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Modelling of Forecast Errors in Geophysical Fluid Flows

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Abstract

A method is sought to decompose errors in numerical forecasts of the atmosphere into components that are uncorrelated. This can simplify the process of representing the probability density function (PDF) of forecast errors, which is needed for data assimilation (DA). A new method based on potential vorticity (PV), and a simpler method, of partitioning errors into balanced and unbalanced parts are investigated. The correlations between these parts in each method are compared. A toy model and an operational forecasting model are used to show that the PV-based variables are usually less correlated than those of the simpler approach.

1 FORECAST ERRORS IN DATA ASSIMILATION

Weather forecasts from numerical weather prediction (NWP) models have improved greatly since they were first produced routinely 50 years ago, partly due to improvements in the model's initial conditions, $\mathbf{x}_{\mathrm{ic}}$. Data assimilation estimates $\mathbf{x}_{\mathrm{ic}}$ by first making a short forecast, $\mathbf{x}_{\mathrm{f}}$ of the current weather, which will be in error, and then making an adjustment, $\mathbf{x}'$, to fit observations. The adjusted state, $\mathbf{x}_{\mathrm{ic}} = \mathbf{x}_{\mathrm{f}} + \mathbf{x}'$, has a smaller error than $\mathbf{x}_{\mathrm{f}}$. The adjustment is subject to an additional constraint prescribed by the forecast error PDF, which restricts the possible $\mathbf{x}'$. Good forecasts depend on accurate characterization of the PDF, and we report on a practical approach that may allow it to be represented compactly.

Let forecast error be defined as ε' in $\mathbf{x}_{\mathrm{f}} = \mathbf{x} + \varepsilon'$, where $\mathbf{x}$ is the 'true state'. The PDF of ε', $P_{\mathrm{f}}(\varepsilon')$, specifies the probability that the forecast has error ε'. A Gaussian with mean zero is the usual choice for $P_{\mathrm{f}}(\varepsilon')$, which is described by the forecast error covariance matrix, $\mathbf{B}_{\epsilon}$

$$P_{\mathrm{f}}(\varepsilon') \sim \exp -\frac{1}{2}\varepsilon'^{\mathrm{T}}\mathbf{B}_{\epsilon}^{-1}\varepsilon'. \tag{1}$$

The PDF, P_{f}, is combined with observational information to give $P_{\mathrm{comb}}(\mathbf{x})$, which is used in the DA problem. By (1) and Bayes' rule [1], $P_{\mathrm{comb}}(\mathbf{x})$ is

$$P_{\mathrm{comb}}(\mathbf{x}) \sim \exp -\frac{1}{2}(\mathbf{x}-\mathbf{x}_{\mathrm{f}})^{\mathrm{T}}\mathbf{B}_{\epsilon}^{-1}(\mathbf{x}-\mathbf{x}_{\mathrm{f}}) \times P_{\mathrm{ob}}(\mathbf{y}|\mathbf{x}). \tag{2}$$

In (2), $P_{\mathrm{ob}}(\mathbf{y}|\mathbf{x})$ is the PDF that the observations (in $\mathbf{y}$) are true given $\mathbf{x}$. In variational DA, $\mathbf{x}$ is found by maximizing $P_{\mathrm{comb}}(\mathbf{x})$ (actually by minimizing $-\ln P_{\mathrm{comb}}(\mathbf{x})$).

In Sec. 2 we describe the models that are used: (i) a toy model and (ii) a NWP model, in Sec. 3 we introduce PV and in Sec. 4 we propose how PV may lead to a compact form of the error covariance matrix. This is demonstrated in Sec. 5 and summarised in Sec. 6.

2 THE NUMERICAL MODELS AND THEIR BALANCE RELATIONS

2.1 One-dimensional shallow water equation model of the atmosphere

The shallow water equations (SWEs) for a rotating fluid describe air motion in a layer. We consider the SWEs for a 1-D atmosphere. Katz et al.[2] introduce the SWEs for u and v (fluid velocities in the x and y-directions), and h (depth of the layer), but we show equations for ψ (streamfunction), χ (velocity potential) and h

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}\right)+\left(\frac{\partial^2\psi}{\partial x^2}+f\right)\left(\frac{\partial^2\chi}{\partial x^2}\right)=0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial^2\chi}{\partial x^2}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial^2\chi}{\partial x^2}\right)+\left(\frac{\partial^2\chi}{\partial x^2}\right)^2-f\left(\frac{\partial^2\psi}{\partial x^2}\right)+g\frac{\partial^2(H+h)}{\partial x^2}=0,\quad \frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0. \tag{3}$$

