M A T H E M A T I C S D E P A R T M E N T

Numerical Methods for the Solution of Systems of Uncertain Steady-State Differential Equations

M.P. Dainton

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U N I V E R S I T Y O F R E A D I N G

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Abstract

In this report, we develop a numerical method to estimate the mean value of a solution to a steady-state partial differential equation containing an unknown parameter. It is assumed that the statistical properties of the uncertain parameter are given explicitly. Various results are obtained for a general admissible realisation, by consideration of simple perturbations in the parameter, k, about the assumed mean value. It is shown that when the mean value is taken over all possible realisations of k, useful resultant terms for the behaviour of the mean value of the numerical solution can be obtained. These terms involve moments of the (discretised) k multivariate distribution function. The solutions can be truncated at second order, to give terms involving the discretised permeability autocorrelation function of k. Estimates of the errors obtained due to this truncation process are made. Results are then evaluated for various different forms of autocorrelation function, with particular interest in the effect of varying the variance, and or correlation length.

Contents

Chapter 1

Introduction to General Problem

This report is concerned with the solution of the model differential equations for oil recovery from an underground reservoir.

Generally, this involves the solution of single- and/or multi-phase flow equations for fluids through a porous medium. An important practical problem associated with this is that a precise knowledge of the properties of the oil-bearing rock is not available, since the oil field may be a considerable distance underground. The only information about the particular makeup of this rock is obtained by experiments performed on samples of the rock brought up from the recovery wells; but the most efficient method of oil extraction involves drilling as few boreholes as possible, ideally just two, an injection well, and a production well. This means that the knowledge of the properties of the intervening rock is extremely limited.

An obvious approach to this problem would be to interpolate the values for the permeabilities between the two well points, using some sort of geological data about the structure of stratigraphic rocks. However this is done, it will mean that there is an infinite number of possibilities, i.e. realisations, that the rock

structure can take, which leads to a probabilistic approach to the problem.

There are two statistical parameters needed when describing an uncertain permeability field in this context:

1. the mean value for the field, $k(\mathbf{r})$

   - in the most general case, this will be a spatially variable function, $\langle k(r)\rangle = k_0(r)$; but in most of this research it is assumed to be homogeneous, i.e. $\langle k(r)\rangle = k_0$.

2. the permeability autocorrelation function (PAF)

   -this relates the statistical properties at different points in the medium. It is defined thus:

$$\rho(\mathbf{r_1}, \mathbf{r_2}) = \langle((k(\mathbf{r_1}) - \langle k(\mathbf{r_1})\rangle)(k(\mathbf{r_2}) - \langle k(\mathbf{r_2})\rangle))\rangle \tag{1.1}$$

   or

$$\rho(\mathbf{r_1}, \mathbf{r_2}) = \langle(k(\mathbf{r_1}) - k_0(\mathbf{r_1}))(k(\mathbf{r_2}) - k_0(\mathbf{r_2}))\rangle \tag{1.2}$$

   Usually, this can be assumed to be a function of separation only

$$\rho(\mathbf{r_1}, \mathbf{r_2}) = \rho(\mathbf{r_1} - \mathbf{r_2}) \tag{1.3}$$

   A further simplification used is that the PAF is (statistically) homogeneous:

$$\rho(\mathbf{r_1} - \mathbf{r_2}) = \rho(\mid \mathbf{r_1} - \mathbf{r_2} \mid) \tag{1.4}$$

   It is important to note that the PAF for the point where $\mathbf{r_1} = \mathbf{r_2} = \mathbf{r}$ is given by:

$$\rho(\mathbf{r}, \mathbf{r}) = \langle(k(\mathbf{r}) - \langle k(\mathbf{r})\rangle)^2\rangle, \tag{1.5}$$

   which is, by definition, equal to the variance, $\sigma^2(\mathbf{r})$.

Darcy's law for fluid flow in a porous medium states that fluid velocity is proportional to the pressure gradient across the fluid:

$$\mathbf{q} = -\frac{k}{\eta} \cdot \nabla p, \tag{1.6}$$

where $\mathbf{q}$ = fluid flow, k = permeability tensor, η = viscosity of fluid, and p = pressure. The work done for this report involves steady-state, single-phase flow which is modelled by the differential equation:

$$\nabla \cdot \mathbf{q} = 0. \tag{1.7}$$

Combining this with equation (1.6), and assuming the viscosity is constant, gives the model equation:

$$\nabla \cdot k \nabla p = 0. \tag{1.8}$$

It is intended that any numerical techniques found to solve this problem, for the mean value and/or standard deviation of pressure, can then be used on more complicated problems, such as non steady-state, or multi-phase flow.

Chapter 2

Properties of a General Realisation

It is illustrative to consider one general realisation as an example; and by investigation of the properties associated with this case, a great deal of information can be obtained about the statistical properties of the complete problem, taken over all realisations. The work is done here in two dimensions, but a generalisation to three dimensions is fairly straightforward.

Consider a discretisation, with a simple five-point difference scheme on a uniform stencil of mesh size h, of equation(1.8) The numerical approximation to the equation is

$$p_{i,j+1}k_{i,j+\frac{1}{2}} + p_{i+1,j}k_{i+\frac{1}{2},j} - (k_{i+\frac{1}{2},j} + k_{i-\frac{1}{2},j} + k_{i,j+\frac{1}{2}} + k_{i,j-\frac{1}{2}})p_{i,j}$$

$$+ p_{i-1,j}k_{i-\frac{1}{2},j} + p_{i,j-1}k_{i,j-\frac{1}{2}} = 0 \qquad (2.1)$$

Note that the discretised values for the permeability field are those evaluated at the half way points, between the nodes.

In the complete statistical problem, the permeability $k(x, y)$ is a random spatial function with a known mean value, which may or may not be assumed to

be homogeneous (i.e. spatially constant), and a spatial autocorrelation function which relates the statistical properties of the function at different points. When the permeability is discretised, as in the case above, the function is represented as a set of statistical variables, each having its own mean value, which are all equal, if the assumption of a homogeneous mean value function is made. Each statistical variable is also correlated to the others, reflecting a discretisation of the original spatial, autocorrelation function. This means that points close together have a high correlation in their discretised permeability values, and points at large distances apart have correlations approaching zero - i.e. the correlation value is related to the separation of the two half grid-points. In the case of an isotropic autocorrelation function the correlation is a function of distance only; and this is the situation investigated here.

