Toeplitz, Hankel, and Berezin transforms on Fock-type spaces

Ghazaleh Asghari

A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
University of Reading
Reading, UK
December 2025

Declaration

This thesis describes the work undertaken at the Department of Mathematics and Statistics of the University of Reading, in fulfillment of the requirements for the degree of Doctor of Philosophy. I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Reading, December 2025

Ghazaleh Asghari

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my supervisor, Prof. Jani Virtanen, for his guidance, patience, and support throughout my PhD journey. I feel incredibly lucky to have worked with someone who not only shared brilliant mathematical insight but also a great sense of humor. Our meetings often involved just as much laughter as mathematics, and his encouragement, especially when I confidently ran in completely the wrong research direction, helped me grow into a far more independent researcher. Thank you for giving me the freedom to explore my own ideas, for always being in my corner, and for proving that doing serious mathematics does not mean being serious all the time.

I would also like to thank my monitoring committee members, Prof. Simon Chandler-Wilde and Prof. Sugata Mondal, for their valuable feedback, advice, and for keeping me on track. Their insightful comments often helped me see my work from a fresh perspective, occasionally even the correct one!

I gratefully acknowledge the EPSRC for funding my PhD. Without their support, this thesis, and quite a few late-night coffees, would not have been possible. I'm also thankful to my friends and colleagues in the Department of Mathematics and Statistics at the University of Reading, including Siavash Sadeghi, David Norrbo, Santeri Miihkinen, Hicham Arroussi, Ryan O'Loughlin, and Inzish Sajid, for the countless discussions, coffee breaks, and shared existential crises about research. You made the experience far more enjoyable.

I would also like to thank my mother, whose love and encouragement have traveled across borders, time zones, and unreliable internet connections. In the past few years, we only managed to see each other once. Yet she never stopped believing in me, sending me strength from afar, and reminding me that no matter how complicated my life became, I still needed to sleep, eat properly, and occasionally "go outside like a normal human being." I have missed her more than I can say.

Finally, my warmest thanks go to my partner, Mohamadreza Farshidfar, for the patience, love, and laughter that kept me sane through the ups and downs of PhD life. Although our mathematical worlds rarely intersect, and our research discussions usually end with one of us saying "I have no idea what that means, but it sounds important", he never stopped listening, asking questions, and cheering me on.

Dedication

To the women of Iran, whose voices rise even when they are silenced, and to the students who dared to dream despite the cost. For “Woman, Life, Freedom”. May their courage be remembered, and may their struggle find its answer.

Abstract

This thesis is a collection of published and submitted papers. It investigates Toeplitz and Hankel operators, as well as Berezin transforms on scalar weighted and vector-valued Fock spaces, focusing on how analytic and geometric properties of the underlying weights influence the boundedness, compactness, Schatten class membership, and spectral behavior of these operators. The results unify and extend several known facts of operator theory on reproducing kernel Hilbert spaces, particularly in the non-Gaussian and vector-valued settings.

In Chapter 2, we introduce weighted Fock spaces, including standard, doubling, scalar weighted, and large vector-valued Fock spaces. Chapter 3 is devoted to a detailed summary of our findings, motivation behind our work, and known results related to our study. More detail is as follows.

In the first part, we study *Hankel operators on doubling Fock spaces* and establish Schatten-class characterizations in terms of integral distance to analytic functions and mean oscillations. By developing a detailed analysis of the geometry associated with the doubling measure and the corresponding radius function, we extend the *Berger–Coburn phenomenon*, i.e., the symmetry between compactness of H_f and $H_{\bar{f}}$ when f is a bounded function on the complex plane, beyond the classical Fock space. These results provide new insight into the structure of Hankel operators on more general weighted Fock spaces.

The second part concerns *Toeplitz operators on large vector-valued Fock spaces*, where functions take values in a separable Hilbert space. For operator-valued symbols G, we derive necessary and sufficient conditions for boundedness, compactness, and Schatten-class membership of the corresponding Toeplitz operators T_G in terms of scalar and operator-valued Berezin transforms, averaging functions, and Carleson conditions. This framework extends known scalar results to infinite-dimensional vector-valued settings and reveals how the fibrewise Hilbert-space structure affects operator-theoretic behavior.

In the final part, we examine *polynomial fixed points of the Berezin transform on Fock-type spaces* associated with weights of the form $e^{-|z|^m}$, $m > 0$. We show that every harmonic polynomial is a fixed point, and depending on the exact form of the weight, most polynomials fixed by the Berezin transform are harmonic.

Collectively, the results contribute to a unified understanding of Toeplitz and Hankel operators and Berezin transforms on generalized Fock spaces, demonstrating the deep interplay between geometric analysis, function theory, and operator theory in both scalar and vector-valued contexts. Full versions of the papers are provided in Appendix A, Appendix B, and Appendix C.

Contents

Chapter 1

Introduction

Operator theory on spaces of holomorphic functions is a central branch of modern functional analysis, linking techniques from complex analysis, harmonic analysis, and geometry. Among the most extensively studied operators in this setting are the *Toeplitz* and *Hankel* operators, which serve as natural models for non-commutative function algebras and Berezin-Toeplitz quantization. Their behavior encodes deep analytic and geometric information about the underlying spaces, and their study continues to be an ongoing source of questions linking operator theory to mathematical physics and the theory of several complex variables.

A particularly rich environment for such investigations is provided by *Fock-type spaces*. The classical Fock space F^2 consists of all entire functions on the complex plane $\mathbb{C}$ that are square-integrable with respect to the Gaussian measure

$$d\lambda(z) = \frac{1}{\pi} e^{-|z|^2}\, dA(z),$$

where dA denotes the Lebesgue area measure. The space F^2, which was originally introduced in quantum mechanics, has long served as an important object of study in functional analysis, complex analysis, and mathematical physics. From the complex analysis point of view, it provides a canonical example of a reproducing kernel Hilbert space, which is a central setting to the study of Toeplitz and Hankel operators. From a geometric perspective, the Fock space can be interpreted as the space of holomorphic sections of a Hermitian line bundle over $\mathbb{C}$, where the Gaussian weight naturally induces a Kähler metric. This interpretation establishes deep connections with complex differential geometry and the framework of geometric quantization.

The Toeplitz operator with symbol f is defined as $T_f(\cdot) = P(f\cdot)$, where P denotes the orthogonal projection from $L^2(\mathbb{C}, d\lambda)$ onto F^2. The Hankel operator is $H_f(\cdot) = (I-P)(f\cdot)$. These operators link pointwise multiplication by f to analytic projection. The study of boundedness, compactness, and Schatten class membership of T_f and H_f has been an important subject of study in operator theory.

In their seminal work, Berger and Coburn [17] observed a notable duality between certain Hankel operators on the classical Fock space: for a bounded function f on $\mathbb{C}$, the Hankel operator H_f is compact if and only if $H_{\bar{f}}$ is compact, where $\bar{f}$ is the complex conjugate of f. This phenomenon, now known as the *Berger-Coburn phenomenon*, has since motivated a substantial body of work examining its validity in more general analytic settings. The first part of the present thesis, based on [10], in collaboration with Hu and Virtanen, extends this analysis to *doubling Fock spaces*, a class of weighted Fock spaces that generalize the Gaussian weight, while keeping much of its analytic structure.

Let $\phi : \mathbb{C} \to \mathbb{R}$ be a subharmonic weight such that the measure $d\mu = \Delta\phi\, dA$ is doubling. That is, there exists $C > 1$ such that

$$0 < \mu(D(z,2r)) \leq C\,\mu(D(z,r)) < \infty, \qquad z \in \mathbb{C},\ r > 0.$$

Taking $H(\mathbb{C})$ as the set of entire functions on $\mathbb{C}$, the corresponding doubling Fock space is

$$F^2_\phi = \{f \in H(\mathbb{C}) : \|f\|^2_{2,\phi} = \int_{\mathbb{C}} |f(z)|^2 e^{-2\phi(z)}\, dA(z) < \infty\}.$$

Such spaces, developed in the works of Christ [24], Marco-Massaneda-Ortega Cerda [64], and Marzo-Ortega Cerda [65], provide a geometric generalization of the Gaussian framework. The Laplacian of the weight defines a radius function $\rho(z)$ via $\mu(D(z,\rho(z))) = 1$, encoding the conformality of the Riemannian metric in the space. Using a notion called integral distance to analytic functions, we obtain characterizations of Schatten class Hankel operators H_f and establish criteria for simultaneous membership of Hankel operators H_f and $H_{\bar{f}}$ in Schatten classes S_p, $p > 0$. In particular, we showed that the Berger-Coburn phenomenon holds for Hilbert-Schmidt Hankel operators on the doubling Fock space F^2_ϕ, and we discuss the validity of the Berger-Coburn phenomenon for Schatten S_p classes, for which $0 < p \leq 1$. Moreover, we discuss why our approach cannot be used to study the validity of the Berger-Coburn phenomenon for Schatten classes S_p with $1 < p < \infty$ and $p \neq 2$. This is since ρ^{p-2} is not necessarily a Muckenhoupt weight. These results extend the scope of known Schatten-class criteria for Hankel operators on less complicated Fock-type spaces.

The second main theme of the thesis concerns *Toeplitz operators on large vector-valued Fock spaces*, based on the joint work with Arroussi and Virtanen [6]. Compared to the long-term interest in the scalar-valued Fock spaces, less attention has been paid to the vector-valued case, where functions take values in finite- or infinite-dimensional Hilbert spaces. Motivated by this, we consider Fock-type spaces of the form $F^2_\phi(\mathbb{C}^n, \mathcal{H})$, consisting of holomorphic functions $f : \mathbb{C}^n \to \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space, with norm

$$\|f\|^2_{2,\phi} = \int_{\mathbb{C}^n} \|f(z)\|^2_{\mathcal{H}} e^{-2\phi(z)}\, dA(z),$$

and where ϕ belongs to the class of admissible weights $\mathcal{W}$ introduced by Dall'Ara [30]. Such spaces generalize scalar weighted Fock spaces $F^2_\phi(\mathbb{C}^n)$ and provide the natural setting for the *vector-valued reproducing kernel Hilbert space*. That is, a reproducing kernel Hilbert space for which the reproducing kernel is not an element of the function space, contrary to the scalar Fock spaces.

For an operator-valued symbol $G : \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$, the Toeplitz operator T_G acts by $T_G f = P(Gf)$, where P is the orthogonal projection from $L^2_\phi(\mathbb{C}^n, \mathcal{H})$ onto $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. We obtain boundedness, compactness, and Schatten-class characterizations for T_G in terms of *Berezin transforms*, *averaging functions*, and *Carleson conditions* adapted to the vector-valued setting. The proofs are based on the geometry induced by the radius function ρ and covering $\mathbb{C}^n$ by suitable lattices associated with admissible weights. These results extend earlier scalar characterizations such as those by Isralowitz-Zhu [53] and Arroussi-He-Li-Tong [8], to the infinite-dimensional vector-valued context and reveal how the fiberwise Hilbert space structure governs the behavior of Toeplitz operators. See [6] for further details. Notice that when $\mathcal{H} = \mathbb{C}^m$, each operator $G(z)$ can be represented by an $m \times m$ matrix, so G becomes a matrix-valued function, which corresponds to the study of block Toeplitz operators. Thus, the matrix-valued symbol corresponds to the finite-dimensional special case of the general operator-valued symbol theory.

The third part of the thesis addresses the *Berezin transform* on Fock-type spaces F^2_m, $m > 0$, based on joint work with Čučković and Şahutoğlu [9]. Note that F^2_m is a doubling Fock space with $\phi(z) = |z|^m/2$. For a symbol f, the Berezin transform of f is

$$B_m f(z) = \langle f k_{m,z}, k_{m,z} \rangle = \int_{\mathbb{C}} f(w)\,|k_{m,z}(w)|^2 e^{-|w|^m}\, dA(w),$$

where $k_{m,z}$ denotes the normalized reproducing kernel of F^2_m. When $m = 2$, the Berezin transform for the classical Fock space is a convolution with a Gaussian function. Note that convolving an image with a Gaussian function is called a Gaussian blur. Mathematically speaking, an image is a bounded function from some coordinate system to the intensity values of pixels. Therefore, one can consider the Berezin transform as a natural smoothing/blurring operator. A classical question, originating in the Gaussian case, is to describe all functions satisfying the fixed-point equation $B_m f = f$. It is known that in the standard Fock space, i.e., when $m = 2$, bounded fixed points are harmonic and thus constant. Intuitively, this is because the only pictures that remain unchanged by blurring, are those with completely uniform colors. This is a famous result by Englis [38], which strongly depends on the relationship between the Berezin transform for the standard Fock space and the heat equation. We are not aware of such a relationship in more general Fock-Type spaces. In [9], we study the polynomial fixed points of the Berezin transform on F^2_m, with $m > 0$. Our results show that when $m = 2$, every polynomial in z and $\bar{z}$ that is fixed by the Berezin transform is harmonic. Notice that this is different from the aforementioned result by Englis, as polynomials are not bounded. Moreover, we showed that when m is different from two, most polynomial fixed points are harmonic, and exceptional behavior is rare.

Taken together, the three components of this thesis contribute to a unified understanding of Toeplitz and Hankel operators, as well as Berezin transforms, on weighted and vector-valued Fock spaces. They illustrate how the geometry induced by the weight function, captured through the doubling property, the admissibility class $\mathcal{W}$, and the associated metric structure, governs the operator-theoretic constructions on these spaces.

The thesis is organized as follows. *Chapter* 2 provides background material and introduces the main examples of Fock-type spaces: the classical, doubling, scalar weighted, and large vector-valued Fock spaces. We will study the radius function, Bergman kernel estimates, and projection formulas, essential for later work. Moreover, we also define the Toeplitz and Hankel operators for each space. *Chapter* 3 presents summaries of the main results, as well as related earlier works: Section 3.1 investigates Schatten-class Hankel operators on doubling Fock spaces and extends the Berger–Coburn phenomenon. Section 3.2 develops the theory of Toeplitz operators on large vector-valued Fock spaces, and Section 3.3 studies fixed points of the Berezin transform on Fock-type spaces. The final section discusses open problems and possible directions for future research. The rest of the thesis contains full versions of each paper [10, 6, 9], given in Appendix A, Appendix B, and Appendix C.

The results presented here demonstrate how analytic, geometric, and operator-theoretic methods interact in the study of holomorphic function spaces and open new avenues for exploration in the theory of Toeplitz and Hankel operators, as well as Berezin transforms, on scalar weighted and vector-valued reproducing-kernel Hilbert spaces.

1.1 List of publications

This thesis is based on the following publications, reproduced in Appendices A-C.

Paper I [10] G. Asghari, Z. J. Hu and J. A. Virtanen, Schatten class Hankel operators on doubling Fock spaces and the Berger-Coburn phenomenon, J. Math. Anal. Appl. **5**40 (2024), no. 2, Paper No. 128596, 32 pp.; MR4764447

Paper II [6] H. Arroussi, G. Asghari, and J. A. Virtanen, Toeplitz operators on large vector-valued Fock spaces. arXiv preprint arXiv:2504.15239.

Paper III [9] G. Asghari, Z. Cuckovic, and S. Sahutoglu, Fixed points of the Berezin transform on Fock-type spaces. arXiv preprint arXiv:2508.13115, to appear in Proceedings of the American Mathematical Society.

The author of this thesis has made a significant contribution to all of the contained publications.

Chapter 2

Definitions and basic results

In this chapter, we introduce Fock-type spaces. This includes standard Fock spaces, doubling Fock spaces, scalar weighted Fock spaces, and large vector-valued Fock spaces. In more detail, we study radius functions, the Bergman kernel and its pointwise estimate, the Bergman projection, and Toeplitz and Hankel operators on each space.

2.1 Standard Fock spaces

Let $\mathbb{C}$ be the complex plane, dA be the Euclidean area measure, and $\alpha > 0$. Consider the Gaussian measure

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z), \qquad z \in \mathbb{C}.$$

The *standard Fock space* F^2_α is the set of all entire functions in $L^2(\mathbb{C}, d\lambda_\alpha)$. Indeed, taking $H(\mathbb{C})$ as the set of entire functions over $\mathbb{C}$, one can consider the standard Fock space as

$$F^2_\alpha = \left\{ f \in H(\mathbb{C}) : \|f\|^2_{2,\alpha} := \int_{\mathbb{C}} |f(z)|^2 \, d\lambda_\alpha(z) < \infty \right\}. \tag{2.1.1}$$

F^2_α is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z) \overline{g(z)} \, d\lambda_\alpha(z),$$

inherited from $L^2(\mathbb{C}, d\lambda_\alpha)$. Moreover, the set of monomials $\{e_n(z) = \sqrt{\alpha^n} z^n / \sqrt{n!} : n = 0, 1, 2, \cdots\}$ is an orthonormal basis for F^2_α. Furthermore, F^2_α is a reproducing kernel Hilbert space. That is, for each $z \in \mathbb{C}$, there is a $K_z \in F^2_\alpha$ such that

$$f(z) = \langle f, K_z \rangle, \quad \forall f \in F^2_\alpha.$$

The reproducing kernel is given by $K_z(w) = K(w, z) = e^{\alpha w \bar{z}}$, for all $w, z \in \mathbb{C}$. The reproducing kernel is conjugate symmetric, i.e., $\overline{K(w,z)} = K(z, w)$ for all $z, w \in \mathbb{C}$. Since $F^2_\alpha \subset L^2(\mathbb{C}, d\lambda_\alpha)$ is closed, there is an orthogonal projection $P : L^2(\mathbb{C}, d\lambda_\alpha) \to F^2_\alpha$ given by

$$P(f)(z) = \langle f, K_z \rangle = \int_{\mathbb{C}} f(w) K(z, w) d\lambda_\alpha(w), \quad \forall f \in L^2(\mathbb{C}, d\lambda_\alpha), \forall z \in \mathbb{C}. \tag{2.1.2}$$

$F^2_1 = F^2$, i.e., F^2_α with $\alpha = 1$, is known as the classical Fock space. A comprehensive reference on classical Fock spaces is [79].

The above establishes the classical framework that we will generalize in the following sections: first to *doubling* and then to *admissible* (Dall'Ara–type) weights, and finally to vector-valued models relevant for Toeplitz/Hankel theory on scalar and vector-valued Fock spaces.

2.2 Doubling Fock spaces

In an attempt to generalize classical Fock spaces, one may consider weighted L^p spaces. Suppose $\phi : \mathbb{C} \to \mathbb{R}$ is subharmonic. For $0 < p < \infty$, $L^p_\phi = L^p(\mathbb{C}, e^{-p\phi} dA)$ is the space of all measurable functions on $\mathbb{C}$ such that

$$\|f\|^p_{p,\phi} = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} dA(z) < \infty,$$

where dA denotes the Lebesgue measure. L^∞_ϕ is the space of all measurable functions f such that

$$\|f\|_{\infty,\phi} = \operatorname*{ess\,sup}_{z\in\mathbb{C}} |f(z)| e^{-\phi(z)} < \infty.$$

A particularly fruitful setting arises when the Laplacian

$$\mu = \Delta\phi$$

defines a *doubling measure*. A positive Borel measure μ on $\mathbb{C}$ is called doubling if there exists a constant $C > 1$ such that

$$0 < \mu(D(z,2r)) \le C\,\mu(D(z,r)) < \infty, \quad z \in \mathbb{C},\ r > 0, \tag{2.2.1}$$

where $D(z,r)$ is the open disk in $\mathbb{C}$ with center z. The smallest constant $C > 1$ satisfying (2.2.1) is called the doubling constant for μ and is denoted by C_μ. This property gives rise to what are known as *doubling Fock spaces*. That is, the doubling Fock space F^p_ϕ is defined by

$$F^p_\phi = L^p_\phi \cap H(\mathbb{C}),$$

whenever ϕ is a subharmonic function, such that $d\mu = \Delta\phi dA$ is a doubling measure. Canonical examples of such weights are $\phi(z) = |z|^m$, with $m > 0$. See Lemma 2.2.6 for more details.

Doubling measures in more detail

Example 2.2.1. 1. Take $\mathbb{R}^n$ with the Euclidean metric $d(x,y) = |x-y|$ and let μ be the Lebesgue measure. Recall that the Lebesgue measure is translation invariant and the measure of the disk $D(x,r)$ is $\pi^{\frac{n}{2}} r^n/\Gamma(\frac{n}{2}+1)$, and hence independent of x. It is easy to see that μ is doubling and $C_\mu = 2^n$.

2. Let $X = (M,g)$ be a complete Riemannian manifold of dimension n, and let μ be the Riemannian volume measure. Assume that the Ricci curvature is non-negative, i.e. $\mathrm{Ric}_M \ge 0$. In the notation of [23], Proposition 4.1, this corresponds to taking $H = 0$, where $H \in \mathbb{R}$ is the constant in the curvature lower bound

$$\mathrm{Ric}_M \ge (n-1)H.$$

For such an H, the quantity V^H_r denotes the volume of a ball of radius r in the simply connected n-dimensional space form of constant sectional curvature H. In particular, when $H = 0$, V^0_r is the volume of a ball of radius r in $\mathbb{R}^n$.

Applying part (iii) of Proposition 4.1 in [23] with $x = p$, $r_1 = r$, and $r_2 = 2r$, we obtain

$$\frac{\mu(B(p,2r))}{\mu(B(p,r))} \le \frac{V^0_{2r}}{V^0_r} = 2^n,$$

which shows that μ is a doubling measure with doubling constant $C_\mu = 2^n$. Note that $B(p,r) = \{x \in M : d_g(p,x) < r\}$, where d_g is the Riemannian distance induced by the metric g.

3. As shown in [3], example 2.2, any Cantor-type set has a structure of a doubling metric measure space. Fix a finite set F of at least two elements, i.e., $|F| = k \geq 2$, and consider the set of sequences of elements of F

$$F^\infty = \{x = (x_i)_{i\in\mathbb{N}} : x_i \in F\}.$$

Fix $a \in (0,1)$, and let us define the distance

$$d_a(x,y) = \begin{cases} 0 & \text{if } x = y, \\ a^j & \text{if } x_i = y_i \text{ for } i < j \text{ and } x_j \neq y_j. \end{cases}$$

One can see that the diameter of F^∞ is equal to a. That is, $\operatorname{diam}(F^\infty) = \sup\{d_a(x,y) : x,y \in F^\infty\} = a$. The measure is constructed as follows. Take the uniformly distributed probability measure ν of F. That is, $\nu(\{f\}) = 1/k$ for all $f \in F$. Define the measure μ on F^∞ as the product measure of ν infinitely many times. Hence, μ is a probability measure over the totally disconnected set F^∞. Notice that for every $x = (x_1,x_2,x_3,\cdots) \in F^\infty$, $\mu(\{x\}) = 0$. To see this, consider the n-neighborhood of x by fixing the first n coordinates.

$$C_n(x) = \{y \in F^\infty : \quad y_i = x_i; i = 1,\cdots,n\}.$$

These are nested. That is, $C_1(x) \supset C_2(x) \supset \cdots$, and $\{x\} = \cap_{n=1}^\infty C_n(x)$. Since μ is the product measure of ν on each coordinate,

$$\mu(C_n(x)) = \prod_{i=1}^n \nu(\{x_i\}) = (\frac{1}{k})^n.$$

Therefore,

$$\mu(\{x\}) = \mu(\cap_{n=1}^\infty C_n(x)) = \lim_{n\to\infty} \mu(C_n(x)) = \lim_{n\to\infty} (\frac{1}{k})^n = 0.$$

Notice that $\mu(\{x\})$ is nonzero only if ν is concentrated on a single point. This is why it is important to take $|F| \geq 2$. It turns out that

$$\mu(D(x,a^j)) = \mu(C_j(x)) = \frac{1}{k^j},$$

meaning that the chance of the first j coordinates equal a given length-j word is k^{-j}. Hence, (F^∞, d_a, μ) is a doubling metric measure space. To see this, notice that for each $0 < r < a$, there exists some integer j with $a^j \leq r < a^{j-1}$. One can see that when $a = 1$, μ is not doubling. This is because when $a = 1$, every two distinct points have the same distance 1. Balls of radius less than 1 are singletons, and balls of radius greater or equal to 1 are the whole space.

Remark 2.2.2. Let μ be a doubling measure on $\mathbb{C}$. It is well known that μ has no point masses, in particular, for any $z \in \mathbb{C}$ and $r > 0$, $\mu(\partial D(z,r)) = \mu(\{z\}) = 0$, see [75], p. 40.

Moreover, every bounded subset of $\mathbb{C}$ has finite μ-measure. Indeed, if $X \subset \mathbb{C}$ is bounded, then $X \subset D(z,r)$ for some $z \in \mathbb{C}$ and $r > 0$, and hence $\mu(X) \leq \mu(D(z,r)) < \infty$.

On the other hand, probability measures with strictly positive densities on $\mathbb{C}$ are not doubling in this sense. Indeed, suppose that μ is a probability measure on $\mathbb{C}$ with strictly positive density and assume, for contradiction, that μ is doubling with doubling constant C_μ. Since the density is strictly positive, every non-empty disk has positive μ-measure. Also, since $\mu(\mathbb{C}) = 1$, we have

$$\mu(D(0,R)) \to 1 \qquad \text{as } R \to \infty.$$

We first note that if two disks $D(a,R)$ and $D(b,R)$ of the same radius intersect, then

$$\mu(D(a,R)) \leq \mu(D(b,4R)) \leq C_\mu^2 \mu(D(b,R)).$$

Hence such disks have comparable measure, with constants depending only on C_μ. Now take $R_k = 3^k$ and consider the disks

$$D(3R_k, R_k), \qquad k \in \mathbb{N}.$$

These disks are pairwise disjoint. Moreover, by applying the preceding comparability along the chain of overlapping disks

$$D(0,R_k), \quad D(R_k,R_k), \quad D(2R_k,R_k), \quad D(3R_k,R_k),$$

we obtain

$$\mu(D(3R_k,R_k)) \geq C_\mu^{-6} \mu(D(0,R_k)).$$

Since $\mu(D(0,R_k)) \to 1$, there exists k_0 such that

$$\mu(D(3R_k,R_k)) \geq \frac{1}{2} C_\mu^{-6}$$

for all $k \geq k_0$. Therefore,

$$1 = \mu(\mathbb{C}) \geq \sum_{k \geq k_0} \mu(D(3R_k,R_k)) = \infty,$$

which is a contradiction. Thus probability measures with strictly positive densities on $\mathbb{C}$ cannot be doubling. See [3], Remark 2.3.

Let μ be a doubling measure over $\mathbb{C}$. Remark **??** implies that $\mu(\mathbb{C}) = \infty$. Fix $z \in \mathbb{C}$. Since $\{D(z,r)\}_{r>0}$ is an increasing exhaustion of $\mathbb{C}$, $\mu(\mathbb{C}) = \lim_{r\to\infty} \mu(D(z,r)) = \infty$. The function $r \mapsto \mu(D(z,r))$ is an increasing homeomorphism from $(0,\infty)$ to itself. Therefore, for every $z \in \mathbb{C}$, there is a unique positive radius $\rho(z)$ such that $\mu(D(z,\rho(z))) = 1$. We call ρ the radius function associated with μ. From now on, we denote $D(z, r\rho(z))$, $r > 0$, by $D^r(z)$.

Notation. We use C to denote positive constants whose value may change from line to line but does not depend on the functions being considered. We say that $A \simeq B$ if there exists a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$. Moreover, $A \lesssim B$ if $A \leq CB$ for some positive constant C.

Radius function for $\mu = \Delta\phi$ and the corresponding geometric features

Assume that ϕ is a subharmonic function, not identically zero on $\mathbb{C}$, and $d\mu = \triangle\phi\, dA$ is a doubling measure. As shown in [64], ρ^{-2} is a regularization of $\Delta\phi$. Indeed, Theorem 14 in [64] states that when ϕ is subharmonic and $\Delta\phi\, dA$ is a doubling measure, there exists a subharmonic function $\psi \in \mathcal{C}^\infty(\mathbb{C})$ and $C > 0$ such that $|\psi - \phi| \leq C$, $\Delta\psi\, dA$ a doubling measure, and $\Delta\psi \simeq \rho_\psi^{-2} \simeq \rho_\phi^{-2}$. Since the spaces of functions and sequences that we consider do not change if

ϕ is replaced by ψ, we will assume that $\phi \in \mathcal{C}^\infty(\mathbb{C})$ and $\Delta\phi dA \simeq dA/\rho^2$ is a doubling measure. Hence, up to normalization by a constant, we can consider $\rho^{-2}(z)dz\otimes d\bar{z}$ to be the metric tensor describing the underlying geometry of our space. Note that the associated Riemannian metric is conformal, with the conformal factor $\rho(z)^{-1}$. Moreover, the corresponding measure associated with the Riemannian metric is dA/ρ^2, which is a doubling measure as already mentioned. To study the function ρ, we first state an important result due to Christ.

Theorem 2.2.3 ([24], Lemma 2.1 & [68], Lemma 2.1)**.** *Let μ be a doubling measure on $\mathbb{C}$. Then there are constants $C > 1$ and $0 < \delta < 1$, only depending on C_μ, such that if D and D' are open disks of radii r and r' respectively, such that $D \cap D' \neq \emptyset$ and $r' < r$, then*

$$C^{-1}(r'/r)^{1/\delta}\mu(D) \leq \mu(D') \leq C(r'/r)^{\delta}\mu(D).$$

Lemma 2.2.4. *Let $\phi \in C^\infty(\mathbb{C})$ and $d\mu = \Delta\phi dA$ be a doubling measure. Then the radius function ρ satisfies the following properties:*

1. *The function ρ is Lipschitz, and in particular continuous. Indeed, for every $z, w \in \mathbb{C}$, $|\rho(z) - \rho(w)| \leq |z - w|$,*
2. *For every $r > 0$, there is a constant $c_r > 1$ depending only on r and the doubling constant for μ such that*
$$c_r^{-1}\rho(z) \leq \rho(w) \leq c_r\rho(z), \quad \forall z \in \mathbb{C} \text{ and } w \in D^r(z), \tag{2.2.2}$$
where $c_r = (1-r)^{-1}$ for every $0 < r < 1$. In other words, $\rho(w)$ and $\rho(z)$ are equivalent on a disk,
3. *There are constants $C, \eta > 0$ and $0 \leq \beta < 1$ such that*
$$C^{-1}|z|^{-\eta} \leq \rho(z) \leq C|z|^{\beta}, \quad \text{when } |z| > 1. \tag{2.2.3}$$

Proof.

1. It is easy to see that $D(z, \rho(z)) \subset D(w, |w-z| + \rho(z))$. Hence $1 \leq \mu(D(w, |w-z| + \rho(z)))$, and thus $\rho(w) \leq \rho(z) + |w-z|$. By symmetry, $|\rho(z) - \rho(w)| \leq |w-z|$.
2. Let $w \in D^r(z)$ and $z \in \mathbb{C}$. Thus, $|w-z| < r\rho(z)$. Then, part 1 implies that
$$(1-r)\rho(z) \leq \rho(w) \leq (1+r)\rho(z).$$
Hence $c_r = (1-r)^{-1}$ satisfies (2.2.2) for $0 < r < 1$. The case $r > 1$ is more complicated, and one needs to use Theorem 2.2.3. We refer the interested reader to [68], Lemma 2.2.
3. Equation (2.2.3) and a partial proof were stated before equation (5) in [64]. Here the full proof is provided for completeness. First, apply Theorem 2.2.3 to $D = D(0,R)$ and $D' = D(0,1)$ with $R > 1$. There are constants $C > 1$ and $0 < \delta < 1$, only depending on C_μ such that
$$C^{-1}(\frac{1}{R})^{1/\delta}\mu(D(0,R)) \leq \mu(D(0,1)) \leq C(\frac{1}{R})^{\delta}\mu(D(0,R)).$$
Define $\mu_1 = \mu(D(0,1))$. We obtain
$$C^{-1}\mu_1 R^{\delta} \leq \mu(D(0,R)) \leq C\mu_1 R^{1/\delta}. \tag{2.2.4}$$
Now assume that $\rho(z) \leq |z|$. Take $D = D(0,|z|)$ and $D' = D^1(z)$ in Theorem 2.2.3. Then
$$C^{-1}\left(\frac{\rho(z)}{|z|}\right)^{1/\delta}\mu(D(0,|z|)) \leq \mu(D^1(z)) \leq C\left(\frac{\rho(z)}{|z|}\right)^{\delta}\mu(D(0,|z|)).$$

Therefore,

$$\frac{C^{-1/\delta}}{\mu(D(0,|z|))^{1/\delta}} \le \frac{\rho(z)}{|z|} \le \frac{C^{\delta}}{\mu(D(0,|z|))^{\delta}}.$$

This is equivalent to

$$\rho(z) \le C^{\delta}\mu(D(0,|z|))^{-\delta}|z|, \tag{2.2.5}$$

and

$$\rho(z) \ge C^{-1/\delta}\mu(D(0,|z|))^{-1/\delta}|z|. \tag{2.2.6}$$

Using (2.2.4) for $R = |z| > 1$, we get $\mu(D(0,|z|)) \ge C^{-1}\mu_1|z|^{\delta}$, which implies that $\mu(D(0,|z|))^{-\delta} \le (C^{-1}\mu_1)^{-\delta}|z|^{-\delta^2}$. Substituting into (2.2.5), $\rho(z) \le C^{\delta}(C^{-1}\mu_1)^{-\delta}|z|^{1-\delta^2} = C_+|z|^{1-\delta^2}$. Therefore,

$$\rho(z) \le C_+|z|^{\beta}, \quad \text{for } |z| > 1 \text{ with } 0 < \beta = 1-\delta^2 < 1.$$

To obtain a lower bound, we use (2.2.4) to get $\mu(D(0,|z|)) \le C\mu_1|z|^{1/\delta}$, which implies that $\mu(D(0,|z|))^{-1/\delta} \ge (C\mu_1)^{-1/\delta}|z|^{-1/\delta^2}$. Substituting into (2.2.6), we obtain that $\rho(z) \ge C^{-1/\delta}(C\mu_1)^{-1/\delta}|z|^{1-1/\delta^2}$. Note that $1-1/\delta^2$ is negative. So,

$$\rho(z) \ge C_-|z|^{-\eta}, \quad \text{for } |z| > 1 \text{ with } \eta = -(1-1/\delta^2) > 0.$$

Finally, take $\rho(z) > |z|$. Then $0 \in D^1(z)$, and (2.2.2) implies that $\rho(z) \simeq \rho(0)$. Thus, $|z| < C_1$ for some constant C_1. Therefore, we are done with the proof of (2.2.3).

□

Consider the distance d_ϕ induced by the metric tensor $\rho^{-2}dz \otimes d\bar{z}$. Indeed, for any $z, w \in \mathbb{C}$,

$$d_\phi(z,w) = \inf_{\gamma}\int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))}dt,$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Lemma 2.2.5 (See [64], Lemma 4). *There exists $\delta > 0$ such that for every $r > 0$ there exists $C_r > 0$ such that*

$$C_r^{-1}\frac{|z-w|}{\rho(z)} \le d_\phi(z,w) \le C_r\frac{|z-w|}{\rho(z)}, \quad \text{for } w \in D^r(z), \tag{2.2.7}$$

and

$$C_r^{-1}\Big(\frac{|z-w|}{\rho(z)}\Big)^{\delta} \le d_\phi(z,w) \le C_r\Big(\frac{|z-w|}{\rho(z)}\Big)^{2-\delta}, \quad \text{for } w \in \mathbb{C}\setminus D^r(z). \tag{2.2.8}$$

Canonical doubling Fock space F^2_m, $m > 0$

Lemma 2.2.6. *Let $\phi(z) = |z|^m$ with $m > 0$. Then $d\mu = \Delta\phi dA$ is a doubling measure. Moreover, there is an $R > 0$ such that*

$$\rho(z) \simeq |z|^{1-m/2}$$

for $|z| > R$. In particular, when $m \ge 2$, ρ is bounded.

Proof. Note that $\Delta\phi(z) = m^2|z|^{m-2}$. To show that $d\mu$ is a doubling measure, it is enough to prove that for any $x \ge 0$ and $r > 0$,

$$\int_{D(x,2r)} |z|^{m-2}dA(z) \le C\int_{D(x,r)} |z|^{m-2}dA(z), \tag{2.2.9}$$

where the constant C is independent of x and r.

We consider $r > \frac{x}{100} \geq 0$ first. Then $D(x,2r) \subset D(0,x+2r)$, so that

$$\frac{1}{m^2}\int_{D(x,2r)} d\mu(\xi) \leq \int_{|\xi|\leq x+2r} |\xi|^{m-2} dA(\xi) \leq \int_{|\xi|\leq 102r} |\xi|^{m-2} dA(\xi) \leq C_1 r^m, \tag{2.2.10}$$

where C_1 only depends on m. On the other hand, if $m \geq 2$,

$$\int_{D(x,r)} d\mu(\xi) \geq \int_{D(x,r)\cap\{\operatorname{Re}\xi\geq x\}} d\mu(\xi) \geq \int_{D(0,r)\cap\{\operatorname{Re}\xi\geq 0\}} d\mu(\xi) \geq C_2 r^m, \tag{2.2.11}$$

where C_2 only depends on m. From (2.2.10) and (2.2.11) we obtain (2.2.9) for $m \geq 2$ and $r > \frac{x}{100}$.

Now take $0 < r < \frac{x}{100}$ and $m > 0$. Let $w = te^{i\theta} \in D(x,2r)$. Then $x - 2r < t < x + 2r$, and $|\operatorname{Im} w| = |t\sin\theta| < 2r$, and thus, $|\sin\theta| < 2r/t \leq 2r/(x-2r)$. Since $r < x/100$, we have

$$D(x,2r) \subset \{te^{i\theta} : x-2r < t < x+2r, |\theta| < \arcsin\frac{2r}{0.98x}\}.$$

For $D(x,r)$, let $|t-x| \leq r/2$ and $|\theta| < r/2x$. Then, taking $w = te^{i\theta}$,

$$\begin{aligned}|w-x| = |te^{i\theta}-x| &= |te^{i\theta} - xe^{i\theta} + xe^{i\theta} - x| \leq |t-x| + x|e^{i\theta/2}||e^{i\theta/2} - e^{-i\theta/2}| \\ &= |t-x| + 2x|\sin\theta/2| \leq |t-x| + 2x|\theta/2| < r/2 + r/2 = r.\end{aligned}$$

Therefore,

$$D(x,r) \supset \{te^{i\theta} : x - r/2 < t < x + r/2, |\theta| < \frac{r}{2x}\}.$$

Hence, for any $m > 0$,

$$\frac{1}{m^2}\int_{D(x,2r)} d\mu \leq \int_{x-2r}^{x+2r} r^{m-1} dr \int_{-\arcsin\frac{2r}{0.98x}}^{\arcsin\frac{2r}{0.98x}} d\theta = \frac{1}{m}[(x+2r)^m - (x-2r)^m] 2\arcsin\frac{2r}{0.98x}.$$

It is easy to see that for $0 \leq u < 1$, $\arcsin u \leq u/\sqrt{1-u^2}$. In fact, take $\phi(u) = u/\sqrt{1-u^2} - \arcsin u$. Then $\phi(0) = 0$, and

$$\phi'(u) = 1/\sqrt{1-u^2} + \frac{u^2}{(1-u^2)^{3/2}} - 1/\sqrt{1-u^2} \geq 0.$$

Thus, ϕ is non-decreasing, and therefore, $\phi(u) \geq u$ for all $0 \leq u < 1$. Let $u = 2r/0.98x$. Then $u/\sqrt{1-u^2} < u/\alpha$, for $\alpha = 1 - 1/(49^2)$. Hence,

$$\frac{1}{m^2}\int_{D(x,2r)} d\mu \lesssim \frac{1}{m}[(x+2r)^m - (x-2r)^m]\frac{r}{x} \simeq r^2 x^{m-2} \tag{2.2.12}$$

Similarly,

$$\begin{aligned}\frac{1}{m^2}\int_{D(x,r)} d\mu &\geq \int_{x-r/2}^{x+r/2} r^{m-1} dr \int_{-\frac{r}{2x}}^{\frac{r}{2x}} d\theta \\ &= \frac{1}{m}[(x+r/2)^m - (x-r/2)^m]\frac{r}{x} \simeq r^2 x^{m-2}.\end{aligned} \tag{2.2.13}$$

Note that constants in the inequalities $\simeq$ are all independent of x and r. Using (2.2.12) and (2.2.13), we obtain (2.2.9).

Let $0 < m < 2$, and $r > \frac{x}{100}$. For $\xi \in D(x,r)$, $|\xi| < x + r$. Since $t \mapsto t^{m-2}$ is decreasing,

$$\int_{D(x,r)} |\xi|^{m-2} dA(\xi) \geq \int_{D(x,r)} (x+r)^{m-2} dA(\xi) = \pi r^2 (x+r)^{m-2} \geq C_3 r^m. \tag{2.2.14}$$

From (2.2.10) and (2.2.14) we obtain (2.2.9) for $0 < m < 2$ and $r > \frac{x}{100}$.