These are the vorticity, divergence and continuity equations, where $H(x)$ is the orographic height, f is the (constant) Coriolis parameter and g is the gravitational acceleration. Generally, ψ and χ are related to u and v by the Helmholtz relations $(u,v,0)^{\mathrm{T}}=\mathbf{k}\times\nabla\psi+\nabla_h\chi$, where ∇ and ∇_h are the 3-D and horizontal differential operators respectively and $\mathbf{k}$ is the vertical unit vector. Orography is present to ensure non-trivial solutions. The grid has 500 points and so the state vector $\mathbf{x}=(\psi,\chi,h)^{\mathrm{T}}$ comprises 1500 elements and $\mathbf{B}_\epsilon$ has 2.25×10^6 matrix elements.

The fields ψ and h are related approximately by the linear balance equation (LBE) $f\partial\psi/\partial x = g\partial(H+h)/\partial x$, which holds well for mid-latitude flow. By linearising the fields about a state $\bar\psi$ and $\bar h$ (which satisfies the LBE) such that $\psi=\bar\psi+\psi'$ and $h=\bar h+h'$ (ψ' and h' are perturbations), the LBE can be written in perturbation form [2] $f\partial\psi'/\partial x=g\partial h'/\partial x$. This is an approximate diagnostic relationship between ψ' and h', or an exact relationship between the parts, ψ'_b and h'_b, that are 'in balance' with each other. We write this in two forms - (i) after integrating (and assuming integration constants are zero) and (ii) after differentiating

$$f\psi'_b-gh'_b=0,\quad f\left(\frac{\partial^2\psi'_b}{\partial x^2}\right)-g\left(\frac{\partial^2 h'_b}{\partial x^2}\right)=0. \tag{4}$$

The parts of ψ' and h' that do not satisfy (4) - and χ' - are said to be 'unbalanced'.

2.2 The Meteorological Office's Unified Model

The model used by the Met Office for NWP is called the Unified Model (UM) [3]. It is based on the primitive equations, which could be specified in terms of ψ, χ, p plus other variables. In the UM, f is variable and pressure, p, is analogous to h in the SWEs. The model domain used has $216\times162\times50$ points. Considering only ψ, χ and p, the state vector $\mathbf{x}=(\psi,\chi,p)^{\mathrm{T}}$ has 5.2×10^6 elements and $\mathbf{B}_\epsilon$ has 2.8×10^{13} elements. For the UM system a LBE is applicable for perturbations between the balanced wind and pressure

$$\nabla_h\cdot(f\bar\rho\nabla_h\psi'_b)-\nabla_h^2p'_b=0. \tag{5}$$

For the UM, $\mathbf{B}_\epsilon$ is too large to use, but the problem can be reduced with a transformation to variables whose errors are decoupled. Our strategy is to find variables whose errors are expected to be decoupled, and the properties of PV are used to do this for the SWEs and UM.

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3 THE POTENTIAL VORTICITY

For the SWEs and UM, a useful quantity called PV, q, can be defined [4]. For the SWEs the PV is $q_{\mathrm{SWE}} = h^{-1}(f + \partial^2\psi/\partial x^2)$. A linearised perturbation form, q'_{SWE}, is used here [2]

$$q'_{\mathrm{SWE}} = \frac{1}{\bar{h}}\left(\frac{\partial^2\psi'}{\partial x^2} - \bar{q}h'\right), \tag{6}$$

where an overbar is a reference state quantity. For the UM, the appropriate PV is called Ertel PV, q_{Ertel}, which is approximated as follows [5] ($\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\eta}$ are specified in [5])

$$q'_{\mathrm{Ertel}} = \bar{\alpha}\nabla_h^2\psi' + \bar{\beta}p' + \bar{\gamma}\frac{\partial p'}{\partial z} + \bar{\eta}\frac{\partial^2 p'}{\partial z^2}. \tag{7}$$

PV is useful because it can be inverted: given PV, suitable boundary conditions and a balance relation, the 'balanced' component of the flow - ie (ψ'_b, χ'_b, h'_b) in the case of the SWEs and (ψ'_b, χ'_b, p'_b) in the case of the UM - can be diagnosed ($\chi'_b = 0$ as χ' does not contribute to PV). It is not possible to make this diagnosis using a LBE alone since this gives h'_b (or p'_b) only if ψ'_b is known (or vice-versa), unless a special assumption is made. A common assumption is that ψ'_b is equal to the total perturbation ψ', which *is* known. We will call this the 'balanced vorticity approximation' (BVA), which avoids the need to use PV (Sec. 4.2). The BVA is good when the horizontal scale of the flow is much less than the Rossby radius [6], $L_R = \sqrt{g\bar{h}}/f$, which holds in the tropics where f is small.