Consider a perturbation representation of the permeability function. This can be written

$$k(x, y) = k_0(x, y) + k_1(x, y), \tag{2.2}$$

where the perturbation $k_1(x, y)$ is taken about the mean value function, $k_0(x, y)$. In the discretised form this can be directly translated to

$$k_{ij} = k^0_{ij} + k^1_{ij}. \tag{2.3}$$

If the simplification that the mean value of the permeability is homogeneous is made then equations (2.2) and (2.3), become respectively

$$k(x, y) = k_0(1 + d(x, y)), \tag{2.4}$$

and

$$k_{ij} = k_0(1 + d_{i,j}), \tag{2.5}$$

where $d(x, y)$ is a random function, with mean value zero, that is assumed small.

Note that the permeability autocorrelation function, as defined in equations (1.1) and (1.2), where the permeability is written as a perturbation and the variance is assumed to be homogeneous, can be written:

$$\begin{aligned}\rho(\mathbf{r},\mathbf{r}') &= \frac{\langle (k(\mathbf{r})-\langle k(\mathbf{r})\rangle)(k(\mathbf{r}')-\langle k(\mathbf{r}')\rangle)\rangle}{\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}} \\ &= \frac{\langle (k_0(1+d(\mathbf{r}))-k_0)(k_0(1+d(\mathbf{r}'))-k_0)\rangle}{\sigma^2} \\ &= \frac{k_0^2}{\sigma^2}\langle d(\mathbf{r})d(\mathbf{r}')\rangle, \end{aligned} \tag{2.6}$$

and for the discretised version, in 2-D

$$\rho_{i,j\ i',j'} = \frac{k_0^2}{\sigma^2}\langle d_{i,j}\ d_{i',j'}\rangle. \tag{2.7}$$

The homogeneous mean assumption is made throughout, for simplicity. However, it is shown later that various conclusions about the generalised case (i.e. with spatially varying mean) can be obtained from consideration of the simplified case.

Substituting equation (2.5) into (2.1), we get

$$\begin{aligned}&-p_{i,j+1}-p_{i+1,j}+4p_{i,j}-p_{i-1,j}-p_{i,j-1}-d_{i+\frac{1}{2},j}p_{i+1,j}-d_{i,j+\frac{1}{2}}p_{i,j+1} \\ &+(d_{i+\frac{1}{2},j}+d_{i-\frac{1}{2},j}+d_{i,j+\frac{1}{2}}+d_{i,j-\frac{1}{2}})p_{i,j}-d_{i-\frac{1}{2},j}p_{i-1,j}-d_{i,j-\frac{1}{2}}p_{i,j-1}=0\ \ . \end{aligned} \tag{2.8}$$

This leads to a system of equations, which, when combined with the relevant boundary conditions for the problem, can be written as the matrix equation

$$A\mathbf{p}+D\mathbf{p}=\mathbf{b}, \tag{2.9}$$

where $\mathbf{b}$ contains the boundary conditions, and is quite sparse. Due to the perturbation formulation, $\mathbf{b}$ can always be split into two parts:

$$\mathbf{b}=\mathbf{b_0}+\mathbf{b_d}, \tag{2.10}$$

where $\mathbf{b_0}$ corresponds to the right hand side vector of the deterministic problem, and $\mathbf{b_d}$ contains various linear combinations of the uncertain statistical variables, $\{d_{ij}\}$.

For the purposes of the following examples, all on a square region, the Dirichlet case has boundary conditions a_S, a_E, a_N, a_W on respective south, east, north, and west boundaries; and the mixed case has a_N, a_S on the north and south boundaries, with normal gradients v_E, and v_W on the east and west ones.

In a general case, the vector $\mathbf{b}$ can be written:

$$\mathbf{b} = \begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \vdots \\ \mathbf{b_i} \\ \vdots \\ \mathbf{b_n} \end{pmatrix} = \begin{pmatrix} \mathbf{b_{1_o}} \\ \mathbf{b_{2_o}} \\ \vdots \\ \mathbf{b_{i_o}} \\ \vdots \\ \mathbf{b_{n_o}} \end{pmatrix} + \begin{pmatrix} \mathbf{b_{1_d}} \\ \mathbf{b_{2_d}} \\ \vdots \\ \mathbf{b_{i_d}} \\ \vdots \\ \mathbf{b_{n_d}} \end{pmatrix} . \tag{2.11}$$

For simple Dirichlet conditions, the component vectors will be:

$$\mathbf{b_1} = \begin{pmatrix} a_S + a_W \\ a_S \\ a_S \\ \vdots \\ a_S \\ a_S + a_E \end{pmatrix} + \begin{pmatrix} d_{\frac{1}{2},1} a_S + d_{1,\frac{1}{2}} a_W \\ d_{\frac{1}{2},2} a_S \\ d_{\frac{1}{2},3} a_S \\ \vdots \\ d_{\frac{1}{2},n-1} a_S \\ d_{\frac{1}{2},n} a_S + d_{1,n+\frac{1}{2}} a_E \end{pmatrix} , \tag{2.12}$$

$$\mathbf{b_n} = \begin{pmatrix} a_W + a_N \\ a_N \\ a_N \\ \vdots \\ a_N \\ a_N + a_E \end{pmatrix} + \begin{pmatrix} d_{n,\frac{1}{2}} a_W + d_{n+\frac{1}{2},1} a_N \\ d_{n+\frac{1}{2},2} a_N \\ d_{n+\frac{1}{2},3} a_N \\ \vdots \\ d_{n+\frac{1}{2},n-1} a_N \\ d_{n+\frac{1}{2},n} a_N + d_{n,n+\frac{1}{2}} a_E \end{pmatrix} , \tag{2.13}$$

and

$$\mathbf{b_i} = \begin{pmatrix} a_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ a_E \end{pmatrix} + \begin{pmatrix} d_{2,\frac{1}{2}} a_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ d_{2,n+\frac{1}{2}} a_E \end{pmatrix}, \quad for \;\; i = 2, 3, ...(n-1). \tag{2.14}$$