Now notice that using (2.2.12) and (2.2.13) and when x is large enough,

$$\int_{D(x,x^{-\frac{m-2}{2}})} |\xi|^{m-2} dA(\xi) \simeq 1. \tag{2.2.15}$$

This, together with the doubling property, implies that there exists $R > 0$ large enough, such that for the Fock space F^2_ϕ, with $\phi(z) = |z|^m$, $m > 0$

$$\rho(z) \simeq |z|^{-\frac{m-2}{2}} = |z|^{1-\frac{m}{2}} \tag{2.2.16}$$

for $|z| \geq R$. □

Definition 2.2.7. Let $d\mu = \Delta\phi dA$ be a doubling measure. We say that ϕ is a doubling weight. For $m > 0$, the function $\phi(z) = |z|^m$ is called the canonical doubling weight. The corresponding doubling Fock spaces will be denoted by F^2_m [1].

Characterizing doubling Fock spaces F^p_ϕ: Hilbert, Banach, and quasi-Banach spaces

Let $0 < p \leq \infty$ and ϕ be a subharmonic function on $\mathbb{C}$, not identically zero, such that $d\mu = \Delta\phi dA$ is a doubling measure. Recall that the doubling Fock space F^p_ϕ is the space of entire functions inside L^p_ϕ.

Lemma 2.2.8 ([64], Lemma 18(a) & [68], Lemma 2.4). *Let $0 < p < \infty$. For any $r > 0$ there exists a constant $C > 0$ such that for any $f \in H(\mathbb{C})$ and $z \in \mathbb{C}$,*

$$|f(z)|^p e^{-p\phi(z)} \leq C \int_{D^r(z)} |f(w)|^p e^{-p\phi(w)} \frac{dA(w)}{\rho(w)^2}. \tag{2.2.17}$$

Note that by Lemma 2.2.4, $\rho(w) \simeq \rho(z)$ for $w \in D^r(z)$. Hence, (2.2.17) can also be written as the following for another constant $\tilde{C}$, only depending on r and the doubling constant:

$$|f(z)|^p e^{-p\phi(z)} \leq \frac{\tilde{C}}{\rho(z)^2} \int_{D^r(z)} |f(w)|^p e^{-p\phi(w)} dA(w).$$

Moreover, Lemma 2.2.8 implies that for any $0 < p \leq \infty$,

$$|f(z)| \lesssim \frac{e^{\phi(z)}}{\rho(z)^{2/p}} \|f\|_{p,\phi}, \quad \forall z \in \mathbb{C}, f \in F^p_\phi. \tag{2.2.18}$$

Let $C(z) = e^{\phi(z)}/\rho(z)^{2/p}$. Notice that since $d\mu = \Delta\phi dA$ is doubling, $\rho(z)$ is never zero. In fact, recalling that $\mu(D(z,\rho(z))) = 1$ for every $z \in \mathbb{C}$, if there exists a point $z \in \mathbb{C}$ with $\rho(z) = 0$, then $\mu(D(z,\rho(z))) = \mu(\{z\}) = 1$, which contradicts the fact that doubling measures cannot have a point mass. One can see that for any compact subset $K \subset \mathbb{C}$ and any $z \in K$, $C(z)$ is continuous and thus bounded. Therefore, (2.2.18) implies that the point evaluation map $f \mapsto f(z)$ is a bounded linear functional on F^p_ϕ, and also uniformly bounded in bounded domains in $\mathbb{C}$. Since locally

---

[1] The notation F^2_m refers to the canonical doubling Fock space with weight $\phi(z) = |z|^m$, and should not be confused with the standard Fock space F^2_α defined in Section 2.1. In particular, $F^2_m \neq F^2_\alpha$ with $\alpha = m$, since the underlying weights are $e^{-|z|^m}$ and $e^{-\alpha|z|^2}$, respectively. This convention is standard in the study of doubling Fock spaces.

uniform limits of holomorphic functions are holomorphic, F^p_ϕ is a closed subspace of L^p_ϕ for every $0 < p \leq \infty$.

Note that L^p_ϕ is a Banach space for $1 \leq p \leq \infty$, a quasi-Banach space for $0 < p < 1$, and a Hilbert space when $p = 2$, with respect to the following inner product:

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-2\phi(z)}dA(z), \quad f, g \in L^2_\phi. \tag{2.2.19}$$

In fact, it is well-known that when $1 \leq p < \infty$, $L^p(\mathbb{C}, d\nu, \|\cdot\|_\nu)$ is a complete metric space with respect to any positive measure ν (See [71], Theorem 3.11). Taking $d\nu = e^{-p\phi}dA$ we obtain L^p_ϕ. When $p = \infty$, consider the linear map $T : L^\infty_\phi \to L^\infty$, with $T(f)(z) = f(z)e^{-\phi(z)}$. Then $\|Tf\|_\infty = \|f\|_{\infty,\phi}$. Since L^∞ is complete and T is an isometry onto its range, the domain L^∞_ϕ is complete as well and hence a Banach space. When $0 < p < 1$, $\|\cdot\|_{p,\phi}$ fails the triangle inequality and therefore is not a norm. It is in fact, a quasi-norm and satisfies the following properties:

1. $\|f\|_{p,\phi} = 0$ if and only if $f = 0$ almost everywhere,
2. $\|\alpha f\|_{p,\phi} = |\alpha|\|f\|_{p,\phi}$, for all $\alpha \in \mathbb{C}$,
3. It satisfies the p-subadditivity inequality. That is, $\|f+g\|^p_{p,\phi} \leq \|f\|^p_{p,\phi} + \|g\|^p_{p,\phi}$. In particular, this implies that $\|\cdot\|_{p,\phi}$ is a quasi-norm, i.e., there exists a constant $K > 0$ such that
$$\|f + g\|_{p,\phi} \leq K\Big(\|f\|_{p,\phi} + \|g\|_{p,\phi}\Big).$$

Note that the p-subadditivity inequality holds due to the fact that when $0 < p < 1$ and $a, b \geq 0$, $(a + b)^p \leq a^p + b^p$.

Recall that a closed subspace of a Banach space is itself a Banach space. Similarly, any closed subspace of a quasi-Banach space is itself a quasi-Banach space. Therefore, $(F^p_\phi, \|\cdot\|_{p,\phi})$ is a Banach space for $1 \leq p \leq \infty$ and a quasi-Banach space for $0 < p < 1$. Take $p = 2$. Further, F^2_ϕ is a Hilbert space over $\mathbb{C}$ such that the point evaluation map $\mathrm{ev}_z : F^2_\phi \to \mathbb{C}$, $f \mapsto f(z)$, is a bounded linear functional. Then the Riesz representation theorem implies that there exists a unique element $K_z \in F^2_\phi$ such that

$$\mathrm{ev}_z(f) = f(z) = \langle f, K_z\rangle, \quad \forall f \in F^2_\phi. \tag{2.2.20}$$

In particular, $K_w(z) = \langle K_w, K_z\rangle = \overline{K_z(w)}$, for every $z, w \in \mathbb{C}$. The function $K_z(\cdot) = K(\cdot, z)$ is called the reproducing kernel or Bergman kernel for F^2_ϕ at $z \in \mathbb{C}$.

Bergman kernel and its pointwise estimates

Notice that, unlike the classical Fock space, and since we do not have enough knowledge about ϕ, there are no explicit formulas for the basis vectors of F^2_ϕ. Similarly, one cannot find a closed form for the reproducing kernel. Later, we will see that taking $\phi(z) = |z|^m$, $m > 0$, such formulas exist. For now, take a generic subharmonic ϕ, which is not identically zero, and such that $\Delta\phi dA$ is a doubling measure. Although an explicit form for the kernel cannot be found, valuable information can still be obtained regarding the behavior of the kernel on and off the diagonal. This is based on the seminal work of Marzo and Ortega-Cerda in [65]. In the following, we state their results on the behavior of the kernel $K(z, w)$ when $w = z$, w is very close to z, and general z and w.

Proposition 2.2.9 ([65], Proposition 2.10). *There exists $C > 0$ such that*

$$C^{-1}\frac{e^{2\phi(z)}}{\rho(z)^2} \leq K(z,z) \leq C\frac{e^{2\phi(z)}}{\rho(z)^2}, \quad \forall z \in \mathbb{C}. \tag{2.2.21}$$

Recall that $K(z,z) = K_z(z) = \langle K_z, K_z\rangle = \|K_z\|_{2,\phi}^2$. Thus, $\|K_z\|_{2,\phi} \simeq e^{\phi(z)}/\rho(z)$ for all $z \in \mathbb{C}$.

Theorem 2.2.10 ([65], Theorem 1.1, Proposition 2.11 & [68], Theorem 2.6). *There exist constants $C > 0$ and $\epsilon > 0$, depending only on the doubling constant, such that*

$$|K(z,w)| \leq C\frac{1}{\rho(z)\rho(w)}\frac{e^{\phi(z)+\phi(w)}}{\exp(d_\phi(z,w)^\epsilon)}, \quad \forall z,w \in \mathbb{C}. \tag{2.2.22}$$

Furthermore, there is $r_0 > 0$ such that

$$|K_z(w)| \simeq \|K_z\|_{2,\phi}\|K_w\|_{2,\phi} \simeq \frac{e^{\phi(z)+\phi(w)}}{\rho(z)^2}, \quad z \in \mathbb{C}, w \in D^{r_0}(z). \tag{2.2.23}$$

Notice that using Lemma 2.2.5, we can write (2.2.22) as

$$|K(z,w)| \leq C\frac{e^{\phi(z)+\phi(w)}}{\rho(z)\rho(w)}e^{-(\frac{|z-w|}{\rho(z)})^\epsilon}, \quad \forall z,w \in \mathbb{C}. \tag{2.2.24}$$

However, the above estimate does not recover the symmetry in the variables z and w at first glance. The following lemma will be used to prove the L^p_ϕ norm estimate of the Bergman kernel.

Lemma 2.2.11 ([68], Lemma 2.7, Lemma 2.8).

1. *For every $\epsilon > 0$, $k \geq 0$, and $r \geq 1$, there is a constant $C_{\epsilon,k}(r) > 0$ such that*

   $$\int_{\mathbb{C}\setminus D^r(z)} \frac{|w-z|^k}{\exp(d_\phi(w,z)^\epsilon)}\frac{dA(w)}{\rho(w)^2} \leq C_{\epsilon,k}(r)\rho(z)^k, \quad \forall z \in \mathbb{C}.$$

   Moreover, $C_{\epsilon,k}(r) \to 0$ as $r \to \infty$, for any $\epsilon > 0$ and $k \geq 0$.

2. *For every $r \geq 1$, there is a constant $C(r) > 0$ such that*

   $$\int_{\mathbb{C}\setminus D^r(z)} |K_z(w)|e^{-\phi(w)}dA(w) \leq C(r)e^{\phi(z)}, \quad \forall z \in \mathbb{C},$$

   and $C(r) \to 0$ as $r \to \infty$.

3. *There exists a constant $C > 0$ such that*

   $$\int_{\mathbb{C}} |K_z(w)|e^{-\phi(w)}dA(w) \leq Ce^{\phi(z)}, \quad \forall z \in \mathbb{C}.$$

Proposition 2.2.12 ([68], Proposition 2.9 & [43], Lemma 2.3). *For any $1 \leq p \leq \infty$, we have*

1. $$\|K_z\|_{p,\phi} \simeq e^{\phi(z)}\rho(z)^{\frac{2}{p}-2}, \quad \forall z \in \mathbb{C}, \tag{2.2.25}$$

2. *Let $k_{p,z} = K_z/\|K_z\|_{p,\phi}$ be the normalized Bergman kernel of F^p_ϕ. The set $\{k_{p,z} : z \in \mathbb{C}\}$ is bounded in F^p_ϕ and $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $|z| \to \infty$.*

Proof. 1. This is Proposition 2.9 in [68], but we state the proof due to its importance. The estimate $\gtrsim$ follows from (2.2.23), (2.2.2), and the observation that the Lebesgue measure of the disk $D^{r_0}(z)$, which from now on, we refer to it by $|D^{r_0}(z)|$ is comparable to $\rho(z)^2$. To prove the other direction, first take $p = 1$. This is Lemma 2.2.11, part 3. When $p = \infty$, notice that using Lemma 2.2.8 and part 3 of Lemma 2.2.11, one obtains

$$|K_z(w)|e^{-\phi(w)} = e^{-\phi(w)+\phi(z)}|K_w(z)|e^{-\phi(z)} \lesssim \frac{e^{-\phi(w)+\phi(z)}}{\rho(z)^2}\int_{D^1(z)}|K_w(\xi)|e^{-\phi(\xi)}dA(\xi) \lesssim \frac{e^{\phi(z)}}{\rho(z)^2}.$$

For $1 < p < \infty$, using the case $p = \infty$ and Lemma 2.2.11, we have

$$\begin{aligned}\int_{\mathbb{C}}|K_z(w)|^p e^{-p\phi(w)}dA(w) &= \int_{\mathbb{C}}|K_z(w)|e^{-\phi(w)}|K_z(w)|^{p-1}e^{-(p-1)\phi(w)}dA(w)\\ &\lesssim \frac{e^{(p-1)\phi(z)}}{\rho(z)^{2p-2}}\int_{\mathbb{C}}|K_z(w)|e^{-\phi(w)}dA(w) = e^{p\phi(z)}\rho(z)^{2-2p}.\end{aligned}$$

2. This is Lemma 2.3 in [43]. By definition, $\|k_{p,z}\| = 1$. Using (2.2.3), write

$$c = \begin{cases} -\eta(1-2/p) & \text{if } p < 2,\\ \beta(1-2/p) & \text{if } p \geq 2. \end{cases}$$

By part 1, (2.2.24), and (2.2.3), we have

$$\begin{aligned}|k_{p,z}(w)| = |K_z(w)|e^{-\phi(z)}\rho(z)^{2-2/p} &\leq Ce^{\phi(w)}\rho(w)^{-1}\rho(z)^{1-2/p}e^{(\frac{|z-w|}{\rho(z)})^{\epsilon}}\\ &\leq Ce^{\phi(w)}\rho(w)^{-1}|z|^{c}e^{(\frac{|z|-|w|}{\rho(w)})^{\epsilon}}.\end{aligned}$$

Hence, $k_{p,z} \to 0$ uniformly on any compact subset of $\mathbb{C}$ as $|z| \to \infty$.

□

Bergman Projection

Recall that F^2_ϕ is a closed subspace of the Hilbert space L^2_ϕ. Therefore, there exists a unique orthogonal projection $P : L^2_\phi \to F^2_\phi$, that is, a bounded linear operator such that for every $f \in L^2_\phi$,

$$Pf \in F^2_\phi \quad \text{and} \quad f - Pf \perp F^2_\phi.$$

In particular, P satisfies $P^2 = P$, $P(L^2_\phi) = F^2_\phi$, and $\|P\| = 1$. We call P the Bergman projection. Recalling the reproducing kernel property of F^2_ϕ, (2.2.20), one can write the Bergman projection $P : L^2_\phi \to F^2_\phi$ as an integral operator given by

$$Pf(z) = \int_{\mathbb{C}} f(w)\overline{K_z(w)}e^{-2\phi(w)}dA(w), \quad f \in L^2_\phi, z \in \mathbb{C}. \tag{2.2.26}$$

Theorem 2.2.13 ([68], Theorem 3.1). *P is a bounded linear operator from L^p_ϕ to F^p_ϕ, for any $1 \leq p \leq \infty$.*

Corollary 2.2.14 ([68], Corollary 3.2). *If $1 \leq p \leq \infty$, and q is the conjugate exponent of p, then*

$$\langle Pf, g\rangle = \langle f, Pg\rangle, \quad \forall f \in L^p_\phi, g \in L^q_\phi.$$

Theorem 2.2.15 ([68], Theorem 3.3). *Let* $1 \le p \le \infty$. *Then* $f = Pf$ *for every* $f \in F^p_\phi$.

Observe that Theorem 2.2.13 together with Theorem 2.2.15 implies that P is a bounded projection of L^p_ϕ onto F^p_ϕ. That is, $P : L^p_\phi \to F^p_\phi$ is a bounded linear operator such that $P \circ P = P$ and $P(L^p_\phi) = F^p_\phi$, for any $1 \le p \le \infty$.

Proposition 2.2.16. *Let* $1 \le p < \infty$ *and* q *be its Hölder conjugate. Then* $(L^p_\phi)^*$ *is isomtrically isomorphic to* L^q_ϕ *by the integral pairing* $\langle \cdot, \cdot \rangle$ *defined by* (2.2.19).

Proof. Define the surjective isometric map

$$U : L^p_\phi \to L^p(\mathbb{C}, dA); \quad f \mapsto f e^{-\phi}.$$

Then for $f \in L^p_\phi$, $\|f\|_{L^p_\phi} = \|Uf\|_{L^p(\mathbb{C},dA)}$. Let us define $\tilde{f} = Uf = fe^{-\phi}$, and $\tilde{g} = Ug = ge^{-\phi}$. Then

$$\langle f, g \rangle = \int_{\mathbb{C}} f \bar{g} e^{-2\phi} dA = \int_{\mathbb{C}} \tilde{f} \bar{\tilde{g}} dA.$$

By the classical duality for the unweighted $L^p(\mathbb{C}, dA)$, we know that

$$(L^p(\mathbb{C}, dA))^* \cong L^q(\mathbb{C}, dA).$$

Therefore, translating back by U^{-1} gives

$$(L^p_\phi)^* \cong L^q_\phi.$$

□

Theorem 2.2.17 ([68], Theorem 3.6). *Let* $1 \le p < \infty$ *and* q *be its Hölder conjugate. Then* $(F^p_\phi)^*$ *can be identified with* F^q_ϕ *(with equivalent norms) by means of the integral pairing* $\langle \cdot, \cdot \rangle$ *given by* (2.2.19). *Namely, the mapping*

$$g \in F^q_\phi \mapsto \langle \cdot, g \rangle \in (F^p_\phi)^*$$

is an antilinear isomorphism.

Theorem 2.2.18. *The linear span of all the reproducing kernels* K_z, $z \in \mathbb{C}$, *is dense in* F^p_ϕ, *for every* $1 \le p < \infty$.

Proof. A partial proof was given in [68], Corollary 3.7. Here a full proof is given for completeness. Let $1 \le p < \infty$. Lemma 3.4 in [68] states that for every $f \in F^p_\phi$, there is a sequence $\{f_n\}_{n \ge 1}$ of functions in $F^p_\phi \cap F^2_\phi$ such that $\lim_{n\to\infty} \|f_n - f\|_{p,\phi} = 0$. By the argument after (2.2.18), the point evaluation map $ev_z : F^p_\phi \to \mathbb{C}$, given by $ev_z(f) = f(z)$ is a bounded linear functional. Take $f \in F^p_\phi$. We can write

$$|f_n(z) - f(z)| = |ev_z(f_n - f)| \le \|ev_z\| \|f_n - f\|_{p,\phi}.$$

Since $\|f_n - f\|_{p,\phi} \to 0$ and ev_z is bounded, $f_n(z) \to f(z)$ as $n \to \infty$.

Moreover, for each $n \ge 1$, since $f_n \in F^2_\phi$, the reproducing property holds. That is, $f_n(z) = \langle f_n, K_z \rangle$, for all $z \in \mathbb{C}$. Now take $g \in F^q_\phi$, where q is the Hölder conjugate of p. Recalling Theorem 2.2.17, and applying Hölder's inequality, one obtains

$$\begin{aligned} |\langle f, g \rangle| &= \left| \int_{\mathbb{C}} f(w) e^{-\phi(w)} \overline{g(w)} e^{-\phi(w)} dA(w) \right| \\ &\le \left(\int_{\mathbb{C}} |f(w)|^p e^{-p\phi(w)} dA(w) \right)^{1/p} \left(\int_{\mathbb{C}} |g(w)|^q e^{-q\phi(w)} dA(w) \right)^{1/q} \qquad (2.2.27) \\ &= \|f\|_{p,\phi} \|g\|_{q,\phi}. \end{aligned}$$

We will show that $\langle\cdot,\cdot\rangle$ is continuous in both variables. Recall that in any Banach space, a functional is continuous if and only if it is bounded. Define $L_f : F^q_\phi \to \mathbb{C}$, by $L_f(g) = \langle f, g\rangle$. By (2.2.27), $|L_f(g)| \le C_1\|g\|_{q,\phi}$, with $C_1 = \|f\|_{p,\phi}$. Hence, L_f is continuous. Similarly, letting $L_g : F^p_\phi \to \mathbb{C}$, by $L_g(f) = \langle f, g\rangle$, we can see that L_g is continuous. Therefore, the pairing $(f,g) \mapsto \langle f, g\rangle$ is continuous in both f and g.

By proposition 2.2.12, $K_z \in F^q_\phi$ for every $z \in \mathbb{C}$. Applying Hölder's inequality once more,

$$|\langle f_n - f, K_z\rangle| \le \|f_n - f\|_{p,\phi}\|K_z\|_{q,\phi} \to 0, \quad \text{as } n \to \infty.$$

Thus, for any $f \in F^p_\phi$, the arguments above imply that

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \langle f_n, K_z\rangle = \langle f, K_z\rangle.$$

One of the consequences of the Hahn-Banach theorem is that if X is a normed vector space and M is a linear subspace of X that is not necessarily closed, then M is dense in X if and only if the only bounded linear functional on X that annihilates M is the zero functional. As a reference, see [26], Corollary 6.14. Furthermore, Theorem 2.2.17 implies that every bounded linear functional on F^p_ϕ is of the form $f \mapsto \langle f, g\rangle$ for some $g \in F^q_\phi$. Now we are ready to show that the linear span of $\{K_z : z \in \mathbb{C}\}$ is dense in F^p_ϕ. Let $g \in (F^p_\phi)^* \cong F^q_\phi$ satisfy $\langle f, g\rangle = 0$ for all f in the linear span of $\{K_z : z \in \mathbb{C}\}$. Then in particular, $g(z) = \langle g, K_z\rangle = 0$ for all $z \in \mathbb{C}$, and thus $g = 0$ identically. Hence, by the Hahn-Banach theorem, the linear span of $\{K_z : z \in \mathbb{C}\}$ is dense in F^p_ϕ. □

Toeplitz and Hankel operators

Recall the Bergman projection $P : L^2_\phi \to F^2_\phi$ given by (2.2.26). Let $\Gamma = \operatorname{span}\{K_z : z \in \mathbb{C}\}$. We have seen in Theorem 2.2.18 that Γ is dense in F^2_ϕ. Consider the class of symbols

$$\mathcal{S} = \{f \text{ measurable on } \mathbb{C} \,:\, fg \in L^2_\phi \text{ for } g \in \Gamma\}.$$

Note that since $K_z \in F^2_\phi$, $L^\infty \subset \mathcal{S}$. Given $f \in \mathcal{S}$ and $g \in \Gamma$, we define the Toeplitz operator T_f and the Hankel operator H_f acting on F^2_ϕ by

$$T_f g(z) = P(fg)(z) = \int_{\mathbb{C}} f(w)g(w)\overline{K_z(w)}e^{-2\phi(w)}dA(w), \quad z \in \mathbb{C}, \tag{2.2.28}$$

and

$$H_f g = (I - P)(fg) = fg - P(fg). \tag{2.2.29}$$

Since $\Gamma \subset F^2_\phi$ is dense, both T_f and H_f are densely defined on F^2_ϕ.

In summary, doubling Fock spaces generalize the classical setting by replacing the strict Gaussian structure with a geometric growth condition on $\Delta\phi$. This condition is flexible enough to allow weights of very different shapes while still supporting a rich function-theoretic and operator-theoretic theory, including sharp Bergman kernel estimates and precise mapping properties of Toeplitz and Hankel operators.

2.3 Scalar weighted Fock spaces- Dall'Ara's weights

To generalize the notion of doubling Fock spaces to the higher-dimensional complex plane, Dall'Ara in [30] introduced the set of admissible weights ϕ as in the definition below. We recall that for $\phi : \mathbb{C}^n \to \mathbb{R}$, the real Laplacian is

$$\Delta\phi(z) = \sum_{j=1}^{n} \left(\frac{\partial^2 \phi}{\partial x_j^2}(z) + \frac{\partial^2 \phi}{\partial y_j^2}(z) \right), \tag{2.3.1}$$

where $z_j = x_j + iy_j$.

Definition 2.3.1. Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ plurisubharmonic function (see, e.g., [55, Chapter 2]). We say that ϕ belongs to the weight class $\mathcal{W}$ of admissible weights if ϕ satisfies the following statements:

(I) There exists $c > 0$ such that

$$\inf_{z \in \mathbb{C}^n} \sup_{\xi \in D(z,c)} \Delta\phi(\xi) > 0, \tag{2.3.2}$$

where $D(z,c)$ is the Euclidean disk centered at z with radius c,

(II) $\Delta\phi$ satisfies the strongest form of the reverse-Hölder inequality. That is, there exists a positive real number C such that

$$\|\Delta\phi\|_{L^\infty(D(z,r))} \le Cr^{-2n} \int_{D(z,r)} \Delta\phi(\xi)\, dA(\xi), \quad \text{for any } z \in \mathbb{C}^n \text{ and } r > 0,$$

(III) the eigenvalues of H_ϕ are comparable (see Part 3 of "Comparing admissible and doubling weights" below for a detailed discussion), i.e., there exists a $\delta_0 > 0$ such that

$$\langle H_\phi(z)u, u\rangle \ge \delta_0 \Delta\phi(z)|u|^2, \quad \text{for any } u, z \in \mathbb{C}^n,$$

where the Hessian matrix of ϕ is given by

$$H_\phi(z) = \left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(z) \right)_{j,k=1}^{n}. \tag{2.3.3}$$

Comparing admissible and doubling weights

1. The lower non-degeneracy condition (I) prevents $\Delta\phi$ from vanishing identically on large regions. Recalling Lemma 2.2.6, $\phi(z) = |z|^m$, $m > 0$ is a doubling weight on $\mathbb{C}$. We will see in Remark 2.3.7 that property (I) implies that ρ is bounded, and that when $m = 1$ the radius function of $\phi(z) = |z|$ is not bounded. Hence, not every doubling measure is admissible.

2. The reverse-Hölder inequality (II) is stronger than doubling. It says that local L^∞ norms of $\Delta\phi$ are controlled by local averages. This condition implies a doubling property. The proof is given in Proposition 2.3.2.

3. To study property (III), recall $\Delta\phi$ given in (2.3.1). To write the Laplacian in the complex form, consider the derivatives

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right). \tag{2.3.4}$$

Then the complex Hessian is given by (2.3.3). Its trace gives

$$\operatorname{Tr} H_\phi(z) = \sum_{j=1}^{n} \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_j}(z) = \frac{1}{4}\Delta\phi(z).$$

Property (III) is specific to higher dimensions $n > 1$. In dimension one, this property is automatic, as the Hessian H_ϕ is just $\Delta\phi/4$. Let us study this property in higher dimensions in more detail. So, take $n > 1$. Note that H_ϕ is an $n \times n$ Hermitian, positive definite matrix. In fact, since ϕ is real-valued and $\mathcal{C}^2$,

$$\overline{\frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z)} = \frac{\partial^2\phi}{\partial z_k \partial \bar{z}_j}(z).$$

Hence, the entries $h_{jk} = \overline{h_{kj}}$, and thus $H_\phi(z)^* = H_\phi(z)$. That is, $H_\phi(z)$ is Hermitian. To show that $H_\phi(z)$ is positive definite, take $u = (u_1, \cdots, u_n) \in \mathbb{C}^n$. Then

$$\langle H_\phi(z)u, u\rangle = \sum_{j,k=1}^{n} \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z) u_j \bar{u}_k.$$

For $z \in \mathbb{C}^n$, consider the complex line in the direction u. That is, take $\gamma(\zeta) = z + \zeta u$, for $\zeta \in \mathbb{C}$. Define the function $\psi(\zeta) = \phi(z + \zeta u)$. Since ϕ is plurisubharmonic, its restriction to any complex line is subharmonic. In particular, $\Delta\psi \geq 0$. Notice that by the chain rule,

$$\frac{\partial\psi}{\partial\zeta}(\zeta) = \sum_{j=1}^{n} \frac{\partial\phi}{\partial z_j}(z + \zeta u)u_j, \quad \text{and} \quad \frac{\partial\psi}{\partial\bar{\zeta}}(\zeta) = \sum_{k=1}^{n} \frac{\partial\phi}{\partial \bar{z}_k}(z + \zeta u)\bar{u}_k,$$

and thus

$$\frac{\partial^2\psi}{\partial\zeta\partial\bar{\zeta}}(\zeta) = \sum_{j,k=1}^{n} \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z + \zeta u)u_j \bar{u}_k.$$

Hence, $\Delta\psi(0) = \langle H_\phi(z)u, u\rangle \geq 0$, and therefore $H_\phi(z)$ is positive definite. Let $\lambda_1(z), \cdots, \lambda_n(z) \geq 0$ be the eigenvalues of $H_\phi(z)$. Hence, $\Delta\phi(z) = 4\sum_{j=1}^{n} \lambda_j(z)$. Interpreting property (III) in terms of eigenvalues,

$$u^* H_\phi(z) u \geq 4\delta_0 \operatorname{Tr} H_\phi(z)|u|^2.$$

The smallest eigenvalue of $H_\phi(z)$ is $\lambda_{min}(z) = \inf_{|u|=1} u^* H_\phi(z) u$. Hence,

$$\lambda_{min} \geq 4\delta_0 \sum_{j=1}^{n} \lambda_j(z).$$

Since $\lambda_{Max}(z) \leq \operatorname{Tr} H_\phi(z) \leq \Delta\phi(z)$, this implies that

$$\lambda_{Max}(z) \lesssim \lambda_{min}(z),$$

so, all eigenvalues are comparable.

4. As an example of Property (III), consider $\phi(z) = |z|^2$ on $\mathbb{C}^n$. Then the complex Hessian satisfies $H_\phi(z) = I_n$, the $n \times n$ identity matrix, so all eigenvalues are equal to 1, and hence

$$\sum_{j=1}^{n} \lambda_j(z) = n.$$

Therefore,

$$\lambda_{\min}(z) = 1 = \frac{1}{n}\sum_{j=1}^{n} \lambda_j(z) = \frac{1}{4n}\Delta\phi(z), \quad \forall z \in \mathbb{C}^n,$$

and Property (III) holds with $\delta_0 = 1/(4n)$.

A simple example for which Property (III) fails is

$$\phi(z_1, z_2) = |z_1|^4 + |z_2|^2, \qquad (z_1, z_2) \in \mathbb{C}^2.$$

Then ϕ is $\mathcal{C}^2$ and plurisubharmonic, and

$$H_\phi(z) = \begin{pmatrix} 4|z_1|^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the eigenvalues of $H_\phi(z)$ are $4|z_1|^2$ and 1, so $\lambda_{\min}(z) = \min\{4|z_1|^2, 1\}$. Moreover, $\Delta\phi(z) = 4\,\mathrm{Tr}\, H_\phi(z) = 16|z_1|^2 + 4$. In particular, when $z_1 = 0$ we have $\lambda_{\min}(z) = 0$ while $\Delta\phi(z) = 4$, so there is no constant $\delta_0 > 0$ such that

$$\langle H_\phi(z)u, u\rangle \geq \delta_0 \Delta\phi(z)|u|^2 \qquad \text{for all } z, u \in \mathbb{C}^2.$$

Therefore, Property (III) does not hold for this weight.

5. Here we give a geometric interpretation of property (III) to the interested readers. Let

$$L_\phi(z, u) := \langle H_\phi(z)u, u\rangle = \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(z) u_j \bar{u}_k.$$

Along a unit vector u, $L_\phi(z, u)$ gives a value between the smallest and the largest eigenvalues of $H_\phi(z)$. This is because $H_\phi(z)$ is Hermitian, and, in particular, self-adjoint and diagonalizable. In fact,

$$\lambda_{min}(z) \leq L_\phi(z, u) \leq \lambda_{Max}(z).$$

So, $L_\phi(z, u)$ can vary a lot in different directions if the eigenvalues are very unequal. However, property (III) guarantees that this cannot happen. Indeed, L_ϕ is closely related to the curvature of the spaces, and property (III) states that the "space" bends the same way in every complex direction, up to a fixed multiplicative constant δ_0. But what is the "space" we are talking about? Weighted Fock spaces can be seen as holomorphic sections of a line bundle. Consider the trivial line bundle $L = \mathbb{C}^n \times \mathbb{C}$. Give it a Hermitian metric $h(z) = e^{-2\phi(z)}$. A holomorphic function f on $\mathbb{C}^n$ can be seen as a section of this bundle. Its L^2-norm is

$$||f||^2 = \int_{\mathbb{C}^n} |f(z)|^2 h(z) dA(z),$$

which is exactly the norm of f in F^2_ϕ. In differential geometry, curvature is given by the curvature form of a connection on a vector bundle. For a complex manifold M

and a Hermitian holomorphic line bundle $L \to M$, there exists a unique connection ∇ on M which is compatible with both the Hermitian and holomorphic structures. This connection ∇ is called the Chern connection of $L \to M$. Hence, the metric $h = e^{-2\phi}$ induces a natural Chern connection on the weighted Fock space. Its curvature is precisely $\theta_h(z) = \bar{\partial}\partial(\log h) = \partial\bar{\partial}(2\phi) = 2\sum_{j,k=1}^n \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z) dz_j \wedge d\bar{z}_k$. For example, when $\phi(z) = |z|^2$, $H_\phi(z) = I_n$, $\theta_h = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, and thus the curvature is constant and the same in every direction. Finally, θ_h and $L_\phi(z,u)$ are two faces of the same objects. Given $u \in \mathbb{C}^n$, let

$$\alpha = \sum_{j=1}^n u_j \frac{\partial}{\partial z_j}, \quad \text{and } \bar{\alpha} = \sum_{k=1}^n \bar{u} \frac{\partial}{\partial \bar{z}_k}.$$

Then

$$\theta_h(z)(\alpha, \bar{\alpha}) = 2L_\phi(z,u).$$

For more information on complex line bundles, Chern connection and curvature, see Chapter 3 of [56].

Proposition 2.3.2. *Let ϕ be an admissible weight as in Definition 2.3.1. Then the reverse-Hölder inequality (II) implies that $d\mu = \Delta\phi dA$ is a doubling measure.*

Proof. To prove this, we use characterizations of A_∞ weights. We denote by A_∞ the Muckenhoupt class

$$A_\infty := \bigcup_{1<p<\infty} A_p,$$

where A_p denotes the Muckenhoupt weight class introduced in Remark 3.1.23. Theorem 9.3.3 in [40] states that a weight ω is in A_∞ if and only if there exist $C > 0$ and $\epsilon < \infty$ such that for all balls B and all measurable subsets E of B,

$$\frac{\omega(E)}{\omega(B)} \leq C\left(\frac{|E|}{|B|}\right)^\epsilon.$$

So, let B be any ball and $E \subset B$ measurable. By the reverse-Hölder inequality,

$$\mu(E) = \int_E \Delta\phi(w) dA(w) \leq |E| \|\Delta\phi\|_{L^\infty(B)} \leq C|E|\frac{\mu(B)}{|B|} = C\mu(B)\frac{|E|}{|B|}.$$

Therefore,

$$\frac{\mu(E)}{\mu(B)} \leq C\left(\frac{|E|}{|B|}\right),$$

implying that $\mu = \Delta\phi$ belongs to A_∞. Proposition 9.3.2 in [40] states that if $\omega \in A_\infty$, then ωdA is doubling. Hence, we can conclude that $d\mu = \Delta\phi dA$ is a doubling measure. □

Remark 2.3.3. The standard symplectic form on $\mathbb{C}^n$ is of the form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$. Let $d = \partial + \bar{\partial}$ be the exterior derivative, and $d^c = i/2(\bar{\partial} - \partial)$. The exterior derivative satisfies $d^2 = 0$. Hence, $\partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0$. Basic differential geometry implies that $\partial^2 = 0$, $\partial\bar{\partial} + \bar{\partial}\partial = 0$, and $\bar{\partial}^2 = 0$. Hence, $dd^c = i\partial\bar{\partial}$. Let $\psi(z) = |z|^2 = \sum_{j=1}^n z_j\bar{z}_j$. Then $\bar{\partial}\psi = \sum_{j=1}^n z_j d\bar{z}_j$, and $\partial\bar{\partial}\psi = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$. Therefore, $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j = i/2\sum_{j=1}^n dz_j \wedge d\bar{z}_j = 1/2dd^c|z|^2$. Now, let $\phi \in \mathcal{C}^2(\mathbb{C}^n)$, with $dd^c\phi \simeq \omega_0$. It is easy to see that ϕ belongs to the weight class $\mathcal{W}$. To see this, note that there are constants $0 < A \leq B < \infty$ such that $A\omega_0 \leq dd^c\phi \leq B\omega_0$ on $\mathbb{C}^n$. Equivalently, there are constants $a, b > 0$ such that for every $z \in \mathbb{C}^n$, the eigenvalues $\lambda_1(z), \cdots, \lambda_n(z)$ of $H_\phi(z)$ satisfy $a \leq \lambda_j(z) \leq b$, for all $1 \leq j \leq n$. This is the case since ω_0 corresponds to the identity matrix in the standard coordinates. Recall that $\Delta\phi(z) = 4\sum_{j=1}^n \lambda_j(z)$. Then for every $z \in \mathbb{C}^n$, $\Delta\phi(z) \geq 4na > 0$, which implies (I) in definition 2.3.1. To see (II), note that for every $z \in \mathbb{C}^n$ and $r > 0$,

$$\|\Delta\phi\|_{L^\infty(D(z,r))} \leq 4nb \leq \frac{b}{a}4na \leq \frac{b}{a}r^{-2n}\int_{D(z,r)} \Delta\phi(\xi)dA(\xi),$$

implying that the reverse Hölder inequality holds with $C = b/a$. Finally, one can see that the eigenvalues are comparable in the sense of (III). In fact, for every $u, z \in \mathbb{C}^n$, $\langle H_\phi(z)u, u\rangle \geq \lambda_{min}(z)|u|^2$ and $\Delta\phi(z) \leq 4n\lambda_{Max}(z) \leq 4nb$. Then

$$\lambda_{min}(z) = \frac{\lambda_{min}(z)}{\Delta\phi(z)}\Delta\phi(z) \geq \frac{a}{4nb}\Delta\phi(z),$$

implying (III) with $\delta_0 = a/(4nb) > 0$.

Scalar weighted Fock spaces, Bergman Kernel, and the orthogonal projection

Suppose that $0 < p < \infty$ and $\phi \in \mathcal{W}$. The space $L^p_\phi(\mathbb{C}^n)$ is the space of all measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{L^p_\phi(\mathbb{C}^n)} = \left(\int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)}dA(z)\right)^{1/p} < \infty,$$

and the space $L^\infty_\phi(\mathbb{C}^n)$ consists of measurable functions endowed with the norm

$$\|f\|_{L^\infty_\phi(\mathbb{C}^n)} = \operatorname*{ess\,sup}_{z\in\mathbb{C}^n} |f(z)|e^{-\phi(z)} < \infty.$$

Denote by $H(\mathbb{C}^n)$ the space of all holomorphic functions on $\mathbb{C}^n$. Then the scalar weighted Fock space is defined as

$$F^p_\phi(\mathbb{C}^n) = L^p_\phi(\mathbb{C}^n) \cap H(\mathbb{C}^n), \tag{2.3.5}$$

with the norm defined above. Similarly to the case of doubling Fock spaces, one can see that the point evaluation map $ev_z : F^2_\phi(\mathbb{C}^n) \to \mathbb{C}$ is a bounded linear functional. In particular, $F^2_\phi(\mathbb{C}^n)$ is a reproducing kernel Hilbert space, with $ev_z(f) = f(z) = \langle f, K_z\rangle$, for all $f \in F^2_\phi(\mathbb{C}^n)$, where the inner product is given by

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-2\phi(z)}dA(z), \quad f, g \in F^2_\phi(\mathbb{C}^n).$$

Moreover, $F^p_\phi(\mathbb{C}^n)$ is a Banach space when $1 \leq p \leq \infty$, and a quasi-Banach space when $0 < p < 1$. Theorem 20 in [30] states that the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$ satisfies the following pointwise estimate. That is, there is a constant $\epsilon > 0$ such that

$$|K(z,w)| \lesssim \frac{e^{\phi(z)}}{\rho(z)^n}\frac{e^{\phi(w)}}{\rho(w)^n}e^{-\epsilon d_\phi(z,w)}, \quad z, w \in \mathbb{C}^n, \tag{2.3.6}$$

where d_ϕ is the distance associated to ϕ and $\rho : \mathbb{C}^n \to (0,\infty)$ is the associated radius function to ϕ defined by

$$\rho(z) = \sup\{r > 0 : \sup_{w\in D(z,r)} \Delta\phi(w) \leq r^{-2}\}. \tag{2.3.7}$$

Note that for a piecewise $\mathcal{C}^1$ curve $\gamma : I \to \mathbb{C}^n$, we define

$$L_\rho(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))}dt.$$

Then $d_\phi(z,w) = \inf_\gamma L_\rho(\gamma)$, where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : I \to \mathbb{C}^n$ with $\gamma(0) = z$ and $\gamma(1) = w$. Moreover, similarly to the doubling case, $d_\phi(z,w) \simeq |z-w|/\rho(z)$. For more details on the radius function ρ and the distance d_ϕ, see [30]. Since $F^2_\phi(\mathbb{C}^n)$ is a closed subspace of the Hilbert space $L^2_\phi(\mathbb{C}^n)$, there is an orthogonal projection $P_{\mathbb{C}} : L^2_\phi(\mathbb{C}^n) \to F^2_\phi(\mathbb{C}^n)$ given by

$$P_{\mathbb{C}}(f)(z) = \int_{\mathbb{C}^n} f(w)\overline{K_z(w)}e^{-2\phi(w)}dA(w), \quad z \in \mathbb{C}^n,$$

which according to [62], Theorem 5, extends to a bounded projection from $L^p_\phi(\mathbb{C}^n)$ to $F^p_\phi(\mathbb{C}^n)$ if $1 \le p \le \infty$. In the following, we state some useful lemmas about the behavior of the radius function and the reproducing kernel. We will not provide the readers with proofs as they can be done similarly to the doubling case.