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4 NEW VARIABLES BASED ON POTENTIAL VORTICITY

Diagnosis of the balanced flow is useful because this component is believed to evolve in a way that is largely decoupled from the unbalanced flow. Let $\mathbf{v}'_{\mathrm{PV}}$ be a representation of forecast error, ε', but in terms of the following balanced/unbalanced variables.

- For the balanced field we choose ψ'_b, which is described entirely in terms of PV.
- For the first unbalanced field we choose χ' which has no associated PV.
- For the second unbalanced field we choose unbalanced height, h'_u, for the SWEs and unbalanced pressure, p'_u, for the UM, which too have no associated PV.

Illustrating for the SWE, $\mathbf{v}'_{\mathrm{PV}} = (\psi'_b, \chi', h'_u)^T$, which has error covariance matrix

$$\mathbf{B}_{\mathrm{PV}} = \begin{pmatrix} \mathbf{B}_{\psi'_b\psi'_b} & \mathbf{B}_{\psi'_b\chi'} & \mathbf{B}_{\psi'_b h'_u} \\ \mathbf{B}^{\mathrm{T}}_{\psi'_b\chi'} & \mathbf{B}_{\chi'\chi'} & \mathbf{B}_{\chi' h'_u} \\ \mathbf{B}^{\mathrm{T}}_{\psi'_b h'_u} & \mathbf{B}^{\mathrm{T}}_{\chi' h'_u} & \mathbf{B}_{h'_u h'_u} \end{pmatrix}. \tag{8}$$

If variables in $\mathbf{v}'_{\mathrm{PV}}$ are uncorrelated then this matrix will simplify: $\mathbf{B}_{\psi'_b\chi'} = 0$, $\mathbf{B}_{\psi'_b h'_u} = 0$, and $\mathbf{B}_{\chi' h'_u} = 0$. Then $\mathbf{v}'_{\mathrm{PV}}$ can be used in a transformed form of (1) with $\mathbf{B}_\epsilon \to \mathbf{B}_{\mathrm{PV}}$ [7]. The hypothesis that $\mathbf{B}_{\mathrm{PV}}$ is block diagonal is expected to hold in a linear system, but neither the SWE or UM systems are linear, and the UM includes parametrizations for radiative, moist and sub-grid-scale processes which lead to correlations. The hypothesis that $\mathbf{B}_{\mathrm{PV}}$ is block diagonal is tested for each system and compared with the simpler choice of variables under the BVA.

4.1 Transformations using potential vorticity

The PV and the LBE are used to compute $\mathbf{v}'_{\mathrm{PV}}$. The method is shown for the SWEs, but the principle extends to the UM. A forecast error from (3) $(\psi', \chi', h')^{\mathrm{T}}$ is to be determined in terms of $(\psi'_b, \chi', h'_u)^{\mathrm{T}}$. The first variable, ψ'_b, is described by PV. This means that (6) can be written as

$$q'_{\mathrm{SWE}} = \frac{1}{\bar{h}}\left(\frac{\partial^2 \psi'}{\partial x^2} - \bar{q}h'\right) = \frac{1}{\bar{h}}\left(\frac{\partial^2 \psi'_b}{\partial x^2} - \bar{q}h'_b\right) = \left(\frac{\partial^2 \psi'_b}{\partial x^2} - \frac{\bar{q}f}{g}\psi'_b\right), \tag{9}$$

using the LBE (4). The solution, ψ'_b, is unique as long as $\bar{q}f$ is positive, which is expected to hold. This equation can be solved by Fourier transforms, but the analogous 3-D equation for the UM is more difficult to solve (we use the Generalised Conjugate Residual solver).