For simple von Neumann/mixed boundary conditions, these would be:

$$\mathbf{b_1} = \begin{pmatrix} a_S + 2hv_W \\ a_S \\ a_S \\ \vdots \\ a_S \\ a_S - 2hv_E \end{pmatrix} + \begin{pmatrix} d_{\frac{1}{2},1} a_S + 2d_{1,1} hv_W \\ d_{\frac{1}{2},2} a_S \\ d_{\frac{1}{2},3} a_S \\ \vdots \\ d_{\frac{1}{2},n+1} a_S \\ d_{\frac{1}{2},n+2} a_S - 2d_{1,n+2} hv_E \end{pmatrix}, \tag{2.15}$$

$$\mathbf{b_n} = \begin{pmatrix} a_N + 2hv_W \\ a_N \\ a_N \\ \vdots \\ a_N \\ a_N - 2d_{n,n+2} hv_E \end{pmatrix} + \begin{pmatrix} d_{n+\frac{1}{2},1} a_N + 2d_{n,1} hv_W \\ d_{n+\frac{1}{2},2} a_N \\ d_{n+\frac{1}{2},3} a_N \\ \vdots \\ d_{n+\frac{1}{2},n+1} a_N \\ d_{n+\frac{1}{2},n+2} a_N - 2d_{n,n+2} hv_E \end{pmatrix}, \tag{2.16}$$

and

$$\mathbf{b_i} = \begin{pmatrix} 2hv_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2hv_E \end{pmatrix} + \begin{pmatrix} 2d_{i,1} hv_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2d_{i,n+2} hv_E \end{pmatrix}, \quad for \;\; i = 2, 3, ...(n-1). \tag{2.17}$$

The matrix A is the usual block tridiagonal matrix for the simple 5-point difference scheme, and can be assumed to be irreducibly diagonally dominant,

with the form,

$$A = \begin{bmatrix} A_1 & -I & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ -I & A_2 & -I & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & -I & A_3 & -I & 0 & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & 0 & -I & A_i & -I & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & 0 & -I & A_{n-2} & -I & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & -I & A_{n-1} & -I \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & -I & A_n \end{bmatrix}. \tag{2.18}$$

For Dirichlet boundary conditions, the component diagonal matrices are $n-by-n$, symmetric, and tridiagonal with the form,

$$A_i = \begin{bmatrix} 4 & -1 & 0 & \dots & \dots \\ -1 & 4 & -1 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -1 & 4 & -1 \\ \dots & \dots & 0 & -1 & 4 \end{bmatrix} \quad for \quad i = 1, 2, ...n, \tag{2.19}$$

and for mixed boundary conditions, they are $(n+2)$ by $(n+2)$, and tridiagonal:

$$A_i = \begin{bmatrix} 4 & -2 & 0 & \dots & \dots \\ -1 & 4 & -1 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -1 & 4 & -1 \\ \dots & \dots & 0 & -2 & 4 \end{bmatrix}, \quad for \quad i = 1, 2, ...n. \tag{2.20}$$

The matrix D has identical structure to A:

$$D = \begin{bmatrix} D_{1,1} & D_{1,2} & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ D_{2,1} & D_{2,2} & D_{2,3} & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & D_{3,2} & D_{3,3} & D_{3,4} & 0 & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & 0 & D_{i,i-1} & D_{i,i} & D_{i,i+1} & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & 0 & D_{n-2,n-3} & D_{n-2,n-2} & D_{n-2,n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & D_{n-1,n-2} & D_{n-1,n-1} & D_{n-1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & D_{n,n-1} & D_{n,n} \end{bmatrix} , \tag{2.21}$$

and the component matrices for the Dirichlet problem are $n - by - n$:

$$D_{i,i} = \begin{bmatrix} \Delta_{i,1} & -d_{i,1\frac{1}{2}} & 0 & \dots & \dots \\ -d_{i,1\frac{1}{2}} & \Delta_{i,2} & -d_{i,2\frac{1}{2}} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -d_{i,n-\frac{3}{2}} & \Delta_{i,n-1} & -d_{i,n-\frac{1}{2}} \\ \dots & \dots & 0 & -d_{i,n-\frac{1}{2}} & \Delta_{i,n} \end{bmatrix}, \quad for \quad i = 1, 2, ...n, \tag{2.22}$$

where

$$\Delta_{i,j} = d_{i-\frac{1}{2},j} + d_{i,j+\frac{1}{2}} + d_{i+\frac{1}{2},j} + d_{i,j-\frac{1}{2}} \quad ,$$

and

$$D_{i,i-1} = D_{i-1,i} = \begin{bmatrix} -d_{i-\frac{1}{2},1} & 0 & \dots & 0 & \dots & \vdots \\ 0 & -d_{i-\frac{1}{2},2} & \dots & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -d_{i-\frac{1}{2},j} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & \dots & -d_{i-\frac{1}{2},n} \end{bmatrix}, \quad for \quad i = 2, 3, ...n. \tag{2.23}$$

For mixed boundary conditions, they are $(n+2)$ by $(n+2)$, and tridiagonal:

$$D_{i,i} = \begin{bmatrix} \Delta_{i,1} & -2d_{i,1\frac{1}{2}} & 0 & \dots & \dots \\ -d_{i,1\frac{1}{2}} & \Delta_{i,2} & -d_{i,2\frac{1}{2}} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -d_{i,n+\frac{1}{2}} & \Delta_{i,n+1} & -d_{i,n+\frac{3}{2}} \\ \dots & \dots & 0 & -2d_{i,n+\frac{3}{2}} & \Delta_{i,n+2} \end{bmatrix}, \quad for \quad i = 1, 2, ...n, \qquad (2.24)$$

where

$$\begin{aligned} \Delta_{i,1} &= d_{i-\frac{1}{2},1} + 2d_{i,1\frac{1}{2}} + d_{i+\frac{1}{2},1}, \\ \Delta_{i,j} &= d_{i-\frac{1}{2},j} + d_{i,j+\frac{1}{2}} + d_{i+\frac{1}{2},j} + d_{i,j-\frac{1}{2}}, \qquad for \quad j = 2, 3, ...(n+1), \\ \Delta_{i,n+2} &= d_{i-\frac{1}{2},n+2} + 2d_{i,n+\frac{3}{2}} + d_{i+\frac{1}{2},1}, \end{aligned}$$

and

$$D_{i,i-1} = D_{i-1,i} = \begin{bmatrix} -d_{i-\frac{1}{2},1} & 0 & \dots & 0 & \dots & \vdots \\ 0 & -d_{i-\frac{1}{2},2} & \dots & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -d_{i-\frac{1}{2},j} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & \dots & -d_{i-\frac{1}{2},n+2} \end{bmatrix}, \quad for \quad i = 2, 3, ...(n+2). \qquad (2.25)$$