Lemma 2.3.4 ([7], Lemma A). *Let ϕ be defined as in Definition 2.3.1. Then the radius function ρ satisfies the following properties.*

(1) There exists $M > 0$ such that

$$\sup_{z\in\mathbb{C}^n} \rho(z) \le M, \tag{2.3.8}$$

(2) The function ρ is Lipschitz. Indeed, for every $z, w \in \mathbb{C}^n$,

$$|\rho(z) - \rho(w)| \le |z-w|, \tag{2.3.9}$$

(3) For $r \in (0,1)$ and $w \in D^r(z)$,

$$(1-r)\rho(z) \le \rho(w) \le (1+r)\rho(z), \tag{2.3.10}$$

(4) There exist $A, B \ge 0$ such that

$$|z|^{-A} \lesssim \rho(z) \lesssim |z|^B, \quad \textit{for } |z| > 1. \tag{2.3.11}$$

By (2.3.10) and the triangle inequality, for any $r \in (0,1)$, there are $m_1 = m_1(r) > 1$ and $m_2 = m_2(r) > 1$ such that

$$D^r(z) \subset D^{m_1 r}(w), \quad \text{and } D^r(w) \subset D^{m_2 r}(z), \quad \text{for every } w \in D^r(z). \tag{2.3.12}$$

Lemma 2.3.5 ([8], Lemma 2.3). *Let $K_z = K(\cdot, z)$ be the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$. The following assertions are true.*

(a) There exists $\alpha \in (0,1]$ such that

$$|K_z(w)| \simeq \|K_z\|_{F^2_\phi(\mathbb{C}^n)} \|K_w\|_{F^2_\phi(\mathbb{C}^n)}, \quad w \in D^\alpha(z), \tag{2.3.13}$$

(b) For $0 < p \le \infty$,

$$\|K_z\|_{F^p_\phi(\mathbb{C}^n)} \simeq e^{\phi(z)}\rho(z)^{2n(1-p)/p}, \quad z \in \mathbb{C}^n, \tag{2.3.14}$$

(c) Let α be as defined in (2.3.13). Then

$$|k_z(w)|^2 e^{-2\phi(w)} \simeq \rho(z)^{-2n}, \quad w \in D^\alpha(z), \tag{2.3.15}$$

(d) *For each $z \in \mathbb{C}^n$, $0 < p \leq \infty$ and $\beta \in \mathbb{R}$,*

$$\int_{\mathbb{C}^n} |K_z(w)|^p e^{-p\phi(w)} \rho(w)^\beta dA(w) \simeq e^{p\phi(z)} \rho(z)^{2n(1-p)+\beta}, \tag{2.3.16}$$

(e) *The set $\{k_z : z \in \mathbb{C}^n\}$ is bounded in $F^2_\phi(\mathbb{C}^n)$ and $k_z \to 0$ uniformly on any compact subsets of $\mathbb{C}^n$ as $|z| \to \infty$.*

Lemma 2.3.6 ([7], Lemma B)**.** *Let $0 < p < \infty$ and define ϕ as in (2.3.2). For any $\delta \in (0,1]$, there exists $C > 0$ such that for any $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$,*

$$|f(z)|^p e^{-p\phi(z)} \leq \frac{C}{\delta^{2n}\rho(z)^{2n}} \int_{D^\delta(z)} |f(w)|^p e^{-p\phi(w)} dA(w). \tag{2.3.17}$$

Remark 2.3.7. We would like to emphasize that property (I) in Definition 2.3.1 implies that the radius function ρ is bounded. In fact, fix $z \in \mathbb{C}^n$, and suppose $r > c$, where c is the constant in (I). Then $D(z,c) \subset D(z,r)$. So, $\sup_{w \in D(z,c)} \Delta\phi(w) \leq \sup_{w \in D(z,r)} \Delta\phi(w) \leq r^{-2}$, where the second inequality comes from the definition of ρ. By (I), there is $m > 0$ such that $\sup_{w \in D(z,c)} \Delta\phi(w) \geq m > 0$. Hence, $m \leq r^{-2}$, and thus $r \leq m^{-1/2}$. Therefore, $\rho(z) \leq M := \max\{c, m^{-1/2}\}$. Note that when $\Delta\phi dA$ is doubling, ρ is defined by integrals/averages of $\Delta\phi$ on balls, and not by the pointwise supremum of $\Delta\phi$. Hence, a doubling condition does not necessarily give any uniform pointwise lower or upper bound on the density $\Delta\phi$. Basically, averages can behave nicely, while pointwise, and therefore supremums, are wild. As an example, one can see that for the doubling weight $\phi(z) = |z|^m$, with $m > 0$, $\rho(z) \simeq |z|^{1-m/2}$ for $|z|$ large enough. Taking $m = 1$, $\rho(z)$ is not bounded.

Remark 2.3.8 (Comparing radius functions for doubling and admissible weights)**.** Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic weight. Following [30], we associate to ϕ the radius function

$$\rho(z) = \sup\Big\{r > 0 : \sup_{w \in D(z,r)} \Delta\phi(w) \leq r^{-2}\Big\}.$$

It is natural to compare this with the radius function ρ_0 used in the theory of doubling weights in one complex dimension, defined by $\mu\big(D(z, \rho_0(z))\big) = 1$, with $\mu = \Delta\phi\, dA$.

- When $n = 1$, the two notions are closely related. Indeed, if $0 < r < \rho(z)$, then

  $$\sup_{w \in D(z,r)} \Delta\phi(w) \leq r^{-2}.$$

  Setting $s = r/\sqrt{\pi}$, we obtain

  $$\mu(D(z,s)) = \int_{D(z,s)} \Delta\phi(w)\, dA(w) \leq |D(z,s)|\, r^{-2} = \pi s^2 r^{-2} = 1.$$

  Hence $s \leq \rho_0(z)$, and therefore

  $$\rho(z) \leq \sqrt{\pi}\, \rho_0(z).$$

  Thus, in one complex dimension, the L^∞-type condition defining ρ implies the L^1-normalization defining ρ_0.

- For $n \geq 2$, however, an analogous definition based on

  $$\int_{D(z,r)} \Delta\phi(w)\, dA(w) = 1$$

  is no longer the appropriate one. The reason is that $\Delta\phi(z) = 4\operatorname{Tr} H_\phi(z)$, so $\Delta\phi$ only records the trace of the complex Hessian, whereas the local geometry is governed by the full Hermitian form $H_\phi(z)$. If $\lambda_1(z), \cdots, \lambda_n(z)$ denote the eigenvalues of $H_\phi(z)$, then $\Delta\phi(z)$ controls only their sum,

  $$\Delta\phi(z) \asymp \lambda_1(z) + \cdots + \lambda_n(z),$$

  and therefore does not distinguish between isotropic and highly anisotropic situations. In particular, an integral condition involving only $\Delta\phi$ yields merely averaged information on the curvature.

  By contrast, the definition of ρ gives uniform control of the curvature on Euclidean balls. Indeed, if $r < \rho(z)$, then

  $$\sup_{w \in D(z,r)} \Delta\phi(w) \leq r^{-2},$$

  and since $H_\phi(w)$ is positive semidefinite, each eigenvalue satisfies

  $$\lambda_j(w) \lesssim r^{-2}, \qquad 1 \leq j \leq n,\ w \in D(z,r).$$

  Hence

  $$H_\phi(w) \lesssim r^{-2} I \qquad \text{for all } w \in D(z,r),$$

  so the curvature is uniformly controlled at scale r. This is precisely the type of local geometric information needed in the pointwise and localization arguments of [30].

Therefore, while ρ_0 is natural in the one-dimensional doubling setting, in several complex variables one works instead with ρ, since it captures the correct local scale of the full complex Hessian rather than only the averaged mass of its trace.

2.4 Large vector-valued Fock spaces

Up to now, we have studied scalar Fock spaces, i.e., spaces of $\mathbb{C}$-valued holomorphic functions with respect to some weights, including the classical Fock space, doubling Fock spaces, and scalar weighted Fock spaces. A possible generalization is to study the space of functions taking values in a finite or infinite-dimensional Hilbert space. So, let $\mathcal{H}$ be a complex separable Hilbert space. We denote by $L^2_\phi(\mathbb{C}^n, \mathcal{H})$ the space of all measurable $\mathcal{H}$-valued functions on $\mathbb{C}^n$ for which

$$\|f\|_{2,\phi}^2 = \int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}}^2 e^{-2\phi(z)}\, dA(z) < \infty, \tag{2.4.1}$$

where dA is the Lebesgue measure on $\mathbb{C}^n$ and ϕ is an *admissible weight* as in Definition 2.3.1. When equipped with the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} \langle f(z), g(z)\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z),$$

$L^2_\phi(\mathbb{C}^n,\mathcal{H})$ becomes a Hilbert space. We say that $f:\mathbb{C}^n\to\mathcal{H}$ is holomorphic if for every continuous linear functional $\phi\in\mathcal{H}^*$, the scalar-valued function $\phi\circ f:\mathbb{C}^n\to\mathbb{C}$ is holomorphic in the usual sense (see, e.g., §3.10 in [42]). The *large vector-valued Fock space* $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is defined by

$$F^2_\phi(\mathbb{C}^n,\mathcal{H})=L^2_\phi(\mathbb{C}^n,\mathcal{H})\cap H(\mathbb{C}^n,\mathcal{H}),$$

where $H(\mathbb{C}^n,\mathcal{H})$ stands for the space of all $\mathcal{H}$-valued holomorphic functions on $\mathbb{C}^n$. Notice that when $\mathcal{H}=\mathbb{C}$, $F^2_\phi(\mathbb{C}^n,\mathbb{C})=F^2_\phi(\mathbb{C}^n)$ as defined in (2.3.5).

$F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is a Hilbert Space

To see that $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is a Hilbert space, we need the following lemmas.

Lemma 2.4.1. *If $f\in F^2_\phi(\mathbb{C}^n,\mathcal{H})$, then $z\mapsto\langle f(z),e\rangle_{\mathcal{H}}\in F^2_\phi(\mathbb{C}^n)$, for any unit element $e\in\mathcal{H}$.*

Proof. By Cauchy-Schwarz inequality,

$$\int_{\mathbb{C}^n}|\langle f(z),e\rangle_{\mathcal{H}}|^2e^{-2\phi(z)}dA(z)\le\int_{\mathbb{C}^n}\|f(z)\|^2_{\mathcal{H}}\|e\|^2_{\mathcal{H}}e^{-2\phi(z)}dA(z)<\infty,$$

which finishes the proof. □

Remark 2.4.2. Lemma 2.4.1 can be generalized for any $h\in H$. Indeed, similarly, one can observe that $z\mapsto\langle f(z),h\rangle_{\mathcal{H}}\in F^2_\phi(\mathbb{C}^n)$, for any element $h\in\mathcal{H}$.

Lemma 2.4.3. *For any $\delta\in(0,1]$, there exists $C>0$ such that for any $f\in F^2_\phi(\mathbb{C}^n,\mathcal{H})$ and $z\in\mathbb{C}^n$,*

$$\|f(z)\|^2_{\mathcal{H}}e^{-2\phi(z)}\le\frac{C}{\delta^{2n}\rho(z)^{2n}}\int_{D^\delta(z)}\|f(w)\|^2_{\mathcal{H}}e^{-2\phi(w)}dA(w).\tag{2.4.2}$$

Proof. Let $f\in F^2_\phi(\mathbb{C}^n,\mathcal{H})$. By Lemma 2.4.1, $\langle f(z),e\rangle_{\mathcal{H}}$ belongs to $F^2_\phi(\mathbb{C}^n)$ and hence holomorphic, for any unit vector $e\in\mathcal{H}$. Hence by Lemma 2.3.6, and applying the Cauchy-Schwarz inequality

$$\begin{aligned}|\langle f(z),e\rangle_{\mathcal{H}}|^2e^{-2\phi(z)}&\le\frac{C}{\delta^{2n}\rho(z)^{2n}}\int_{D^\delta(z)}|\langle f(w),e\rangle_{\mathcal{H}}|^2e^{-2\phi(w)}dA(w)\\&\le\frac{C}{\delta^{2n}\rho(z)^{2n}}\int_{D^\delta(z)}\|f(w)\|^2_{\mathcal{H}}\|e\|^2_{\mathcal{H}}e^{-2\phi(w)}dA(w).\end{aligned}$$

Since $\|e\|_{\mathcal{H}}=1$, we can use $\|f(z)\|_{\mathcal{H}}=\sup_{\|e\|=1}|\langle f(z),e\rangle|$ to obtain (2.4.2) and the proof is complete. Indeed, by the Cauchy-Schwarz inequality, for $\|e\|_{\mathcal{H}}=1$, $|\langle f(z),e\rangle|\le\|f(z)\|_{\mathcal{H}}\|e\|_{\mathcal{H}}=\|f(z)\|_{\mathcal{H}}$. So, $\sup_{\|e\|=1}|\langle f(z),e\rangle|\le\|f(z)\|_{\mathcal{H}}$. Conversely, take $e=f(z)/\|f(z)\|_{\mathcal{H}}$. When $f(z)\ne0$, $\|e\|=1$, and $|\langle f(z),e\rangle|=\langle f(z),f(z)/\|f(z)\|_{\mathcal{H}}\rangle=\|f(z)\|_{\mathcal{H}}$. Hence, $\|f(z)\|_{\mathcal{H}}=\sup_{\|e\|=1}|\langle f(z),e\rangle|$. □

Remark 2.4.4. Let $z\in\mathbb{C}^n$. Then by Lemma 2.4.3, $\|f(z)\|_{\mathcal{H}}\le C\frac{e^{\phi(z)}}{\rho(z)^n}\|f\|_{2,\phi}$, and hence the point evaluation map $f\mapsto f(z)$ is a bounded linear homomorphism from $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ to $\mathcal{H}$. Let $C(z)$ be the bounding constant, depending only on z, ϕ, and n. One can see that for any compact subset $K\subset\mathbb{C}^n$, and any $z\in K$, $C(z)$ is bounded. To see this, first take K not overlapping the unit disk centered at the origin. Then (2.3.11) implies that $C(z)\simeq e^{\phi(z)}|z|^{nA}$ for some $A>0$, and thus bounded. Now, assume that K overlaps the unit disk $D(0,1)$. By (2.3.9), ρ is continuous, and since ϕ is $\mathcal{C}^2$, it is enough to show that ρ never vanishes on the unit disk, to conclude that $C(z)$ is continuous and thus bounded on K. Let $z\in D(0,1)$. By continuity of $\Delta\phi$, and since ϕ is plurisubharmonic, there is some constant $M > 0$ such that $\sup_{w\in D(0,2)} \Delta\phi(w) = M$. Let $N = \max\{M, 1\}$. Then $\frac{1}{N} \le 1$, and thus $D(z, \frac{1}{N}) \subset D(0,2)$, for every $z \in D(0,1)$. Hence, $\sup_{w\in D(z,\frac{1}{N})} \Delta\phi(w) \le N \le N^2$. Using (2.3.7), we can conclude that $\rho(z) \ge \frac{1}{N}$ for every $z \in D(0,1)$, and in particular $\rho(z) \neq 0$.

Hence, given $z \in \mathbb{C}^n$, Lemma 2.4.3 and Remark 2.4.4 imply that there is a constant $C(z)$, bounded on compact subsets of $\mathbb{C}^n$ such that

$$\|f(z)\|_{\mathcal{H}} \le C(z)\|f\|_{2,\phi}, \quad \text{for } f \in F^2_\phi(\mathbb{C}^n, \mathcal{H}). \tag{2.4.3}$$

Therefore, the point evaluation map $f \mapsto f(z)$ is a bounded linear homomorphism from $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ to $\mathcal{H}$ and uniformly bounded in bounded domains of $\mathbb{C}^n$. Since locally uniform limits of holomorphic functions are holomorphic, we conclude that $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is a closed subspace of $L^2_\phi(\mathbb{C}^n, \mathcal{H})$, and thus a Hilbert space.

$F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is a reproducing kernel Hilbert Space

Let us compare the point evaluation maps of F^2_ϕ and $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. In the former, the point evaluation map $ev_z : F^2_\phi \to \mathbb{C}$ is a bounded linear functional, and for the latter, the point evaluation map $ev_z : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to \mathcal{H}$ is a bounded linear homomorphism, and not a functional, since $\mathcal{H}$ is not necessarily the complex plane. This is the reason behind the fact that we cannot apply the Riesz representation theorem to conclude the existence of the reproducing kernel as an element of $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. In the following, we generalize the notion of the reproducing kernel to when the functions take values in a general Hilbert space.

Definition 2.4.5. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{H}^*$ be its dual, and let $\mathcal{F}$ be a Hilbert space of functions $f : \mathbb{C}^n \to \mathcal{H}$. We say that $\mathcal{F}$ is a *vector-valued reproducing kernel Hilbert* space if there is a map $K^{\mathcal{H}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{H} \otimes \mathcal{H}^*$ with $K^{\mathcal{H}}(z,w)^* \cong K^{\mathcal{H}}(w,z)$, and

$$f(z) = \int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w) f(w)\, dV(w) \quad \text{for } f \in \mathcal{F}, \tag{2.4.4}$$

where dV is a measure on $\mathbb{C}^n$.

Note that here $\cong$ stands for the natural isomorphism $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{H}^* \otimes \mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Then there is a natural isomorphism $\mathcal{L}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$. In fact, using the map $B : \mathcal{H}^* \times \mathcal{H} \to \mathcal{L}(\mathcal{H})$ defined by $B(\lambda, w)(v) = \lambda(v)w$, and the universal property of the tensor products, we obtain a linear map $T_B : \mathcal{H}^* \otimes \mathcal{H} \to \mathcal{L}(\mathcal{H})$. This map is an isomorphism with inverse $S(L) = \sum_{i=1}^\infty e^i \otimes Le_i$, where $\{e_i\}_{i=1}^\infty$ is an orthonormal basis of $\mathcal{H}$ and $\{e^i\}_{i=1}^\infty$ is the dual basis of $\mathcal{H}^*$. Therefore, the vector-valued reproducing kernel $K^{\mathcal{H}}$ can be viewed as a map $K^{\mathcal{H}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$.

Write $K^{\mathcal{H}}_z(\cdot) = K^{\mathcal{H}}(\cdot, z)$. As a special case of Definition 2.4.5, consider $K^{\mathcal{H}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$ and $K^{\mathcal{H}}(z,w)^* = K^{\mathcal{H}}(w,z)$. Let us consider the inner product of $\mathcal{F}$ as

$$\langle f, g\rangle_{\mathcal{F}} = \int_{\mathbb{C}^n} \langle f(z), g(z)\rangle_{\mathcal{H}}\, dV(z).$$

It follows that $\langle f(z), h\rangle_{\mathcal{H}} = \langle f, K^{\mathcal{H}}_z h\rangle_{\mathcal{F}}$ for every $h \in \mathcal{H}$ and $z \in \mathbb{C}^n$. This can be seen as an analog to $f(z) = \langle f, K_z\rangle$ in the scalar setting.

Taking $dV = e^{-2\phi}dA$, $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is a vector-valued reproducing kernel Hilbert space, and its reproducing kernel $K^{\mathcal{H}}_z$ is a map from $\mathbb{C}^n$ to $\mathcal{H}\otimes\mathcal{H}^*$. The reproducing kernel property takes the form

$$f(z) = \int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w)f(w)e^{-2\phi(w)}dA(w), \quad f \in F^2_\phi(\mathbb{C}^n,\mathcal{H}).$$

When $\mathcal{H} = \mathbb{C}$, the above integral is equivalent to the scalar reproducing kernel property of $F^2_\phi(\mathbb{C}^n)$, where the action of the reproducing kernel in the scalar case $F^2_\phi(\mathbb{C}^n)$ is given by the usual multiplication. Being an element of $\mathcal{H}\otimes\mathcal{H}^*$, the most general $K^{\mathcal{H}}(z,w)$ is of the form $\sum_{m,n=1}^\infty K_{mn}(z,w)e_m\otimes e^n$, where $K_{mn}(z,w)$ are some complex scalars. The following lemma shows that the reproducing kernel of $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is obtained by taking $K_{mn}(z,w) = \delta_{mn}K(z,w)$, where $K(z,w)$ is the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$. That is,

$$K^{\mathcal{H}}_w(z) = K^{\mathcal{H}}(z,w) = \sum_{n=1}^\infty K(z,w)e_n\otimes e^n.$$

Lemma 2.4.6. *Let ϕ be as in Definition 2.3.1, and let $\mathcal{H}$ be a separable Hilbert space. The reproducing kernel of $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is of the form*

$$K^{\mathcal{H}}_w(z) = K^{\mathcal{H}}(z,w) = \sum_{n=1}^\infty K(z,w)e_n\otimes e^n,$$

where $K(z,w)$ is the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$.

Proof. Applying Lemma 2.4.1, we can write

$$\begin{aligned}
&\int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w)f(w)e^{-2\phi(w)}dA(w) = \\
&\int_{\mathbb{C}^n}\sum_{n=1}^\infty \langle f(w),e_n\rangle_{\mathcal{H}}e_nK(z,w)e^{-2\phi(w)}dA(w) \\
&= \sum_{n=1}^\infty \langle f(z),e_n\rangle_{\mathcal{H}}e_n \\
&= f(z),
\end{aligned}$$

showing that the choice we made for the reproducing kernel does make sense. Moreover, since $K(z,w)$ is conjugate symmetric, and by the natural isomorphism $\mathcal{H}\otimes\mathcal{H}^* \cong \mathcal{H}^*\otimes\mathcal{H}$,

$$K^{\mathcal{H}}(z,w)^* = \sum_{n=1}^\infty \overline{K(z,w)}e^n\otimes e_n \cong \sum_{n=1}^\infty K(w,z)e_n\otimes e^n = K^{\mathcal{H}}(w,z).$$

□

Remark 2.4.7. The reproducing kernel in Lemma 2.4.6 is unique when viewed as an $\mathcal{L}(\mathcal{H})$-valued kernel. More precisely, if $K^{\mathcal{H}}_1$, $K^{\mathcal{H}}_2 : \mathbb{C}^n\times\mathbb{C}^n \to \mathcal{L}(\mathcal{H})$ both satisfy the reproducing kernel property

$$\langle f(z),h\rangle_{\mathcal{H}} = \langle f, K^{\mathcal{H}}_{j,z}h\rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})}, \qquad f\in F^2_\phi(\mathbb{C}^n,\mathcal{H}),\ h\in\mathcal{H},\ z\in\mathbb{C}^n,$$

for $j = 1,2$, then $K^{\mathcal{H}}_1 = K^{\mathcal{H}}_2$. Indeed, for fixed $z\in\mathbb{C}^n$ and $h\in\mathcal{H}$, the point evaluation map $ev_z : F^2_\phi(\mathbb{C}^n,\mathcal{H}) \to \mathcal{H}$, with $f\mapsto f(z)$, is bounded (see the discussion preceding Definition 2.4.5). Hence, the scalar-valued map

$$L_{z,h} : F^2_\phi(\mathbb{C}^n,\mathcal{H}) \to \mathbb{C}, \qquad L_{z,h}(f) = \langle f(z),h\rangle_{\mathcal{H}},$$

is a bounded linear functional. Since $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is a Hilbert space, the Riesz representation theorem yields a unique vector $g_{z,h} \in F^2_\phi(\mathbb{C}^n,\mathcal{H})$ such that

$$\langle f(z),h\rangle_{\mathcal{H}} = \langle f, g_{z,h}\rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} \qquad \text{for all } f \in F^2_\phi(\mathbb{C}^n,\mathcal{H}).$$

Thus, for each reproducing kernel $K^{\mathcal{H}}_j$, the vector $K^{\mathcal{H}}_{j,z}h$ is precisely this unique Riesz representer, and in particular

$$K^{\mathcal{H}}_{j,z}h \in F^2_\phi(\mathbb{C}^n,\mathcal{H}).$$

Therefore,

$$\langle f,(K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z})h\rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} = 0 \qquad \text{for all } f \in F^2_\phi(\mathbb{C}^n,\mathcal{H}).$$

Choosing

$$f = (K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z})h \in F^2_\phi(\mathbb{C}^n,\mathcal{H}),$$

we obtain

$$\|(K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z})h\|^2_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} = 0.$$

Hence

$$(K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z})h = 0 \qquad \text{for every } h \in \mathcal{H},$$

which implies

$$K^{\mathcal{H}}_{1,z} = K^{\mathcal{H}}_{2,z}.$$

Since z was arbitrary, it follows that

$$K^{\mathcal{H}}_1 = K^{\mathcal{H}}_2.$$

Consequently, the reproducing kernel is genuinely unique as an $\mathcal{L}(\mathcal{H})$-valued kernel. The formula in Lemma 2.4.6,

$$K^{\mathcal{H}}(z,w) = \sum_{n=1}^{\infty} K(z,w)e_n \otimes e^n,$$

depends on the chosen orthonormal basis $\{e_n\}_{n\geq 1}$ only at the level of coordinates. Under the natural identification $\mathcal{H}\otimes\mathcal{H}^* \cong \mathcal{L}(\mathcal{H})$, one has

$$\sum_{n=1}^{\infty} e_n \otimes e^n = I_{\mathcal{H}},$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. So the above expression is simply

$$K^{\mathcal{H}}(z,w) = K(z,w)I_{\mathcal{H}}.$$

Thus the kernel is not merely unique up to isomorphism of $\mathcal{H}\otimes\mathcal{H}^*$, but it is uniquely determined as the operator-valued kernel $K(z,w)I_{\mathcal{H}}$. Different orthonormal bases only give different tensor-coordinate expressions for the same operator.

Orthogonal projection and the vectorial Toeplitz operator

Since $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is a closed subset of the Hilbert space $L^2_\phi(\mathbb{C}^n,\mathcal{H})$, there is an orthogonal projection $P : L^2_\phi(\mathbb{C}^n,\mathcal{H}) \to F^2_\phi(\mathbb{C}^n,\mathcal{H})$, given by the following lemma.

Lemma 2.4.8. *Let ϕ be as in Definition 2.3.1, and $\mathcal{H}$ be a separable Hilbert space. The integral operator*

$$P(f)(z) = \int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w)f(w)e^{-2\phi(w)}dA(w) = \int_{\mathbb{C}^n} f(w)K(z,w)e^{-2\phi(w)}dA(w), \quad z \in \mathbb{C}^n,$$

is the orthogonal projection of $L^2_\phi(\mathbb{C}^n,\mathcal{H})$ onto $F^2_\phi(\mathbb{C}^n,\mathcal{H})$.

Proof. Let $\{e_m\}_{m=1}^\infty$ be an orthonormal basis of $\mathcal{H}$. For $f \in L^2_\phi(\mathbb{C}^n,\mathcal{H})$, define the scalar-valued functions $f_m^*(z) := \langle f(z), e_m\rangle_{\mathcal{H}}$, for $z \in \mathbb{C}^n$. Then

$$f(z) = \sum_{m=1}^\infty f_m^*(z)e_m$$

in $\mathcal{H}$ for a.e. z, and

$$\|f\|^2_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} = \sum_{m=1}^\infty \|f_m^*\|^2_{L^2_\phi(\mathbb{C}^n)}.$$

By Lemma 2.4.6, for each $z \in \mathbb{C}^n$,

$$P(f)(z) = \sum_{m=1}^\infty e_m P_{\mathbb{C}}(f_m^*)(z),$$

where $P_{\mathbb{C}} : L^2_\phi(\mathbb{C}^n) \to F^2_\phi(\mathbb{C}^n)$ is the scalar Bergman projection. Since $P_{\mathbb{C}}(f_m^*) \in F^2_\phi(\mathbb{C}^n)$ for every m, it follows that $P(f)$ is $\mathcal{H}$-valued holomorphic. Moreover,

$$\|P(f)\|^2_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} = \sum_{m=1}^\infty \|P_{\mathbb{C}}(f_m^*)\|^2_{L^2_\phi(\mathbb{C}^n)} \le \sum_{m=1}^\infty \|f_m^*\|^2_{L^2_\phi(\mathbb{C}^n)} = \|f\|^2_{L^2_\phi(\mathbb{C}^n,\mathcal{H})},$$

so $P(f) \in F^2_\phi(\mathbb{C}^n,\mathcal{H})$. Next, for $f \in L^2_\phi(\mathbb{C}^n,\mathcal{H})$,

$$P(Pf)(z) = \sum_{m=1}^\infty e_m P_{\mathbb{C}}\Big((Pf)_m^*\Big)(z).$$

But

$$(Pf)_m^*(z) = \langle Pf(z), e_m\rangle_{\mathcal{H}} = P_{\mathbb{C}}(f_m^*)(z),$$

hence, since $P_{\mathbb{C}}$ is a projection,

$$P(Pf)(z) = \sum_{m=1}^\infty e_m P_{\mathbb{C}}(P_{\mathbb{C}}(f_m^*))(z) = \sum_{m=1}^\infty e_m P_{\mathbb{C}}(f_m^*)(z) = Pf(z).$$

Therefore $P^2 = P$, so P is a projection onto $F^2_\phi(\mathbb{C}^n,\mathcal{H})$. It remains to show that P is orthogonal. Let $f \in L^2_\phi(\mathbb{C}^n,\mathcal{H})$. Then $f = (f - Pf) + Pf$. We claim that $\langle f - Pf, Pf\rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} = 0$. Indeed, using the orthonormal basis expansion,

$$\begin{aligned}
\langle f, Pf\rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} &= \int_{\mathbb{C}^n} \langle f(z), Pf(z)\rangle_{\mathcal{H}} e^{-2\phi(z)}\,dA(z) \\
&= \int_{\mathbb{C}^n} \left\langle \sum_{m=1}^\infty f_m^*(z)e_m, \sum_{k=1}^\infty P_{\mathbb{C}}(f_k^*)(z)e_k \right\rangle_{\mathcal{H}} e^{-2\phi(z)}\,dA(z) \\
&= \int_{\mathbb{C}^n} \sum_{m=1}^\infty f_m^*(z)\,\overline{P_{\mathbb{C}}(f_m^*)(z)}\,e^{-2\phi(z)}\,dA(z) \\
&= \sum_{m=1}^\infty \langle f_m^*, P_{\mathbb{C}}(f_m^*)\rangle_{L^2_\phi(\mathbb{C}^n)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\langle Pf, Pf\rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} &= \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} P_{\mathbb{C}}(f_m^*)(z)\overline{P_{\mathbb{C}}(f_m^*)(z)}\, e^{-2\phi(z)}\, dA(z) \\ &= \sum_{m=1}^{\infty} \langle P_{\mathbb{C}}(f_m^*), P_{\mathbb{C}}(f_m^*)\rangle_{L^2_\phi(\mathbb{C}^n)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle f - Pf, Pf\rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} &= \langle f, Pf\rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} - \langle Pf, Pf\rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} \\ &= \sum_{m=1}^{\infty}\Big(\langle f_m^*, P_{\mathbb{C}}(f_m^*)\rangle_{L^2_\phi(\mathbb{C}^n)} - \langle P_{\mathbb{C}}(f_m^*), P_{\mathbb{C}}(f_m^*)\rangle_{L^2_\phi(\mathbb{C}^n)}\Big) = 0,\end{aligned}$$

because $P_{\mathbb{C}}$ is the orthogonal projection of $L^2_\phi(\mathbb{C}^n)$ onto $F^2_\phi(\mathbb{C}^n)$. Thus we can conclude that P is the orthogonal projection of $L^2_\phi(\mathbb{C}^n,\mathcal{H})$ onto $F^2_\phi(\mathbb{C}^n,\mathcal{H})$. □

Lemma 2.4.9. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$, and $K_z^{\mathcal{H}} = \sum_{j=1}^{\infty} K_z e_j \otimes e^j$ be the reproducing kernel of $F^2_\phi(\mathbb{C}^n,\mathcal{H})$, with K_z the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$. Then the linear span of $\{K_z^{\mathcal{H}} e_i : z \in \mathbb{C}^n, i \geq 1\}$ is dense in $F^2_\phi(\mathbb{C}^n,\mathcal{H})$.*

Proof. First, notice that for any $z,w \in \mathbb{C}^n$ and $i \geq 1$, $K_z^{\mathcal{H}}(w)e_i = K_z(w)e_i \in \mathcal{H}$. Moreover, it is easy to see that for any $z \in \mathbb{C}^n$ and $i \geq 1$, $K_z e_i \in F^2_\phi(\mathbb{C}^n,\mathcal{H})$. Let $\Gamma = \operatorname{span}\{K_z^{\mathcal{H}} e_i : z \in \mathbb{C}^n, i \geq 1\}$. Since $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is a Hilbert space, the density of Γ is equivalent to $\Gamma^{\perp} = \{0\}$, where $\perp$ represents the orthogonal complement. So, let $f \in \Gamma^{\perp}$. Then by definition, for any $z \in \mathbb{C}^n$ and any $i \geq 1$, $\langle f, K_z^{\mathcal{H}} e_i\rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} = 0$. Consider the natural isomorphism $\mathcal{H}\otimes\mathcal{H}^* \cong \mathcal{L}(\mathcal{H})$. By the reproducing kernel property,

$$0 = \langle f, K_z^{\mathcal{H}} e_i\rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} = \langle f(z), e_i\rangle_{\mathcal{H}}.$$

Hence, $\langle f(z), h\rangle_{\mathcal{H}} = 0$ for every h is the linear span of $\{e_i\}_{i\geq 1}$. Because this span is dense in $\mathcal{H}$, we get $\langle f(z), h\rangle_{\mathcal{H}} = 0$, for every $h \in \mathcal{H}$. Thus, $f(z) = 0$ for all $z \in \mathbb{C}^n$. Hence, f is identically zero, and we can conclude that $\Gamma^{\perp} = \{0\}$. Therefore, Γ is a dense subset of $F^2_\phi(\mathbb{C}^n,\mathcal{H})$. □

Definition 2.4.10. Denote by $T_\phi(\mathcal{L}(\mathcal{H}))$ the space of holomorphic operator-valued functions $G : \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$ such that each $G(z)$ is positive and satisfies

$$K_z(\cdot)\|G(\cdot)\|_{\mathcal{L}(\mathcal{H})} \in L^2_\phi(\mathbb{C}^n), \quad z \in \mathbb{C}^n. \tag{2.4.5}$$

Let $\Gamma = \operatorname{span}\{K_z e_i : z \in \mathbb{C}^n, i \geq 1\}$. By Lemma 2.4.9, Γ is a dense subset of $F^2_\phi(\mathbb{C}^n,\mathcal{H})$. Let $f \in \Gamma$. Using $\|G(w)f(w)\|^2_{\mathcal{H}} \leq \|G(w)\|^2_{\mathcal{L}(\mathcal{H})}\|f(w)\|^2_{\mathcal{H}}$ and (2.4.5), one can conclude that $Gf \in L^2_\phi(\mathbb{C}^n,\mathcal{H})$. For $G \in T_\phi(\mathcal{L}(\mathcal{H}))$, the *vectorial Toeplitz operator* T_G is densely defined by

$$T_G f(z) = P(Gf)(z) = \int_{\mathbb{C}^n} G(w)f(w)K(z,w)e^{-2\phi(w)}dA(w),$$

for $f \in F^2_\phi(\mathbb{C}^n,\mathcal{H})$. For each fixed $z \in \mathbb{C}^n$ and $f \in \Gamma$, the integral above is understood as a Bochner integral with values in $\mathcal{H}$.

Remark 2.4.11. Let G be a map from $\mathbb{C}^n$ to the Banach space of bounded linear operators on $\mathcal{H}$. Definition 3.10.1 in [42] states that G is a holomorphic operator-valued function if for every $u,v \in \mathcal{H}$, $z \mapsto \langle G(z)u, v\rangle_{\mathcal{H}}$ is holomorphic.

Remark 2.4.12. For the convenience of the reader, we justify that the integral in Definition 2.4.10 is indeed a Bochner integral. We use the standard definition from Diestel-Uhl [32], Chapter II: if (X,Σ,μ) is a measure space and E is a Banach space, then a map $F : X \to E$ is Bochner integrable provided it is strongly measurable and

$$\int_X \|F(x)\|_E \, d\mu(x) < \infty.$$

In that case $\int_X F\,d\mu$ is defined as the limit of integrals of E-valued simple functions converging to F.

Fix $z \in \mathbb{C}^n$ and let $f \in \Gamma$. We claim that $w \mapsto G(w)f(w)K(z,w)$ is Bochner integrable as an $\mathcal{H}$-valued function on $\mathbb{C}^n$ with respect to the measure $d\mu_\phi(w) := e^{-2\phi(w)}\,dA(w)$. To see this, notice that since $f \in \Gamma = \operatorname{span}\{K_\zeta e_i : \zeta \in \mathbb{C}^n,\ i \geq 1\}$, there exist $\zeta_1,\cdots,\zeta_m \in \mathbb{C}^n$, $i_1,\cdots,i_m \in \mathbb{N}$, and scalars $c_1,\cdots,c_m$ such that

$$f(w) = \sum_{j=1}^{m} c_j K(w,\zeta_j)e_{i_j}, \qquad w \in \mathbb{C}^n.$$

Hence

$$G(w)f(w)K(z,w) = \sum_{j=1}^{m} c_j\, K(w,\zeta_j)K(z,w)\, G(w)e_{i_j}.$$

We first check strong measurability. For each fixed j and each $v \in \mathcal{H}$, the scalar function

$$w \mapsto \langle G(w)e_{i_j}, v\rangle_{\mathcal{H}}$$

is holomorphic by the definition of holomorphy of the operator-valued map G. In particular, it is measurable. Thus $w \mapsto G(w)e_{i_j}$ is weakly measurable. Since $\mathcal{H}$ is separable, Pettis' measurability theorem implies that $w \mapsto G(w)e_{i_j}$ is strongly measurable. Multiplication by the scalar measurable function $K(w,\zeta_j)K(z,w)$ preserves strong measurability, and finite sums of strongly measurable functions are strongly measurable. Therefore $w \mapsto G(w)f(w)K(z,w)$ is strongly measurable.