The second variable, χ', is already a prognostic variable in (3) and so needs no processing. For the third variable, h'_u, substitute in (4) ψ' and h'. This will not give zero since only the balanced parts will satisfy (4). The residual is called the 'linear imbalance' or 'anti-PV', ζ'_a

$$\zeta'_a = f\frac{\partial^2 \psi'}{\partial x^2} - g\frac{\partial^2 h'}{\partial x^2} = f\frac{\partial^2 \psi'_u}{\partial x^2} - g\frac{\partial^2 h'_u}{\partial x^2}. \tag{10}$$

The full perturbations have balanced and unbalanced parts, $\psi' = \psi'_b + \psi'_u$ and $h' = h'_b + h'_u$, and since the balanced parts satisfy (4), ζ'_a is equivalently expressed with the unbalanced parts only, as has been done in (10). The unbalanced fields have zero PV and so from (6), $\partial^2\psi'_u/\partial x^2 = \bar{q}h'_u$. The term $\partial^2\psi'_u/\partial x^2$ can be eliminated from (10) giving

$$f\frac{\partial^2 \psi'}{\partial x^2} - g\frac{\partial^2 h'}{\partial x^2} = f\bar{q}h'_u - g\frac{\partial^2 h'_u}{\partial x^2}. \tag{11}$$

The solution, h'_u, is unique as long as $\bar{q}f$ is positive. This is similar to (9) and is solved in a similar way. An analogous 3-D equation for p'_u exists for the UM system.

4.2 Transformations using the balanced vorticity approximation

Unlike in Sec. 4.1, the streamfunction is taken to be completely balanced under the BVA. Then the following set of fields are used to describe forecast errors.

- The 'balanced' variable is ψ' - this is already a forecast perturbation field.
- The first unbalanced variable is χ' - this is also already a forecast perturbation field.
- The second unbalanced variable is called h'_r. It is the unbalanced height under the BVA, and is found from the residual of the LBE (4), $h'_r = h' - h'_b = h' - (f/g)\psi'$.

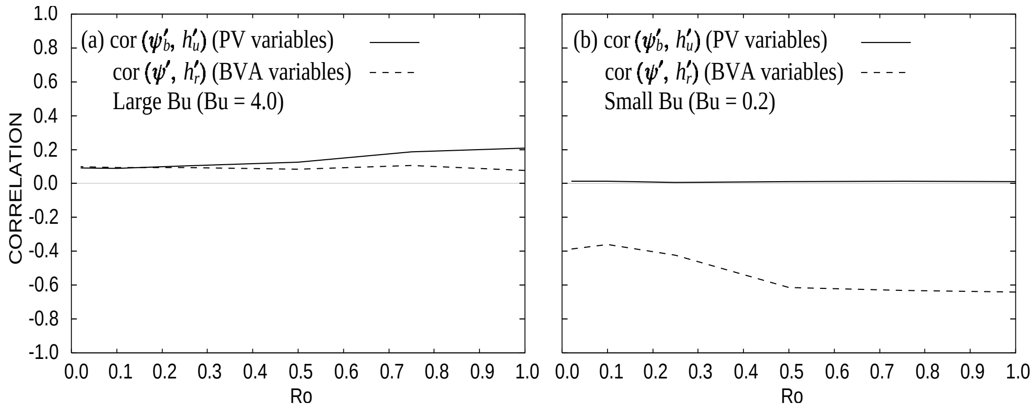

Figure 1: Correlations between balanced wind and unbalanced height errors for the SWEs.

For the BVA, $\mathbf{v}'_{\mathrm{BVA}} = (\psi', \chi', h'_r)^{\mathrm{T}}$, (with $h'_r \to p'_r$ for the UM system). The assumption that ψ', rather than ψ'_b, describes the 'balance' is often unrealistic. In Sec. 5 the correlations between elements of $\mathbf{v}'_{\mathrm{BVA}}$ are compared to those between $\mathbf{v}'_{\mathrm{PV}}$. This is important for the UM because the BVA is used to represent forecast errors in the Met Office's operational DA system [8].

5 NUMERICAL EXPERIMENTS

A sample of errors, $\{\varepsilon'\}$, are transformed into $\{\mathbf{v}'_{\mathrm{PV}}\}$ and $\{\mathbf{v}'_{\mathrm{BVA}}\}$ for the SWE and UM systems and used to compute correlations between the variables.