Consider one specific realisation for the set of perturbations where the matrix D is denoted by D_n, vector $\mathbf{b}$ by $\mathbf{b_n}$, and $\mathbf{b_d}$ by $\mathbf{b_{d_n}}$. The corresponding expression to equation (2.9) is then

$$A\mathbf{p_n} + D_n\mathbf{p_n} = \mathbf{b_0} + \mathbf{b_{d_n}}. \qquad (2.26)$$

This can be re-arranged to give

$$\mathbf{p_n} = A^{-1}(\mathbf{b_0} + \mathbf{b_{d_n}}) - A^{-1}D_n\mathbf{p_n} \qquad (2.27)$$

This can then be formed into a perturbation series, and, due to fact that D_n and $\mathbf{b_{d_n}}$ are both linear in the $\{d_{ij}\}$ terms, the second order approximation to

the perturbation series is

$$\mathbf{p_{s_n}} = A^{-1}(\mathbf{b_0} + \mathbf{b_{d_n}}) - A^{-1}D_nA^{-1}(\mathbf{b_0} + \mathbf{b_{d_n}}) + A^{-1}D_nA^{-1}D_nA^{-1}\mathbf{b_0}. \quad (2.28)$$

The equivalent full perturbation series is

$$\mathbf{p_n} = A^{-1}\sum_{i=0}^{\infty}(-D_nA^{-1})^i\mathbf{b_n}. \quad (2.29)$$

This series is convergent if, and only if

$$Sr(D_nA^{-1}) < 1, \quad (2.30)$$

where $Sr(M)$ denotes the spectral radius of the matrix M [4].

Alternatively, convergence with respect to a general norm, is satisfied for realisation n, if

$$\|(D_nA^{-1})^i\| \longrightarrow 0 \qquad as\ i \to \infty \ ,$$

So a sufficient condition for convergence in realisation n is

$$\|D_nA^{-1}\| < 1,$$

or,

$$\|D_n\| < \|A^{-1}\|^{-1}. \quad (2.31)$$

This implies a restriction on the size of all possible admissible realisations for the perturbations - they must be at least finite, and bounded.

Now consider the effect of truncating the series (2.29) to second order

$$\mathbf{p_{s_n}} = A^{-1}\sum_{j=0}^{2}(-D_nA^{-1})^j\mathbf{b_n}. \quad (2.32)$$

This introduces an error between this expression and (2.29) given by,

$$\begin{aligned}\mathbf{e_n} &= A^{-1}\sum_{j=3}^{\infty}(-D_nA^{-1})^j\mathbf{b_n} \\ &= A^{-1}(-D_nA^{-1})^3\sum_{j=0}^{\infty}(-D_nA^{-1})^j\mathbf{b_n}. \quad (2.33)\end{aligned}$$

If it can be shown that there exists a bound on $D_n A^{-1}$ in realisation n such that:

$$\|D_n A^{-1}\| \leq \nu_n \quad , \tag{2.34}$$

where $0 \leq \nu_n < 1$, then a bound on the error expression in (2.33) can be obtained quite straightforwardly:

$$\|\mathbf{e_n}\| < \frac{\|A^{-1}\| \quad \nu_n^3 \quad \|\mathbf{b_n}\|}{1 - \nu_n}. \tag{2.35}$$

Let us assume there is a bound on the maximum relative value that each perturbation can take, given by δ, where $\delta < 1$. This gives an absolute bound on the entries of a general D_n-type matrix:

$$|D_n| \leq \delta |A|, \qquad \forall \quad n. \tag{2.36}$$

Here we are using the convention that if for two matrices M and N, of the same order, we have

$$M_{ij} \leq N_{ij}, \qquad \forall \quad i, j,$$

then we write

$$M \leq N,$$

and we define the absolute value of any matrix to be such that,

$$|M|_{ij} = |M_{ij}| \qquad \forall \quad i, j.$$

Now consider $\|D_n A^{-1}\|_p$ for general $p \in \mathbb{N}$, and for all admissible realisations, n, defined in the usual way:

$$\|D_n A^{-1}\|_p = \max_{\mathbf{x}} \frac{\|D_n A^{-1} \mathbf{x}\|_p}{\|\mathbf{x}\|_p}. \tag{2.37}$$

Because of the properties for a general p-norm [5], we know that

$$\|D_n A^{-1}\|_p \leq \|D_n\|_p \|A^{-1}\|_p$$

Now, consider $\|D_n\|$, for all n. We have

$$\begin{aligned}
\|D_n\|_p^p &= \max_{\mathbf{x}} \frac{\|D_n\mathbf{x}\|_p^p}{\|\mathbf{x}\|_p^p} \\
&= \max_{\mathbf{x}} \frac{\sum_i |\sum_j (D_n)_{ij} x_j|^p}{\sum_i |x_i|^p} \\
&\leq \max_{\mathbf{x}} \frac{\sum_i (\sum_j |(D_n)_{ij} x_j|)^p}{\sum_i |x_i|^p} \\
&\leq \max_{\mathbf{x}} \frac{\sum_i |\sum_j |(D_n)_{ij}||x_j||^p}{\sum_i |x_i|^p} \\
&= \max_{\mathbf{x}} \frac{\||D_n||\mathbf{x}|\|_p^p}{\|\mathbf{x}\|_p^p} \\
&\leq \max_{\mathbf{x}} \frac{\||D_n|\|_p^p \|\mathbf{x}|\|_p^p}{\|\mathbf{x}\|_p^p} \\
&= \||D_n|\|_p^p,
\end{aligned}$$

since $\|\mathbf{x}\|_p = \||\mathbf{x}|\|_p$. Therefore,

$$\begin{aligned}
\|D_n A^{-1}\|_p &\leq \|D_n\|_p \|A^{-1}\|_p \\
&\leq \||D_n|\|_p \|A^{-1}\|_p
\end{aligned}$$

Combining this result with (2.36), we have

$$\|D_n A^{-1}\|_p \leq \delta \||A|\|_p \|A^{-1}\|_p, \qquad \forall \ \ n. \tag{2.38}$$

So, from (2.31), we find that

$$\delta < \frac{1}{\||A|\|_p \|A^{-1}\|_p} \quad ,$$

is a sufficient condition for convergence of the approximate series for all the relevant realisations.