Next, we verify integrability of the norm. Using $\|G(w)f(w)\|_{\mathcal{H}} \leq \|G(w)\|_{\mathcal{L}(\mathcal{H})}\|f(w)\|_{\mathcal{H}}$, we obtain

$$\|G(w)f(w)K(z,w)\|_{\mathcal{H}} \leq |K(z,w)|\,\|G(w)\|_{\mathcal{L}(\mathcal{H})}\,\|f(w)\|_{\mathcal{H}}.$$

Hence

$$\int_{\mathbb{C}^n} \|G(w)f(w)K(z,w)\|_{\mathcal{H}}\, e^{-2\phi(w)}\,dA(w) \leq \int_{\mathbb{C}^n} |K(z,w)|\,\|G(w)\|_{\mathcal{L}(\mathcal{H})}\,\|f(w)\|_{\mathcal{H}}\, e^{-2\phi(w)}\,dA(w).$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_{\mathbb{C}^n} |K(z,w)|\,\|G(w)\|_{\mathcal{L}(\mathcal{H})}\,\|f(w)\|_{\mathcal{H}}\, e^{-2\phi(w)}\,dA(w)\\ &\qquad \leq \left(\int_{\mathbb{C}^n} |K(z,w)|^2\|G(w)\|^2_{\mathcal{L}(\mathcal{H})} e^{-2\phi(w)}\,dA(w)\right)^{1/2}\left(\int_{\mathbb{C}^n} \|f(w)\|^2_{\mathcal{H}} e^{-2\phi(w)}\,dA(w)\right)^{1/2}.\end{aligned}$$

The first factor is finite by condition (2.4.5), and the second factor is finite because $f \in \Gamma \subset F^2_\phi(\mathbb{C}^n,\mathcal{H})$. Therefore

$$\int_{\mathbb{C}^n} \|G(w)f(w)K(z,w)\|_{\mathcal{H}}\, e^{-2\phi(w)}\,dA(w) < \infty.$$

We have shown that $w \mapsto G(w)f(w)K(z,w)$ is strongly measurable and integrable in norm with respect to $d\mu_\phi$. Hence, by the definition of the Bochner integral, for every fixed $z \in \mathbb{C}^n$ and every $f \in \Gamma$,

$$\int_{\mathbb{C}^n} G(w)f(w)K(z,w)e^{-2\phi(w)}\,dA(w)$$

is a well-defined Bochner integral with values in $\mathcal{H}$.

Chapter 3

Motivation and summary of results

This chapter provides a detailed summary of the articles [10, 6, 9], on which this thesis is based. Copies of each article are provided in Appendices A-C. Motivation behind our work and similar earlier results are also discussed in an attempt to provide a self-contained manuscript to interested readers.

3.1 Berger-Coburn phenomenon for Hankel operators on Fock-type spaces

In this section, we focus on the Berger-Coburn phenomenon for Hankel operators on Fock-type spaces. We begin our discussion by defining the Berger-Coburn phenomenon and the famous operator ideals of bounded linear operators. Then we recall known results about the Berger-Coburn phenomenon for Hankel operators, to prepare the readers for a summary of our paper [10] (equivalently, Appendix A) on the Berger-Coburn phenomenon for Schatten class Hankel operators on doubling Fock spaces.

Berger-Coburn phenomenon for Hankel operator on Fock-type spaces

Let F^2_ϕ be a Fock-type space. One can take $\phi(z) = \alpha|z|^2$, with $\alpha = 1$ or $\alpha > 0$, ϕ can satisfy $0 < m < \Delta\phi < M$ for $m, M > 0$, or ϕ can be a doubling weight. For an arbitrary symbol f, in general, very little about $H_{\bar{f}}$ can be inferred from the properties of H_f. Berger and Coburn proved in [17] that for a bounded function f in $\mathbb{C}^n$, H_f is compact on the classical Fock space, if and only if $H_{\bar{f}}$ is compact. Let S be an operator ideal inside bounded linear operators over F^2_ϕ, and let f be a bounded function on $\mathbb{C}$ or $\mathbb{C}^n$, depending on the function space. The following is referred to as the Berger-Coburn phenomenon for Hankel operators.

Berger-Coburn phenomenon for Hankel operators: For a bounded function f, $H_f \in S$ if and only if $H_{\bar{f}} \in S$.

Operator ideals: Compact and Schatten class operators

Let X and Y be normed vector spaces. Thus, one can take X and Y to be Banach or Hilbert spaces. A compact operator $T : X \to Y$ is a linear operator that sends bounded subsets of X to relatively compact subsets of Y, i.e., sets that have compact closure in Y. Such an operator is necessarily bounded and so continuous.

Definition 3.1.1 ([26], Definition 3.2)**.** If X and Y are Banach spaces, and $A \in \mathcal{B}(X,Y)$ is a bounded linear operator from X to Y, then A is completely continuous if for any sequence $\{x_n\}$ in X such that $x_n \rightharpoonup x$, i.e., x_n converges weakly to x, $\|Ax_n - Ax\| \to 0$.

Proposition 3.1.2 ([26], Proposition 3.3)**.** *Let X and Y be Banach spaces, and $A \in \mathcal{B}(X,Y)$. Then*

1. *If A is compact, it is completely continuous,*
2. *If X is reflexive and A is completely continuous, then A is compact.*

Recall that a Banach space X is reflexive if its canonical embedding into its bidual is onto. That is, consider the natural map $J : X \to X^{**}$, defined by $J(x)(x^*) = x^*(x)$. J is linear and isometric. In fact, $\|J(x)\|_{X^{**}} = \sup_{\|x^*\|_{X^*} \le 1} |J(x)(x^*)| = \sup_{\|x^*\|_{X^*} \le 1} |x^*(x)|$. By the Hahn-Banach theorem, we know that $\|x\|_X = \sup_{\|x^*\|_{X^*} \le 1} |x^*(x)|$. Therefore $\|J(x)\|_{X^{**}} = \|x\|_X$, and J is an isometry. If J is onto, i.e., every element of X^{**} is of the form $J(x)$ for some $x \in X$, we say that X is reflexive.

Remark 3.1.3. Note that every Hilbert space is reflexive. In fact, let $\mathcal{H}$ be a Hilbert space. The Riesz representation theorem gives an isomorphic isomorphism between $\mathcal{H}$ and its dual $\mathcal{H}^*$, via $\Phi : \mathcal{H} \to \mathcal{H}^*$, such that $\Phi(x)(y) = \langle y, x\rangle$. Hence $\mathcal{H} \cong \mathcal{H}^*$. Taking the dual again gives $\mathcal{H}^{**} \cong (\mathcal{H}^*)^* \cong \mathcal{H}$. The canonical embedding $J : \mathcal{H} \to \mathcal{H}^{**}$ coincides, up to these identifications, with the identity map. So, it is surjective, and $\mathcal{H}$ is reflexive.

Remark 3.1.4. For any positive measure μ and $1 < p < \infty$, $L^p(\mu)$ is reflexive. This is due to the fact that $(L^p_\mu)^* \cong L^q(\mu)$, where $1/p + 1/q = 1$. However, $L^1(\mu)$ is not reflexive, since $(L^1(\mu))^* \cong L^\infty(\mu)$, but $(L^\infty(\mu))^* \ncong L^1(\mu)$.

Proposition 3.1.5 ([26], Proposition 3.5)**.** *Let X, Y and Z be Banach spaces, and $K(X,Y)$ be the set of compact operators from X to Y. Then*

1. *$K(X,Y)$ is a closed linear subspace of $\mathcal{B}(X,Y)$,*
2. *If $T \in K(X,Y)$ and $A \in \mathcal{B}(Y,Z)$, then $AT \in K(X,Z)$,*
3. *If $T \in K(Y,Z)$ and $A \in \mathcal{B}(X,Y)$, then $TA \in K(X,Z)$.*

Hence, $K(X,X)$ is a closed two-sided ideal in $B(X,X)$.

Corollary 3.1.6. *Since $K(X,Y)$ is a closed linear subspace of $\mathcal{B}(X,Y)$, if (T_n) is a sequence in $K(X,Y)$ such that $\|T_n - T\|_{\mathcal{B}(X,Y)} \to 0$ for some $T \in \mathcal{B}(X,Y)$, then $T \in K(X,Y)$. In particular, T is compact.*

Theorem 3.1.7 ([70], Theorem 4.19)**.** *Let X and Y be Banach spaces and $T \in \mathcal{B}(X,Y)$. Then T is compact if and only if its adjoint T^* is compact.*

Restricting to a Hilbert space $\mathcal{H}$, one obtains more compactness characterizations compared to Banach spaces. For example,

1. Suppose that $\{e_n\}$ and $\{\sigma_n\}$ are orthonormal sets in $\mathcal{H}$, and that $\{\lambda_n\}$ is a sequence of complex numbers such that
$$|\lambda_1| \ge |\lambda_2| \ge \cdots \ge 0, \qquad |\lambda_n| \to 0.$$
Let T be the linear operator on $\mathcal{H}$ defined by
$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle \sigma_n, \qquad x \in \mathcal{H},$$

then T is compact ([80], Proposition 1.19) and its singular values are precisely

$$s_n(T) = |\lambda_n|, \qquad n \geq 1.$$

Conversely, every compact operator T on $\mathcal{H}$ admits such a representation, called a canonical decomposition or singular value decomposition. See, for example, [80, Section 1.3]. Although this representation is not unique, the sequence of singular values $\{s_n(T)\}$ is uniquely determined by T (counting multiplicities). See Lemma 3.1.8 for more details.

2. If T is a self-adjoint compact operator on $\mathcal{H}$, then there exists a sequence of real numbers $\{\lambda_n\}$ tending to 0 and there exists an orthonormal set $\{e_n\}$ in $\mathcal{H}$ such that

   $$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n,$$

   for all $x \in \mathcal{H}$, where for each $n \in \mathbb{N}$, e_n is an eigenvector of T corresponding to the eigenvalue λ_n ([80], Theorem 1.20),

3. A linear operator T on $\mathcal{H}$ is compact if and only if there is a sequence of finite rank operators T_n such that $\|T_n - T\| \to 0$ as $n \to \infty$ ([80], Theorem 1.21).

Lemma 3.1.8. *Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a compact operator. Suppose that*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \sigma_n, \qquad x \in \mathcal{H},$$

where $\{e_n\}$ and $\{\sigma_n\}$ are orthonormal systems in $\mathcal{H}$, and $\{\lambda_n\}$ is a sequence of complex numbers satisfying

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0, \qquad |\lambda_n| \to 0.$$

Then the sequence $\{|\lambda_n|\}$ is uniquely determined by T, counting multiplicities. In particular, the singular values of T are well-defined.

Proof. First observe that T^*T is a positive compact operator. Indeed, since T is compact and T^* is bounded, the composition $T^*T = T^* \circ T$ is compact. Moreover, for every $x \in \mathcal{H}$,

$$\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 \geq 0,$$

so T^*T is positive. Using the representation of T, we have

$$T^*y = \sum_{n=1}^{\infty} \overline{\lambda_n} \langle y, \sigma_n \rangle e_n, \qquad y \in \mathcal{H}.$$

Hence,

$$T^*Tx = \sum_{n=1}^{\infty} |\lambda_n|^2 \langle x, e_n \rangle e_n.$$

Therefore each e_n is an eigenvector of T^*T with eigenvalue $|\lambda_n|^2$. Hence, the numbers $|\lambda_n|^2$ are precisely the non-zero eigenvalues of T^*T. Since eigenvalues are defined by the condition that $T^*T - \lambda I$ is not invertible, they are intrinsic to the operator, and their multiplicities are given by the dimensions of the corresponding eigenspaces. Hence the non-zero eigenvalues of T^*T, counted with multiplicity, are uniquely determined by T^*T, and therefore by T. It follows that the sequence $\{|\lambda_n|^2\}$ is uniquely determined by T, counting multiplicities. Taking square roots shows that $\{|\lambda_n|\}$ is uniquely determined by T, counting multiplicities.

Thus, although the representation of T above is not necessarily unique, every such representation gives the same sequence $\{|\lambda_n|\}$. These numbers are called the singular values of T. $\square$

Definition 3.1.9. Let $\mathcal{H}$ be a Hilbert space. Given $0 < p < \infty$, we define the Schatten p-class of $\mathcal{H}$, denoted $S_p(\mathcal{H})$, to be the space of all compact operators T on $\mathcal{H}$ with its singular value sequence $\{\lambda_n\}$ belonging to l_p, the p-summable sequence space ([80], Section 1.4).

When $1 \le p < \infty$, S_p is a Banach space with the norm $\|T\|_p = [\sum_{n=1}^{\infty} |\lambda_n|^p]^{1/p}$. For $p = 1$, S_1 is called the trace class, and when $p = 2$, S_2 is called the Hilbert-Schmidt class. In the following, we mention a few results about the Schatten class operators.

1. If $Tx = \sum_n \lambda_n \langle x, e_n\rangle \sigma_n$ is a canonical decomposition of a compact operator T, then $T^*x = \sum_n \overline{\lambda_n}\langle x, \sigma_n\rangle e_n$ is the canonical decomposition of T^*. Thus $T \in S_p$ if and only if $T^* \in S_p$. Moreover, $\|T\|_p = \|T^*\|_p$ for all $T \in S_p$,

2. If T is a compact operator on $\mathcal{H}$ with singular values $\{\lambda_n\}$, then

$$\sum_{n=1}^{\infty} |\lambda_n|^2 = \sum_{n=1}^{\infty} \|Tx_n\|^2 = \sum_{n,m=1}^{\infty} |\langle Tx_n, x_m\rangle|^2,$$

   for any orthonormal basis $\{x_n\}$ of $\mathcal{H}$ ([80], Theorem 1.22),

3. Suppose T is a positive compact operator on $\mathcal{H}$ and $p > 0$, then $T \in S_p$ if and only if $T^p \in S_1$. Moreover, $\|T\|_p^p = \|T^p\|_1$ ([80], Lemma 1.25),

4. If T is a compact operator on $\mathcal{H}$ and $p > 0$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{p/2} \in S_1$, if and only if $T^*T \in S_{p/2}$. Moreover,

$$\|T\|_{S_p}^p = \||T|\|_{S_p}^p = \||T|^p\|_{S_1} = \|T^*T\|_{S_{p/2}}^{p/2}.$$

   Consequently, $T \in S_p$ if and only if $|T| \in S_p$ ([80], Theorem 1.26),

5. Suppose T is a compact operator on $\mathcal{H}$ and $p \ge 1$. Then T is in S_p if and only if

$$\sum |\langle Te_n, \sigma_n\rangle|^p < \infty, \tag{3.1.1}$$

   for all orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$. If T is positive, we also have ([80], Theorem 1.28)

$$\|T\|_{S_p} = \sup\left\{\left[\sum |\langle Te_n, \sigma_n\rangle|^p\right]^{1/p} : \quad \{e_n\} \text{ and } \{\sigma_n\} \text{ are orthonormal}\right\},$$

6. Suppose T is a compact operator on $\mathcal{H}$ and $p \ge 2$. Then T is in S_p if and only if $\sum_{n=1}^{\infty} \|Te_n\|^p < \infty$ for all orthonormal sets $\{e_n\}$ in $\mathcal{H}$ ([80], Theorem 1.33),

7. S_p is a two-sided ideal in $\mathcal{L}(\mathcal{H})$ ([80], Corollary 1.35).

Known results about the Berger-Coburn phenomenon for Hankel operators

Before we state older results on the Berger-Coburn phenomenon, we give an example to show the importance of the boundedness assumption on f.

Example 3.1.10 ([14])**.** Let $g(z) = z$, and thus unbounded over $\mathbb{C}$. Consider $H_g : F^2 \to L^2$. Since g is holomorphic, $H_g = 0$, and belongs to the Hilbert-Schmidt class. Let $e_j(z) = z^j/\sqrt{j!}$ be the orthonormal basis vectors of F^2, where j's are non-negative integers. Recall that

$$Pf(z) = \int_{\mathbb{C}} f(w)K(z,w)d\lambda, \quad \text{where } K(z,w) = e^{z\bar{w}} \text{ and } d\lambda(w) = \frac{1}{\pi}e^{-|w|^2}dA(w).$$

By definition of Hankel operators, $H_{\bar{g}}e_j = \bar{g}e_j - P(\bar{g}e_j)$. Then

$$P(\bar{g}e_j)(z) = \frac{1}{\sqrt{j!}}\sum_{l=0}^{\infty}\frac{\langle w^j, w^{l+1}\rangle}{l!}z^l = \begin{cases} 0 & \text{if } j = 0, \\ \sqrt{j}e_{j-1}(z) & \text{if } j > 0. \end{cases}$$

Recalling that $T \in S_2$ if and only if $\sum_{n=0}^{\infty}\|Te_n\|^2 < \infty$, one can see that $H_{\bar{g}}$ is not in S_2. Indeed, by orthogonality of $H_{\bar{g}}e_j$ and $P(\bar{g}e_j)$,

$$\|H_{\bar{g}}e_j\|^2 = \|\bar{g}e_j\|^2 - \|P(\bar{g}e_j)\|^2 = \|\frac{z^{j+1}}{\sqrt{j!}}\|^2 - j = 1.$$

Thus $\sum\|H_{\bar{g}}e_j\|^2 = \infty$, and $H_{\bar{g}} \notin S_2(F^2, L^2)$.

Recall the standard Fock space F^2_α defined in (2.1.1). Similarly, for $0 < p < \infty$, we denote by F^p_α the space of entire functions $f \in L^p(\mathbb{C}, d\lambda_\alpha)$, that is,

$$F^p_\alpha := \{f \in H(\mathbb{C}) : \|f\|^p_{p,\alpha} = \int_{\mathbb{C}} |f(z)|^p d\lambda_\alpha(z) < \infty\}.$$

We note that some of the notation appearing in Theorem 3.1.11 and Theorem 3.1.12 is introduced later in Remark 3.1.13.

Theorem 3.1.11 (Berger-Coburn phenomenon for compact Hankel operators)**.** *Let $f \in L^\infty(\mathbb{C}^n)$. Then the following hold.*

1. *H_f is compact on the classical Fock space F^2 if and only if $H_{\bar{f}}$ is compact. This is the result of Berger and Coburn in [17], Theorem B,*
2. *Let $1 < p < \infty$. Then $H_f : F^p_\alpha \to L^p_\alpha$ is compact if and only if $H_{\bar{f}} : F^p_\alpha \to L^p_\alpha$ is compact, where F^p_α is the standard Fock space. This is the result of Hagger and Virtanen in [41], Theorem 8,*
3. *Let $\phi \in C^2(\mathbb{C}^n)$. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and assume that there are positive real numbers m and M such that $m\,\mathrm{Id}_{2n} \le \mathrm{Hess}_{\mathbb{R}}\phi \le M\,\mathrm{Id}_{2n}$, where Id_{2n} is the $2n \times 2n$ identity operator. Take $0 < p \le q < \infty$ or $1 \le q < p < \infty$. Then $H_f : F^p_\phi(\mathbb{C}^n) \to L^q_\phi(\mathbb{C}^n)$ is compact if and only if $H_{\bar{f}}$ is compact. This is the result of Hu and Virtanen in [49], Theorem 1.2.*

Theorem 3.1.12 (Berger-Coburn phenomenon for Schatten class Hankel operators)**.** *Let $f \in L^\infty(\mathbb{C}^n)$. Then the following hold.*

1. *Assume that H_f is a Hilbert-Schmidt operator on the classical Fock space F^2. Then $H_{\bar{f}}$ is also a Hilbert-Schmidt operator, and $\|H_{\bar{f}}\|_{S_2} \le 2\|H_f\|_{S_2}$. This is the result of Bauer in [14], Theorem 2.4,*

2. *Suppose $\phi \in \mathcal{C}^2(\mathbb{C}^n)$ is real valued, $dd^c\phi \simeq \omega_0$, and $1 < p < \infty$. Then $H_f : F^2_\phi(\mathbb{C}^n) \to L^2_\phi(\mathbb{C}^n)$ is in the Schatten class S_p if and only if $H_{\bar{f}} \in S_p$, and $\|H_{\bar{f}}\|_{S_p} \leq C\|H_f\|_{S_p}$, where the constant C is independent of f. Here $d = \partial + \bar{\partial}$ is the exterior derivative, $d^c = \frac{i}{2}(\bar{\partial} - \partial)$, and $\omega_0 = \frac{1}{2}dd^c|z|^2$ is the Euclidean Kähler form on $\mathbb{C}^n$. When $n = 1$, this is equivalent to $\Delta\phi \simeq m$, for some positive constant m. This is the result of Hu and Virtanen in [51], Theorem 1.2.*

Remark 3.1.13. For symmetric matrices A and B, we use the convention that $A \leq B$ if $B - A$ is positive semidefinite. That is, for all x, $\langle (B-A)x, x\rangle \geq 0$. Notice that for $\phi \in \mathcal{C}^2(\mathbb{C}^n)$, the requirement $m\,\mathrm{Id}_{2n} \leq \mathrm{Hess}_{\mathbb{R}}\,\phi \leq M\,\mathrm{Id}_{2n}$ in Theorem 3.1.11 implies $dd^c\phi \simeq \omega_0$ as in the assumption of Theorem 3.1.12. In fact, recall from Remark 2.3.3 that $dd^c\phi = i\partial\bar{\partial}\phi = i\sum_{j,k=1}^n \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z) dz_j \wedge d\bar{z}_k$. For $z \in \mathbb{C}^n$, let $v \in T_z\mathbb{C}^n \cong \mathbb{R}^{2n}$. Write $v = (a_1, \cdots, a_n, b_1, \cdots, b_n)$, and define $u_j = a_j + ib_j$, for each $1 \leq j \leq n$. Recall that the real Hessian is the $2n \times 2n$ real symmetric matrix

$$\mathrm{Hess}_{\mathbb{R}}\,\phi = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where

$$A_{jk} = \frac{\partial^2\phi}{\partial x_j \partial x_k}, \qquad B_{jk} = \frac{\partial^2\phi}{\partial x_j \partial y_k}, \qquad C_{jk} = \frac{\partial^2\phi}{\partial y_j \partial y_k}.$$

Associated to $\mathrm{Hess}_{\mathbb{R}}\,\phi$, one can obtain the quadratic form $Q(v) = v^T(\mathrm{Hess}_{\mathbb{R}}\,\phi)v$. In coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$, $Q(v)$ can be written as

$$Q(v) = \sum_{j,k=1}^n Q_{jk} = \sum_{j,k=1}^n \left(\frac{\partial^2\phi}{\partial x_j \partial x_k} a_j a_k + 2\frac{\partial^2\phi}{\partial x_j \partial y_k} a_j b_k + \frac{\partial^2\phi}{\partial y_j \partial y_k} b_j b_k \right).$$

Using (2.3.4), one can write $\partial_{x_j} = \partial_{z_j} + \partial_{\bar{z}_j}$, and $\partial_{y_j} = i(\partial_{z_j} - \partial_{\bar{z}_j})$. For simplicity, use the notation $\phi_{z_j z_k} := \frac{\partial^2\phi}{\partial z_j \partial z_k}$, and similarly for other partial derivatives. Since ϕ is real-valued, $\phi_{\bar{z}_j \bar{z}_k} = \overline{\phi_{z_j z_k}}$, and $\phi_{\bar{z}_j z_k} = \overline{\phi_{z_j \bar{z}_k}}$. Therefore,

$$\begin{aligned}
\phi_{x_j x_k} &= \phi_{z_j z_k} + \phi_{\bar{z}_j \bar{z}_k} + \phi_{z_j \bar{z}_k} + \phi_{\bar{z}_j z_k} = 2\,\mathrm{Re}(\phi_{z_j z_k}) + 2\,\mathrm{Re}(\phi_{z_j \bar{z}_k}), \\
\phi_{x_j y_k} &= i[\phi_{z_j z_k} - \phi_{z_j \bar{z}_k} + \phi_{\bar{z}_j z_k} - \phi_{\bar{z}_j \bar{z}_k}] = -2\,\mathrm{Im}(\phi_{z_j z_k}) + 2\,\mathrm{Im}(\phi_{z_j \bar{z}_k}), \\
\phi_{y_j y_k} &= -\phi_{z_j z_k} + \phi_{z_j \bar{z}_k} + \phi_{\bar{z}_j z_k} - \phi_{\bar{z}_j \bar{z}_k} = -2\,\mathrm{Re}(\phi_{z_j z_k}) + 2\,\mathrm{Re}(\phi_{z_j \bar{z}_k}),
\end{aligned}$$

where Re and Im stand for the real and imaginary part, respectively. Using $\phi_{z_k z_j} = \phi_{z_j z_k}$, $\phi_{z_k \bar{z}_j} = \overline{\phi_{z_j \bar{z}_k}}$, $\mathrm{Re}(\overline{\phi_{z_j \bar{z}_k}}) = \mathrm{Re}(\phi_{z_j \bar{z}_k})$, and $\mathrm{Im}(\overline{\phi_{z_j \bar{z}_k}}) = -\mathrm{Im}(\phi_{z_j \bar{z}_k})$, one can write

$$\begin{aligned}
Q_{jk} + Q_{kj} = {} & 4\,\mathrm{Re}(\phi_{z_j z_k})(a_j a_k - b_j b_k) - 4\,\mathrm{Im}(\phi_{z_j z_k})(a_j b_k + a_k b_j) \\
& + 4\,\mathrm{Re}(\phi_{z_j \bar{z}_k})(a_j a_k + b_j b_k) + 4\,\mathrm{Im}(\phi_{z_j \bar{z}_k})(a_j b_k - a_k b_j).
\end{aligned}$$

This implies that

$$Q(v) = 2\,\mathrm{Re}[\sum_{j,k=1}^n \phi_{z_j \bar{z}_k} u_j \bar{u}_k + \phi_{z_j z_k} u_j u_k].$$

Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the standard complex structure on $\mathbb{C}^n$. That is, $J^2 = -\mathrm{Id}$, and J acts on the vector v by the 90 degree rotation. Hence, $Jv = (-b_1, \cdots, -b_n, a_1, \cdots, a_n)$. Then

$$\begin{aligned} Q(Jv) &= \sum_{j,k=1}^{n} \left(\frac{\partial^2 \phi}{\partial x_j \partial x_k} b_j b_k - 2 \frac{\partial^2 \phi}{\partial x_j \partial y_k} b_j a_k + \frac{\partial^2 \phi}{\partial y_j \partial y_k} a_j a_k \right) \\ &= 2\,\mathrm{Re}[\sum_{j,k=1}^{n} \phi_{z_j \bar{z}_k} u_j \bar{u}_k - \phi_{z_j z_k} u_j u_k]. \end{aligned}$$

Hence,

$$Q(v) + Q(Jv) = 4\,\mathrm{Re}[\sum_{j,k=1}^{n} \phi_{z_j \bar{z}_k} u_j \bar{u}_k].$$

On the other hand

$$dd^c \phi(v, Jv) = i \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(z) dz_j \wedge d\bar{z}_k(v, Jv).$$

Notice that $z_j = x_j + iy_j$, $dz_j \wedge d\bar{z}_k(v, Jv) = dz_j(v) d\bar{z}_k(Jv) - dz_j(Jv) d\bar{z}_k(v)$, and $dz_j(v) = \frac{\partial z_j}{\partial x_1} a_1 + \cdots + \frac{\partial z_j}{\partial x_n} a_n + \frac{\partial z_j}{\partial y_1} b_1 + \cdots + \frac{\partial z_j}{\partial y_n} b_n = a_j + ib_j = u_j$. Similarly, $d\bar{z}_j(v) = a_j - ib_j = \bar{u}_j$, $dz_j(Jv) = iu_j$, and $d\bar{z}_j(Jv) = -i\bar{u}_j$. Thus, $dz_j \wedge d\bar{z}_k(v, Jv) = -2iu_j \bar{u}_k$. Hence,

$$dd^c \phi(v, Jv) = 2 \sum_{j,k=1}^{n} \phi_{z_j \bar{z}_k} u_j \bar{u}_k.$$

Note that since ϕ is real-valued, $\phi_{\bar{z}_j z_k} = \overline{\phi_{z_j \bar{z}_k}}$. This implies that $dd^c \phi(v, Jv)$ is real. In fact,

$$\overline{\sum_{j,k=1}^{n} \phi_{z_j \bar{z}_k} u_j \bar{u}_k} = \sum_{j,k=1}^{n} \overline{\phi_{z_j \bar{z}_k}} \bar{u}_j u_k = \phi_{\bar{z}_j z_k} \bar{u}_j u_k = \sum_{j,k=1}^{n} \phi_{z_j \bar{z}_k} u_j \bar{u}_k,$$

where in the last equality, we reindexed the double sum over j and k. Therefore, one can conclude that

$$dd^c \phi(v, Jv) = \frac{1}{2}[Q(v) + Q(Jv)]. \tag{3.1.2}$$

We use (3.1.2) to show that if $m\,\mathrm{Id}_{2n} \le \mathrm{Hess}_{\mathbb{R}} \phi \le M\,\mathrm{Id}_{2n}$, then $dd^c \phi \simeq \omega_0$. We proceed as follows. By assumption, for every real $v \in \mathbb{R}^{2n}$, $m|v|^2 \le Q(v) \le M|v|^2$. Using $|v| = |Jv|$, we also have $m|v|^2 \le Q(Jv) \le M|v|^2$, which implies that

$$m|v|^2 \le \frac{1}{2}[Q(v) + Q(Jv)] \le M|v|^2.$$

Using (3.1.2) and the fact that $\omega_0(v, Jv) = |v|^2$, we can finally conclude that $m\omega_0(v, Jv) \le dd^c \phi(v, Jv) \le M\omega_0(v, Jv)$.

Assume that A is a positive semidefinite matrix. Recall that we cannot say much about the inner product $\langle Ax, y \rangle$, except for the Cauchy-Schwarz inequality $|\langle Ax, y \rangle|^2 \le \langle Ax, x \rangle \langle Ay, y \rangle$. Now since A is positive definite, one can use the inequality $\sqrt{ab} \le (a+b)/2$ for $a, b > 0$, to conclude that $|\langle Ax, y \rangle| \le 1/2(\langle Ax, x \rangle + \langle Ay, y \rangle)$, which gives us an upper bound. However, we cannot find a universal lower bound for the inner product $\langle Ax, y \rangle$. Accordingly, one can consider the positive semidefinite matrices $A = M\,\mathrm{Id}_{2n} - \mathrm{Hess}_{\mathbb{R}} \phi$ and $B = \mathrm{Hess}_{\mathbb{R}} \phi - m\,\mathrm{Id}_{2n}$. Similarly, by $dd^c \phi \simeq \omega_0$, we exactly mean that for any $v \in \mathbb{R}^{2n}$, $m\omega_0(v, Jv) \le dd^c \phi(v, Jv) \le M\omega_0(v, Jv)$. Similarly to the

$|\langle Ax,y\rangle| \leq 1/2(\langle Ax,x\rangle + \langle Ay,y\rangle)$, we can find an upper bound for $dd^c\phi(v,Jw)$. To obtain this, we use the following symmetric bilinear form. For the real $(1,1)$-form $\omega = dd^c\phi$, define the bilinear form

$$B(v,w) := \omega(v,Jw), \quad v,w \in \mathbb{R}^{2n}.$$

It is easy to see that $dd^c\phi(Jv,Jw) = dd^c\phi(v,w)$, and $\omega = dd^c\phi$ is skew-symmetric. Using this, one can see that B is symmetric. In fact, $B(w,v) = \omega(w,Jv) = \omega(Jw,J^2v) = \omega(Jw,-v) = -\omega(Jw,v) = \omega(v,Jw) = B(v,w)$. The quadratic form associated to B is

$$\tilde{Q}(v) = B(v,v) = \omega(v,Jv),$$

which is exactly the quantity we have already controlled. For any symmetric bilinear form, $B(v,w) = 1/2[\tilde{Q}(v+w) - \tilde{Q}(v) - \tilde{Q}(w)]$. So any control of $\tilde{Q}$ gives control of the full bilinear form $B(v,w)$ from above. Because of the negative terms, we cannot control B from below. Indeed,

$$\begin{aligned}|dd^c\phi(v,Jw)| = |\omega(v,Jw)| = |B(v,w)| &= \frac{1}{2}|\tilde{Q}(v+w) - \tilde{Q}(v) - \tilde{Q}(w)| \leq \frac{1}{2}|\tilde{Q}(v+w) + \tilde{Q}(v) + \tilde{Q}(w)| \\ &\leq \frac{M}{2}(|v+w|^2 + |v|^2 + |w|^2) \leq M(|v|^2 + |w|^2) = M(\omega_0(v,Jv) + \omega_0(w,Jw)) \\ &\leq \frac{M}{m}(dd^c\phi(v,Jv) + dd^c\phi(w,Jw)).\end{aligned}$$

Berger-Coburn phenomenon for Hankel operators on doubling Fock spaces

The Berger-Coburn phenomenon for compact Hankel operators on doubling Fock spaces is still open. In the following, we will give a summary of our paper [10] (equivalently, Appendix A), regarding our approach to studying the Berger-Coburn phenomenon for Schatten class Hankel operators. What we do here is a generalization of Hu and Virtanen's works in [49, 51].

Definition 3.1.14. For $f \in L^q_{loc}$, $q \geq 1$, and $r > 0$, define

$$G_{q,r}(f)(z) = \inf\left\{\left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f-h|^q dA\right)^{1/q} : h \in H(D^r(z))\right\},$$

where $D^r(z) = D(z, r\rho(z))$, and $|E|$ is the Lebesgue measure of $E \subset \mathbb{C}$. Then for $0 < p \leq \infty$, $q \geq 1$, $r > 0$, and $\alpha \in \mathbb{R}$, the space $\mathrm{IDA}^{p,q,\alpha}_r$, integral distance to analytic function, consists of all $f \in L^q_{loc}$ such that

$$\|f\|_{\mathrm{IDA}^{p,q,\alpha}_r} = \|\rho^\alpha G_{q,r}(f)\|_{L^p} < \infty.$$

Remark 3.1.15. Let $0 < p \leq \infty$, $q \geq 1$, and $\alpha \in \mathbb{R}$. For any $r_1, r_2 > 0$, we have

$$\mathrm{IDA}^{p,q,\alpha}_{r_1} = \mathrm{IDA}^{p,q,\alpha}_{r_2},$$

with equivalence of norms. In particular, the space $\mathrm{IDA}^{p,q,\alpha}_r$ is independent of the choice of $r > 0$. This follows from the proof of Theorem A.1.1.

Note that $\mathrm{IDA}^{p,2} := \mathrm{IDA}^{p,2,0}_1$ type conditions were first introduced by Luecking [60] in his study of Schatten class Hankel operators on the Bergman space $A^2(D)$ of the unit disk. Later on, Hu and Virtanen [49, 51] generalized it to $\mathrm{IDA}^{s,q} := \mathrm{IDA}^{s,q,0}_1$ to study compact Hankel operators $H_f : F^p_\phi(\mathbb{C}^n) \to L^q_\phi(\mathbb{C}^n)$, for $p > q$, and $s = pq/(p-q)$, where $m\,\mathrm{Id}_{2n} \leq \mathrm{Hess}_{\mathbb{R}}\,\phi \leq M\,\mathrm{Id}_{2n}$.

Definition 3.1.16. For $f \in L^1_{loc}$, define the average function on $\mathbb{C}$ by

$$\hat{f}_r(z) = \frac{1}{|D^r(z)|}\int_{D^r(z)} f\,dA.$$

The above is the usual average function, but modified according to the doubling property of the measure under consideration.

Given a sequence $\{a_j\}_{j=1}^{\infty} \subset \mathbb{C}$, and $r > 0$, we call $\{a_j\}_{j=1}^{\infty}$ an r-lattice if $\{D^r(a_j)\}_{j=1}^{\infty}$ covers $\mathbb{C}$ and the disks of $\{D^{r/5}(a_j)\}_{j=1}^{\infty}$ are pairwise disjoint. Moreover, for an r-lattice $\{a_j\}_{j=1}^{\infty}$, and a real number $m > 1$, there exists an integer N such that

$$1 \le \sum_{j=1}^{\infty} \chi_{D^{mr}(a_j)(z)} \le N, \quad \forall z \in \mathbb{C}, \tag{3.1.3}$$

where χ_E is the characteristic function of a subset E of $\mathbb{C}$.

Note that using (2.2.2) and the triangle inequality, there exists $m \in (0,1)$ such that $D^{mr}(w) \subset D^r(z)$, whenever $w \in D^{mr}(z)$. For $r > 0$, let $\{a_j\}_{j=1}^{\infty}$ be a mr-lattice, and let $J_z := \{j : z \in D^r(a_j)\}$, so that $|J_z| = \sum_{j=1}^{\infty} \chi_{D^r(a_j)}(z) \le N$, for some integer N. Let $\eta : \mathbb{C} \to [0,1]$ be the following smooth function with bounded derivatives.

$$\eta(z) = \begin{cases} 1, & \text{if } |z| \le 1/2, \\ 0, & \text{if } |z| \ge 1. \end{cases}$$

For each $j \ge 1$ we define $\eta_j(z) = \eta(\frac{z-a_j}{mr\rho(a_j)})$. We can normalize η_j such that $\int_{\mathbb{C}} \eta_j dA = 1$, for each $j \ge 1$. Define $\psi_j(z) = \frac{\eta_j(z)}{\sum_{k=1}^{\infty}\eta_k(z)}$. Then one can see that $\{\psi_j\}_{j=1}^{\infty}$ is a partition of unity subordinate to $\{D^{mr}(a_j)\}_{j\ge 1}$, satisfying the following properties:

$$\operatorname{Supp}\psi_j \subset D^{mr}(a_j), \quad \psi_j(z) \ge 0, \quad \sum_{j=1}^{\infty}\psi_j(z) = 1,$$

$$|\rho(a_j)\bar{\partial}\psi_j| \le C, \quad \sum_{j=1}^{\infty}\bar{\partial}\psi_j(z) = 0,$$

where the constant C may depend on r. For $j = 1, 2, \cdots$, we can pick $h_j \in H(D^r(a_j))$ such that

$$\widehat{|f-h_j|^q}_r(a_j) = \frac{1}{|D^r(a_j)|}\int_{D^r(a_j)} |f-h_j|^q dA = G_{q,r}(f)(a_j)^q.$$

To see why the infimum is attained, please see Lemma A.3.1.

In the following, we give a characterization of $\mathrm{IDA}_r^{p,q,\alpha}$.

Theorem 3.1.17 (Theorem A.1.1, [10], Theorem 1.1). *Let $\phi \in \mathcal{C}^{\infty}(\mathbb{C})$ be subharmonic such that $d\mu = \Delta\phi dA$ is a doubling measure. Suppose that $1 \le q \le \infty$, $0 < p < \infty$, $\alpha \in \mathbb{R}$, $r > 0$, and $f \in L^q_{loc}$. Then if $f \in \mathrm{IDA}_r^{p,q,\alpha}$, $f = f_1 + f_2$, where $f_1 \in \mathcal{C}^2(\mathbb{C})$ and*

$$\rho^{1+\alpha}|\bar{\partial}f_1| + \rho^{1+\alpha}(\widehat{|\bar{\partial}f_1|^q}_s)^{1/q} + \rho^{\alpha}(\widehat{|f_2|^q}_s)^{1/q} \in L^p, \tag{3.1.4}$$

for some (equivalent any) $s > 0$, and

$$\|f\|_{\mathrm{IDA}_r^{p,q,\alpha}} \simeq \inf\left\{\|\rho^{1+\alpha}(\widehat{|\bar{\partial}f_1|^q}_r)^{1/q}\|_{L^p} + \|\rho^{\alpha}(\widehat{|f_2|^q}_r)^{1/q}\|_{L^p}\right\}, \tag{3.1.5}$$

where the infimum is taken over all possible decompositions $f = f_1 + f_2$, with f_1 and f_2 satisfying the conditions $f_1(z) := \sum_{j=1}^{\infty} h_j(z)\psi_j(z)$, and $f_2(z) = f(z) - f_1(z)$, where h_j and ψ_j are given above.

Recall the set $\mathcal{S}$ and the Hankel operator H_f defined in (2.2.29). The following result characterizes the Schatten class Hankel operators on doubling Fock spaces in terms of IDA. A similar characterization for Schatten class Hankel operators on the Bergman space $A^2(D)$ was obtained by Luecking [60], Theorem 4. Moreover, Hu and Virtanen [51] proved a similar result on Fock spaces F^2_ϕ with $dd^c\phi \simeq \omega_0$, in Theorem 1.1.

Theorem 3.1.18 (Theorem A.1.2, [10], Theorem 1.2)**.** *Let $0 < p \leq \infty$, and $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic such that $d\mu := \Delta\phi dA$ is a doubling measure. Then for $f \in \mathcal{S}$, the following are equivalent:*

(1) $H_f : F^2_\phi \to L^2_\phi$ is in S_p,

(2) $f \in \mathrm{IDA}^{p,2,-2/p}_r$, for some (equivalent any) $r > 0$.

Moreover,

$$\|H_f\|_{S_p} \simeq \|f\|_{\mathrm{IDA}^{p,2,-2/p}_r}. \tag{3.1.6}$$

Our goal is to study the Berger-Coburn phenomenon. So, the next step is to characterize the simultaneous membership of H_f and $H_{\bar{f}}$ in $S_p(F^2_\phi, L^2_\phi)$. For this purpose, we define the space IMO of integral mean oscillations.