5.1 Correlations for the one-dimensional shallow water equation model

It is possible to integrate the SWEs under different flow regimes, described here by two parameters. The Burger number, Bu, describes the ratio L_R/L where L is the horizontal lengthscale. At large Bu (small horizontal scales) the BVA is expected to be good and so we expect similar results from $\mathbf{v}'_{\mathrm{PV}}$ and $\mathbf{v}'_{\mathrm{BVA}}$. The Rossby number is Ro $= U/fL$, where U is the characteristic wind. A small Ro indicates that mass and wind variables are related well by (4). As Ro increases, the LBE becomes more approximate and nonlinear terms become more important.

Figure 1 shows correlations cor(ψ'_b, h'_u) for $\mathbf{v}'_{\mathrm{PV}}$ and cor(ψ', h'_r) for $\mathbf{v}'_{\mathrm{BVA}}$ as a function of Ro for high and low Bu. For high Bu (Fig. 1a), where the BVA is good, the correlations are small for $\mathbf{v}'_{\mathrm{PV}}$ (solid line) and $\mathbf{v}'_{\mathrm{BVA}}$ (dashed line). The correlations for PV variables are higher than for BVA variables at the larger Ro, probably due to the increasing nonlinearity (the PV transformations rely on linearity). For low Bu (Fig. 1b), where the BVA is not good, the PV variables show consistently small correlations at all Ro shown, unlike the BVA variables. Correlations involving χ' (not shown) are small for all cases. These results confirm that, overall, the PV variables are better than the BVA variables at representing weakly correlated errors.

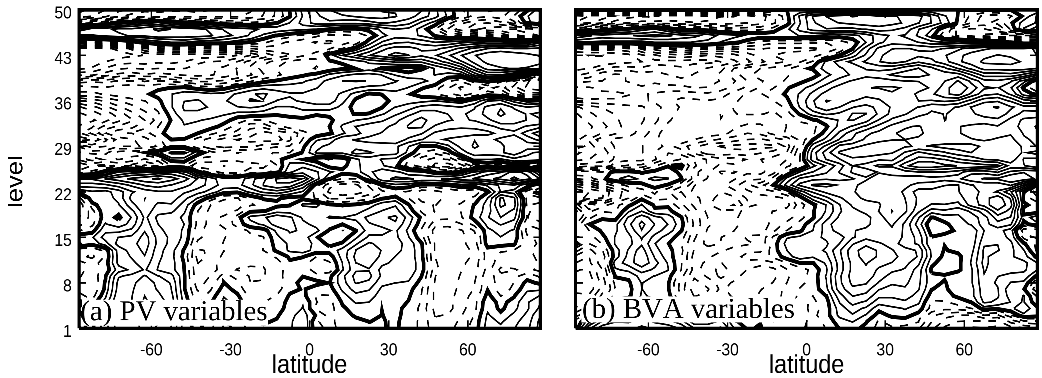

Figure 2: Correlations between balanced wind and unbalanced pressure errors for the UM. Negative values are dotted and the zero line is thick. Contours are spaced every 0.1.

5.2 Correlations for the Unified Model

In the UM it is not easy to control Bu or Ro. In Fig. 2 are latitude/height sections of $\mathrm{cor}(\psi'_b, p'_u)$ in $\mathbf{v}'_{\mathrm{PV}}$ and $\mathrm{cor}(\psi', p'_r)$ in $\mathbf{v}'_{\mathrm{BVA}}$. Correlations for PV variables (Fig. 2a) are not small. It is unclear whether this is because of the nature of the UM, or whether the solutions of the UM's equivalent of (9) and (11) have not been achieved to sufficient accuracy (the 3-D solver left residuals). Correlations for BVA variables (Fig. 2b) are even larger, showing that there is an advantage to using PV variables. On average, $\mathrm{cor}(\psi'_b, p'_u)$ are smaller than $\mathrm{cor}(\psi', p'_r)$ by 0.1.

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6 SUMMARY

Potential vorticity, used with a balance relation, can help define a new set of variables that partition forecast errors into balanced and unbalanced parts. The hypothesis that these are uncorrelated is tested in a simple 1-D and a full 3-D model of the atmosphere. The correlations between PV variables are compared to those between quasi-balanced and unbalanced variables which inappropriately label the rotational wind as balanced under all flow regimes (the BVA). In the SWEs the PV variables, unlike the BVA variables, have small correlations over all flow regimes tested. This shows that the PV variables adapt appropriately to the flow regime. In the UM correlations are relatively large for the PV and BVA sets, but the PV set shows generally smaller correlations. In solving UM analogues of (9) and (11) we found large residuals using the 3-D solver, leaving scope to improve the UM results with an alternative solver.

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