In equations (2.11) to (2.17), it is seen that the vector $\mathbf{b_n}$ can be written as $\mathbf{b_0} + \mathbf{b_{d_n}}$, where the vector $\mathbf{b_{d_n}}$ contains linear combinations of the $\{d_{ij}\}$ perturbations as its components. By again taking the maximal value for all these perturbations, each component of $\mathbf{b_n}$ can be shown to satisfy

$$-2(\mathbf{b_0})_i \ \leq \ (\mathbf{b_n})_i \ \leq \ 2(\mathbf{b_0})_i.$$

We can therefore conclude that

$$\|\mathbf{b_n}\| \quad \leq \quad 2\ \|\mathbf{b_0}\| \qquad\qquad \forall \ \ n.$$

The errors for each realisation in expression(2.32) thus satisfy

$$\|\mathbf{e_n}\| \leq \frac{2\ \delta^3 \||A|\|^3 \|A^{-1}\|^4 \|\mathbf{b_0}\|}{1 - \delta \||A|\| \|A^{-1}\|} \qquad\qquad \forall \ \ n. \tag{2.39}$$

This expression is very useful when considering the error, obtained when truncating the series, in the mean value of the numerical solution.

Chapter 3

Statistical Approach to the Numerical Problem

One method of approach to the full statistical problem might be to take the mean value over all possible realisations of the numerical scheme in equation(2.1)

$$-\langle p_{i,j+1}k_{i,j+\frac{1}{2}}\rangle - \langle p_{i+1,j}k_{i+\frac{1}{2},j}\rangle + \langle p_{i,j}(k_{i+\frac{1}{2},j} + k_{i-\frac{1}{2},j} + k_{i,j+\frac{1}{2}} + k_{i,j-\frac{1}{2}})\rangle$$
$$- \langle p_{i-1,j}k_{i-\frac{1}{2},j}\rangle - \langle p_{i,j-1}k_{i,j-\frac{1}{2}}\rangle = 0 \quad (3.1)$$

Whilst this equation is precisely true, it contains no useful information, as the discretised pressure function cannot now be separated out from the cross terms, which contain it. Indeed, equation(3.1) leads nowhere.

If we return to equation(2.28)

$$\mathbf{p_{sn}} = A^{-1}(\mathbf{b_0} + \mathbf{b_{dn}}) - A^{-1}D_nA^{-1}(\mathbf{b_0} + \mathbf{b_{dn}}) + A^{-1}D_nA^{-1}D_nA^{-1}\mathbf{b_0},$$

taking the mean value, over all realisations, element-by-element, of both sides gives

$$\langle \mathbf{p_s}\rangle = A^{-1}\mathbf{b_0} + A^{-1}\langle \mathbf{b_d}\rangle - A^{-1}\langle DA^{-1}\rangle \mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle + A^{-1}\langle DA^{-1}D\rangle A^{-1}\mathbf{b_0}, \quad (3.2)$$

where the convention is that

$$(\langle Vector \rangle)_i = \langle (Vector)_i \rangle,$$

and

$$(\langle Matrix \rangle)_{ij} = \langle (Matrix)_{ij} \rangle.$$

These uncertain matrices and vectors obey all the usual rules of matrix algebra, in that

$$\begin{aligned}(\langle AB \rangle)_{ij} &= \langle (AB)_{ij} \rangle \\ &= \langle \sum_k A_{ik} B_{kj} \rangle \\ &= \sum_k \langle A_{ik} B_{kj} \rangle.\end{aligned}$$

Also if A is known exactly, and B is uncertain, then

$$\begin{aligned}(\langle AB \rangle)_{ij} &= \sum_k \langle A_{ik} B_{kj} \rangle \\ &= \sum_k A_{ik} \langle B_{kj} \rangle \\ &= \sum_k A_{ik} (\langle B \rangle)_{kj},\end{aligned}$$

so that $\langle AB \rangle = A \langle B \rangle$ under matrix algebra rules. Similar rules will apply for vector operations also,

$$\begin{aligned}(\langle A\mathbf{v} \rangle)_i &= \langle (A\mathbf{v})_i \rangle \\ &= \langle \sum_j A_{ij} v_j \rangle \\ &= \sum_j \langle A_{ij} v_j \rangle,\end{aligned}$$

and if, for example, A is known and $\mathbf{v}$ is uncertain, then,

$$(\langle A\mathbf{v} \rangle)_i = \sum_j \langle A_{ij} v_j \rangle = \sum_j A_{ij} \langle v_j \rangle \quad ,$$

so that

$$\langle A\mathbf{v} \rangle = A \langle \mathbf{v} \rangle.$$

It is fairly trivial to prove all the other algebraic rules.

The vector $\mathbf{b_d}$, and the matrix D only contain linear terms in the perturbations; and since we have already assumed that the mean value of all the perturbations, over all realisations, is zero, taking the mean value of any first order terms in $\{d_{ij}\}$ gives zero.

Equation(3.2) can therefore be re-written:

$$\langle \mathbf{p_s} \rangle = A^{-1}\mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle + A^{-1}\langle DA^{-1}D\rangle A^{-1}\mathbf{b_0}. \tag{3.3}$$

If we were tackling this problem in a deterministic way, we might make the intuitive assumption that a good approximation to the mean value of the numerical solution would be obtained by solving the problem using just the mean value of the permeability field as data. This is equivalent to solving the problem:

$$A\mathbf{p_A} = \mathbf{b_0}, \tag{3.4}$$

with solution

$$\mathbf{p_A} = A^{-1}\mathbf{b_0}. \tag{3.5}$$

It can be seen, by comparison of (3.5) with (3.3), that this deterministic solution is, in effect, a first order approximation to the exact mean value of the numerical solution. This means that $\mathbf{p_A}$ can be thought of as an approximation to $\langle \mathbf{p} \rangle$ that effectively contains information about the mean value of the permeability field. The information about the second order terms in $\{d_{ij}\}$ has been discarded, which means that two aspects of the statistical information have been lost:

1. variance of the field .

   Terms like $\langle d_{i,j}^2 \rangle$ represent the variance, or the dispersion of the permeability field. For a single variable, qualitatively, the variance represents the spread of possible values that it can take, away from the

mean value. For this reason, if this information is excluded, it makes the mean value approximation meaningless in a statistical sense, i.e. equation(3.5) could be an approximation to the solution for either a field with zero dispersion (that is, one which is known precisely), or for a field with an arbitrarily large dispersion (one that is highly uncertain).

2. correlation of the field .

   Terms like $\langle d_{i,j} d_{i',j'} \rangle$ as seen in equation(2.7), are equal to the discretised correlation function for the separated points (i, j), and (i', j'), and give a measure of how similar the statistical properties at the two points are. It is very important to take this measure into account, as was mentioned in Chapter 1, because this is a fundamental property of the uncertainty in the permeability field. Leaving out these terms would give the result for a 'static' uncertain field, that is, one where the permeability is uncertain at all sample points with the values at each point being unrelated statistically to each other.