Definition 3.1.19. For $f \in L^2_{loc}$ and $r > 0$, the mean oscillation of f is defined by

$$MO_{2,r}(f)(z) = \left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f - \hat{f}_r(z)|^2 dA\right)^{1/2}.$$

Given $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$, we define the space $\mathrm{IMO}^{p,2,\alpha}_r$ to be the family of those $f \in L^2_{loc}$ such that

$$\|f\|_{\mathrm{IMO}^{p,2,\alpha}_r} = \|\rho^\alpha MO_{2,r}(f)\|_{L^p} < \infty.$$

Theorem 3.1.20 (Theorem A.1.4, [10], Theorem 1.3)**.** *Let $0 < p < \infty$ and assume that $\phi \in \mathcal{C}^\infty(\mathbb{C})$ is subharmonic such that $d\mu = \Delta\phi dA$ is a doubling measure. Then the following are equivalent.*

(1) Both H_f and $H_{\bar{f}} \in S_p(F^2_\phi, L^2_\phi)$,

(2) $f \in \mathrm{IMO}^{p,2,-2/p}_r$, for some (equivalent any) $r > 0$. Moreover,

$$\|H_f\|_{S_p} + \|H_{\bar{f}}\|_{S_p} \simeq \|f\|_{\mathrm{IMO}^{p,2,-2/p}_r}.$$

Note that Hu and Virtanen [51] found a similar characterization for when $dd^c\phi \simeq \omega_0$ in Theorem 6.2. Moreover, Xia and Zheng [77] found a characterization for the simultaneous membership of Schatten class Hankel operators on the classical Fock space over $\mathbb{C}^n$ in terms of the standard deviation. For more information see Theorem 1.1 in [77].

Recalling Example 3.1.10, one can show that taking $f(z) = z^k$, where k is a positive integer, $m > 0$, and $\phi(z) = |z|^m$, H_f belongs to the Hilbert-Schmidt class of operators from F^2_ϕ to L^2_ϕ, while $H_{\bar{f}}$ is not [72]. Indeed, one can prove a more general result based on the characterization given in Theorem 3.1.20 to emphasize the importance of the boundedness assumption in the Berger-Coburn phenomenon for Hankel operators.

Example 3.1.21 (Theorem A.5.4, [10], Theorem 5.4)**.** Let f be a non-constant entire function and F^2_ϕ be a doubling Fock space. Then $H_{\bar{f}}$ is not in $S_2(F^2_\phi, L^2_\phi)$.

Proof. Since f is holomorphic, $H_{\bar{f}} = 0$, and thus belongs to the Hilbert-Schmidt class. Applying Theorem 3.1.20, it is enough to show that $f \notin \mathrm{IMO}_1^{2,2,-1}$. First note that f is harmonic on $D^1(z)$ and by the mean-value property of harmonic functions,

$$\widehat{f_1}(z) = \frac{1}{|D^1(z)|}\int_{D^1(z)} f\,dA = f(z).$$

Hence,

$$MO_{2,1}(f)(z) = \left(\frac{1}{|D^1(z)|}\int_{D^1(z)} |f(w) - f(z)|^2 dA(w)\right)^{1/2}.$$

Since f is entire, it is homolorphic on $\overline{D(z,\rho(z))}$. Set $M = \sup_{|\zeta - z| = \rho(z)} |f(\zeta)|$. By the Cauchy estimate, $|f'(z)| \le M/\rho(z)$. Using the Taylor expansion, for every w with $|w - z| < \rho(z)$, there is ζ on the segment between z and w such that

$$f(w) - f(z) = f'(z)(w - z) + \frac{1}{2}f''(\zeta)(w - z)^2. \tag{3.1.7}$$

Note that $|\zeta - z| \le |w - z| = s\rho(z)$, for some $0 < s < 1$. The disk centered at ζ of radius $r := \rho(z) - |\zeta - z|$ is contained in $D^1(z) = D(z,\rho(z))$. Hence, the Cauchy estimate at ζ gives

$$|f''(\zeta)| \le \frac{2!}{r^2} \sup_{|x-\zeta|=r} |f(x)|.$$

Moreover,

$$\sup_{|x-\zeta|=r} |f(x)| \le \sup_{|y-z|\le\rho(z)} |f(y)| = \sup_{|y-z|=\rho(z)} |f(y)| = M,$$

because the supremum over the closed disk is equal to the supremum on the boundary by the maximum modulus principle. Therefore, the Cauchy estimate at ζ with radius r yields

$$|f''(\zeta)| \le \frac{2M}{r^2} \le \frac{2M}{\rho(z)^2(1-s)^2}.$$

Then (3.1.7) implies that

$$\begin{aligned}
|f(w) - f(z)| &\ge \left| |f'(z)||w - z| - \frac{1}{2}|f''(\zeta)||w - z|^2 \right| \\
&\ge \left| \rho(z)|f'(z)|\frac{s^2}{(1-s)^2} - \rho(z)s|f'(z)| \right| \\
&= \rho(z)|f'(z)| \left| \frac{s^2}{(1-s)^2} - s \right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{|D^1(z)|}\int_{D^1(z)} |f(w) - f(z)|^2 dA(w) &= \frac{1}{|D^1(z)|}\int_0^{2\pi}\int_0^1 |f(z + s\rho(z)e^{i\theta}) - f(z)|^2 s\rho(z)^2 ds d\theta \\
&= 2\pi \int_0^1 |f(z + s\rho(z)e^{i\theta}) - f(z)|^2 s ds \\
&\ge 2\pi(\rho(z)|f'(z)|)^2 \int_0^1 \left| \frac{s^2}{(1-s)^2} - s \right|^2 s ds \\
&\ge 2\pi(\rho(z)|f'(z)|)^2 \int_{1/2}^{3/4} \left| \frac{s^2}{(1-s)^2} - s \right|^2 s ds.
\end{aligned}$$

Take $\psi(s) = s^2/(1-s)^2 - s$. Then $\psi'(s) = 2s/(1-s)^3 - 1 > 0$ for $s \geq 1/2$, and thus ψ is increasing on $[1/2, 3/4]$. Moreover, $\psi(1/2) = 1/2$. So, $\psi(s)^2 \geq \psi(1/2)^2 = 1/4$ for all $s \in [1/2, 3/4]$. Hence,

$$\int_{1/2}^{3/4} \left| \frac{s^2}{(1-s)^2} - s \right|^2 s ds \geq \int_{1/2}^{3/4} \frac{s}{4} ds = \frac{5}{128}.$$

Therefore, we can conclude that

$$\begin{aligned} MO_{2,1}(f)(z) &= \left(\frac{1}{|D^1(z)|} \int_{D^1(z)} |f(w) - f(z)|^2 dA(w) \right)^{1/2} \\ &\geq \sqrt{\frac{5\pi}{64}} \rho(z) |f'(z)|. \end{aligned}$$

Hence,

$$\begin{aligned} \|f\|_{\mathrm{IMO}_1^{2,2,-1}} &= \left(\int_{\mathbb{C}} \rho(z)^{-2} MO_{2,1}(f)(z)^2 dA(z) \right)^{1/2} \\ &\geq \sqrt{\frac{5\pi}{64}} \left(\int_{\mathbb{C}} \rho(z)^{-2} |f'(z)|^2 \rho(z)^2 dA(z) \right)^{1/2}. \end{aligned}$$

So, since f is entire and non-constant, it follows that $f \notin \mathrm{IMO}_1^{2,2,-1}$, and thus $H_{\bar{f}}$ is not Hilbert-Schmidt. □

In the following, we state the Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators on doubling Fock spaces, and discuss the open problem for the other Schatten classes S_p, when $1 < p < \infty$.

Theorem 3.1.22 (Theorem A.1.5, [10], Theorem 1.4). *Let $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic and suppose that $d\mu = \Delta\phi \, dA$ is a doubling measure. Then for $f \in L^\infty$, $H_f \in S_2(F^2_\phi, L^2_\phi)$ if and only if $H_{\bar{f}} \in S_2(F^2_\phi, L^2_\phi)$, with*

$$\|H_{\bar{f}}\|_{S_2} \simeq \|H_f\|_{S_2}.$$

Sketch of proof. Let $H_f \in S_2$. Since $f \in L^\infty$, it is in particular in L^2_{loc}. Then by Theorem 3.1.18, $f \in \mathrm{IDA}_r^{2,2,-1}$, for some (equivalent any) $r > 0$, and $\|H_f\|_{S_2} \simeq \|f\|_{\mathrm{IDA}_r^{2,2,-1}} < \infty$. Then taking, $f = f_1 + f_2$ as in Theorem 3.1.17, by (3.1.6) and (3.1.5), we see that

$$\|H_{\bar{f}_2}\|_{S_2} = \|\bar{f}_2\|_{\mathrm{IDA}_r^{2,2,-1}} \lesssim \|\rho^{-1}(\widehat{|\bar{f}_2|^2}_r)^{1/2}\|_{L^2} = \|\rho^{-1}(\widehat{|f_2|^2}_r)^{1/2}\|_{L^2} \lesssim \|f\|_{\mathrm{IDA}_r^{2,2,-1}}.$$

Hence, $H_{\bar{f}_2} \in S_2$. To show that $H_{\bar{f}_1} \in S_2$, let $\{a_j\}_{j=1}^\infty$ be a fixed $m_1 r$-lattice for some $m_1 \in (0,1)$ and $r > 0$. Choose a partition of unity $\{\psi_j\}_{j=1}^\infty$ subordinate to $\{D^{m_1 r}(a_j)\}$. Take $f = f_1 + f_2$ with $f_1 = \sum_{j=1}^\infty h_j \psi_j$ as in Theorem 3.1.17. Then $\bar\partial \bar{f}_1 = F + H$, where $F = \sum_{j=1}^\infty \bar{h}_j \bar\partial \psi_j$ and $H = \sum_{j=1}^\infty \psi_j \bar\partial \bar{h}_j$. One can see that $|F(z)| \lesssim \rho^{-1}(z) G_{2,r}(f)(z)$, implying that $\|F\|_{L^2} \leq \|f\|_{\mathrm{IDA}_r^{2,2,-1}}$. Besides, $\|H\|_{L^2} \leq \|\bar\partial \bar{f}_1\|_{L^2} + \|F\|_{L^2}$. Lemma 7.1 in [51] states that for $f \in \mathcal{C}^2(\mathbb{C}) \cap L^\infty$, there is a constant C, independent of f such that $\|\partial f\|_{L^2} \leq C \|\bar\partial f\|_{L^2}$. Therefore,

$$\|\bar\partial \bar{f}_1\|_{L^2} = \|\partial f_1\|_{L^2} \leq C \|\bar\partial f_1\|_{L^2} \leq C \|f\|_{\mathrm{IDA}_r^{2,2,-1}}, \tag{3.1.8}$$

and thus $\|H\|_{L^2} \lesssim \|f\|_{\mathrm{IDA}_r^{2,2,-1}}$.

Now, we are left to find a suitable upper bound for $\|H_{\bar{f}_1}\|_{S_2}$ in terms of $\|H\|_{L^2}$ and $\|f\|_{\mathrm{IDA}_r^{2,2,-1}}$. To do this, we proceed as follows. For $m_1, m_2 \in (0,1)$, (3.1.4) implies that

$$\begin{aligned}\|H_{\bar{f}_1}\|_{S_2}^2 &\lesssim \int_{\mathbb{C}} \left[(\widehat{|F|^2}_{m_1m_2r})^{1/2}\right]^2 dA + \int_{\mathbb{C}} \left[(\widehat{|H|^2}_{m_1m_2r})^{1/2}\right]^2 dA \\ &\lesssim \|f\|_{\mathrm{IDA}_r^{2,2,-1}}^2 + \int_{\mathbb{C}} \left[(\widehat{|H|^2}_{m_1m_2r})^{1/2}\right]^2 dA.\end{aligned}$$

Let $z \in D^r(a_j) \cap D^r(a_k)$. Applying the Cauchy estimate, one can see that

$$|\bar{\partial}(\bar{h}_k(z) - \bar{h}_j(z))| \lesssim \frac{1}{\rho(z)} G_{2,R}(f)(z), \tag{3.1.9}$$

for some $R > m_1m_2r$. Using (3.1.9) and the identity $\bar{\partial}\bar{h}_k = \sum_{j=1}^{\infty} \psi_j \bar{\partial}(\bar{h}_k - \bar{h}_j) + H$,

$$|\bar{\partial}\bar{h}_k(z)|^2 \lesssim \left(\rho^{-1}(z) G_{2,R}(f)(z)\right)^2 + |H(z)|^2.$$

Since h_k is holomorphic, so is ∂h_k, and thus $|\partial h_k|^2$ is subharmonic. Then, one can show that for $z \in D^{m_1r}(a_k)$,

$$\begin{aligned}|\bar{\partial}\bar{h}_k(z)|^2 &\leq \frac{1}{|D^{m_1m_2r}(z)|} \int_{D^{m_1m_2r}(z)} |\bar{\partial}\bar{h}_k(w)|^2 dA(w) \\ &\lesssim (\rho^{-1}(z))^2 G_{2,\tilde{R}}(f)(z)^2 + \widehat{|H|^2}_{m_1m_2r}(z),\end{aligned}$$

for some $\tilde{R} > R$. Moreover, one can see that

$$\left[(\widehat{|H|^2}_{m_1m_2r}(z))^{1/2}\right]^2 \leq \left|\sum_{k=1}^{\infty} \psi_k \bar{\partial}\bar{h}_k\right|^2 \lesssim \left(\rho^{-1}(z))G_{2,s}(f)(z)\right)^2 + \widehat{|H|^2}_{m_1r}(z).$$

Let $w \in D^{m_1r}(z)$, and define $U_w = \{z \in \mathbb{C} : w \in D^{m_1r}(z)\}$. Note that by (2.2.2), there are constants $\alpha, \beta > 0$, only depending on the doubling constant, such that for every $z \in U_w$, $D^{m_1r}(z)$ contains $D^{\alpha r}(w)$, and $U_w \subset D^{\beta r}(w)$. Then Fubini's theorem implies that

$$\begin{aligned}\int_{\mathbb{C}} \widehat{|H|^2}_{m_1r}(z) dA(z) &= \int_{\mathbb{C}} \frac{1}{|D^{m_1r}(z)|} \int_{D^{m_1r}(z)} |H(w)|^2 dA(w) dA(z) \\ &= \int_{\mathbb{C}} |H(w)|^2 \left(\int_{\mathbb{C}} \frac{\chi_{D^{m_1r}(z)}(w)}{|D^{m_1r}(z)|} dA(z)\right) dA(w) \\ &= \int_{\mathbb{C}} |H(w)|^2 \left(\int_{U_w} \frac{dA(z)}{|D^{m_1r}(z)|}\right) dA(w) \\ &\leq \int_{\mathbb{C}} |H(w)|^2 \left(\frac{|D^{\beta r}(w)|}{|D^{\alpha r}(w)|}\right) dA(w) \lesssim \|H\|_{L^2}^2.\end{aligned}$$

Hence, we can conclude that $\|H_{\bar{f}_1}\| \lesssim \|f\|_{\mathrm{IDA}_r^{2,2,-1}}$. ☐

Remark 3.1.23. A natural question to ask is if the Berger-Coburn phenomenon for Hankel operators on doubling Fock spaces holds for other Schatten classes S_p with $1 < p < \infty$. This is an open problem, and is discussed in [10], Remark 6.1 (equivalently, Remark A.6.1). For $1 < p < \infty$ we say that ω is a Muckenhoupt weight and write $\omega \in A_p$ if there is a constant $C > 0$ such that for any disk $B \subset \mathbb{C}$, we have

$$\left(\frac{1}{|B|} \int_B \omega dA\right)\left(\frac{1}{|B|} \int_B \omega^{-q/p} dA\right)^{p/q} \leq C < \infty,$$

where q is the Hölder conjugate of p and $|B|$ is the Lebesgue measure of B. As shown in [34], if $\omega \in A_p$ and $1 < p < \infty$, then the Ahlfors-Beurling operator

$$\mathcal{I}(f)(z) = p.v. - \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2} dA(z)$$

is bounded on $L^p(\omega)$. Hence, similarly to the proof of Lemma 7.1 in [51], we can show that when f is bounded,

$$\|\partial f\|_{L^p(\omega)} \leq C\|\bar{\partial} f\|_{L^p(\omega)},$$

where C is a constant depending only on p.

To generalize Theorem 3.1.22 to the other values of $1 < p < \infty$, our approach would require the single additional ingredient that $\omega = \rho^{p-2}$ is a Muckenhoupt weight (see (3.1.8)). However, we have not been able to prove this condition. Notice that when $p = 2$, $\omega = 1$ is trivially a Muckenhoupt weight.

Berger-Coburn phenomenon fails

On Hardy and Bergman spaces, there are bounded functions f, for which H_f is compact, while $H_{\bar{f}}$ is not compact, and therefore the Berger-Coburn phenomenon for compact Hankel operators fails [41]. As an example on the Bergman space, take a Blaschke product b with zeros at $\alpha_k = 1 - 1/2^k$. That is,

$$b(z) = \prod_{k=1}^{\infty} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z}\frac{|\alpha_k|}{\alpha_k}, \quad z \in \mathbb{D}.$$

Because b is a bounded analytic function, $H_b = 0$ is trivially compact, but it can be shown that b is not in the little Bloch space [11] and hence $H_{\bar{b}}$ is not compact according to Axler's characterization of compact Hankel operators with conjugate analytic symbols [12].

One can ask if there is a similar phenomenon for Toeplitz operators. One of the simple properties of Toeplitz operators is that the boundedness of the symbol implies the boundedness of the operator. However, the converse is generally false.

Example 3.1.24. We show that the converse statement, namely that boundedness of the Toeplitz operator implies boundedness of the symbol, fails even in the classical Fock space F^2. Consider the family of symbols

$$\phi_\zeta(z) = \frac{1}{\zeta} e^{-\pi|z|^2/\zeta}, \qquad z \in \mathbb{C},$$

where $\zeta \in \mathbb{C} \setminus \{0\}$. It is known (see [31, Example 4.3]) that the Toeplitz operator T_{ϕ_ζ} is bounded on F^2 if and only if

$$\left|1 + \frac{1}{\zeta}\right| \geq 1.$$

Choose ζ such that

$$\frac{1}{\zeta} = -\frac{1}{4} + i.$$

Then

$$\phi_\zeta(z) = \left(-\frac{1}{4} + i\right)\exp\left(-\pi\left(-\frac{1}{4} + i\right)|z|^2\right) = \left(-\frac{1}{4} + i\right)\exp\left(\left(\frac{\pi}{4} - i\pi\right)|z|^2\right).$$

Hence,

$$|\phi_\zeta(z)| = \left|-\frac{1}{4} + i\right| e^{\frac{\pi}{4}|z|^2} \longrightarrow \infty \quad \text{as } |z| \to \infty,$$

so $\phi_\zeta \notin L^\infty(\mathbb{C})$. On the other hand,

$$\left|1 + \frac{1}{\zeta}\right| = \left|\frac{3}{4} + i\right| = \frac{5}{4} > 1,$$

and therefore T_{ϕ_ζ} is bounded on F^2. This provides an explicit example of an unbounded symbol whose associated Toeplitz operator is bounded.

Example 3.1.25. An unbounded but Lebesgue square-integrable symbol f gives even rise to a Hilbert-Schmidt Toeplitz operator T_f. Indeed, let $f \in L^2(\mathbb{C}, dA)$, $\{e_n(z)\}_{n=0}^\infty$ be the standard orthonormal basis of F^2, and $P : L^2(\mathbb{C}, d\lambda) \to F^2$ be the orthogonal projection given in (2.1.2) with $\alpha = 1$. Then

$$\|T_f\|_{S_2}^2 = \sum_{n=0}^{\infty} \|P(f e_n)\|_{L^2(\mathbb{C},d\lambda)}^2 \leq \sum_{n=0}^{\infty} \|f e_n\|_{L^2(\mathbb{C},d\lambda)}^2.$$

Using $d\lambda(z) = \pi^{-1} e^{-|z|^2} dA(z)$ and since the Taylor series expansion of the exponential function implies that

$$\sum_{n=0}^{\infty} |e_n(z)|^2 = e^{|z|^2},$$

we obtain

$$\|T_f\|_{S_2}^2 \leq \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \, dA(z) < \infty,$$

and thus T_f belongs to the Hilbert-Schmidt class. For example, take $\mathbb{D}$ to be the unit disk centered at the origin. Then $f(z) = |z|^{-1/4} \chi_{\mathbb{D}}(z)$ is unbounded but belongs to $L^2(\mathbb{C}, dA)$. Hence T_f is Hilbert-Schmidt, and in particular bounded, although $f \notin L^\infty(\mathbb{C})$.

Consider the classical Fock space over $\mathbb{C}^n$, and let $d\mu(z) = e^{-|z|^2/2} dA(z)/(2\pi)^n$. To characterize the boundedness of a Toeplitz operator on the classical Fock space $F^2(\mathbb{C}^n, d\mu)$, instead of looking at the boundedness of the symbol, it is convenient to look at the heat transform of the symbol. For $t > 0$, the heat transform of a symbol g is given by

$$\tilde{g}^{(t)}(a) = (4\pi t)^{-n} \int_{\mathbb{C}^n} g(w) e^{-|w-a|^2/4t} dA(w),$$

whenever the integral exists. Let k_a be the normalized reproducing kernel at $a \in \mathbb{C}^n$. Then $\langle T_g k_a, k_a \rangle = g^{(1/2)}(a)$, and hence boundedness of T_g implies boundedness of $g^{(1/2)}$. Comparing with Theorem 3.2.1 in the next section, we see that this necessary condition is also sufficient if g is a positive function. For general complex-valued symbols, Berger and Coburn [16] showed the following two complementary norm estimates. First, for $t \in (0, 1/4)$, there exists $C_1(t)$ such that $\|T_g\| \leq C_1(t) \|g^{(t)}\|_\infty$. Next, for every $t \in (1/4, 1)$ there exists $C_2(t)$ such that $\|g^{(t)}\|_\infty \leq C_2(t) \|T_g\|$. Given these two different regimes, the single endpoint $t = 1/4$ remains unresolved. Berger and Coburn [16] conjectured that for g with $g k_a \in L^2(\mathbb{C}^n, d\mu)$ for all $a \in \mathbb{C}^n$, T_g is bounded if and only if $\tilde{g}^{(1/4)}$ is bounded.

A counterexample to the above conjecture was recently found in [59, Theorem 1.2], which reads as follows. There exists a measurable symbol g on $\mathbb{C}^n$ such that $g k_a \in L^2(\mathbb{C}^n, d\mu)$ for all $a \in \mathbb{C}^n$, the Toeplitz operator T_g extends to a bounded operator on $F^2(\mathbb{C}^n, d\mu)$, and the heat transform $\tilde{g}^{(1/4)}$ is unbounded on $\mathbb{C}^n$.

For the rest of this section, our main goal is to investigate the Berger-Coburn phenomenon for Schatten class Hankel operators on doubling Fock spaces when $0 < p \leq 1$.

Consider the Schatten class Hankel operators acting on Fock-type spaces, with $0 < p \leq 1$. Xia [76] used the following simple function

$$f(z) = \begin{cases} \frac{1}{z} & \text{if } |z| \geq 1, \\ 0 & \text{if } |z| < 1. \end{cases} \tag{3.1.10}$$

to show that the Berger-Coburn phenomenon does not hold for trace-class Hankel operators on the classical Fock space. Hu and Virtanen [50] noticed that when $0 < p \leq 1$, the same example shows that there is no Berger-Coburn for Schatten class Hankel operators on generalized Fock spaces F^2_ϕ with $m < \Delta\phi < M$, where $m, M > 0$. In [10], Theorem 1.6 (equivalently Theorem A.1.7), we used Xia's example again to prove that the Berger-Coburn phenomenon fails for some $S_p(F^2_\phi, L^2_\phi)$ while it remains open whether it fails for the remaining doubling Fock spaces. The idea behind the proof is to use Theorem 3.1.18, Theorem 3.1.20, and the growth of the radius function outside the unit disk centered at the origin (2.2.3). Indeed, taking f as in (3.1.10), one can show that $f \in \mathrm{IDA}^{p,2,-2/p}_r$, but $f \notin \mathrm{IMO}^{p,2,-2/p}_r$, provided the following doubling condition holds.

Theorem 3.1.26 (Theorem A.1.7, [10], Theorem 1.6)**.** *Let $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic with $d\mu = \Delta\phi\, dA$ a doubling measure. Recall the growth of the radius function* (2.2.3)*, stating that there are constants $C, \eta > 0$ and $0 \leq \beta < 1$, such that $C^{-1}|z|^{-\eta} \leq \rho(z) \leq C|z|^\beta$ when $|z| > 1$. Then, for $0 < p \leq 1$ with $\beta \leq \frac{1-p}{1-p/2}$, the Berger-Coburn phenomenon for Schatten class Hankel operators fails; that is, there is an $f \in L^\infty(\mathbb{C})$ such that $H_f \in S_p(F^2_\phi, L^2_\phi)$ but $H_{\bar{f}} \notin S_p(F^2_\phi, L^2_\phi)$.*

In particular, when ρ is bounded, the Berger-Coburn phenomenon fails for all $0 < p \leq 1$.

A simple consequence of the preceding theorem is that if F^2_ϕ is a doubling Fock space, then the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ provided that p is sufficiently small. Moreover, the closer p is to 1, i.e., the trace class, the less we know about the Berger-Coburn phenomenon. Because when p is very close to 1, β must be very close to zero to conclude that the Berger-Coburn phenomenon fails. However, it will still include a very large class of doubling Fock spaces, including standard Fock spaces, and F^2_ϕ with $\Delta\phi$ bounded from below and above. This is, of course compatible with the known results mentioned above.

Recalling Lemma 2.2.6, it is easy to see that for the canonical doubling weights $\phi(z) = |z|^m$, $\beta = 1 - m/2$. Hence, one can conclude the following.

Corollary 3.1.27 (Corollary A.1.8, [10], Corollary 1.7)**.** *Let $\phi(z) = |z|^m$, $m > 0$, and $0 < p \leq 1$. Then the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ if*

$$m \geq \frac{p}{1 - \frac{p}{2}}.$$

In particular, if $m \geq 2$, then the phenomenon fails for all Schatten classes S_p with $0 < p \leq 1$.

3.2 Boundedness, compactness, and Schatten class membership of Toeplitz operators on Fock-type spaces

In this section, we first recall some known results on the boundedness, compactness, and Schatten class membership of Toeplitz operators acting on Fock-type spaces, including standard, doubling, and scalar weighted Fock spaces. Then we investigate the boundedness, compactness, and Schatten class membership of vectorial Toeplitz operators acting on large vector-valued Fock spaces, based on the results in [6]. A copy of this article is provided in Appendix B.

Known results about boundedness and compactness of Toeplitz operators on Fock-type spaces

Consider the standard Fock space F^2_α over $\mathbb{C}$, and let μ be a positive Borel measure on $\mathbb{C}$. Define the Toeplitz operator T_μ by

$$T_\mu(f)(z) = \int_{\mathbb{C}} K(z,w) f(w) e^{-\alpha|w|^2} d\mu(w), \quad z \in \mathbb{C}. \tag{3.2.1}$$

Note that T_μ is very loosely defined here, since it is not clear when the above integral will converge. Assume that μ satisfies

$$\int_{\mathbb{C}} |K(z,w)|^2 e^{-\alpha|w|^2} d\mu(w) < \infty, \quad \forall z \in \mathbb{C}. \tag{3.2.2}$$

Then T_μ is well defined on $\operatorname{span}\{K_z : z \in \mathbb{C}\}$, which is a dense subset of F^2_α. Define the Berezin transform of the measure μ on $\mathbb{C}$ as

$$\tilde{\mu}(z) = \int_{\mathbb{C}} |k_z(w)|^2 e^{-\alpha|w|^2} d\mu(w) = \int_{\mathbb{C}} e^{-\alpha|z-w|^2} d\mu(w), \quad z \in \mathbb{C}, \tag{3.2.3}$$

where $k_z(w) = e^{\alpha w\bar{z} - \frac{\alpha}{2}|z|^2}$ is the normalized Bergman kernel of F^2_α. Moreover, define the averaging function $\hat{\mu}_r$ by the average of μ over balls $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$. That is $\hat{\mu}_r(z) = \mu(B(z,r))/|B(z,r)|$, where $|B(z,r)| = \pi r^2$ is the Lebesgue measure of the disk $B(z,r)$.

Theorem 3.2.1 (Isralowitz & Zhu [53], Theorem A). *Let μ be a positive measure, $r > 0$, and $\{a_n\}_{n\geq 1}$ be a lattice generated by points r and ri. Then the following conditions are equivalent.*

1. *T_μ is bounded on F^2_α,*
2. *$\tilde{\mu}$ is bounded on $\mathbb{C}$,*
3. *$\mu(B(z,r))$ is bounded on $\mathbb{C}$,*
4. *The averaging sequence $\{\mu(B(a_n,r))\}_{n\geq 1}$ is bounded.*

Theorem 3.2.2 (Isralowitz & Zhu [53], Theorem B). *Let μ be a positive measure, $r > 0$, and $\{a_n\}_{n\geq 1}$ be a lattice generated by points r and ri. Then the following conditions are equivalent.*

1. *T_μ is compact on F^2_α,*
2. *$\tilde{\mu}(z) \to 0$ as $|z| \to \infty$,*
3. *$\mu(B(z,r)) \to 0$ as $|z| \to \infty$,*
4. *$\mu(B(a_n,r)) \to 0$ as $n \to \infty$.*

Later, Hu and Lv [44] generalized the above results to $T_\mu : F^p_\alpha \to F^q_\alpha$ for different values of $1 < p,q < \infty$. In the following, we give a summary of their work. For a Borel measure μ on $\mathbb{C}^n$, and $t > 0$, define the t-Berezin transform of μ on $\mathbb{C}^n$ by

$$\tilde{\mu}_t(z) = \int_{\mathbb{C}} |k_z(w)|^t e^{-\frac{\alpha t}{2}|w|^2} d\mu(w) = \int_{\mathbb{C}} e^{-\frac{\alpha t}{2}|z-w|^2} d\mu(w), \quad z \in \mathbb{C}. \tag{3.2.4}$$

Notice that $\tilde{\mu}_2$ is just the Berezin transform (3.2.3).

Definition 3.2.3. Let $0 < p,q < \infty$, and take μ to be a positive measure. We call μ a (p,q)-Fock Carleson measure, if the embedding operator $i : F^p_\alpha \to L^q_\alpha(d\mu)$ is bounded, i.e., there is a constant C such that for all $f \in F^p_\alpha$,

$$\left(\int_{\mathbb{C}^n} \left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^q d\mu(z)\right)^{1/q} \leq C\|f\|_{p,\alpha}.$$

We call μ a vanishing (p,q)-Fock Carleson measure, if

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} \left|f_j(z)e^{-\frac{\alpha}{2}|z|^2}\right|^q d\mu(z) = 0,$$

whenever $\{f_j\}_{j\geq 1}$ is a bounded sequence in F^p_α that converges to zero uniformly on compact subsets of $\mathbb{C}^n$ as $j \to \infty$.

We say that μ satisfies the *integral property* if μ is a positive measure satisfying a property similar to (3.2.2) on $\mathbb{C}^n$. Moreover, for $r > 0$, a sequence $\{a_k\}_{k\geq 1}$ in $\mathbb{C}^n$ is called an r-lattice if $\{B(a_k, r)\}_{k\geq 1}$ covers $\mathbb{C}^n$ and $\{B(a_k, r/2)\}_{k\geq 1}$ are pairwise disjoint. The following theorems classify bounded and compact Toeplitz operators with the symbol μ for different values of p and q.

Theorem 3.2.4 (Hu & Lv [44], Theorem 3.1 & Theorem 4.2)**.** *Let $1 < p \leq q < \infty$ and μ satisfy the integral property. Then the following statements are equivalent.*

1. *$T_\mu : F^p_\alpha \to F^q_\alpha$ is bounded,*
2. *$\mu(B(\cdot, \delta)) \in L^\infty$ for some (any) $\delta > 0$,*
3. *$\tilde{\mu}_t$ is bounded on $\mathbb{C}^n$ for some (any) $t > 0$,*
4. *The sequence $\{\mu(B(a_k, r))\}_{k\geq 1}$ is bounded,*
5. *μ is a (p,q)-Fock Carleson measure.*

Theorem 3.2.5 (Hu & Lv [44], Theorem 3.2 & Theorem 4.3)**.** *Let $1 < p \leq q < \infty$ and μ satisfy the integral property. Then the following statements are equivalent.*

1. *$T_\mu : F^p_\alpha \to F^q_\alpha$ is compact,*
2. *$\mu(B(z, \delta)) \to 0$ as $|z| \to \infty$ for some (any) $\delta > 0$,*
3. *$\tilde{\mu}_t(z) \to 0$ as $|z| \to \infty$ for some (any) $t > 0$,*
4. *The sequence $\mu(B(a_k, r)) \to 0$ as $k \to \infty$ for some (any) $r > 0$,*
5. *μ is a vanishing (p,q)-Fock Carleson measure.*

Theorem 3.2.6 (Hu & Lv [44], Theorem 3.3 & Theorem 4.4). *Let $1 < q < p < \infty$ and μ satisfy the integral property. Then the following statements are equivalent.*

1. *$T_\mu : F^p_\alpha \to F^q_\alpha$ is bounded,*
2. *$T_\mu : F^p_\alpha \to F^q_\alpha$ is compact,*
3. *$\mu(B(\cdot, \delta)) \in L^{pq/(p-q)}$ for some (any) $\delta > 0$,*
4. *$\tilde{\mu}_t \in L^{pq/(p-q)}$ for some (any) $t > 0$,*
5. *$\sum_{k=1}^{\infty} \mu(B(a_k, r))^{\frac{pq}{p-q}} < \infty$ for some (any) $r > 0$,*
6. *μ is a $(pq, pq - p + q)$-Fock Carleson measure,*
7. *μ is a vanishing $(pq, pq - p + q)$-Fock Carleson measure.*

Schuster and Varolin [73] later studied the boundedness and compactness of T_μ on F^p_ϕ for $1 \leq p < \infty$ over $\mathbb{C}^n$, where $\phi \in \mathcal{C}^2(\mathbb{C}^n)$ and $dd^c\phi \simeq \omega_0 = dd^c|z|^2$, using the following generalization of Carleson measures.

Definition 3.2.7. Let μ be a positive measure on $\mathbb{C}^n$ and fix $1 \leq p < \infty$. We say that μ is Carleson for F^p_ϕ if there exists a positive constant C such that

$$\int_{\mathbb{C}^n} |f|^p e^{-p\phi} d\mu \leq C \int_{\mathbb{C}^n} |f|^p e^{-p\phi} dA, \quad \forall f \in F^p_\phi.$$

That is, the inclusion $i : F^p_\phi \to L^p(e^{-p\phi} d\mu)$ is bounded. Moreover, μ is vanishing Carleson if the inclusion $i : F^p_\phi \to L^p(e^{-p\phi} d\mu)$ is compact.

Theorem 3.2.8 (Schuster & Varolin [73], Theorem 1 & Theorem 2). *Let $\phi \in \mathcal{C}^2(\mathbb{C}^n)$, $dd^c\phi \simeq \omega_0$, and $1 \leq p < \infty$. Let μ be a positive measure on $\mathbb{C}^n$. The Toeplitz T_μ is given by*

$$T_\mu f(z) = \int_{\mathbb{C}^n} K(z, w) f(w) e^{-2\phi(w)} d\mu(w). \tag{3.2.5}$$

Then $T_\mu : F^p_\phi \to F^p_\phi$ is everywhere defined and bounded if and only if μ is a Carleson measure. Moreover, $T_\mu : F^p_\phi \to F^p_\phi$ is compact if and only if μ is a vanishing Carleson measure.

The above theorem was generalized to $T_\mu : F^p_\phi \to F^q_\phi$, for $\phi \in \mathcal{C}^2(\mathbb{C}^n)$, $dd^c\phi \simeq \omega_0$, and $0 < p, q < \infty$ by Hu and Lv [45]. Recalling Remark 2.3.3, we will not state their result here, as we will state the generalization later for the set of admissible weights as in Definition 2.3.1. For $n = 1$, this was also generalized to doubling Fock spaces by Hu and Lv [43] as follows. First, notice that given an auxiliary parameter $t > 0$, similarly to (3.2.4), the t-Berezin transform of μ is defined by

$$\tilde{\mu}_t = \int_{\mathbb{C}} |k_{t,z}(w)|^t e^{-t\phi(w)} d\mu(w), \quad z \in \mathbb{C}, \tag{3.2.6}$$

where $k_{t,z}$ is the normalized Bergman kernel of the doubling Fock space F^t_ϕ. For $r > 0$, the r-averaging transform of μ is defined by

$$\hat{\mu}_r(z) = \frac{\mu(D^r(z))}{|D^r(z)|} \simeq \frac{\mu(D^r(z))}{\rho(z)^2}, \quad z \in \mathbb{C}. \tag{3.2.7}$$

Moreover, Given $r > 0$, we call a sequence $\{a_k\}_{k\geq 1}$ in $\mathbb{C}$ an r-lattice if $\{D^r(a_k)\}_{k\geq 1}$ covers $\mathbb{C}$ and the disks $\{D^{r/5}(a_k)\}_{k\geq 1}$ are pairwise disjoint. Then, the boundedness and compactness of T_μ, (3.2.5), can be characterized as follows.

Theorem 3.2.9 (Hu & Lv [43], Theorem 3.2). *Let $0 < p \leq q < \infty$ and μ a positive Borel measure on $\mathbb{C}$. Then the following statements are equivalent.*

1. *$T_\mu : F^p_\phi \to F^q_\phi$ is bounded,*
2. *$\tilde{\mu}_t \rho^{2(p-q)/pq} \in L^\infty$ for some (or any) $t > 0$,*
3. *$\hat{\mu}_\delta \rho^{2(p-q)/pq} \in L^\infty$ for some (or any) $\delta > 0$,*
4. *The sequence $\{\hat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/pq}\}_{k\geq 1} \in l^\infty$ for some (or any) r-lattice $\{a_k\}_{k\geq 1}$.*

Theorem 3.2.10 (Hu & Lv [43], Theorem 3.3). *Let $0 < p \leq q < \infty$ and μ a positive Borel measure on $\mathbb{C}$. Then the following statements are equivalent.*

1. *$T_\mu : F^p_\phi \to F^q_\phi$ is compact,*
2. *$\tilde{\mu}_t(z)\rho(z)^{2(p-q)/pq} \to 0$ as $|z| \to \infty$ for some (or any) $t > 0$,*
3. *$\hat{\mu}_\delta(z)\rho(z)^{2(p-q)/pq} \to 0$ as $|z| \to \infty$ for some (or any) $\delta > 0$,*
4. *$\hat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/pq} \to 0$ as $k \to \infty$ for some (or any) r-lattice $\{a_k\}_{k\geq 1}$.*

Theorem 3.2.11 (Hu & Lv [43], Theorem 3.4). *Let $0 < q < p < \infty$ and μ a positive Borel measure on $\mathbb{C}$. Then the following statements are equivalent.*

1. *$T_\mu : F^p_\phi \to F^q_\phi$ is bounded,*
2. *$T_\mu : F^p_\phi \to F^q_\phi$ is compact,*
3. *$\tilde{\mu}_t \in L^{pq/(p-q)}$ for some (or any) $t > 0$,*
4. *$\hat{\mu}_\delta \in L^{pq/(p-q)}$ for some (or any) $\delta > 0$,*
5. *The sequence $\{\hat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/pq}\}_{k\geq 1} \in l^{pq/(p-q)}$ for some (or any) r-lattice $\{a_k\}_{k\geq 1}$.*

Finally, Arrousi, He, Li, and Tong [8] generalized the above theorems to study the boundedness and compactness of $T_\mu : F^p_\phi \to F^q_\phi$, when $\phi \in \mathcal{C}^2(\mathbb{C}^n)$ is an admissible weight as in Definition 2.3.1. Note that $\tilde{\mu}_t$ is defined as in (3.2.6), while similarly to (3.2.7), $\hat{\mu}_r(z) = \mu(D^r(z))/|D^r(z)| \simeq \mu(D^r(z))/\rho(z)^{2n}$.

Theorem 3.2.12 (Arrousi, He, Li, & Tong [8], Theorem 1.1). *Let $0 < p \leq q < \infty$, μ a finite positive Borel measure on $\mathbb{C}^n$, $\phi \in \mathcal{C}^2(\mathbb{C}^n)$ is an admissible weight, and α as in* (2.3.13). *Then the following statements are equivalent.*

1. *$T_\mu : F^p_\phi \to F^q_\phi$ is bounded,*
2. *$\tilde{\mu}_t \rho^{2n(p-q)/pq} \in L^\infty$ for some (or any) $t > 0$,*
3. *$\hat{\mu}_\delta \rho^{2n(p-q)/pq} \in L^\infty$ for some (or any) $o < \delta \leq \alpha$,*

4. *The sequence* $\{\hat{\mu}_r(a_k)\rho(a_k)^{2n(p-q)/pq}\}_{k\geq 1} \in l^\infty$ *for some (or any) r-lattice* $\{a_k\}_{k\geq 1}$ *with* $0 < r \leq \alpha$.

