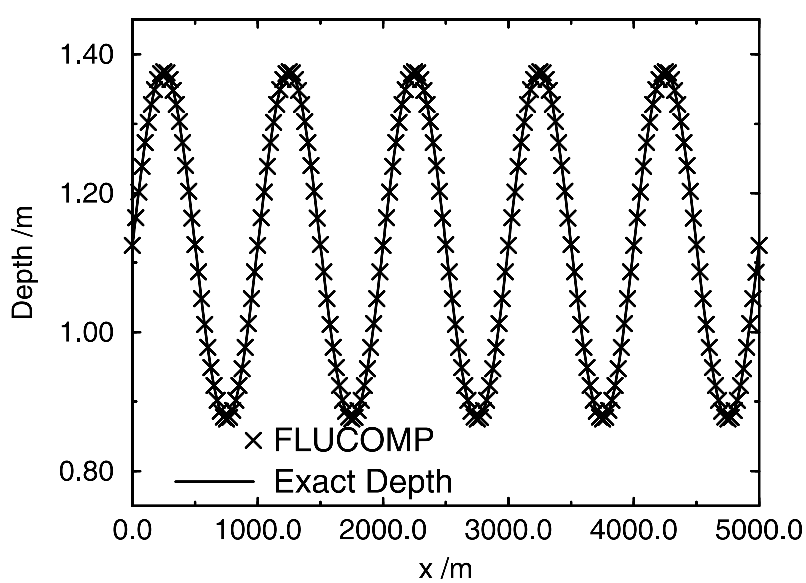

Figure 8: FLUCOMP Results against Exact Solution for Example 5

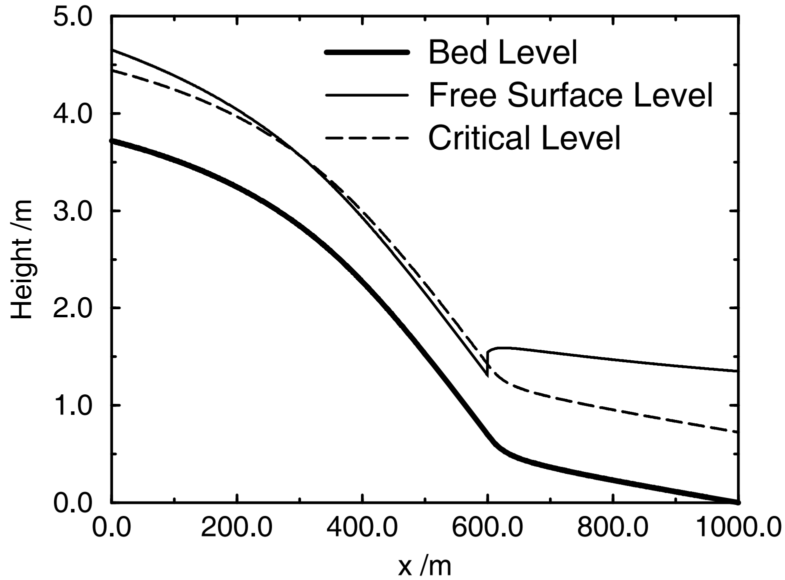

Figure 9: Surface Level and Bed Level for Example 6

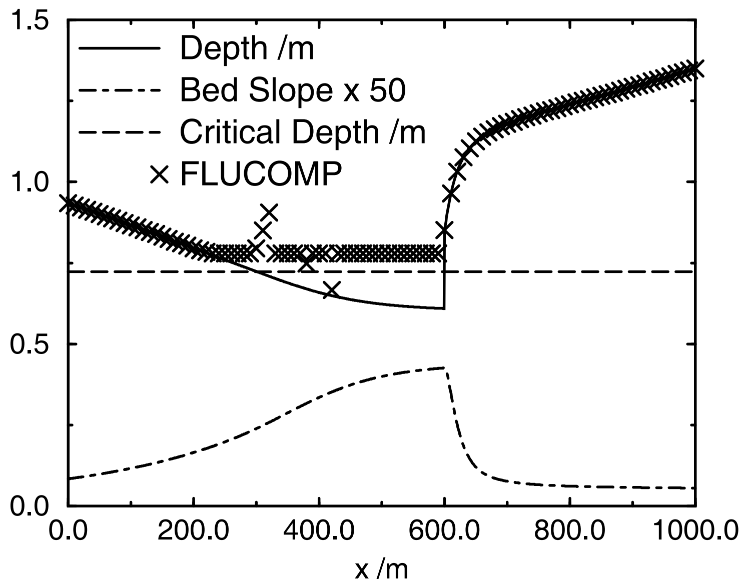

Figure 10: FLUCOMP Results against Exact Solution for Example 6