So the approximation from (3.3) for the mean value of the numerical solution, when written

$$\langle \mathbf{p_s} \rangle = A^{-1}\mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d} \rangle + A^{-1}\langle DA^{-1}D \rangle A^{-1}\mathbf{b_0}, \tag{3.6}$$

contains information concerning the mean, variance, and autocorrelation function of the permeability field, which is, essentially, what is required from the original problem.

Clearly, the error introduced by making the approximation is given by:

$$\langle \mathbf{e} \rangle = A^{-1} \sum_{m=3}^{\infty} \langle (-DA^{-1})^m \mathbf{b} \rangle \tag{3.7}$$

i.e. the error between the approximation (3.6), and the exact mean value of the numerical solution, as in equation(3.3).

Consider a general element of this error vector:

$$(\langle \mathbf{e} \rangle)_i = \langle (\mathbf{e_n})_i \rangle, \tag{3.8}$$

where $\mathbf{e_n}$ is as defined in (2.37).

Now consider the expected value of a general functional of a number of uncertain statistical variables, assuming the functional f, and the set of all possible realisations R_m have suitable properties, then,

$$\langle f(x_1, x_2, x_3, \ldots, x_m) \rangle = \int_{R_m} f(x_1, x_2, x_3, \ldots, x_m) \rho(x_1, x_2, x_3, \ldots, x_m) d^m\mathbf{x}, \tag{3.9}$$

where $\rho(x_1, x_2, x_3, \ldots, x_m)$ is the (joint) multivariate distribution for the variables, and,

$$\begin{aligned}
&| \langle f(x_1, x_2, x_3, \ldots, x_m) \rangle | \\
&\qquad = \; | \int_{R_m} f(x_1, x_2, x_3, \ldots, x_m) \rho(x_1, x_2, x_3, \ldots, x_m) d^m\mathbf{x} \; | \\
&\qquad \leq \; \int_{R_m} | \, f(x_1, x_2, x_3, \ldots, x_m) \rho(x_1, x_2, x_3, \ldots, x_m) \, | \, d^m\mathbf{x} \\
&\qquad \leq \; \int_{R_m} | \, f(x_1, x_2, x_3, \ldots, x_m) \, | \cdot | \, \rho(x_1, x_2, x_3, \ldots, x_m) \, | \, d^m\mathbf{x} \\
&\qquad \leq \; \max_{R_m} | \, f(x_1, x_2, x_3, \ldots, x_m) \, | \cdot \int_{R_m} | \, \rho(x_1, x_2, x_3, \ldots, x_m) \, | \, d^m\mathbf{x}.
\end{aligned}$$

Since $\rho(x_1, x_2, x_3, \ldots, x_m)$ is the m-dimensional multivariate distribution function, it is, by convention, always positive, and is assumed to be normalised, so that

$$\int_{R_m} | \, \rho(x_1, x_2, x_3, \ldots, x_m) \, | \, d^m\mathbf{x} = 1. \tag{3.10}$$

Therefore, substituting (3.10) into the inequality expression above gives,

$$| \langle f(x_1, x_2, x_3, \ldots, x_m) \rangle | \quad \leq \quad \max_{R_m} | \, f(x_1, x_2, x_3, \ldots, x_m) \, | \, . \tag{3.11}$$

It can therefore be deduced that the mean value of any function or functional of m statistical variables, irrespective of whether they are correlated, must lie within the extremal values of that function or functional over its admissible space.

Also,

$$\begin{aligned} |\langle \mathbf{v}(x_1,\ldots,x_m)\rangle| &= |\int_{R_m} \mathbf{v}(x_1,\ldots,x_m)\ \ \rho(x_1,\ldots,x_m)d^m\mathbf{x}| \\ &\leq \int_{R_m} |\mathbf{v}(x_1,\ldots,x_m)|\ \ |\rho(x_1,\ldots,x_m)|d^m\mathbf{x} \end{aligned}$$

and since $\rho \geq 0$ over R_m,

$$|\langle \mathbf{v}(x_1,\ldots,x_m)\rangle| \leq \int_{R_m} |\mathbf{v}(x_1,\ldots,x_m)|\ \ \rho(x_1,\ldots,x_m)d^m\mathbf{x}.$$

Hence for any l_p norm $|\cdot|$,

$$|\langle \mathbf{v}(x_1,\ldots,x_m)\rangle| \leq \langle|\mathbf{v}(x_1,\ldots,x_m)|\rangle, \tag{3.12}$$

irrespective of the distribution function for the variables $\{x_j\}$.

We can therefore establish a bound on the error that has been introduced when using expression(3.6) as an approximation to the mean value of the numerical solution, $\langle \mathbf{e}\rangle$.

First, substituting $\langle \mathbf{e}\rangle$ into expression(3.12) gives

$$\|\langle \mathbf{e}\rangle\| \leq \langle\|\mathbf{e}\|\rangle. \tag{3.13}$$

Now, since

$$\mathbf{e_n} = A^{-1}\sum_{m=3}^{\infty}(-D_nA^{-1})^m(\mathbf{b_0}+\mathbf{b_n}),$$

$\mathbf{e_n}$ is a vector function, in some specified way, of all the $\{d_{ij}\}$ terms, which are themselves uncertain statistical variables. Therefore, the process of taking the norm of a general vector $\mathbf{e_n}$ can be thought of as a functional of $\mathbf{e_n}$ - which means it is itself a function of the $\{d_{ij}\}$ type variables. So, the function $\|\mathbf{e}\|$ must satisfy the expression(3.11).

Therefore,

$$\langle \|\mathbf{e}\| \rangle \quad \leq \quad \max_{all\ realisations,n} \{\|\mathbf{e_n}\|\}. \tag{3.14}$$

Then, by combining (3.13) with (3.15), we can see that

$$\|\langle \mathbf{e} \rangle\| \quad \leq \quad \max_{all\ realisations,n} \{\|\mathbf{e_n}\|\}, \tag{3.15}$$

which is a bound for the norm on the error introduced, when expression(3.6) is used to approximate the mean value of the numerical solution.

Therefore, by combining equation(3.15), with equation(2.39), gives,

$$\|\langle \mathbf{e} \rangle\| \quad \leq \frac{2\ \delta^3 \||A|\|^3 \|A^{-1}\|^4 \|\mathbf{b_0}\|}{1 - \delta \||A|\| \|A^{-1}\|}, \tag{3.16}$$

the required bound on the approximation to the mean value of the numerical solution.