Theorem 3.2.13 (Arrousi, He, Li, & Tong [8], Theorem 1.2). *Let* $0 < p \leq q < \infty$, μ *a finite positive Borel measure on* $\mathbb{C}^n$, $\phi \in C^2(\mathbb{C}^n)$ *is an admissible weight, and* α *as in* (2.3.13). *Then the following statements are equivalent.*

1. $T_\mu : F^p_\phi \to F^q_\phi$ *is compact,*
2. $\tilde{\mu}_t(z)\rho(z)^{2n(p-q)/pq} \to 0$ *as* $|z| \to \infty$ *for some (or any)* $t > 0$,
3. $\hat{\mu}_\delta(z)\rho(z)^{2n(p-q)/pq} \to 0$ *as* $|z| \to \infty$ *for some (or any)* $0 < \delta \leq \alpha$,
4. $\hat{\mu}_r(a_k)\rho(a_k)^{2n(p-q)/pq} \to 0$ *as* $k \to \infty$ *for some (or any) r-lattice* $\{a_k\}_{k\geq 1}$ *with* $0 < r \leq \alpha$.

Theorem 3.2.14 (Arrousi, He, Li, & Tong [8], Theorem 1.4). *Let* $0 < q < p < \infty$, μ *a finite positive Borel measure on* $\mathbb{C}^n$, $\phi \in C^2(\mathbb{C}^n)$ *is an admissible weight, and* α *as in* (2.3.13). *Then the following statements are equivalent.*

1. $T_\mu : F^p_\phi \to F^q_\phi$ *is bounded,*
2. $T_\mu : F^p_\phi \to F^q_\phi$ *is compact,*
3. $\tilde{\mu}_t \in L^{pq/(p-q)}$ *for some (or any)* $t > 0$,
4. $\hat{\mu}_\delta \in L^{pq/(p-q)}$ *for some (or any)* $0 < \delta \leq \alpha$,
5. *The sequence* $\{\hat{\mu}_r(a_k)\rho(a_k)^{2n(p-q)/pq}\}_{k\geq 1} \in l^{pq/(p-q)}$ *for some (or any) r-lattice* $\{a_k\}_{k\geq 1}$ *with* $0 < r \leq \alpha$.

Known results about Schatten class membership of Toeplitz operators on Fock-type spaces

In the following, we state some results regarding the Schatten class membership of the Toeplitz operator T_μ acting on standard Fock spaces, doubling Fock spaces, and scalar weighted Fock spaces.

Theorem 3.2.15 (Isralowitz & Zhu [53], Theorem 4.4 & Theorem 5.4). *Let* $\mu \geq 0$, $0 < p < \infty$, $r > 0$, *and* $\{a_n\}_{n\geq 1}$ *be the lattice in* $\mathbb{C}$ *generated by r and ri. Then the following are equivalent.*

1. T_μ *is in the Schatten class* $S_p(F^2_\alpha)$,
2. $\tilde{\mu} \in L^p$,
3. $\mu(B(\cdot, r)) \in L^P$,
4. *The sequence* $\{\mu(B(a_n, r))\}_{n\geq 1} \in l^P$.

Theorem 3.2.16 (Oliver & Pascuas [68], Theorem 6.1). *Let* μ *be a locally finite positive Borel measure on* $\mathbb{C}$, $0 < p < \infty$, *and* ϕ *a doubling weight. Then the following are equivalent.*

1. T_μ *is in the Schatten class* $S_p(F^2_\phi)$,
2. *There is* $r_0 > 0$ *such that any r-lattice* $\{z_j\}_{j\geq 1}$ *with* $0 < r < r_0$ *satisfies* $\{\hat{\mu}_r(z_j)\}_{j\geq 1} \in l^p$,

3. *There an r-lattice $\{z_j\}_{j\geq 1}$ such that $\{\hat{\mu}_r(z_j)\}_{j\geq 1} \in l^p$,*

4. *There is $r > 0$ such that $\hat{\mu}_r \in L^p(\mathbb{C}, dA/\rho^2)$,*

5. *$\tilde{\mu}_2 \in L^p(\mathbb{C}, dA/\rho^2)$.*

Theorem 3.2.17 (Arrousi, He, Li, & Tong [8], Theorem 1.9). *Let μ be a locally finite positive Borel measure on $\mathbb{C}^n$, $\phi \in \mathcal{C}^2(\mathbb{C}^n)$ is an admissible weight, and $0 < p < \infty$. Then $T_\mu \in S_p(F^2_\phi)$ if and only if $\tilde{\mu}_2 \in L^p(\mathbb{C}^n, dA/\rho^{2n})$.*

Boundedness, compactness, and Schatten class membership of vectorial Toeplitz operators on Large vector-valued Fock spaces

For the rest of this section, we give a summary of our main results [6] (equivalently, Appendix B) regarding the boundedness, compactness and Schatten class memberships of the vectorial Toeplitz operator T_G. Recalling Definition 2.4.10 and Definition 2.3.1, for $G \in T_\phi(\mathcal{L}(\mathcal{H}))$, one can define the vectorial Toeplitz operator $T_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to F^2_\phi(\mathbb{C}^n, \mathcal{H})$ as

$$T_G f(z) = \int_{\mathbb{C}^n} G(w) f(w) K(z,w) e^{-2\phi(w)} dA(w),$$

where $K(z,w)$ is the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$. To characterize the boundedness and compactness of T_G, we define the Berezin transform $\tilde{G}$ by

$$\tilde{G}(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w), \quad z \in \mathbb{C}^n,$$

where for each $z \in \mathbb{C}^n$, $k_z = K_z / \|K_z\|_{F^2_\phi(\mathbb{C}^n)}$ is the normalized Bergman kernel of $F^2_\phi(\mathbb{C}^n)$. For $r > 0$, the corresponding averaging function $\widehat{G}_r$ is defined by

$$\widehat{G}_r(z) = \frac{\int_{D^r(z)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w)}{|D^r(z)|} \simeq \frac{\int_{D^r(z)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w)}{\rho(z)^{2n}}.$$

Definition 3.2.18. We say that G satisfies the Carleson condition if the inclusion map $I_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to L^2_\phi(\mathbb{C}^n, \mathcal{H}, \|G\|_{\mathcal{L}(\mathcal{H})} dA)$ is bounded, that is, there is a constant C such that

$$\left(\int_{\mathbb{C}^n} \|f(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) \right)^{1/2} \leq C \|f\|_{2,\phi}, \quad \text{for } f \in F^2_\phi(\mathbb{C}^n, \mathcal{H}). \tag{3.2.8}$$

We say that G satisfies the vanishing Carleson condition if the embedding operator $I_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to L^2_\phi(\mathbb{C}^n, \mathcal{H}, \|G\|_{\mathcal{L}(\mathcal{H})} dA)$ is compact, that is, for any bounded sequence $\{f_j\}_{j=1}^\infty$ in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ that converges to zero uniformly on any compact subset of $\mathbb{C}^n$ as $j \to \infty$,

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} \|f_j(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) = 0. \tag{3.2.9}$$

Remark 3.2.19. For an r-lattice $\{z_k\}_{k\geq 1}$ and a real number $m \geq 1$, there exists some integer N, only depending on m and r, such that each $z \in \mathbb{C}^n$ can be in at most N balls of the form $D^{mr}(z_k)$. That is,

$$\sum_{k=1}^{\infty} \chi_{D^{mr}(z_k)}(z) \leq N, \quad \text{for } z \in \mathbb{C}^n, \tag{3.2.10}$$

where χ_E is a characteristic function of a subset E of $\mathbb{C}^n$.

Lemma 3.2.20 (Lemma B.2.6, [6], Lemma 2.6)**.** *Let ϕ be an admissible weight, $e \in \mathcal{H}$ be a unit element, and k_z the normalized reproducing kernel of $F^2_\phi(\mathbb{C}^n)$. The set $\{k_z(\cdot)e : z \in \mathbb{C}^n\}$ is bounded in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ and $k_z(\cdot)e \to 0$ uniformly on any compact subsets of $\mathbb{C}^n$ as $|z| \to \infty$.*

Theorem 3.2.21 (Theorem B.1.2, [6], Theorem 1.2)**.** *Let $G \in T_\phi(\mathcal{L}(\mathcal{H}))$ and α be as in* (2.3.13)*. Then the following conditions are equivalent:*

1. *$T_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is bounded;*
2. *$\widetilde{G} \in L^\infty(\mathbb{C}^n, dA)$;*
3. *$\widehat{G}_\delta \in L^\infty(\mathbb{C}^n, dA)$ for some (or any) $0 < \delta \leq \alpha$;*
4. *$\{\widehat{G}_\delta(z_k)\}_k$ is a bounded sequence for some (or any) δ-lattice $\{z_k\}_k$ with $0 < \delta \leq \alpha$;*
5. *G satisfies a Carleson condition.*

Moreover,

$$\|T_G\| \simeq \|\widetilde{G}\|_{L^\infty(\mathbb{C}^n,dA)} \simeq \|\widehat{G}_\delta\|_{L^\infty(\mathbb{C}^n,dA)} \simeq \|\{\widehat{G}_\delta(z_k)\}_k\|_{l^\infty}. \tag{3.2.11}$$

Sketch of proof. Using (2.3.15), (3.2.10), and Lemma 2.3.6, one can show that (2), (3), and (4) are equivalent. To show that (1) implies (2), use (2.3.14), positivity of $G(w)$ for every $w \in \mathbb{C}^n$, Lemma 3.2.20, and Lemma 2.4.3, to see

$$\begin{aligned}\tilde{G}(z) &\simeq \rho(z)^n e^{-\phi(z)} \int_{\mathbb{C}^n} \sup_{\|e\|=1} \langle G(w)k_z(w)e, e\rangle_{\mathcal{H}} K(z,w) e^{-2\phi(w)} dA(w) \\ &\simeq \rho(z)^n e^{-\phi(z)} \sup_{\|e\|=1} \langle T_G k_z(z)e, e\rangle_{\mathcal{H}} \\ &\lesssim \|T_G k_z(\cdot)e\|_{2,\varphi} \lesssim \|T_G\|.\end{aligned} \tag{3.2.12}$$

To show that (3) implies (1), we first prove that there is a constant $C > 0$ such that

$$\|T_G f\|^2_{2,\phi} \leq C \int_{\mathbb{C}^n} \|f(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} \hat{G}_\delta(w)^2 dA(w), \quad \forall f \in F^2_\phi(\mathbb{C}^n, \mathcal{H}), \forall \delta > 0. \tag{3.2.13}$$

Then Lemma 2.4.3 implies that $\|T_G f\|^2_{2,\phi} \lesssim \|\hat{G}_\delta\|^2_{L^\infty(\mathbb{C}^n,dA)} \|f\|^2_{2,\phi}$, and thus T_G is bounded. To show that (4) implies (5), note that using Lemma 2.4.3, (2.3.10), (2.3.12), and (3.2.10), there is $m > 1$ such that

$$\begin{aligned}&\int_{\mathbb{C}^n} \|f(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{\rho(z_k)^{2n}} \int_{D^\delta(z_k)} \left(\int_{D^{m\delta}(z_k)} \|f(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} dA(w) \right) \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) \\ &\lesssim \|\{\hat{G}_\delta(z_k)\}_k\|_{l^\infty} \|f\|^2_{2,\phi}.\end{aligned} \tag{3.2.14}$$

Finally, taking $f = k_z e$ in (3.2.8), $\tilde{G}(z) \lesssim \|k_z\|^2_{F^2_\phi(\mathbb{C}^n)} = 1$, and therefore (5) implies (2). ☐

Theorem 3.2.22 (Theorem B.1.3, [6], Theorem 1.3)**.** *Let $G \in T_\phi(\mathcal{L}(\mathcal{H}))$ and α be as in* (2.3.13)*. Then the following conditions are equivalent:*

1. *$T_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is compact;*

2. $\widetilde{G}(z) \to 0$ *as* $|z| \to \infty$;

3. $\widehat{G}_\delta(z) \to 0$ *as* $|z| \to \infty$ *for some (or any)* $0 < \delta \leq \alpha$;

4. $\widehat{G}_\delta(z_k) \to 0$ *as* $k \to \infty$ *for some (or any)* δ*-lattice* $\{z_k\}_k$ *with* $0 < \delta \leq \alpha$;

5. *G satisfies a vanishing Carleson condition.*

Sketch of proof. It is easy to show that (2) implies (3) and (3) implies (4). Let us show that (4) implies (2). Assuming (4), for every $\epsilon > 0$, there is $K \in \mathbb{N}$ such that whenever $k > K$, $\hat{G}_\delta(z_k) < \epsilon$. Take m as in (2.3.12). Then

$$\tilde{G}(z) \lesssim \int_{\overline{\cup_{k=1}^K D^{m\delta}(z_k)}} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) \\ + \sum_{k=K+1}^{\infty} \rho(z_k)^{2n} \hat{G}_\delta(z_k) \left(\sup_{w \in D^\delta(z_k)} |k_z(w)|^2 e^{-2\phi(w)} \right).$$

Using Lemma 2.3.6 and (3.2.10) we obtain

$$\begin{aligned} &\sum_{k=K+1}^{\infty} \rho(z_k)^{2n} \hat{G}_\delta(z_k) \left(\sup_{w \in D^\delta(z_k)} |k_z(w)|^2 e^{-2\phi(w)} \right) \\ &\lesssim \sup_{k \geq K+1} \hat{G}_\delta(z_k) \left(\sum_{k=K+1}^{\infty} \int_{D^{m\delta}(z_k)} |k_z(w)|^2 e^{-2\phi(w)} dA(w) \right) \\ &\lesssim \epsilon N \|k_z\|^2_{F^2_\phi(\mathbb{C}^n)} \lesssim \epsilon. \end{aligned} \tag{3.2.15}$$

Moreover, part (e) of Lemma 2.3.5 implies that as $|z| \to \infty$,

$$\int_{\overline{\cup_{k=1}^K D^{m\delta}(z_k)}} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) < \epsilon,$$

which together with (3.2.15) implies (2). To show that (1) implies (2), use Lemma 3.2.20, compactness of T_G, and (3.2.12), to conclude that as $|z| \to \infty$, $\tilde{G}(z) \lesssim \|T_G k_z e\|_{2,\phi} \to 0$. To show that (3) implies (1), let $\epsilon > 0$ be arbitrary and $\{f_j\}_{j=1}^\infty$ be a sequence in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ that converges to zero uniformly on any compact subset of $\mathbb{C}^n$. By assumption, there is some $R > 0$ such that $\hat{G}_\delta(z) < \sqrt{\epsilon}$, whenever $|z| > R$. This together with (3.2.13), for big enough j we have

$$\begin{aligned} \|T_G f_j\|^2_{2,\phi} &\lesssim \int_{|z| \leq R} \hat{G}_\delta(w)^2 \|f_j(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} dA(w) \\ &\quad + \int_{|z| > R} \hat{G}_\delta(w)^2 \|f_j(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} dA(w) \\ &\lesssim \epsilon + \epsilon \|f_j\|^2_{2,\phi} \lesssim \epsilon, \end{aligned} \tag{3.2.16}$$

and thus (1) holds. To show that (4) implies (5), let $\{f_k\}_k$ be a bounded sequence in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ that converges to zero uniformly on compact subsets of $\mathbb{C}^n$. By our assumption, letting $\epsilon > 0$, there exists $r_0 > 0$ such that $\sup_{|z_k| > r_o} \hat{G}_\delta(z_k) < \epsilon$. Observe that there is $r_0 \leq r_1$ such that if a point z_k of the sequence $\{z_k\}_k$ belongs to $\{|z| \leq r_0\}$, then $D^\delta(z_k) \subset \{|z| \leq r_1\}$. So take k big enough such that $\sup_{\{|z|\leq r_1\}} \|f_k(z)\|_{\mathcal{H}} < \epsilon$. This together with (3.2.10), we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n} \|f_k(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) \\
&\lesssim \int_{\{|\zeta|\leq r_1\}} \|f_k(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) \\
&+ \sup_{|z_k|>r_o} \hat{G}_\delta(z_k) \sum_{|z_k|>r_0} \int_{D^{m\delta}(z_k)} \|f_k(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} dA(w) \\
&\lesssim \epsilon + \epsilon \|f_k\|^2_{2,\phi} \lesssim \epsilon.
\end{aligned}$$

This implies (5). Finally, assuming (5), $I_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to L^2_\phi(\mathbb{C}^n, \mathcal{H}, \|G\|_{\mathcal{L}(\mathcal{H})} dA)$ is compact. Using (3.2.9) and Lemma 3.2.20, we have

$$\int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) \to 0, \quad \text{as } |z| \to \infty,$$

and therefore (2) holds. This proves the desired result and completes the proof. □

To characterize the Schatten class membership of the vectorial Toeplitz operator T_G, we define the operator-valued Berezin transform of G by

$$\tilde{G}^{op}(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)}\, G(w)\, dA(w), \quad z \in \mathbb{C}^n,$$

and the corresponding averaging operator by

$$\hat{G}^{op}_r(z) = \frac{\int_{D^r(z)} G(w)\, dA(w)}{|D^r(z)|} \simeq \frac{\int_{D^r(z)} G(w) dA(w)}{\rho(z)^{2n}}, \quad z \in \mathbb{C}^n.$$

Theorem 3.2.23 (Theorem B.1.4, [6], Theorem 1.4). *Let $1 \leq p < \infty$, and $0 < \delta < \alpha$, where α is as in (2.3.13). Then for any orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, the following statements are equivalent:*

1. *The operator T_G belongs to $S_p(F^2_\phi(\mathbb{C}^n, \mathcal{H}))$;*
2. $$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z)\, e_m, e_m \rangle_{\mathcal{H}} \Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$
3. $$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \hat{G}^{op}_\delta(z)\, e_m, e_m \rangle_{\mathcal{H}} \Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$
4. *Let $\{z_j\}_{j\geq 1}$ be a δ-lattice. Then*
   $$\sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_\delta(z_j)\, e_m, e_m \rangle_{\mathcal{H}} \Big)^p < \infty.$$

The characterization of Schatten class membership of Toeplitz operators T_G, for $0 < p < 1$, is more complicated. As one can see in Theorem 3.2.26, to get a full charaterization, we need to add an extra condition about the symbol $G(z)$, that is, $G(z)$ is a compact operator on $\mathcal{H}$, for every $z \in \mathbb{C}^n$. However, sufficient conditions for $T_G \in S_p(F^2_\phi(\mathbb{C}^n, \mathcal{H}))$ is exactly as those when $1 \leq p < \infty$, as explained in Proposition 3.2.25 below.

Proposition 3.2.24 (Proposition B.1.5, [6], Proposition 1.5). *Let $0 < p < 1$, and $0 < \delta < \alpha$, where α is as in (2.3.13). Then for any orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, the following statements are equivalent:*

1.
$$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z)\, e_m, e_m \rangle_{\mathcal{H}} \Big)^p \, \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$

2.
$$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \hat{G}^{op}_{\delta}(z)\, e_m, e_m \rangle_{\mathcal{H}} \Big)^p \, \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$

3. Let $\{z_j\}_{j\geq 1}$ be a δ-lattice. Then
$$\sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_{\delta}(z_j)\, e_m, e_m \rangle_{\mathcal{H}} \Big)^p < \infty.$$

Proposition 3.2.25 (Proposition B.1.6, [6], Proposition 1.6). *Let $0 < p < 1$, and $0 < \delta < \alpha$, where α is as in (2.3.13). If there is an orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, such that*
$$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z)\, e_m, e_m \rangle_{\mathcal{H}} \Big)^p \, \frac{dA(z)}{\rho(z)^{2n}} < \infty,$$
then the operator T_G belongs to $S_p(F^2_{\phi}(\mathbb{C}^n, \mathcal{H}))$.

The following theorem gives the necessary condition for the Schatten class membership of T_G by assuming that $G(z)$ is a compact operator on $\mathcal{H}$.

Theorem 3.2.26 (Theorem B.1.7, [6], Theorem 1.7). *Let $0 < p < 1$, $\delta < \min(1/2, \alpha)$, where α is as in (2.3.13), and $\{z_j\}_{j\geq 1}$ be a δ-lattice. Assume that $G(z)$ is compact for every $z \in \mathbb{C}^n$, and $T_G \in S_p(F^2_{\phi}(\mathbb{C}^n, \mathcal{H}))$. Then there is a family of orthonormal bases $\{e^j_m\}_{m\geq 1}$ of $\mathcal{H}$, possibly depending on $z_j \in \mathbb{C}^n$, such that*
$$\sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_{\delta}(z_j)\, e^j_m, e^j_m \rangle_{\mathcal{H}} \Big)^p < \infty,$$
where $\{e^j_m\}_{m\geq 1}$ is the basis of $\mathcal{H}$, obtained by eigenvectors of $\hat{G}^{op}_{\delta}(z_j)$, for each $j \geq 1$.

Here we will not give proofs of Theorem 3.2.23, and Proposition 3.2.24. We encourage the interested readers to check [6] (equivalently, Appendix B). The proof of Theorem 3.2.26 is more complicated, and hence it is nice to see a sketch of the proof. To do this, we give the following lemmas. The idea of the proof originally comes from the work of Luecking [61] in studying the Schatten class Toeplitz operators on Hardy and Bergman spaces. Later, an analogous idea was applied to studying the Schatten class Toeplitz operators with measure symbols on doubling Fock spaces in [68]. This approach has also been used to study the Schatten class Hankel operators acting on generalized Fock spaces in [10, 51].

Lemma 3.2.27 (Lemma B.2.16, [6], Lemma 2.15). *Assume that the vectorial Toeplitz operator $T_G : F^2_{\phi}(\mathbb{C}^n, \mathcal{H}) \to F^2_{\phi}(\mathbb{C}^n, \mathcal{H})$ is compact. Moreover, take $G(w) : \mathcal{H} \to \mathcal{H}$ to be compact for every $w \in \mathbb{C}^n$ and let $\delta > 0$. Then for every fixed $z \in \mathbb{C}^n$, the average operator*
$$\widehat{G}^{op}_{\delta}(z) \simeq \int_{D^{\delta}(z)} G(w) \frac{dA(w)}{\rho(w)^{2n}} : \mathcal{H} \to \mathcal{H}$$

is compact.

Lemma 3.2.28 (Lemma B.4.1, [6], Lemma 4.1). *For $R > 0$ and any finite sequence $\{z_j\}_{j=1}^m$ of different points in $\mathbb{C}^n$, let*

$$M_R(\{z_j\}_{j=1}^m) := \max_{1\le j\le m} \#\{k \in \{1,\cdots,m\} : |z_j - z_k| < R\min(\rho(z_j),\rho(z_k))\}.$$

Then $\{z_j\}_{j=1}^m$ can be partitioned into at most $M_R(\{z_j\}_{j=1}^m)$ subsequences such that any two different points z_j and z_k in the same subsequence satisfy either $z_j \notin D^R(z_k)$, or $z_k \notin D^R(z_j)$. That is, $|z_j - z_k| \ge R\min(\rho(z_j),\rho(z_k))$.

Lemma 3.2.29 (Lemma B.4.2, [6], Lemma 4.2). *Let $\delta > 0$, $R > 1$, and $\{z_j\}_{j\ge1}$ be a δ-lattice. Then $M_R(\{z_j\}_{j=1}^m) \le 6^{2n}R^{4n}\delta^{-2n}N_\delta$, for every finite sublattice $\{z_j\}_{j=1}^m$, where as in (3.2.10), $N_\delta = \sup_{z\in\mathbb{C}^n}\sum_{j=1}^\infty \chi_{D^\delta(z_j)}(z)$.*

Sketch of proof of Theorem 3.2.26. Assume that $0 < \delta < 1/2$, $R > 1$, and fix $M \in \mathbb{N}$. Let $\{z_j\}_{j=1}^M$ be the finite sublattice obtained by considering the first M elements of the δ-lattice $\{z_j\}_{j\ge1}$. Then Lemma 3.2.28 implies that $\{z_j\}_{j=1}^M$ can be partitioned into $M_R(\{z_j\}_{j=1}^M)$ subsequences such that any two different points a_j and a_k in the same subsequence satisfy $|a_j - a_k| \ge R\min(\rho(a_j),\rho(a_k))$. Let $\{a_j\}_{j=1}^s$ be one such subsequence. Let $\{B_{k,m}(z) = f_k(z)e_m\}_{k,m\ge1}$ be an orthonormal basis of $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ with $\{f_k\}_{k\ge1}$ being an orthonormal basis of $F^2_\phi(\mathbb{C}^n)$, and $\{e_m\}_{m\ge1}$ any orthonormal basis of $\mathcal{H}$. We can construct a bounded linear operator A on $F^2_\phi(\mathbb{C}^n,\mathcal{H})$ by $Af(z) = \sum_{k,m=1}^s\langle f, B_{k,m}\rangle k_{a_k}(z)e_m$. Let $U(z) = (\sum_{j=1}^s \chi_{D^\delta(a_j)}(z))G(z)$. Then $U \le NG$, where N is as in (3.2.10). That is, $NG(z) - U(z)$ is a positive operator on $\mathcal{H}$ for every $z \in \mathbb{C}^n$. Since $T_G \in S_p$, we can conclude that $T_U \in S_p$, and $\|T_U\|_{S_p} \le N\|T_G\|_{S_p}$. Set $T = A^*T_UA$ such that $\|T\|_{S_p} \lesssim \|T_U\|_{S_p}$. It is easy to see that when $k,m > s$, $\langle TB_{k,m}, B_{k,m}\rangle = \langle T_UAB_{k,m}, AB_{k,m}\rangle = 0$. We can split T as $T = D_s + M_s$, where D_s is the diagonal operator defined by

$$D_sf = \sum_{k,m=1}^s \langle TB_{k,m}, B_{k,m}\rangle\langle f, B_{k,m}\rangle B_{k,m}, \quad \text{where } f \in F^2_\phi(\mathbb{C}^n,\mathcal{H}), \tag{3.2.17}$$

and M_s is the off-diagonal operator defined by

$$\begin{aligned} M_sf = &\sum_{k,m=1}^s \sum_{\substack{r,n=1\\ r\ne k, m\ne n}}^s \langle TB_{k,m}, B_{r,n}\rangle\langle f, B_{k,m}\rangle B_{r,n} + \sum_{k,m=1}^s\sum_{\substack{r=1\\ r\ne k,}}^s\langle TB_{k,m}, B_{r,m}\rangle\langle f, B_{k,m}\rangle B_{r,m} \\ &+ \sum_{k,m=1}^s\sum_{\substack{n=1\\ m\ne n,}}^s\langle TB_{k,m}, B_{k,n}\rangle\langle f, B_{k,m}\rangle B_{k,n}, \quad \text{where } f \in F^2_\phi(\mathbb{C}^n,\mathcal{H}). \end{aligned} \tag{3.2.18}$$

Recall that $U(z) = 0$ if $z \notin \cup_{j=1}^s D^\delta(a_j)$. Then using (2.3.15), and positivity of $G(z)$ and $\hat{G}^{op}_\delta(z)$, there is a constant $C_1 > 0$, only depending on δ such that

$$\|D_s\|^p_{S_p} \ge C_1\sum_{m=1}^s\sum_{k=1}^s\Big(\langle \hat{G}^{op}_\delta(a_k)e_m, e_m\rangle_{\mathcal{H}}\Big)^p. \tag{3.2.19}$$

On the other hand, by Proposition 1.29 in [80], the fact that $(x+y)^p \leq x^p + y^p$ for $p \leq 1$, definition of $U(z)$, and positivity of $G(z)$, we obtain

$$\begin{aligned} \|M_s\|_{S_p}^p \leq N^p \sum_{\substack{r,k=1\\ r\neq k}}^{s} \sum_{\substack{m,n=1\\ m\neq n}}^{s} & \left| \sum_{j=1}^{s} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)||k_{a_r}(z)| e^{-2\phi(z)} dA(z) \right|^p \\ + N^p \sum_{\substack{k,r=1\\ r\neq k}}^{s} \sum_{m=1}^{s} & \left(\sum_{j=1}^{s} \int_{D^\delta(a_j)} \langle G(z)e_m, e_m\rangle_{\mathcal{H}} |k_{a_k}(z)||k_{a_r}(z)| e^{-2\phi(z)} dA(z) \right)^p \\ + N^p \sum_{k=1}^{s} \sum_{\substack{n=1\\ m\neq n}}^{s} & \left| \sum_{j=1}^{s} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)|^2 e^{-2\phi(z)} dA(z) \right|^p . \end{aligned} \tag{3.2.20}$$

Define

$$J_{k,r}^{m,n}(G,s) = \sum_{j=1}^{s} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)||k_{a_r}(z)| e^{-2\phi(z)} dA(z).$$

Since $k \neq r$, then $|a_k - a_r| \geq R \min(\rho(a_k), \rho(a_r))$. Thus for $z \in D^\delta(a_j)$ it is easy to see that either

$$|z - a_k| \geq \tilde{R} \min(\rho(z), \rho(a_k)), \tag{3.2.21}$$

or

$$|z - a_r| \geq \tilde{R} \min(\rho(z), \rho(a_r)), \tag{3.2.22}$$

where $\tilde{R} = \frac{R-1}{3}$. Using (2.3.6) and (2.3.14), we can write

$$|k_{a_k}(z)||k_{a_r}(z)| e^{-2\phi(z)} \lesssim \frac{e^{\frac{-\epsilon}{2}[d_\phi(z,a_k) + d_\phi(z,a_r)]}}{\rho(z)^n} |k_{a_k}(z)|^{1/2} |k_{a_r}(z)|^{1/2} e^{-\phi(z)}.$$

Furthermore, since $z \in D^\delta(a_j)$, and $r \neq k$, we can assume that $j \neq k$, and by the above argument we have $d_\phi(z,a_k) + d_\phi(z,a_r) \geq d_\phi(a_r,a_k) \geq \tilde{R}$. Hence, for $z \in D^\delta(a_j)$, we can conclude that $e^{\frac{-\epsilon}{2}[d_\phi(z,a_k) + d_\phi(z,a_r)]} \leq e^{\frac{-\epsilon}{2}\tilde{R}}$. Hence, using $\rho(z) \simeq \rho(a_j)$, we obtain

$$J_{k,r}^{m,n}(G,s) \lesssim e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \frac{1}{\rho(a_j)^n} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)|^{1/2} |k_{a_r}(z)|^{1/2} e^{-\phi(z)} dA(z).$$

By Lemma 2.3.6 we have

$$|k_{a_k}(z)|^{1/2} e^{-\phi(z)/2} \lesssim \left(\frac{1}{\rho(z)^{2n}} \int_{D^\delta(z)} |k_{a_k}(\xi)|^{p/2} e^{-\frac{p}{2}\phi(\xi)} dA(\xi) \right)^{1/p}.$$

Since $z \in D^\delta(a_j)$, there exists some $m_1 > 1$ such that $D^\delta(z) \subset D^{m_1\delta}(a_j)$. Therefore,

$$|k_{a_k}(z)|^{1/2} e^{-\phi(z)/2} \lesssim \frac{1}{\rho(z)^{2n/p}} \left(\int_{D^{m_1\delta}(a_j)} |k_{a_k}(\xi)|^{p/2} e^{-\frac{p}{2}\phi(\xi)} dA(\xi) \right)^{1/p} = \frac{1}{\rho(z)^{2n/p}} S_k(a_j)^{1/p},$$

where

$$S_k(z) = \int_{D^{m_1\delta}(z)} |k_{a_k}(\xi)|^{p/2} e^{-\frac{p}{2}\phi(\xi)} dA(\xi).$$

Similarly,

$$|k_{a_r}(z)|^{1/2} e^{-\phi(z)/2} \lesssim \frac{1}{\rho(z)^{2n/p}} S_r(a_j)^{1/p}.$$

Therefore,

$$J^{m,n}_{k,r}(G,s) \lesssim e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \frac{1}{\rho(a_j)^n} \int_{D^{\delta}(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} \frac{1}{\rho(z)^{4n/p}} S_k(a_j)^{1/p} S_r(a_j)^{1/p} dA(z).$$

Since $0 < p < 1$, $4/p - 1 > 1$, and

$$J^{m,n}_{k,r}(G,s) \lesssim e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \rho(a_j)^{n(1-\frac{4}{p})} S_k(a_j)^{1/p} S_r(a_j)^{1/p} \langle \hat{G}^{op}_{\delta}(a_j)e_m, e_n\rangle_{\mathcal{H}}.$$

Then since $0 < p < 1$,

$$\left|J^{m,n}_{k,r}(G,s)\right|^p \lesssim e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e_m, e_n\rangle_{\mathcal{H}}|^p \rho(a_j)^{n(p-4)} S_k(a_j) S_r(a_j). \tag{3.2.23}$$

Using (3.2.10), (2.3.14), (2.3.16), and Lemma 2.3.6, we can show that

$$\sum_{k=1}^{s} S_k(a_j) \lesssim N \int_{D^{m_1\delta}(a_j)} \rho(\xi)^{\frac{-np}{2}} dA(\xi) \simeq \rho(a_j)^{2n-\frac{np}{2}},$$

where the constant only depends on δ. This together with (3.2.23) implies that

$$\sum_{\substack{k,r=1\\ r\neq k}}^{s} \left|J^{m,n}_{k,r}(G,s)\right|^p \lesssim e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e_m, e_n\rangle_{\mathcal{H}}|^p, \tag{3.2.24}$$

where the constant only depends on $0 < \delta < 1/2$.

Since $T_G \in S_p(F^2_{\phi}(\mathbb{C}^n, \mathcal{H}))$, it is in particular compact. Lemma 3.2.27 along with compactness of $G(w)$ for every $w \in \mathbb{C}^n$ then implies that $\hat{G}^{op}_{\delta}(a_j)$ is a compact operator on $\mathcal{H}$ for every $a_j \in \mathbb{C}^n$. Since $\hat{G}^{op}_{\delta}(a_j)$ is positive, it is, in particular, self-adjoint. Then the spectral Theorem for self-adjoint and compact operators implies that for each $j \geq 1$, there exists an orthonormal basis $\{e^j_m\}_{m\geq 1}$ of $\mathcal{H}$ consisting of eigenvectors of $\hat{G}^{op}_{\delta}(a_j)$. That is,

$$\mathcal{H} = \overline{\operatorname{span}\{e^j_m\}}, \quad \text{where } m \geq 1.$$

Hence, $\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}} = 0$, for $m \neq n$. Comparing (3.2.20) with (3.2.23), and by the positivity of $\hat{G}^{op}_{\delta}(a_j)$, we can see that

$$\begin{aligned}
\|M_s\|^p_{S_p} &\lesssim \sum_{k,m=1}^{s} \sum_{\substack{r,n=1\\ r\neq k, m\neq n}}^{s} \left|J^{m,n}_{k,r}(G,s)\right|^p + \sum_{k,m=1}^{s} \sum_{\substack{r=1\\ r\neq k}}^{s} \left(J^{m,m}_{k,r}(G,s)\right)^p + \sum_{k,m=1}^{s} \sum_{\substack{n=1\\ m\neq n}}^{s} \left|J^{m,n}_{k,k}(G,s)\right|^p \\
&\lesssim e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \sum_{\substack{m,n=1\\ m\neq n}}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}}|^p + e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{m=1}^{s} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_m\rangle_{\mathcal{H}}|^p \\
&\quad + e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \sum_{\substack{m,n=1\\ m\neq n}}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}}|^p \\
&= e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j,m=1}^{s} \left(\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}}\right)^p,
\end{aligned} \tag{3.2.25}$$

where the first and the third terms vanish because of the compactness argument above. Note that inequality (3.2.24) was established for an *arbitrary* orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, and the constant involved does not depend on the chosen basis. Hence, for each fixed index j, we may apply (3.2.24) with a possibly different orthonormal basis $\{e_m^j\}_{m\geq 1}$, for instance, the eigenbasis of $\hat{G}_\delta^{op}(a_j)$. Doing so yields a valid inequality for each j, and summing these inequalities over j gives (3.2.25). In other words, the passage from (3.2.24) to (3.2.25) does not require a single global orthonormal basis.

Therefore, by (3.2.19) and (3.2.25), we can conclude that

$$
\begin{aligned}
\|T_G\|_{S_p}^p &\gtrsim \|T\|_{S_p}^p \geq \|D_s\|_{S_p}^p - \|M_s\|_{S_p}^p \\
&\geq (C_1 - Q(N)e^{\frac{-p\epsilon}{2}\tilde{R}}) \sum_{m,j=1}^{s} \left(\langle \hat{G}_\delta^{op}(z_j) e_m^j, e_m^j \rangle_{\mathcal{H}} \right)^p,
\end{aligned}
\tag{3.2.26}
$$

where $Q(N)$ is some power of N, not depending on s. Since $e^{\frac{-p\epsilon}{2}\tilde{R}} \to 0$ as $R \to \infty$, there is always a constant $R = R(p,\delta,N)$ such that $C(p,\delta,N) = C_1 - Q(N)e^{\frac{-p\epsilon}{2}\tilde{R}} > 0$.

Fix M to be a positive integer. Then Lemma 3.2.29 implies that $\{z_j\}_{j=1}^M$ can be partitioned into at most $6^{2n}R^{4n}\delta^{-2n}N_\delta$ subsequences such that any different points z_j and z_k in the same subsequence satisfy $|z_j - z_k| \geq R\min(\rho(z_j), \rho(z_k))$. Then by (3.2.26), we obtain

$$
\sum_{m,j=1}^{M} \left(\langle \hat{G}_\delta^{op}(z_j) e_m^j, e_m^j \rangle_{\mathcal{H}} \right)^p \leq C(p,\delta,N_\delta)^{-1} 6^{2n} R^{4n} \delta^{-2n} N_\delta \|T_G\|_{S_p}^p < \infty.
$$

Since the RHS does not depend on M, we are done with the proof of Theorem 3.2.23. ☐

3.3 The Berezin transform on Fock-type spaces and fixed point theorems

We begin this section by defining the Berezin transform for standard Fock spaces. Investigating the relationship between the Berezin transform and the heat equation shows that the bounded fixed points of the Berezin transform for standard Fock spaces are harmonic. We then study the Berezin transform on the Fock-type space F_m^2, $m > 0$[1], and give a detailed summary of our paper [9] (equivalently, Appendix C) on fixed points of the Berezin transform on Fock-type spaces. The main result is that when $m = 2$, i.e., the classical Fock space, every polynomial fixed point is harmonic.

For general $m > 0$, the situation is more complicated. While in the classical case $m = 2$ all polynomial fixed points are harmonic, for general m non-harmonic polynomial fixed points may occur. However, these exceptional cases are highly constrained, and in a precise sense (made explicit later) harmonicity remains the generic behavior.

---

[1] In this section, we use a different normalization of the weight and consider the space of functions in $L^2(\mathbb{C}, e^{-|z|^m} dA)$. This corresponds to the doubling Fock space $F^2_{\tilde{\phi}}$ with $\tilde{\phi}(z) = \frac{1}{2}|z|^m$, and hence differs from the canonical doubling Fock space F_m^2 defined in Chapter 2, which is based on the doubling weight $\phi(z) = |z|^m$. This distinction only affects normalization constants and does not change the qualitative behavior of the Berezin transform.

Berezin transform on standard Fock spaces and famous fixed point results

Recall the standard Fock space F^2_α and let $k_z = K_z/\|K_z\|_{2,\alpha}$ be the normalized reproducing kernel. Given a Lebesgue measurable function f with $f|k_z|^2 \in L^1(\mathbb{C}, d\lambda_\alpha)$, we define the Berezin transform $\tilde{f}$ on $\mathbb{C}$ as

$$\begin{aligned} B_\alpha f(z) = \tilde{f}(z) = \langle f k_z, k_z\rangle &= \int_{\mathbb{C}} |k_z(w)|^2 f(w) d\lambda_\alpha \\ &= \int_{\mathbb{C}} \left| e^{\alpha \bar{z} w - \frac{\alpha}{2}|z|^2}\right|^2 f(w) d\lambda_\alpha(w) \\ &= \frac{\alpha}{\pi}\int_{\mathbb{C}} f(w) e^{-\alpha|z-w|^2} dA(w) \\ &= \int_{\mathbb{C}} f(z \pm w) d\lambda_\alpha(w). \end{aligned} \tag{3.3.1}$$

Therefore, the Berezin transform for standard Fock spaces is a convolution with a Gaussian function.

Now, we explain the relationship between the Berezin transform over standard Fock spaces and the heat equation, and use it to show that the bounded fixed points of the Berezin transform must be harmonic.