Chapter 4

Numerical Approach to the Problem

At first sight, evaluation of the statistical terms in equation (3.6) :

$$\langle \mathbf{p_s} \rangle = A^{-1}\mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle + A^{-1}\langle DA^{-1}D\rangle A^{-1}\mathbf{b_0},$$

are very awkward to deal with, because, due to the presence of A^{-1} between the two matrices in each term, the terms $\langle DA^{-1}\mathbf{b_d}\rangle$, and $\langle DA^{-1}D\rangle A^{-1}\mathbf{b_0}$ involve very complicated linear combinations of the correlation terms, $\{\langle d_{i,j}d_{i',j'}\rangle\}$, dependent on the inverse of A. This problem can be resolved by considering the structure of the D_n matrices, and noticing that each term like $\{d_{i,j}\}$ occurs either twice, if it is next to a boundary, or four times if it is on an internal half grid-point.

Suppose (i, j) is an internal half grid-point lying between grid-points labelled y, and z. It can be seen from equations (2.21) to (2.24) that the $d_{i,j}$ term only occurs at the matrix positions (y, y), (y, z), (z, y), and (z, z) in the following way:

$$
\begin{array}{c} \\ y \\ \\ z \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & y & & z & & \end{array} \\
\left[\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & d_{i,j} & \ldots & -d_{i,j} & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & -d_{i,j} & \ldots & d_{i,j} & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & &
\end{array}\right]
\end{array}
. \tag{4.1}
$$

This contribution can be thought of as $d_{i,j}E^{ij}$, where

$$
E^{ij} = \left[\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & 1 & \ldots & -1 & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & -1 & \ldots & 1 & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & &
\end{array}\right]
. \tag{4.2}
$$

The full matrix D_n can then be written as a weighted sum of simple elemental matrices like E^{ij} that contain either only four or two non-zero elements:

$$
D_n = \sum_{halfgrid-points} d^n_{i,j} E^{ij} \tag{4.3}
$$

If a sum over all half grid-points is written as $\sum_{i,j}$ (each half grid-point being labelled (i,j)), then the expression (3.6) can be written:

$$
\langle DA^{-1}D \rangle = \langle \sum_{i,j} d_{i,j} E^{ij} A^{-1} \sum_{i',j'} d_{i',j'} E^{i'j'} \rangle, \tag{4.4}
$$

where the mean is taken over the joint multivariate distribution function for all the values of $d_{i,j}$ at half grid-points. Consider a general element of this matrix:

$$\begin{aligned}
\langle DA^{-1}D\rangle_{kl} &= \langle\sum_{i,j} d_{i,j}E^{ij}A^{-1}\sum_{i',j'} d_{i',j'}E^{i'j'}\rangle_{kl} \\
&= \langle(\sum_{i,j} d_{i,j}E^{ij}A^{-1}\sum_{i',j'} d_{i',j'}E^{i'j'})_{kl}\rangle \\
&= \langle(\sum_{i,j} d_{i,j}E^{ij})_{km}A^{-1}_{mn}(\sum_{i',j'} d_{i',j'}E^{i'j'})_{nl}\rangle \\
&= \langle\sum_{i,j} d_{i,j}E^{ij}_{km}A^{-1}_{mn}\sum_{i',j'} d_{i',j'}E^{i'j'}_{nl}\rangle \\
&= \langle\sum_{i,j}\sum_{i',j'}(d_{i,j}E^{ij}_{km})A^{-1}_{mn}(d_{i',j'}E^{i'j'}_{nl})\rangle \\
&= \langle\sum_{i,j}\sum_{i',j'}(d_{i,j}d_{i',j'}E^{ij}_{km}A^{-1}_{mn}E^{i'j'}_{nl})\rangle \\
&= \sum_{i,j}\sum_{i',j'}(\langle d_{i,j}d_{i',j'}\rangle E^{ij}_{km}A^{-1}_{mn}E^{i'j'}_{nl}) \\
&= \sum_{i,j,i',j'}(\langle d_{i,j}d_{i',j'}\rangle E^{ij}_{km}A^{-1}_{mn}E^{i'j'}_{nl}), \qquad (4.5)
\end{aligned}$$

where $\sum_{i,j,i',j'}$ has the meaning of a sum over all possible ways that each half grid-point can correlate with each other half grid-point. Of course, if there are N of these half grid-points, then there will be $(N-1)*N$ possible ways that they can correlate with each other.

A similar procedure can be done with the second term in equation (3.6), $A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle$ to obtain a similar series term which involves a weighted sum. In this case the sum is performed over half grid-points that are adjacent to the boundary only, correlating with all other half grid-points within the region.

The $\langle d_{i,j}d_{i',j'}\rangle$ terms are now just equal to the normalised discretised correlation function between points (i,j) and (i',j'), as seen in equation (2.7), and is, in the most general case, a vector function of the separation of the two points.

An isotropic assumption can be made about the correlation, in which case, the terms $\langle d_{i,j}d_{i',j'}\rangle$ are just a function of the absolute distance of separation of the points (i,j) and (i',j'), A more interesting case would be one in which the correlation function has a different form in different directions. There might be

a strong correlation in the properties of the rocks in a horizontal direction, and much weaker correlation in the vertical direction, representing the usual layered structure of oil bearing strata. A further development in an anisotropic correlation function might be a periodicity in the vertical direction, which would represent a repetition in the layered structure of the rock properties. This would introduce an extra parameter into the correlation function, in addition to the two correlation lengths, which would be the spatial periodicity of the rock layering.

Typical possibilities for models for the spatial correlation function would be an exponential or a Gaussian-type decay. An isotropic exponential decay would have the form:

$$\rho(\mathbf{r}-\mathbf{r}') = \sigma^2 \exp(-|\mathbf{r}-\mathbf{r}'|/\lambda), \tag{4.6}$$

where λ is the (characteristic) correlation length.

For a general correlation function, the correlation length, λ_α, associated with direction α is defined to be

$$\lambda_\alpha = \int_0^\infty \rho_\alpha(x)dx, \tag{4.7}$$

[5]. So a Gaussian-type correlation function, takes the form:

$$\rho(\mathbf{r},\mathbf{r}') = \sigma^2 \exp(-\frac{\pi|\mathbf{r}-\mathbf{r}'|^2}{4\lambda^2}), \tag{4.8}$$

where λ is the (in this case, isotropic) correlation length.