Theorem 3.3.1 ([79], Theorem 3.13). *Let $H_t = B_{1/t}$ for any positive parameter t. Then we have the following semigroup property.*

$$H_s H_t = H_{s+t}, \quad \forall s, t > 0.$$

Theorem 3.3.2 ([79], Theorem 3.14). *Let f be a measurable function on $\mathbb{C}$ such that*

$$\int_{\mathbb{C}} |f(w)| e^{-\alpha|z-w|^2} dA(w) < \infty$$

for every $\alpha > 0$ and every $z \in \mathbb{C}$. For $t > 0$, define

$$u(z,t) := H_t f(z) = \frac{1}{\pi t}\int_{\mathbb{C}} f(w) e^{-|z-w|^2/t} dA(w).$$

Then u is defined on $\mathbb{C} \times (0,\infty)$. The function u, written in real variables as $u(x,y,t)$ with $z = x+iy$, is in $C^2(\mathbb{R}^2 \times (0,\infty))$ and satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial u}{\partial t}, \tag{3.3.2}$$

for all $(x,y,t) \in \mathbb{R}^2 \times (0,\infty)$. Moreover, if f is bounded and continuous on $\mathbb{C}$, then u also satisfies the initial condition

$$\lim_{t\to 0^+} H_t f(z) = f(z), \quad \forall z \in \mathbb{C}.$$

The above assumption on f defines a large class of functions. For example, it is satisfied by every function in $L^p(\mathbb{C})$, $1 \le p \le \infty$, and in particular by compactly supported functions and Schwartz functions. More generally, it is satisfied by locally integrable functions whose growth at infinity is slower than every Gaussian.

Note that in the heat equation (3.3.2), the value $u(x,y,t)$ represents the temperature at the point $(x,y) \in \mathbb{C}$ at time t. Thus, the function $f(z)$ represents the initial temperature distribution in the complex plane at time $t = 0$. With this interpretation, the assumption that f be bounded and continuous is reasonable.

Corollary 3.3.3. *For any positive α and β,*

$$B_\alpha B_\beta = B_{\frac{\alpha\beta}{\alpha+\beta}} = B_\beta B_\alpha.$$

In particular, $B_\alpha^2 = B_{\alpha/2}$. If f is bounded and continuous, then

$$\lim_{\alpha\to\infty} B_\alpha f(z) = f(z), \quad \forall z \in \mathbb{C}.$$

Proof. It is a direct consequence of Theorem 3.3.1 and Theorem 3.3.2. □

Theorem 3.3.4 ([79], Theorem 3.25)**.** *Let $f \in L^\infty(\mathbb{C})$, and n is a positive integer. Then*

$$|B_\alpha^n f(z) - B_\alpha^n f(w)| \leq \frac{C\|f\|_\infty}{\sqrt{n}}|z-w|,$$

for all z and w in $\mathbb{C}$, where $C = \sqrt{\alpha/\pi}$. In particular, the Berezin transform $B_\alpha f$ is Lipschitz continuous as a function of z, i.e., there exists $\widetilde{C} > 0$ such that

$$|B_\alpha f(z) - B_\alpha f(w)| \leq \widetilde{C}|z-w|, \quad z, w \in \mathbb{C}.$$

Theorem 3.3.5 ([79], Proposition 3.27)**.** *If $f \in L^\infty(\mathbb{C})$, then the following conditions are equivalent.*

1. *$\tilde{f} = f$,*
2. *f is harmonic,*
3. *f is constant.*

Proof. First, we show that (2) is equivalent to (3). Trivially, every constant function is harmonic. To see that (2) implies (3), write $f(z) = u(x,y) + iv(x,y)$. Since f is bounded, both u and v are bounded, as real-valued functions over $\mathbb{C}$. Moreover, since f is harmonic, $\Delta f = \Delta u + i\Delta v = 0$, implying that both u and v are harmonic. Let $\tilde{v}$ be the harmonic conjugate of u. That is, $g(z) = u(z) + i\tilde{v}(z)$ is an entire function. Let $h(z) = e^{g(z)}$. Then h is entire, and its modulus $|h(z)| = e^{\mathrm{Real}\, g(z)} = e^{u(z)}$ is bounded. Liouville's theorem states that any bounded entire function is constant. Thus, h is constant, implying that u is constant. Similarly, one can show that v is constant, and therefore f is constant. To show that (3) implies (1), let f be constant. Then $\tilde{f} = \langle f k_z, k_z\rangle = f\|k_z\|_{2,\alpha}^2 = f$. Finally, to see that (1) implies (3), let $\tilde{f} = f$. Then $(\tilde{f})^n = f$ for all positive integers n. By Theorem 3.3.4, there is a positive constant C such that

$$|(\tilde{f})^n(z) - (\tilde{f})^n(w)| = |f(z) - f(w)| \leq \frac{C\|f\|_\infty}{\sqrt{n}}|z-w|,$$

for all $z, w \in \mathbb{C}$ with $z \neq w$. Letting $n \to \infty$, we see that f must be a constant. □

Remark 3.3.6. Engliš [38, 36] (see also [79, section 3.3]) showed that on the classical Fock space F^2, there are non harmonic fixed points of the corresponding Berezin transform. For example $f(x+iy) = e^{ax+by}$ is fixed in F^2 for any $a, b \in \mathbb{C}$ such that $a^2+b^2 = 8\pi i$, however f is not harmonic.

Berezin transform on F^2_m, $m > 0$: Fock-type space

Let m be a positive real number. Consider the space $L^2_m = L^2(\mathbb{C}, e^{-|z|^m} dA)$, where $dA = r dr d\theta/2\pi$ is the Lebesgue measure on $\mathbb{C}$. L^2_m is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^m} dA(z), \quad f, g \in L^2_m.$$

Then the Fock-type space F^2_m, is defined as the subspace of holomorphic functions in $L^2(\mathbb{C}, e^{-|z|^m} dA)$.

We emphasize that this normalization differs from Definition 2.2.7, where the canonical doubling Fock space F^2_m is defined using the weight $e^{-2|z|^m}$. The present space corresponds instead to the doubling Fock space associated with the weight $\tilde{\phi}(z) = \frac{1}{2}|z|^m$.

An important observation is that the Bergman kernel of F^2_m has an explicit form. Given a function $f \in F^2_m$, its Taylor series $f(z) = \sum_{j=0}^{\infty} f_j z^j$ converges uniformly on compact subsets of $\mathbb{C}$. Furthermore,

$$\begin{aligned}
\|f\|^2 &= \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} dA(z) \\
&= \int_{\mathbb{C}} \sum_{j,k=0}^{\infty} f_j z^j \overline{f}_k \bar{z}^k e^{-|z|^m} dA(z) \\
&= \int_0^{2\pi} \int_0^{\infty} \sum_{j,k=0}^{\infty} f_j \overline{f}_k r^{j+k} e^{i\theta(j-k)} e^{-r^m} \frac{r dr d\theta}{2\pi} \\
&= \sum_{j=0}^{\infty} |f_j|^2 \int_0^{\infty} r^{2j+1} e^{-r^m} dr \\
&= \sum_{j=0}^{\infty} |f_j|^2 \frac{1}{m} \Gamma\left(\frac{2j+2}{m}\right).
\end{aligned}$$

We note that the scalar product for $f, g \in F^2_m$ is defined by

$$\begin{aligned}
\langle f, g\rangle &= \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^m} dA(z) \\
&= \int_{\mathbb{C}} \sum_{j,k=0}^{\infty} f_j z^j \overline{g_k} \bar{z}^k e^{-|z|^m} dA(z) \\
&= \int_0^{2\pi} \int_0^{\infty} \sum_{j,k=0}^{\infty} f_j \overline{g}_k r^{j+k+1} e^{i\theta(j-k)} e^{-r^m} \frac{dr d\theta}{2\pi} \\
&= \sum_{j=0}^{\infty} f_j \overline{g}_j \frac{1}{m} \Gamma\left(\frac{2j+2}{m}\right).
\end{aligned}$$

Hence, the monomials z^n, with $n = 0, 1, 2, \cdots$ form an orthogonal basis for F^2_m. In particular,

$$\left\{\sqrt{m d_j} z^j; \quad j = 0, 1, 2, \cdots\right\}$$

is an orthonormal basis for F^2_m where

$$d_j = \frac{1}{\Gamma\left(\frac{2j+2}{m}\right)}. \tag{3.3.3}$$

To find the Bergman kernel, we proceed as follows. Let $f \in F^2_m$ and $z \in \mathbb{C}$. Then

$$\begin{aligned}|f(z)| = |\sum_{j=0}^{\infty} f_j z^j| \leq \sum_{j=0}^{\infty} |f_j||z|^j = \sum_{j=0}^{\infty} \frac{|f_j|}{\sqrt{md_j}} \sqrt{md_j}|z|^j \\ \leq \left(\sum_{j=0}^{\infty} \frac{|f_j|^2}{md_j}\right)^{1/2} \left(\sum_{j=0}^{\infty} md_j|z|^{2j}\right)^{1/2}.\end{aligned}$$

The first sum on the right hand side equals $\|f\|$ and the ratio test shows that the second sum above converge uniformly on compact subsets. Thus, the evaluation map $f \mapsto f(z)$ is a bounded linear functional on $\mathbb{C}$ and uniformly bounded for z. Furthermore, F^2_m is closed inside $L^2(\mathbb{C}, e^{-|z|^m}dA)$, and hence a Hilbert space. Thus, there exists $K_{m,z} \in F^2_m$ such that for any $f \in F^2_m$, $f(z) = \langle f, K_{m,z}\rangle$. Indeed,

$$f(z) = \sum_{j=0}^{\infty} f_j z^j = \sum_{j=0}^{\infty} f_j md_j z^j \frac{1}{md_j} = \langle f, K_{m,z}\rangle,$$

where, $K_{m,z}(w) = m\sum_{j=0}^{\infty} d_j w^j \bar{z}^j$, for any $z, w \in \mathbb{C}$. From now on, we will write

$$K_m(w,z) = K_{m,z}(w) = m\sum_{j=0}^{\infty} d_j w^j \bar{z}^j = m\sum_{j=0}^{\infty} \frac{w^j \bar{z}^j}{\Gamma(\frac{2j+2}{m})}.$$

Definition 3.3.7 (Berezin transform on F^2_m)**.** Let $k_{m,z} = \frac{K_{m,z}}{\|K_{m,z}\|}$ be the normalized Bergman kernel of F^2_m. One can define the Berezin transform of a function f as

$$B_m f(z) = \langle f k_{m,z}, k_{m,z}\rangle = \int_{\mathbb{C}} f(w)|k_{m,z}(w)|^2 e^{-|w|^m} dA(w),$$

whenever the above integral exists.

Proposition 3.3.8. *Let $m > 0$. The Berezin transform B_m is defined on any polynomial in z and $\bar{z}$.*

Proof. Let $f = f(z,\bar{z})$ be a polynomial of degree n. That is, f can be written as

$$f(z,\bar{z}) = \sum_{\substack{i,j\geq 0 \\ i+j\leq n}} a_{ij} z^i \bar{z}^j$$

for constants a_{ij}. Then

$$\begin{aligned}B_m f(z) &= \int_{\mathbb{C}} f(w,\bar{w})|k_{m,z}(w)|^2 e^{-|w|^m} dA(w) \\ &= \frac{1}{K_m(z,z)} \int_{\mathbb{C}} (\sum_{\substack{i,j\geq 0 \\ i+j\leq n}} a_{ij} w^i \bar{w}^j)|K_m(z,w)|^2 e^{-|w|^m} dA(w) \\ &= \frac{m^2}{K_m(z,z)} \sum_{\substack{i,j\geq 0 \\ i+j\leq n}} a_{ij} \sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \int_{\mathbb{C}} w^i \bar{w}^j \bar{w}^k w^l e^{-|w|^m} dA(w).\end{aligned} \tag{3.3.4}$$

Let

$$A = \int_{\mathbb{C}} w^{i+l} \bar{w}^{j+k} e^{-|w|^m} dA(w).$$

Then

$$A=\int_0^\infty\int_0^{2\pi} r^{i+l+j+k+1}e^{i\theta(i+l-j-k)}e^{-r^m}drd\theta/2\pi.$$

When $i \geq j$, the above integral is nonzero only for $k=i+l-j$, and when $j>i$, A is nonzero only for $l=j+k-i$. First, take $i\geq j$. Then

$$A=\int_0^\infty r^{2(i+l)+1}e^{-r^m}dr=\frac{1}{m}\Gamma(\frac{2(i+l)+2}{m})=\frac{1}{m}\frac{1}{d_{i+l}}.$$

Therefore,

$$\begin{aligned}B_mf(z)&=\frac{m}{K_m(z,z)}\sum_{\substack{i,j\geq 0\\ i+j\leq n}}a_{ij}\sum_{l=0}^\infty\frac{d_{i+l-j}d_l}{d_{i+l}}z^{i+l-j}\bar{z}^l\\ &\leq d_0^{-1}\sum_{\substack{i,j\geq 0\\ i+j\leq n}}a_{ij}\sum_{l=0}^\infty\frac{d_{i+l-j}d_l}{d_{i+l}}z^{i+l-j}\bar{z}^l.\end{aligned}$$

We are left to show that the above sum converges for every $z\in\mathbb{C}$. Let

$$S(z,\bar{z})=\sum_{l=0}^\infty a_l z^{i+l-j}\bar{z}^l,\quad\text{where } a_l=\frac{d_{i+l-j}d_l}{d_{i+l}}.$$

Using the Stirling formula for gamma functions, we will show that $a_l\to 0$ as $l\to\infty$ exponentially fast. Indeed, let $\alpha=2/m$, and recall the Stirling formula $\Gamma(x)\sim\sqrt{2\pi}x^{x-1/2}e^{-x}$ as $x\to\infty$. Then

$$a_l=\frac{\Gamma(\alpha(i+l+1))}{\Gamma(\alpha(i+l-j+1))\Gamma(\alpha(l+1))}=\frac{\Gamma(\alpha l+C_1)}{\Gamma(\alpha l+C_2)\Gamma(\alpha l+C_3)},$$

Where $C_1=\alpha(i+1)$, $C_2=\alpha(i-j+1)$, and $C_3=\alpha$. Applying Stirling's formula to $\alpha l\to\infty$,

$$\begin{aligned}a_l&\sim\frac{(\alpha l+C_1)^{(\alpha l+C_1-1/2)}e^{-(\alpha l+C_1)}}{\sqrt{2\pi}(\alpha l+C_2)^{(\alpha l+C_2-1/2)}e^{-(\alpha l+C_2)}(\alpha l+C_3)^{(\alpha l+C_3-1/2)}e^{-(\alpha l+C_3)}}\\ &\to\frac{ce^{\alpha l}}{(\alpha l)^{\alpha l}},\quad\text{as } l\to\infty,\end{aligned}$$

for some constant c. We want to show that $S(z,\bar{z})=\sum_{l=0}^\infty a_l z^{i-j}|z|^{2l}$ converges absolutely for every $z\in\mathbb{C}$. Using the root test for convergence, take $b_l=|a_l||z|^{2l}$. Then

$$\sqrt[l]{|b_l|}=|a_l|^{1/l}|z|^2\to c^{1/l}e^\alpha(\alpha l)^{-\alpha}|z|^2\to 0,\quad\text{as } l\to\infty.$$

Thus, $S(z,\bar{z})$ is absolutely convergent for any $z\in\mathbb{C}$. The case of $i<j$ can be done similarly, and we can conclude that for any polynomial f, and any $z\in\mathbb{C}$, $B_mf(z)$ is finite. □

Polynomial fixed points of the Berezin transform on Fock-Type spaces

In the following, we give a summary of our paper [9] (equivalently, Appendix C) about the polynomial fixed points of the Berezin transform on F_m^2, where $m>0$. We will see that most of the polynomials which are fixed under the Berezin transform are harmonic.

Proposition 3.3.9 (Lemma C.3.1, [9], Lemma 2). *Let $m>0$ and B_m be the Berezin transform on F_m^2. Then harmonic polynomials in z and $\bar{z}$ are fixed points of the Berezin transform.*

Proof. Assume that f is a holomorphic polynomial. Since $\{z^n : n = 0, 1, \cdots\}$ is an orthogonal basis for F^2_m, one can see that $f \in F^2_m$. Moreover, Lemma 5.2 in [20] states that when f is a polynomial, $fK_{m,z} \in L^2_m$ for every $z \in \mathbb{C}$. Hence, the reproducing kernel property implies that

$$\begin{aligned} B_m(f)(z) &= \int_{\mathbb{C}} f(w)|k_{m,z}(w)|^2 e^{-|w|^m} dA(w) = \frac{1}{K_m(z,z)} \int_{\mathbb{C}} f(w)K_{m,z}(w)\overline{K_{m,z}(w)} e^{-|w|^m} dA(w) \\ &= \frac{1}{\|K_{m,z}\|^2} \int_{\mathbb{C}} K_m(z,w)[K_{m,z}(w)f(w)] e^{-|w|^m} dA(w) = \frac{K_{m,z}(z)f(z)}{\|K_{m,z}\|^2} = f(z). \end{aligned}$$

The same holds with f replaced by $\bar{f}$, and thus $f + \bar{g}$ is fixed under the Berezin transformation whenever f and g are both holomorphic polynomials. Since harmonic functions can be written as a sum of holomorphic and conjugate holomorphic functions, any harmonic polynomial is a fixed point of the Berezin transformation. □

Now that we know that harmonic polynomials are fixed points of the Berezin transform B_m for every positive m, our focus now is to investigate whether every polynomial f which is fixed under B_m, i.e., $B_m f = f$, is harmonic and how the value of m affects the picture.

Lemma 3.3.10 (Lemma C.2.1, [9], Lemma 1)**.** *Let $\beta(\cdot,\cdot)$ be the beta function, and k a positive integer. The function $x \to \beta(x, k-x)$ is convex on $(0,k)$ and attains its minimum at $x = k/2$.*

Let us define

$$H_{n,\tau} = \left\{ \sum_{j=0}^{n} a_j z^{j+\tau} \bar{z}^j : a_j \in \mathbb{C} \right\}$$

for $n, \tau \in \mathbb{N}_0$, and

$$H_{n,\tau} = \left\{ \sum_{j=0}^{n} a_j z^j \bar{z}^{j-\tau} : a_j \in \mathbb{C} \right\}$$

for $\tau \in \mathbb{Z} \setminus \mathbb{N}_0$ and $n \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of non-negative integers.

Lemma 3.3.11 (Lemma C.3.3, [9], Lemma 4)**.** *Let f be a polynomial (of z and $\bar{z}$) of degree n such that $B_m f = f$. Then*

$$f = \sum_{\tau=-n}^{n} f_\tau,$$

where $f_\tau \in H_{n,\tau}$ and $B_m f_\tau = f_\tau$ for $\tau = -n, \ldots, n$.

Theorem 3.3.12 (Theorem C.1.1, [9], Theorem 1)**.** *Assume that $B_m f = f$ for a polynomial f of z and $\bar{z}$ with nonnegative coefficients and for some $m > 0$. Then f is harmonic.*

Sketch of proof. Using Lemma 3.3.11 and by contradiction, it is enough to show that the non-harmonic polynomial $f(z) = \sum_{j=1}^{n} a_j z^{j+\tau} \bar{z}^j$, with $n \in \mathbb{N}_0$, $0 \le \tau \le n$, $a_j \ge 0$, and $a_n > 0$ is not a fixed point. Similarly to (3.3.4), one can see that

$$K_m(z,z) B_m f(z) = m \sum_{l=0}^{\infty} \sum_{j=1}^{n} a_j \frac{d_{l+\tau} d_l}{d_{j+l+\tau}} z^{l+\tau} \bar{z}^l,$$

and

$$K_m(z,z) f(z) = m \sum_{l=0}^{\infty} \sum_{j=1}^{n} a_j d_l z^{j+\tau+l} \bar{z}^{j+l} = m \sum_{l=j}^{\infty} \sum_{j=1}^{n} a_j d_{l-j} z^{l+\tau} \bar{z}^l.$$

Assuming $B_m f = f$,

$$\begin{aligned} 0 = K_m(z,z)B_m f(z) - K_m(z,z)f(z) =& m\sum_{l=0}^{j-1}\sum_{j=1}^{n} a_j \frac{d_{l+\tau}d_l}{d_{j+\tau+l}} z^{l+\tau}\bar{z}^l \\ &+ m\sum_{l=j}^{\infty}\sum_{j=1}^{n} a_j \left(\frac{d_{l+\tau}d_l}{d_{j+\tau+l}} - d_{l-j}\right) z^{l+\tau}\bar{z}^l, \end{aligned} \tag{3.3.5}$$

for all z. Using Lemma 3.3.10, one can see that $\frac{d_{l+\tau}d_l}{d_{j+\tau+l}} - d_{l-j} > 0$. Hence, $a_n > 0$ implies $K_m(z,z)B_m f(z) - K_m(z,z)f(z) \not\equiv 0$, which contradicts (3.3.5). Thus, f is not a fixed point. □

When $m = 2$, one can write the Berezin transform as

$$B_2 f(z) = 2\int_{\mathbb{C}} f(z+\xi)e^{-|\xi|^2}\,dA(\xi),$$

where $dA = r dr d\theta/2\pi$. Notice that since we normalize dA by 2π, the above formula has a multiple of 2 instead of $1/\pi$ as in (3.3.1). Computing $B_2 f$ for $f = z^{j+\tau}\bar{z}^j$, where $j \in \mathbb{N}$ and $\tau \in \mathbb{N}_0$, obtaining the matrix representation of B_2 over a suitable basis, and applying the rank-nullity theorem, one can conclude the following theorem.

Theorem 3.3.13 (Theorem C.1.2, [9], Theorem 2). *Let f be a polynomial of z and $\bar{z}$ such that $B_2 f = f$. Then f is harmonic.*

Notice that the above theorem is different from Theorem 3.3.5, as polynomials are not bounded.

Remark 3.3.14. The conclusion of Theorem 3.3.13 can be viewed as a Liouville-type result for fixed points of the Berezin transform. It is worth noting that every polynomial in z and $\bar{z}$ on $\mathbb{C}^n$ defines a tempered distribution, i.e., $\mathbb{C}[z,\bar{z}] \subset \mathcal{S}'(\mathbb{C}^n)$. In this broader setting, a stronger statement is available: if $u \in \mathcal{S}'(\mathbb{C}^n)$ satisfies $B[u] = u$ in the classical Fock space, then u is a harmonic polynomial; see Lemma 2.8 in [15]. This can be interpreted as a Liouville theorem for tempered distributions and thus constitutes a generalization of Theorem 3.3.13.

However, the proof of Theorem 3.3.13 given in Appendix C (See the proof of Theorem C.1.2) is of a different nature. The argument in which Lemma 2.8 in [15] is based on, relies on the identification of the Berezin transform with a Gaussian convolution (equivalently, the heat semigroup) and uses Fourier-analytic methods in $\mathcal{S}'(\mathbb{C}^n)$. In contrast, our proof does not pass through the Laplacian or distributional techniques. Instead, it is based on the algebraic structure of the Berezin transform acting on polynomials and the decomposition results developed in Appendix C, in particular Lemmas C.3.3 and C.5.1, together with the argument in the proof of Theorem C.1.2. This approach works directly within the function-theoretic framework of Fock-type spaces and does not rely on the heat semigroup structure, which in general is not available beyond the classical case.

For $m \neq 2$, the situation is more difficult, since the Bergman kernels do not have a closed form and the computations involve many gamma functions. Our main result shows that $B_m f = f$ for a polynomial f implies that f is harmonic for all m, except possibly countably many m.

Theorem 3.3.15 (Theorem C.1.3, [9], Theorem 3). *For $n \in \mathbb{N}_0$, there exists a discrete (possibly empty) set $Z_n \subset (0,\infty)$ with no cluster points in $(0,\infty)$ such that if $m \in (0,\infty)\setminus Z_n$ and $B_m f = f$ for a polynomial f of degree at most n, then f is harmonic.*

Sketch of proof. By Lemma 3.3.11, it is enough to prove the theorem for functions $f \in H_{n,\tau}$, where $0 \leq \tau \leq n$. For any $z \in \mathbb{C}$ and $f \in H_{n,\tau}$, define

$$T_{m,n}f(z) = K_m(z,z)B_m f(z) - K_m(z,z)f(z).$$

Then $\ker(B_m - I) = \ker(T_{m,n}) \supseteq \operatorname{span}\{z^\tau\} = \ker(T_{2,n})$, and $\operatorname{rank}(B_m - I) = \operatorname{rank}(T_{m,n}) \leq n = \operatorname{rank}(T_{2,n})$. For any $m > 0$, the rank-nullity theorem implies that $\operatorname{rank}(T_{m,n}) + \dim(\ker(T_{m,n})) = \dim(H_{n,\tau}) = n+1$. Then, similarly to (3.3.5),

$$\begin{aligned} T_{m,n}(z^{j+\tau}\bar{z}^j) &= m\sum_{l=0}^{j-1} \frac{d_{l+\tau}d_l}{d_{j+\tau+l}} z^{l+\tau}\bar{z}^l + m\sum_{l=j}^{\infty}\left(\frac{d_{l+\tau}d_l}{d_{j+\tau+l}} - d_{l-j}\right) z^{l+\tau}\bar{z}^l \\ &= \sum_{k=0}^{\infty} a_{j,k}(m) z^{k+\tau}\bar{z}^k, \end{aligned}$$

where each $a_{j,k}$ is holomorphic on $U = \{z \in \mathbb{C} : Re(z) > 0\}$, by properties of the gamma function. Note that the matrix $[a_{j,k}(m)]$ is of size $\infty \times (n+1)$. Let $A = [a_{j,k}(2)]_{\infty\times(n+1)}$. By Theorem 3.3.13, $\operatorname{rank}(T_{2,n}) = n$, implying that $\operatorname{rank}(A) = n$. Recall from linear algebra that the rank of a matrix equals the size, i.e., the number of rows or columns, of the largest square submatrix that has a non-zero determinant. Thus, there exists an $n\times n$ submatrix $S_n(2)$ of A such that $\det(S_n(2)) \neq 0$. Choosing the same fixed row/column indices that give the nonzero minor at $m = 2$, produces a finite matrix $S_n(m)$ whose entries are holomorphic functions on U. This is because each $a_{j,k}$ is holomorphic on U. Therefore, $S(m) := \det S_n(m)$ is holomorphic on U. Since $S(2) \neq 0$, S is a holomorphic function which is not identically zero. So its zeros are isolated. Let $Z_{n,\tau}$ be the zero set of S. $Z_{n,\tau}$ is a discrete set with no accumulation point in U. Since $\det(S_n(m)) \neq 0$ for $z \in U \setminus Z_{n,\tau}$, $\operatorname{rank}(T_{m,n}) \geq n$ for $z \in U \setminus Z_{n,\tau}$. However, we showed earlier that for all $0 < m < \infty$, $\operatorname{rank}(T_{m,n}) \leq n$ as $\dim(\ker(T_{m,n})) \geq 1$. Hence, $\operatorname{rank}(T_{m,n}) = n$ and $\dim(\ker(T_{m,n})) = 1$ for $m \in (0,\infty) \setminus Z_{n,\tau}$. Namely, $\operatorname{span}\{z^\tau\} = \ker(T_{m,n})$ for $m \in (0,\infty) \setminus Z_{n,\tau}$.

Let $Z_n^1 = \cup_{\tau=0}^n Z_{n,\tau}$. Hence Z_n^1 is a discrete set with no cluster in $(0,\infty)$. Furthermore, we showed that only the holomorphic polynomials are fixed whenever $0 \leq \tau \leq n$ and $m \in (0,\infty)\setminus Z_n^1$. Similarly, we can show that there is a discrete set Z_n^2, such that when $-n \leq \tau < 0$ and $m \in (0,\infty) \setminus Z_n^2$, only the conjugate holomorphic polynomials are fixed. Let $Z_n = Z_n^1 \cup Z_n^2$. Therefore, if $m \in (0,\infty) \setminus Z_n$ and $B_m f = f$ for a polynomial f of degree at most n, then f is harmonic. □

Remark 3.3.16. Recall that any finite union of discrete sets without accumulation points is discrete with no accumulation points. For example, let $Z_1, Z_2 \in (0,\infty)$ be discrete and without cluster points in $(0,\infty)$. Suppose for contradiction that $Z = Z_1 \cup Z_2$ has an accumulation point $x \in (0,\infty)$. Then there exists a sequence $x_n \subset Z$, all distinct, with $x_n \to x$. Because each x_n lies in either Z_1 or Z_2, by the pigenhole principle, infinitely many x_n lies in one of the two sets, say Z_1. But then x would be an accumulation point of Z_1, contradicting the hypothesis. Therefore, no such x exists, so Z has no accumulation points in $(0,\infty)$. Equivalently, every point of Z is isolated, so Z is discrete.

Corollary 3.3.17 (Corollary C.1.4, [9], Corollary 1)**.** *Let f be a polynomial of z and $\bar{z}$ of degree at most n. Then the following are equivalent.*

1. *$B_m f = f$ for some $m \in (0,\infty) \setminus Z_n$,*
2. *$B_m f = f$ for all $m \in (0,\infty) \setminus Z_n$.*

Proof. (1) $\Rightarrow$ (2). Assume f is a polynomial in $z, \bar{z}$ with $\deg f \le n$ and there exists some $m_0 \in (0,\infty) \setminus Z_n$ with $B_{m_0} f = f$. By Theorem 3.3.15, this forces f to be harmonic. But by Proposition 3.3.9, every harmonic polynomial is a fixed point of the Berezin transform for all parameters m. Hence for every $m \in (0,\infty)$ (and in particular for every $m \in (0,\infty) \setminus Z_n$) we have $B_m f = f$. Thus (2) holds.

(2) $\Rightarrow$ (1). This is immediate: if $B_m f = f$ for all $m \in (0,\infty) \setminus Z_n$, then in particular there exists some $m \in (0,\infty) \setminus Z_n$ with $B_m f = f$.

Therefore, (1) and (2) are equivalent. □

Since the countable union of countable sets is countable, we have the following corollary.

Corollary 3.3.18 (Corollary C.1.5, [9], Corollary 2)**.** *There exists a countable, possibly empty, set $Z \subset (0,\infty)$ such that if $B_m f = f$ for a polynomial f of z and $\bar{z}$ and $m \in (0,\infty) \setminus Z$, then f is harmonic.*

Using a more computational approach, we also show that if $B_m f = f$ for a binomial function f, then f is harmonic.

Theorem 3.3.19 (Theorem C.1.6, [9], Theorem 4)**.** *Let $m > 0$ and $f(z) = c_1 z^a \bar{z}^b + c_2 z^c \bar{z}^d$, where a, b, c, and d are positive integers. Then f is a fixed point of the Berezin transformation B_m if and only if $c_1 = c_2 = 0$.*

Proof for monomials. Let $m > 0$ and $f(z) = z^p \bar{z}^q$ with $p, q > 0$, be a fixed point of the Berezin transform. By Proposition 3.3.9, when either p or q are zero, f is harmonic and is fixed under B_m. So, assume that $p, q \ge 1$ and $B_m f = f$. Then

$$K_m(z,z) z^p \bar{z}^q = K_m(z,z) B_m f(z) = \int_{\mathbb{C}} w^p \bar{w}^q |K_m(w,z)|^2 e^{-|w|^m} \, dA(w).$$

Hence,

$$\begin{aligned} m \sum_{k=0}^{\infty} d_k z^p \bar{z}^q z^k \bar{z}^k &= m^2 \sum_{k,l=0}^{\infty} d_k d_l \int_{\mathbb{C}} w^p \bar{w}^q z^k \bar{w}^k \bar{z}^l w^l e^{-|w|^m} \, dA(w) \\ &= m^2 \sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \int_{\mathbb{C}} w^{p+l} \bar{w}^{q+k} e^{-|w|^m} \, dA(w). \end{aligned}$$

First, assume that $p \ge q$. The above integral is nonzero only if $k = p + l - q$. Therefore,

$$\begin{aligned} m \sum_{k=0}^{\infty} d_k z^{p+k} \bar{z}^{q+k} &= m^2 \sum_{l=0}^{\infty} d_{p+l-q} d_l z^{p+l-q} \bar{z}^l \int_0^{\infty} r^{2(p+l)+1} e^{-r^m} \, dr \\ &= m^2 \sum_{l=0}^{\infty} d_{p+l-q} d_l z^{p+l-q} \bar{z}^l \frac{1}{m} \Gamma\left(\frac{2(p+l)+2}{m} \right) \\ &= m \sum_{l=0}^{\infty} d_{p+l-q} d_l z^{p+l-q} \bar{z}^l \frac{1}{d_{p+l}}. \end{aligned} \tag{3.3.6}$$

For the above equation to hold, every corresponding l-term on each side must agree. Note that the zeroth term on the right hand side is holomorphic, while there is no holomorphic term on the left hand side. Hence, they can not be equal.

When $p < q$, similarly we obtain

$$m\sum_{k=0}^{\infty} d_k z^{p+k}\bar{z}^{q+k} = m\sum_{k=0}^{\infty} d_k d_{q-p+k} z^k \bar{z}^{q-p+k}\frac{1}{d_{q+k}}.$$

The zeroth term on the right-hand side is conjugate holomorphic, while there is no conjugate holomorphic term on the left-hand side. Hence, they are not equal.

Therefore, we can conclude that any monomial which is a fixed point of the Berezin transformation should be either of the form z^p or $\bar{z}^q$. □

Remark 3.3.20. The idea behind the proof of Theorem 3.3.19 for binomial fixed points of the Berezin transform is similar to the monomial case explained above. However, the computations are much more complicated and involve properties of Beta and Gamma functions. Interested readers are encouraged to check out the paper [9], where a copy is provided in Appendix C.

It would be interesting to know if the set Z in Corollary 3.3.18 can be non-empty. So, we finish this section with the following conjecture.

Conjecture 3.3.21. Let f be a polynomial in z and $\bar{z}$. If for any $m > 0$ we have $B_m f = f$, then f is harmonic.

3.4 Open problems

In what follows, we give a few open problems, proposing future possible directions related to papers [10, 6, 9], i.e., Appendices A-C.

Regarding [10] (equivalently, Appendix A), one can study the following.

1. Exploring the Berger-Coburn phenomenon for Schatten class Hankel operators with $p \neq 2$ on doubling Fock spaces, and other function spaces.
2. Exploring the Berger-Coburn phenomenon for compact Hankel operators on doubling Fock spaces, and other function spaces.
3. Exploring the Berger-Coburn conjecture for bounded, compact, and Schatten class Toeplitz operators on classical, doubling, and other Fock-type spaces.

Regarding [6] (equivalently, Appendix B), one can study the following.

1. Can we prove the result of Theorem 3.2.26 without assuming that the symbol $G(z)$ is a compact operator on $\mathcal{H}$, for every $z \in \mathbb{C}^n$?
2. Studying the Fredholm property of Toeplitz operators T_f acting on scalar weighted Fock spaces $F^p_\phi(\mathbb{C}^n)$.
3. Study of boundedness and compactness of the vectorial Toeplitz operator T_G acting on large vector-valued Fock spaces $F^p_\phi(\mathcal{H})$, with $p \neq 2$.
4. Studying the Fredholm property of vectorial Toeplitz operators T_G acting on large vector-valued Fock spaces $F^p_\phi(\mathcal{H})$.

Regarding [9] (equivalently, Appendix C), one can study the following.

1. Exploring Conjecture 3.3.21 for $m \neq 2$.
2. Studying the fixed points of the Berezin transform B_m for non-polynomial functions and more general doubling Fock spaces.
3. Exploring the possible relationship between the Berezin transform B_m and the Laplace-Beltrami operator, as well as the corresponding heat equation. Doing that may not only result in a more general fixed-point theorem, but also may help us to quantize the complex plane equipped with a more general metric.

Appendices

Appendix A

Schatten class Hankel operators on doubling Fock spaces and the Berger-Coburn phenomenon[1]

Abstract

Using the notion of integral distance to analytic function, we give a characterization of Schatten class Hankel operators acting on doubling Fock spaces on the complex plane and use it to show that for $f \in L^\infty$, if H_f is Hilbert-Schmidt, then so is $H_{\bar{f}}$. This property is known as the Berger-Coburn phenomenon. When $0 < p \le 1$, we show that the Berger-Coburn phenomenon fails for a large class of doubling Fock spaces. Along the way, we illustrate our results for the canonical weights $|z|^m$ when $m > 0$.

A.1 Introduction and main results

Let $dA = \frac{1}{2i} dz \wedge d\bar{z}$ be the Lebesgue measure on $\mathbb{C}$, and ϕ be a subharmonic function. For $0 < p < \infty$, $L^p_\phi = L^p(\mathbb{C}, e^{-p\phi} dA)$ is the space of all measurable functions on $\mathbb{C}$ such that

$$\|f\|^p_{p,\phi} = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} dA(z) < \infty, \tag{A.1.1}$$

and L^∞_ϕ is the space of measurable functions f such that

$$\|f\|_{\infty,\phi} = \operatorname*{ess\,sup}_{z\in\mathbb{C}} |f(z)| e^{-\phi(z)} < \infty. \tag{A.1.2}$$

Moreover, we write $L^p(\Omega)$ for the space $L^p(\Omega, dA)$ where $\Omega \subset \mathbb{C}$, and we abbreviate $L^p(\mathbb{C}, dA)$ as L^p. A positive Borel measure μ on $\mathbb{C}$ is called doubling if there exists some constant $C > 1$ such that

$$\mu(D(z, 2r)) \le C\mu(D(z, r)) \tag{A.1.3}$$

for all $z \in \mathbb{C}$ and $r > 0$, where $D(z, r)$ is the open disk in $\mathbb{C}$ with center z and radius r. The smallest $C > 1$ is called the doubling constant for μ. Hence, for each $z \in \mathbb{C}$, $\lim_{r\to\infty} \mu(D(z, r)) =$

---

[1] This appendix reproduces the paper *"Schatten class Hankel operators on doubling Fock spaces and the Berger–Coburn phenomenon"* by G. Asghari, Z. J. Hu, and J. A. Virtanen, *Journal of Mathematical Analysis and Applications* 540 (2024), no. 2, Paper No. 128596.

Apart from formatting adjustments, this appendix coincides with the published version.

∞. It is well known that μ has no point mass, i.e.,

$$\mu(\partial D(z,r)) = \mu(\{z\}) = 0 \quad \text{for every } z \in \mathbb{C} \text{ and } r > 0, \tag{A.1.4}$$

and is nonzero and locally finite. That is,

$$0 < \mu(D(z,r)) < \infty \quad \text{for every } z \in \mathbb{C} \text{ and } r > 0. \tag{A.1.5}$$

Note that since for each $z \in \mathbb{C}$, $\lim_{r\to\infty} \mu(D(z,r)) = \infty$, the function $r \mapsto \mu(D(z,r))$ is an increasing homeomorphism from $(0,\infty)$ to itself. Therefore, for every $z \in \mathbb{C}$, there is a unique positive radius $\rho(z)$ such that $\mu(z,\rho(z)) = 1$. For more information on doubling measures see [75]. Denote by $H(\mathbb{C})$ the space of holomorphic functions on $\mathbb{C}$. Then the doubling Fock space F^p_ϕ is defined by

$$F^p_\phi = L^p_\phi \cap H(\mathbb{C}) \tag{A.1.6}$$

where ϕ is a subharmonic function, not identically zero on $\mathbb{C}$, and $d\mu = \triangle\phi\, dA$ is a doubling measure. As shown in [64], ρ^{-2} is a regularization of $\Delta\phi$. Indeed, Theorem 14 in [64] states that when ϕ is subharmonic and $\Delta\phi\, dA$ is a doubling measure, there exists a subharmonic function $\psi \in \mathcal{C}^\infty(\mathbb{C})$ and $C > 0$ such that $|\psi - \phi| \le C$, $\Delta\psi\, dA$ a doubling measure, and $\Delta\psi \simeq \rho_\psi^{-2} \simeq \rho_\phi^{-2}$. The comparability relation $\simeq$ is explained at the beginning of Section 2. Since the spaces of functions and sequences that we consider do not change if ϕ is replaced by ψ, we will assume that $\phi \in \mathcal{C}^\infty(\mathbb{C})$ and $\Delta\phi dA \simeq dA/\rho^2$ is a doubling measure. Hence, up to normalization by a constant, we can consider $\rho^{-2}(z) dz \otimes d\bar{z}$ to be the metric tensor describing the underlying geometry of our space.