A Gaussian-type form for the correlation function is more useful mathematically when considering anisotropic correlations, because the distance squared terms can be separated out straightforwardly - for example, in two dimensions:

$$|\mathbf{r}-\mathbf{r}'|^2 = |(x-x')^2+(y-y')^2| = (x-x')^2+(y-y')^2,$$

and so the correlation function becomes a simple product:

$$\rho(\mathbf{r}-\mathbf{r}') = \sigma^2 \exp(-\frac{\pi(x-x')^2}{4\lambda_x^2})\exp(-\frac{\pi(y-y')^2}{4\lambda_y^2}). \tag{4.9}$$

An introduction of periodicity can also be made by an extra product. For example, if there is periodicity assumed in the x-direction, then

$$\rho(\mathbf{r}-\mathbf{r}') = \sigma^2 \exp\{-\frac{\pi(x-x')^2}{4\lambda_x^2}\} \exp\{-\frac{\pi(y-y')^2}{4\lambda_y^2}\} \cos(\omega(x-x')), \qquad (4.10)$$

where $\frac{2\pi}{\omega}$ is the model for the spatial periodicity, an approximate length equivalent to the repetition distance in the rock structure.

Chapter 5

Results Section

The following section shows results for different types of imposed boundary conditions.

Figures 5.1(a)-(e) have enforced flow at one end, no flow at the opposite end, and zero pressure conditions along the two sides. 5.1(a) is the deterministic solution with homogeneous permeability values; 5.1(b) is the result with variance equal to 0.1; 5.1(c) has variance 0.2; 5.1(d) has variance 0.4; and for 5.1(e) the variance is equal to 1.0.

Figures 5.2(a)-(e) have no flow conditions at either end and a pressure difference of 1.0 across the region horizontally. Again, figures (a), (b), (c), (d), and (e) are the results for increasing covariance values 0.0 (that is the deterministic solution), 0.1, 0.2, 0.4, and 1.0.

Figures 5.3(a)-(d) show the effects of increasing correlation length in relation to grid size, and overall scale of the region, with the same boundary conditions as 5.1. Figure (a) shows results for a correlation length of 0.01, which is approximately one third of the grid size; figures (b), (c), and (d) are results for correlation lengths 0.1, 0.5, and 10.0, respectively, which are all larger than the grid size. The variance for all these results was 0.2.

The final figures 5.4(a)-(b) show results for anisotropic correlation lengths. The boundary conditions were, again the same as those in 5.1. Figure (a) has correlation length 0.1 in the x direction, and 1.0 in the y direction, and figure (b) has correlation length 1.0 in the x, and 0.1 in the y direction. Variance was 0.2 throughout.

Figure 5.1(a)

Figure 5.1(b)

Figure 5.1(c)

Figure 5.1(d)

Figure 5.1(e)

Figure 5.2(a)

Figure 5.2(b)

Figure 5.2(c)

Figure 5.2(d)

Figure 5.2(e)

Figure 5.3(a)

Figure 5.3(b)

Figure 5.3(c)

Figure 5.3(d)

Figure 5.4(a)

Figure 5.4(b)

Figures 5.1(a) to 5.1(e) show how increasing the variance of the permeability field increases the difference between the second order approximation to the mean value of the pressure field, and its deterministic solution. Particular features, such as gradients, seem to be distorted, and exaggerated.

Figures 5.2(a) to 5.1(e) show a similar effect on a different set of boundary conditions, with no flow at either ends, and an induced pressure difference across the region. The deterministic case just results in a simple constant flow across the pressure difference. When the non-deterministic cases were considered for small variance, the flow (as in the gradient of pressure) is induced to increase, but because of the necessity of satisfying the boundary conditions, the gradient is forced to decrease close to the higher boundary condition. With increasing variance, this effect becomes more pronounced with the decrease in gradient becoming more sharp, as the boundary conditions always have to be satisfied. Eventually, this effect becomes so great that in 5.2(d), the maximum principle for pressure is violated, and this roughly corresponds with the series expansion (2.29) becoming invalid due to large stochastic perturbation in permeability.

Figures 5.3(a) to 5.3(d) show effects of changing the correlation length, relative to the grid size. For the first case, the correlation length is considerably less than the grid size, meaning that the statistical properties at each grid-point have virtually no correlation with each other. The values are therefore virtually the same as the equivalent deterministic case figure 5.1(a). In figure 5.3(b), the correlation length is roughly three times the grid size, meaning that the properties of nearby grid-points are correlated, much more strongly than between arbitrary grid-points. This manifests itself as a slight distortion in the shape of the solution, with respect to the deterministic solution. For figure (c) the correlation length is 0.5, meaning that the statistical properties of the pressure at the grid-points are

correlated over roughly half the region, producing slightly more distortion. In the last figure, the correlation length is ten times the dimensions of the dimensions of the region, meaning that all grid-points are roughly equally correlated with each other. This results in a large distortion compared to the deterministic case, as expected.

Figures 5.4(a) and 5.4(b) are supposed to show differences in cases where the correlation function is spatially anisotropic. However, the results for several tests of this type were not very interesting; and tended to give results similar to the isotropic cases with the smaller correlation length. That is, the shorter correlation length always tended to dominate the observed behaviour, and this does not lead to very interesting results. The reason for this could be that the more interesting behaviour is contained in the higher order moment term, and it will be particularly interesting to see how these results behave in time.

Chapter 6

Conclusions

We have developed a method for estimating mean values for numerical solutions to systems of steady-state partial differential equations which contain a spatially varying uncertain parameter, that takes into account terms up to second order in the multivariate distribution function. This gives a significant insight into the behaviour of the mean when dispersion of the parameter values is taken into account, as opposed to using just the deterministic solution, which only includes information about the mean of the uncertain parameter, and consequently has no real statistical information. Bounds on the accuracy of this approach have been found and could be developed further, by evaluation and comparison for different multivariate distribution functions.

The problem with many of these results is the non-physicality of the model equations, and the need to impose artificial boundary conditions, which do not allow a proper development of the flow behaviour to be done. It is, however always interesting to observe the behaviour of the solutions with respect to the deterministic solution, and this is where the bulk of the work has been performed.

The equivalent results to those shown here, for time-varying systems of equations should prove to be much more interesting. It will be particularly interesting

to observe the time-varying behaviour of the variance of the pressure solution as a spatial function. Also, the effect of using more physical (that is, time-varying) boundary conditions will be interesting to observe. Work is, at present being done in this area, and the results should soon be available in a further report.

Chapter 7

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