It is well known that $(F^p_\phi, \|\cdot\|_{p,\phi})$ is a Banach space for $1 \le p \le \infty$ and a quasi-Banach space for $0 < p < 1$. Let $K_z = K(\cdot,z)$ be the reproducing kernel of F^2_ϕ. Then the orthogonal projection $P : L^2_\phi \to F^2_\phi$ is given by

$$Pf(z) = \int_{\mathbb{C}} f(w)\overline{K_z(w)} e^{-2\phi(w)} dA(w). \tag{A.1.7}$$

Then, as shown in [68], for any $1 \le p \le \infty$, P is a bounded linear operator from L^p_ϕ to F^p_ϕ, and for any $f \in F^p_\phi$, $f = Pf$. Let $\Gamma = \operatorname{span}\{K_z : z \in \mathbb{C}\}$, and consider the class of symbols

$$\mathcal{S} = \{f \text{ measurable} : fg \in L^2_\phi \text{ for } g \in \Gamma\}.$$

Note that $L^\infty \subset \mathcal{S}$. Given $f \in \mathcal{S}$, define the Toeplitz operator T_f and the Hankel operator H_f on F^p_ϕ by

$$T_f g = P(fg), \quad H_f g = (I - P)(fg) = fg - P(fg). \tag{A.1.8}$$

The doubling Fock spaces as well as some pointwise estimates of the Bergman kernel have been studied in seminal papers of Christ [24], and Marco, Massaneda and Ortega-Ceda [64, 65]. Oliver and Pascuas [68] studied the characterization of boundedness, compactness and the Schatten class membership of Toeplitz operators on doubling Fock spaces. In [49], Hu and Virtanen introduced a new space IDA of locally integrable functions whose integral distance to holomorphic functions is finite and used it to characterize boundedness and compactness of Hankel operators on weighted Fock spaces. Using the same notion, in [51] they characterized Schatten class Hankel operators acting on weighted Fock spaces F^2_Φ, where $m \le \triangle\Phi \le M$ for some $m, M > 0$. Recently, their characterizations of bounded and compact Hankel operators was extended to the setting of doubling Fock spaces in [63].

In the present work, we use a generalized version of IDA to study the Schatten class membership of Hankel operators on doubling Fock spaces. Of particular interest is the result of Berger and Coburn [17] which says that, for $f \in L^\infty$, if H_f is a compact operator acting on the classical Fock space F^2, then so is $H_{\bar{f}}$. We refer to this property as the Berger-Coburn phenomenon and note that an analogous statement fails both in the Hardy and Bergman spaces (see, e.g., [41]). More recently, Berger and Coburn's result has been extended to Fock spaces with standard weights by Hagger and Virtanen [41] (using limit operator techniques as opposed to C^*-algebra techniques and Hilbert space methods) and to generalized Fock spaces F^p_Φ by Hu and Virtanen [49]. Our approach is similar to that of [49] except that we need to deal with more complicated geometry induced by the function ρ arising in the study of doubling Fock spaces.

It is natural to ask whether the Berger-Coburn phenomenon also holds for Schatten class Hankel operators. Indeed, Bauer [14] was the first to show that this property holds for Hilbert-Schmidt Hankel operators on F^2. Recently, Hu and Virtanen in [51] proved that when $1 < p < \infty$, H_f acting on F^2_Φ is in the Schatten class S_p if and only if $H_{\bar{f}}$ is in S_p. This was followed by the work of Xia [76], in which he showed also that if $f(z) = 1/z$ for $|z| > 1$ and $f = 0$ elsewhere, then H_f acting on the classical Fock space F^2 is in the trace class while $H_{\bar{f}}$ is not. In his work, Xia employed a rather long and involved calculations using the standard basis vectors $e_k(z) = z^k/\sqrt{k!}$ and the reproducing kernel $K(z,w) = e^{z\bar{w}}$. Observe that for non-standard weighted Fock spaces, there are no explicit formulas for the basis vectors or the reproducing kernel. To overcome this, Hu and Virtanen [50] used their characterizations of Schatten class Hankel operators to verify that Xia's example shows that the Berger-Coburn phenomenon fails for $S_p(F^2_\varphi, L^2_\varphi)$ when $0 < m < \Delta\varphi < M$ and $0 < p \leq 1$. Here, we use an analogous approach on doubling Fock spaces to prove the existence of the Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators. When $0 < p \leq 1$, we show that the Berger-Coburn phenomenon fails for some doubling Fock spaces—the larger the value of p, the fewer Fock spaces we can cover.

To state our main results, following [49, 60] with a modification according to the doubling property of the measure under consideration, we define

$$G_{q,r}(f)(z) = \inf\left\{ \Big(\frac{1}{|D^r(z)|} \int_{D^r(z)} |f - h|^q \, dA \Big)^{1/q} : h \in H(D^r(z)) \right\}, \tag{A.1.9}$$

for $f \in L^q_{loc}$, $q \geq 1$ and $r > 0$. Here $|D^r(z)|$ is the Lebesgue measure of $D^r(z) := D(z, r\rho(z))$. Now, for $0 < p \leq \infty$, $1 \leq q \leq \infty$, and $\alpha \in \mathbb{R}$, the space $\mathrm{IDA}^{p,q,\alpha}_r$ consists of all $f \in L^q_{loc}$ such that $\|f\|_{\mathrm{IDA}^{p,q,\alpha}_r} = \|\rho^\alpha G_{q,r}(f)\|_{L^p} < \infty$. Besides, for $f \in L^1_{loc}$, define $\hat{f}_r(z) := |D^r(z)|^{-1} \int_{D^r(z)} f \, dA$.

Theorem A.1.1 (IDA decomposition). *Let $\phi \in C^\infty(\mathbb{C})$ be subharmonic such that $d\mu = \Delta\phi \, dA$ is a doubling measure. Suppose that $1 \leq q \leq \infty$, $0 < p < \infty$, $\alpha \in \mathbb{R}$, and $f \in L^q_{loc}$. Then for $f \in \mathrm{IDA}^{p,q,\alpha}_r$, $f = f_1 + f_2$ where $f_1 \in C^2(\mathbb{C})$ and*

$$\rho^{1+\alpha}|\bar{\partial} f_1| + \rho^{1+\alpha}(\widehat{|\bar{\partial} f_1|^q}_r)^{1/q} + \rho^\alpha(\widehat{|f_2|^q}_r)^{1/q} \in L^p, \tag{A.1.10}$$

for some (equivalent any) $r > 0$, and

$$\|f\|_{\mathrm{IDA}^{p,q,\alpha}_r} \simeq \inf\left\{ \|\rho^{1+\alpha}(\widehat{|\bar{\partial} f_1|^q}_r)^{1/q}\|_{L^p} + \|\rho^\alpha(\widehat{|f_2|^q}_r)^{1/q}\|_{L^p} \right\}, \tag{A.1.11}$$

where the infimum is taken over all possible decompositions $f = f_1 + f_2$, with f_1 and f_2 satisfying the conditions in (A.3.11).

Theorem A.1.1 was stated in [63] without proof. We believe that the proof is rather technical and not trivial at all. It appears that this theorem should be a natural extension of Theorem 3.8 in [49]. However, bounding a solution to the $\bar{\partial}$-equation in the doubling Fock space is problematic.

Theorem A.1.2 (Schatten class membership of Hankel operators). *Let $0 < p \leq \infty$, and $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic such that $d\mu := \Delta\phi dA$ is a doubling measure. Then for $f \in \mathcal{S}$, the following are equivalent:*

(1) $H_f : F^2_\phi \to L^2_\phi$ is in S_p,

(2) $f \in \mathrm{IDA}^{p,2,-2/p}_r$, for some (equivalent any) $r > 0$.

Moreover,

$$\|H_f\|_{S_p} \simeq \|f\|_{\mathrm{IDA}^{p,2,-2/p}_r}. \tag{A.1.12}$$

Remark A.1.3. Assuming smoothness of ρ^{-2}, the condition for the S_p membership of the Hankel operator on the doubling Fock space is equivalent to the condition that $G_{2,r}(f)$ belongs to the space of L^p functions on $\mathbb{C}$ with the conformal metric $\rho^{-2}dz \otimes d\bar{z}$.

To characterize the simultaneous membership of H_f and $H_{\bar{f}}$ in S_p, we need to define the space of integral mean oscillation. First, for $f \in L^2_{loc}$ and $r > 0$, the mean oscillation of f is defined by

$$MO_{2,r}(f)(z) = \left(\frac{1}{|D^r(z)|} \int_{D^r(z)} |f - \hat{f}_r(z)|^2 dA \right)^{1/2}. \tag{A.1.13}$$

Given $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$, we define the space $\mathrm{IMO}^{p,2,\alpha}_r$ to be the family of those $f \in L^2_{loc}$ such that

$$\|f\|_{\mathrm{IMO}^{p,2,\alpha}_r} = \|\rho^\alpha MO_{2,r}(f)\|_{L^p} < \infty. \tag{A.1.14}$$

Theorem A.1.4. *Let $0 < p < \infty$ and assume that $\phi \in \mathcal{C}^\infty(\mathbb{C})$ is subharmonic such that $d\mu = \Delta\phi dA$ is a doubling measure. Then the following are equivalent.*

(1) Both H_f and $H_{\bar{f}} \in S_p(F^2_\phi, L^2_\phi)$,

(2) $f \in \mathrm{IMO}^{p,2,-2/p}_r$, for some (equivalent any) $r > 0$. Moreover,

$$\|H_f\|_{S_p} + \|H_{\bar{f}}\|_{S_p} \simeq \|f\|_{\mathrm{IMO}^{p,2,-2/p}_r}. \tag{A.1.15}$$

Using the preceding result, it is easy to show that $H_{\bar{f}}$ is not Hilbert-Schmidt on F^2_ϕ when f is a non-constant entire function (see Theorem A.5.4), which implies an analogous result of Schneider [72] for the canonical weights $\phi(z) = |z|^m$ and $f(z) = z^k$ when k is a positive integer and $m > 0$. However, when we restrict our study to bounded symbols, it turns out that $H_{\bar{f}} \in S_2$ whenever $H_f \in S_2$ as seen in the following theorem.

Theorem A.1.5 (Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators). *Let $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic and suppose that that $d\mu = \Delta\phi\, dA$ is a doubling measure. Then for $f \in L^\infty$, $H_f \in S_2(F^2_\phi, L^2_\phi)$ if and only if $H_{\bar{f}} \in S_2(F^2_\phi, L^2_\phi)$, with*

$$\|H_{\bar{f}}\|_{S_2} \simeq \|H_f\|_{S_2}. \tag{A.1.16}$$

It is worth emphasizing that the preceding theorem for Hilbert-Schmidt Hankel operators was proved by Bauer [14] in 2004, and it took almost two decades until it was proved for other Schatten classes by Hu and Virtanen [51]. This leads to the following question.

Open Problem A.1.6. Does the Berger-Coburn phenomenon hold true for other Schatten classes S_p when $1 < p < \infty$?

For a discussion on the preceding open problem (involving the Muckenhoupt condition for the boundedness of the Beurling-Ahlfors operator), see Remark A.6.1 in Section 6.

Before stating our last theorem, we recall the following growth condition for the function ρ. Given a doubling Fock space F^2_ϕ, there are constants $C, \eta > 0$ and $0 \le \beta < 1$ such that

$$C^{-1}|z|^{-\eta} \le \rho(z) \le C|z|^{\beta} \tag{A.1.17}$$

for $|z| > 1$ (see Equation (5) of [64]); we denote the smallest β that satisfies (A.1.17) by β_ϕ.

The following result shows the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ provided that β_ϕ is sufficiently small in comparison with the value of p.

Theorem A.1.7. *Let $\phi \in C^\infty(\mathbb{C})$ be subharmonic with $d\mu = \Delta\phi \, dA$ a doubling measure. Then, for $0 < p \le 1$ with $\beta_\phi \le \frac{1-p}{1-p/2}$, the Berger-Coburn phenomenon for Schatten class Hankel operators fails; that is, there is an $f \in L^\infty(\mathbb{C})$ such that $H_f \in S_p(F^2_\phi, L^2_\phi)$ but $H_{\bar{f}} \notin S_p(F^2_\phi, L^2_\phi)$.*

In particular, when ρ is bounded, the Berger-Coburn phenomenon fails for all $0 < p \le 1$.

A simple consequence of the preceding theorem is that if F^2_ϕ is a doubling Fock space, then the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ provided that p is sufficiently small.

Another consequence is the following corollary, in which we consider again the canonical doubling weights $\phi(z) = |z|^m$ and determine when the Berger-Coburn phenomenon fails for these weights.

Corollary A.1.8. *Let $m > 0$ and $0 < p \le 1$. Then the Berger-Coburn phenomenon fails for $S_p(F^2_{|z|^m}, L^2_{|z|^m})$ if*

$$m \ge \frac{p}{1 - \frac{p}{2}}.$$

In particular, if $m \ge 2$, then the phenomenon fails for all Schatten classes S_p with $0 < p \le 1$.

Theorem A.1.7 and its corollary lead to the following question.

Open Problem A.1.9. Determine whether the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ when $0 < p \le 1$ and $\Delta\phi \, dA$ is doubling.

The paper is organized as follows. In the next section, we provide preliminaries on the reproducing kernel, including global and local estimates, and elaborate more on the radius function ρ and the induced metric on the complex plane. In Section 3, we provide useful lemmas and use them to prove Theorem A.1.1 (IDA decomposition). In Section 4, we use Toeplitz operators with locally finite positive Borel measures to prove Theorem A.1.2, which characterizes the Schatten class membership of Hankel operators. Section 5 is devoted to the study of the function space IMO of integral mean oscillation, which we use to prove Theorem A.1.4. Finally, in Section 6, we prove the Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators on general doubling Fock spaces as stated in Theorem A.1.5. We finish the last section with the proofs of Theorem A.1.7 and Corollary A.1.8.

A.2 Preliminaries

In this section we recall and prove some key lemmas on the function ρ, the reproducing kernel of F^2_ϕ, the space $\mathrm{IDA}^{p,q,\alpha}_r$, and their related integral and norm estimates.

Notation. We use C to denote positive constants whose value may change from line to line but does not depend on the functions being considered. We say that $A \simeq B$ if there exists a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$. Moreover, $A \lesssim B$ if $A \leq CB$ for some positive constant C.

Let ϕ be a subharmonic function on $\mathbb{C}$ such that $d\mu = \Delta\phi dA$ is a doubling measure. Recall that there is a function ρ such that $\mu(D(z,\rho(z))) = 1$, for every point $z \in \mathbb{C}$. In other words, the radius of a disk with unit measure depends on the center of the disk. As shown in the Fig 1, $D(z,\rho(z)) \subset D(w,|w-z|+\rho(z))$. Hence, $1 \leq \mu(D(w,|w-z|+\rho(z)))$, and thus $\rho(w) \leq \rho(z)+|w-z|$. By symmetry,

$$|\rho(w)-\rho(z)| \leq |w-z|, \quad \text{for every } z,w \in \mathbb{C}. \tag{A.2.1}$$

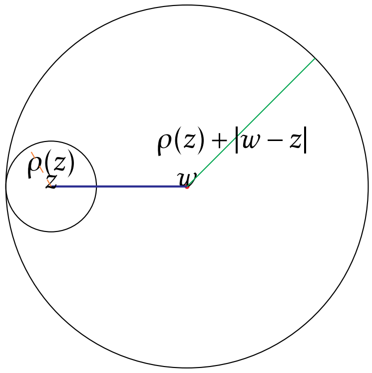

Figure A.1: Relation between $\rho(z)$ and $\rho(w)$

Lemma A.2.1 (See [68], Lemma 2.2). *For every $r > 0$ there is a constant $c_r \geq 1$, depending only on r and the doubling constant for μ, such that*

$$c_r^{-1}\rho(z) \leq \rho(w) \leq c_r\rho(z), \quad \textit{for every } z \in \mathbb{C} \textit{ and } w \in D^r(z). \tag{A.2.2}$$

Namely, $c_r = (1-r)^{-1}$, for every $0 < r < 1$. In other words, $\rho(w)$ and $\rho(z)$ are equivalent on a disk.

Consider the distance d_ϕ induced by the metric $\rho^{-2}dz \otimes d\bar{z}$. Indeed, for any $z,w \in \mathbb{C}$,

$$d_\phi(z,w) = \inf_\gamma \int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))}dt, \tag{A.2.3}$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Lemma A.2.2 (See [64], Lemma 4). *There exists $\delta > 0$ such that for every $r > 0$ there exists $C_r > 0$ such that*

$$C_r^{-1}\frac{|z-w|}{\rho(z)} \leq d_\phi(z,w) \leq C_r\frac{|z-w|}{\rho(z)}, \quad \textit{for } w \in D^r(z), \tag{A.2.4}$$

and

$$C_r^{-1}\Big(\frac{|z-w|}{\rho(z)}\Big)^\delta \leq d_\phi(z,w) \leq C_r\Big(\frac{|z-w|}{\rho(z)}\Big)^{2-\delta}, \quad \textit{for } w \in \mathbb{C} \setminus D^r(z), \tag{A.2.5}$$

Now we can state the following pointwise estimate for the Bergman kernel.

Lemma A.2.3. *(1) There exist $C, \epsilon > 0$ such that*

$$|K(w,z)| \le C\frac{e^{\phi(w)+\phi(z)}}{\rho(w)\rho(z)}e^{-\left(\frac{|z-w|}{\rho(z)}\right)^{\epsilon}}, \quad w,z \in \mathbb{C}, \tag{A.2.6}$$

(2) There exists some $r_0 > 0$ such that for $z \in \mathbb{C}$ and $w \in D^{r_0}(z)$, we have

$$|K(w,z)| \simeq \frac{e^{\phi(w)+\phi(z)}}{\rho(z)^2}. \tag{A.2.7}$$

(3) $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z \to \infty$, where $k_{p,z} := \frac{K_z}{\|K_z\|_{p,\phi}}$ is the normalized Bergman kernel of F^p_ϕ.

(4) For any $1 \le p \le \infty$, we have that

$$\|K_z\|_{p,\phi} \simeq e^{\phi(z)}\rho(z)^{2/p-2}. \tag{A.2.8}$$

Proof. See Theorem 1.1 and Proposition 2.11 of [65] respectively for parts (1) and (2), Lemma 2.3 of [43] for part (3), and Proposition 2.9 of [68] for part (4). □

Given a sequence $\{a_j\}_{j=1}^\infty \subset \mathbb{C}$, and $r > 0$, we call $\{a_j\}_{j=1}^\infty$ an r-lattice if $\{D^r(a_j)\}_{j=1}^\infty$ covers $\mathbb{C}$ and the disks of $\{D^{r/5}(a_j)\}_{j=1}^\infty$ are pairwise disjoint. Moreover, for an r-lattice $\{a_j\}_{j=1}^\infty$, and a real number $m > 1$, there exists an integer N such that

$$1 \le \sum_{j=1}^{\infty} \chi_{D^{mr}(a_j)(z)} \le N \tag{A.2.9}$$

where χ_E is the characteristic function of a subset E of $\mathbb{C}$. For $f, e \in L^2_\phi$, the tensor product $f \otimes e$ as a rank one operator on L^2_ϕ is defined by

$$f \otimes e(g) = \langle g, e\rangle f, \quad g \in L^2_\phi. \tag{A.2.10}$$

Lemma A.2.4. *Given $r > 0$, there is some constant $C > 0$ such that if Γ is an r-lattice in $\mathbb{C}$, and if $\{e_a : a \in \Gamma\}$ is an orthonormal set in L^2_ϕ, then*

$$\left\|\sum_{a\in\Gamma} k_{2,a} \otimes e_a\right\|_{L^2_\phi \to L^2_\phi} \le C, \tag{A.2.11}$$

where $k_{2,a} := \frac{K_a}{\|K_a\|_{2,\phi}}$ is the normalized Bergman kernel.

Proof. Note that $\{\lambda_a = \langle g, e_a\rangle_{2,\phi}\}_{a\in\Gamma} \in l^2$. Then similar to the proof of Lemma 2.4 in [45],

$$\left\|\sum_{a\in\Gamma} \lambda_a k_{2,a}\right\|_{2,\phi} \le C\|\{\lambda_a\}_{a\in\Gamma}\|_{l^2}, \tag{A.2.12}$$

where the constant C only depends on r. Then similar to the proof of Lemma 2.4 in [51], we have

$$\left\|\left(\sum_{a\in\Gamma} k_{2,a} \otimes e_a\right)(g)\right\|^2_{2,\phi} = \left\|\sum_{a\in\Gamma} \langle g, e_a\rangle k_{2,a}\right\|^2_{2,\phi} \le C\sum_{a\in\Gamma} |\langle g, e_a\rangle|^2 \le C\|g\|^2_{2,\phi}. \tag{A.2.13}$$

□

We finish this section with a description of ρ for the canonical weights $|z|^m$ with $m > 0$.

Lemma A.2.5. *Let $\phi(z) = |z|^m$ with $m > 0$. Then $d\mu = \Delta\phi dA$ is a doubling measure. Moreover, there is an $R > 0$ such that*

$$\rho(z) \simeq |z|^{1-m/2}$$

for $|z| > R$. In particular, when $m \geq 2$, ρ is bounded.

Proof. Note that $\Delta\phi(z) = m^2|z|^{m-2}$. To show that $d\mu$ is a doubling weight, it is enough to prove that for any $x \geq 0$ and $r > 0$,

$$\int_{D(x,2r)} |z|^{m-2} dA(z) \leq C \int_{D(x,r)} |z|^{m-2} dA(z), \tag{A.2.14}$$

where the constant C is independent of x and r.

We consider $r > \frac{x}{100} \geq 0$ first. Then $D(x,2r) \subset D(0, x+2r)$, so that

$$\int_{D(x,2r)} d\mu(\xi) \leq \int_{|\xi|\leq x+2r} |\xi|^{m-2} dA(\xi) \leq \int_{|\xi|\leq 102r} |\xi|^{m-2} dA(\xi) \leq C_1 r^m. \tag{A.2.15}$$

On the other hand, if $m \geq 2$,

$$\int_{D(x,r)} d\mu(\xi) \geq \int_{D(x,r)\cap\{\operatorname{Re}\xi\geq x\}} d\mu(\xi) \geq \int_{D(0,r)\cap\{\operatorname{Re}\xi\geq 0\}} d\mu(\xi) \geq C_2 r^m. \tag{A.2.16}$$

From (A.2.15) and (A.2.16) we obtain (A.2.14) for $m \geq 2$ and $r > \frac{x}{100}$.

Now we suppose $0 < r < \frac{x}{100}$. Then

$$\begin{aligned} D(x,2r) &\subset \{te^{i\theta} : x - 2r < t < x + 2r, |\theta| < \arcsin \frac{2r}{x}\}, \\ D(x,r) &\supset \{te^{i\theta} : x - c_1 r < t < x + c_2 r, |\theta| < \arcsin \frac{r}{2x}\}, \end{aligned}$$

where c_1 and c_2 are positive constants independent of x are r. Hence,

$$\begin{aligned} \int_{D(x,2r)} d\mu \leq \int_{x-2r}^{x+2r} r^{m-1} dr \int_{-\arcsin\frac{2r}{x}}^{\arcsin\frac{2r}{x}} d\theta &\simeq \frac{r}{x}[(x+2r)^m - (x-2r)^m] \\ \simeq \frac{r}{x} r x^{m-1} = r^2 x^{m-2}, \end{aligned} \tag{A.2.17}$$

where the constant in the inequalities $\simeq$ are all independent of x and r. Similarly,

$$\begin{aligned} \int_{D(x,r)} d\mu &\geq \int_{x-c_1 r}^{x+c_2 r} r^{m-1} dr \int_{-\arcsin\frac{r}{2x}}^{\arcsin\frac{r}{2x}} d\theta \\ &\simeq \frac{r}{x}[(x+c_2 r)^m - (x-c_1 r)^m] \simeq r^2 x^{m-2}. \end{aligned} \tag{A.2.18}$$

Using (A.2.17) and (A.2.18), we obtain (A.2.14).

For $0 < m < 2$, and $r > \frac{x}{100}$,

$$\int_{D(x,r)} |\xi|^{m-2} dA(\xi) = \int_{D(0,r)} |\xi + x|^{m-2} dA(\xi) \geq \int_{D(0,r)} |\xi|^{m-2} dA(\xi) \geq C_3 r^m. \tag{A.2.19}$$

From (A.2.15) and (A.2.19) we obtain (A.2.14) for $0 < m < 2$ and $r > \frac{x}{100}$.

Now notice that using (A.2.17) and (A.2.18) and when x is large enough,

$$\int_{(x,x^{-\frac{m-2}{2}})} |\xi|^{m-2} dA(\xi) \simeq 1. \tag{A.2.20}$$

This, together with the doubling property implies that there exists $R > 0$ large enough, such that for the Fock space $F^2_{|z|^m}$,

$$\rho(z) \simeq |z|^{-\frac{m-2}{2}} = |z|^{1-\frac{m}{2}} \tag{A.2.21}$$

for $|z| \geq R$. $\square$

A.3 The space IDA

The goal of this section is to prove the IDA decomposition Theorem A.1.1. Before proving the theorem, we need to see some definitions and lemmas.

Lemma A.3.1. *Suppose* $1 \leq q < \infty$. *Then for* $f \in L^q_{loc}$, $z \in \mathbb{C}$, *and* $r > 0$, *there is* $h \in H(D^r(z))$ *such that*

$$\left(\widehat{|f-h|^q}_r(z)\right)^{1/q} = G_{q,r}(f)(z), \tag{A.3.1}$$

and for $s < r$,

$$\sup_{w \in D^s(z)} |h(w)| \leq C\|f\|_{L^q(D^r(z),dA)}, \tag{A.3.2}$$

where the constant C *is independent of* f *and* r.

Proof. This proof is similar to the proof of Lemma 3.3 in [49]. Taking $h = 0$,

$$G_{q,r}(f)(z) \leq \left(\widehat{|f|^q}_r(z)\right)^{1/q} < \infty. \tag{A.3.3}$$

Then for $j = 1, 2, \cdots$, pick $h_j \in H(D^r(z))$ such that

$$\left(\widehat{|f-h_j|^q}_r(z)\right)^{1/q} \to G_{q,r}(f)(z) \quad \text{as } j \to \infty. \tag{A.3.4}$$

Hence for sufficiently large j,

$$\left(\widehat{|h_j|^q}_r(z)\right)^{1/q} \leq C\{\left(\widehat{|f-h_j|^q}_r(z)\right)^{1/q} + \left(\widehat{|f|^q}_r(z)\right)^{1/q}\} \leq C\left(\widehat{|f|^q}_r(z)\right)^{1/q}. \tag{A.3.5}$$

Thus, we can find a subsequence $\{h_{j_k}\}_{k=1}^\infty$ and a function $h \in H(D^r(z))$ such that $\lim_{k\to\infty} h_{j_k}(w) = h(w)$ for $w \in D^r(z)$. By (A.3.4),

$$G_{q,r}(f)(z) \leq \left(\widehat{|f-h|^q}_r(z)\right)^{1/q} \leq \liminf_{k\to\infty} \left(\widehat{|f-h_{j_k}|^q}_r(z)\right)^{1/q} = G_{q,r}(f)(z) \tag{A.3.6}$$

where in the RHS inequality we have used Fatou's Lemma. This gives us (A.3.1).
Now for $w \in D^s(z)$, by the mean value Theorem,

$$|h(w)| \leq \left(\widehat{|h|^q}_s(z)\right)^{1/q} \leq C\left(\widehat{|h|^q}_r(z)\right)^{1/q} \leq \left(\widehat{|f|^q}_r(z)\right)^{1/q} = C\|f\|_{L^q(D^r(z),dA)}. \tag{A.3.7}$$

$\square$

Now we are ready to define f_1 and f_2 in Theorem A.1.1. Using (A.2.2) and the triangle inequality, there exists $m \in (0,1)$ such that $D^{mr}(w) \subset D^r(z)$, whenever $w \in D^{mr}(z)$. For $r > 0$, let $\{a_j\}_{j=1}^{\infty}$ be a mr-lattice, and let $J_z := \{j : z \in D^r(a_j)\}$, so that $|J_z| = \sum_{j=1}^{\infty} \chi_{D^r(a_j)}(z) \leq N$, for some integer N. Let $\eta : \mathbb{C} \to [0,1]$ be the following smooth function with bounded derivatives.

$$\eta(z) = \begin{cases} 1 & \text{if } |z| \leq 1/2, \\ 0 & \text{if } |z| \geq 1. \end{cases} \tag{A.3.8}$$

For each $j \geq 1$ we define $\eta_j(z) = \eta(\frac{z-a_j}{mr\rho(a_j)})$. We can normalize η_j such that $\int_{\mathbb{C}} \eta_j dA = 1$, for each $j \geq 1$. Define $\psi_j(z) = \frac{\eta_j(z)}{\sum_{k=1}^{\infty} \eta_k(z)}$. Then one can see that $\{\psi_j\}_{j=1}^{\infty}$ is a partition of unity subordinate to $\{D^{mr}(a_j)\}_{j\geq 1}$, satisfying the following properties:

$$\begin{aligned} \operatorname{Supp} \psi_j \subset D^{mr}(a_j), \quad \psi_j(z) \geq 0, \quad \sum_{j=1}^{\infty} \psi_j(z) = 1, \\ |\rho(a_j)\bar{\partial}\psi_j| \leq C, \quad \sum_{j=1}^{\infty} \bar{\partial}\psi_j(z) = 0, \end{aligned} \tag{A.3.9}$$

where the constant C may depend on r.

By Lemma A.3.1, for $j = 1, 2, \cdots$, we can pick $h_j \in H(D^r(a_j))$ such that

$$|\widehat{f-h_j}|^q{}_r(a_j) = \frac{1}{|D^r(a_j)|} \int_{D^r(a_j)} |f-h_j|^q dA = G_{q,r}(f)(a_j)^q. \tag{A.3.10}$$

For $1 \leq q < \infty$ and $f \in L^q_{loc}$, decompose $f = f_1 + f_2$ as

$$f_1(z) := \sum_{j=1}^{\infty} h_j(z)\psi_j(z), \quad f_2(z) := f(z) - f_1(z). \tag{A.3.11}$$

Lemma A.3.2. *Let $1 \leq q < \infty$, $f \in L^q_{loc}$, and $r > 0$. Decomposing $f = f_1 + f_2$ as in (A.3.11), we have $f_1 \in \mathcal{C}^2(\mathbb{C})$ and*

$$\rho(z)|\bar{\partial} f_1(z)| + \rho(z)(\widehat{|\bar{\partial} f_1|^q}_{mr})^{1/q} + (\widehat{|f_2|^q}_{mr})^{1/q} \leq C G_{q,R}(f)(z), \tag{A.3.12}$$

for some $R > r$ and $m \in (0,1)$.

Proof. Using the properties of h_j and ψ_j we can easily see that $f_1 \in \mathcal{C}^2(\mathbb{C})$. Let $z \in \mathbb{C}$, and $J_z = \{j : z \in D^r(a_j)\}$. We know that if $z \in D^r(a_j)$, then $\rho(z) \leq C\rho(a_j)$. Therefore, knowing $\sum_{j=1}^{\infty} \bar{\partial}\psi_j = 0$, using (A.3.9), the triangle inequality, and since $|h_j - h_1|^q$ is plurisubharmonic on $D^r(a_j)$,

$$\begin{aligned} \rho(z)|\bar{\partial} f_1(z)| &= \rho(z)\left|\bar{\partial}(\sum_{j=1}^{\infty} h_j(z)\psi_j(z))\right| \leq \rho(z)\sum_{j=1}^{\infty} |h_j(z) - h_1(z)||\bar{\partial}\psi_j(z)| \\ &\leq C \sum_{j\in J_z} \left[\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)} |h_j - h_1|^q dA\right]^{1/q} \rho(a_j)|\bar{\partial}\psi_j(z)| \\ &\leq C \sum_{j\in J_z} \left[\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)} \{|f-h_j|^q + |f-h_1|^q\} dA\right]^{1/q} \\ &\leq C \sum_{j\in J_z} \left(|\widehat{f-h_j}|^q{}_r(a_j)\right)^{1/q} + \left(|\widehat{f-h_1}|^q{}_r(a_j)\right)^{1/q} \\ &\leq C \sum_{j\in J_z} G_{q,r}(a_j) \leq C G_{q,s}(f)(z), \end{aligned} \tag{A.3.13}$$

for some $s > r$, where the last inequality can be shown similarly to Corollary 3.4 in [49], and using the fact that $|J_z|$ is finite.

Moreover, note that

$$\begin{aligned}\rho(z)\Big(\widehat{|\bar\partial f_1|^q}_{mr}(z)\Big)^{1/q} &= \rho(z)\bigg[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}|\bar\partial f_1(w)|^q dA(w)\bigg]^{1/q}\\ &\le C\bigg[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}\rho(w)^q|\bar\partial f_1(w)|^q dA(w)\bigg]^{1/q}\\ &\le C\bigg[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}G_{q,s}(f)(w)^q dA(w)\bigg]^{1/q}\\ &\le C\sup_{w\in D^{mr}(z)}G_{q,s}(f)(w)\le CG_{q,R}(f)(z), \qquad (A.3.14)\end{aligned}$$

for some $R > s$, where again for the last inequality we use Corollary 3.4 in [49]. Similarly, since $\sum_{j=1}^{\infty}\psi_j = 1$,

$$|f_2(w)|^q = |f(w)-\sum_{j=1}^{\infty}h_j(w)\psi_j(w)|^q \le \sum_{j=1}^{\infty}|f(w)-h_j(w)|^q|\psi_j(w)|^q. \qquad (A.3.15)$$

Hence, using $|\psi_j|\le 1$,

$$\begin{aligned}\Big(\widehat{|f_2|^q}_{mr}(z)\Big)^{1/q} &\le \sum_{j=1}^{\infty}\bigg[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}|f-h_j|^q|\psi_j|^q dA\bigg]^{1/q}\\ &\le C\sum_{j\in J_z}G_{q,r}(f)(a_j)\le CG_{q,R}(f)(z), \qquad (A.3.16)\end{aligned}$$

similar to the previous part for $\rho|\bar\partial f_1|$. Putting everything together, we can find a big enough $R > r$ such that (A.3.12) holds. □

Proof of Theorem A.1.1. First, we show that if (A.1.10) holds for some r, then it holds for any r. Let $R > 0$. For $0 < r < R$ take $t = \frac{r}{2C_2R}$ and take $z_1,\cdots,z_N$ in the unit disk $D(0,1)$ so that $D(0,1)\subset\cup_{j=1}^N D(z_j,t)$. Set $a_j(z) = z+R\rho(z)z_j$. Then

$$\begin{aligned}D^R(z) &\subset \cup_{j=1}^N D(z+R\rho(z)z_j, tR\rho(z))\subset\cup_{j=1}^N D(a_j(z),\frac{r}{2}\rho(a_j(z)))\\ &= \cup_{j=1}^N D^{r/2}(a_j(z)). \qquad (A.3.17)\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{C}}\Big(\widehat{|g|^q}_R(z)\Big)^s dA(z) &\le C\int_{\mathbb{C}}\sum_{j=1}^{N}\Big(\widehat{|g|^q}_{r/2}(a_j(z))\Big)^s dA(z)\\ &\le C\int_{\mathbb{C}}dA(z)\sum_{j=1}^{N}\frac{1}{|D^{cr}(a_j(z))|}\int_{D^{cr}(a_j(z))}\Big(\widehat{|g|^q}_r(u)\Big)^s dA(u)\\ &= C\int_{\mathbb{C}}\Big(\widehat{|g|^q}_r(u)\Big)^s dA(u)\sum_{j=1}^{N}\int_{\mathbb{C}}\chi_{D^{cr}(a_j(z))}(u)\frac{1}{|D^{cr}(a_j(z))|}dA(z)\\ &\le C\int_{\mathbb{C}}\Big(\widehat{|g|^q}_r(u)\Big)^s dA(u), \qquad (A.3.18)\end{aligned}$$

where for the second inequality take $c > 0$ such that $D^{cr}(a_j(z)) \subset \cap_{u\in D^{cr}(a_j(z))} D^r(u)$. Taking $s = p/q$ implies that (A.1.10) holds for some $r > 0$, if and only if it holds for any r.

Now assume that $f \in \mathrm{IDA}_r^{p,q,\alpha}$. That is, $f \in L^q_{loc}$ with $\|\rho^\alpha G_{q,r}(f)\|_{L^p} < \infty$. Decompose $f = f_1 + f_2$ as in Lemma A.3.2. Then $f_1 \in \mathcal{C}^2(\mathbb{C})$, and (A.3.12) holds. Multiplying both sides with ρ^α and taking the L^p-norm, we obtain (A.1.10).

□

A.4 Schatten class Hankel operators on doubling Fock spaces

Recall that for a bounded linear operator $T : H_1 \to H_2$ between two Hilbert spaces, the singular values λ_n are defined by

$$\lambda_n = \lambda_n(T) = \inf\{\|T - K\| : K : H_1 \to H_2, \operatorname{rank} K \le n\}. \tag{A.4.1}$$

The operator T is compact if and only if $\lambda_n \to 0$. Given $0 < p < \infty$, we say that T is in the Schatten class S_p and write $T \in S_p(H_1, H_2)$, if its singular value sequence $\{\lambda_n\}$ belongs to l^p. Then $\|T\|^p_{S_p} = \sum_{n=0}^\infty |\lambda_n|^p$ defines a norm when $1 \le p < \infty$ and a quasinorm when $0 < p < 1$. Moreover, for the quasi-Banach case, we have the triangle inequality.

$$\|T + S\|^p_{S_p} \le \|T\|^p_{S_p} + \|S\|^p_{S_p}, \quad \text{when } T, S \in S_p,\ 0 < p < 1, \tag{A.4.2}$$

which is called the Rotfel'd inequality. For a positive compact operators T on H and $p > 0$, $T \in S_p$ if and only if $T^p \in S_1$. Moreover, $\|T\|^p_{S_p} = \|T^p\|_{S_1}$. See [80] for further details on the properties of Schatten class operators, as well as the proof of the next two theorems.

Theorem A.4.1 (See [80], Theorem 1.26). *If T is a compact operator on H and $p > 0$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{p/2} \in S_1$, if and only if $T^*T \in S_{p/2}$. Moreover,*

$$\|T\|^p_{S_p} = \||T|\|^p_{S_p} = \||T|^p\|_{S_1} = \|T^*T\|^{p/2}_{S_{p/2}}. \tag{A.4.3}$$

Consequently, $T \in S_p$ if and only if $|T| \in S_p$.

Theorem A.4.2 (See [80], Theorem 1.28). *Suppose T is a compact operator on H and $p \ge 1$. Then T is in S_p if and only if*

$$\sum |\langle Te_n, \sigma_n\rangle|^p < \infty, \tag{A.4.4}$$

for all orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$. If T is positive, we also have

$$\|T\|_{S_p} = \sup\Big\{\Big[\sum |\langle Te_n, \sigma_n\rangle|^p\Big]^{1/p} : \quad \{e_n\} \text{ and } \{\sigma_n\} \text{ are orthonormal}\Big\}. \tag{A.4.5}$$

Given a locally finite positive Borel measure μ on $\mathbb{C}$, we define the Toeplitz operator T_μ with symbol μ as

$$T_\mu f(z) = \int_{\mathbb{C}} f(w)\overline{K_z(w)}e^{-2\phi(w)}d\mu(w). \tag{A.4.6}$$

Moreover, for every $r > 0$, the r-averaging transform of μ is defined by

$$\hat{\mu}_r(z) := \frac{\mu(D^r(z))}{|D^r(z)|} \simeq \frac{\mu(D^r(z))}{\rho(z)^2}. \tag{A.4.7}$$

Theorem A.4.3 (See [68], Theorem 4.1). *Let μ be a locally finite positive Borel measure on $\mathbb{C}$, and let $0 < p < \infty$. Then the following are equivalent.*

(1) $T_\mu \in S_p(F^2_\phi)$,

(2) There is $r_0 > 0$ *such that any r-lattice* $\{z_j\}_{j\geq 1}$ *with* $r \in (0, r_0)$ *satisfies* $\{\hat{\mu}_r(z_j)\}_{j\geq 1} \in l^p$,

(3) There is an r-lattice $\{z_j\}_{j\geq 1}$ *such that* $\{\hat{\mu}_r(z_j)\}_{j\geq 1} \in l^p$,

(4) There is $r > 0$ *such that* $\hat{\mu}_r \in L^p(\mathbb{C}, d\sigma)$,

Moreover, $\|T_\mu\|^p_{S_p} \simeq \|\hat{\mu}_r\|_{L^p(\mathbb{C},d\sigma)}$, *where* $d\sigma = dA/\rho^2$.

The rest of this section is devoted to the proof of the Schatten class membership of the Hankel operators Theorem A.1.2. For this purpose, let $a \in \mathbb{C}$ and $r > 0$. Let $A^2(D^r(a), e^{-2\phi}dA)$ be the weighted Bergman space containing the holomorphic functions in $L^2(D^r(a), e^{-2\phi}dA)$. Let $P_{a,r} : L^2(D^r(a), e^{-2\phi}dA) \to A^2(D^r(a), e^{-2\phi}dA)$ be the orthogonal projection, and for $f \in L^2(D^r(a), e^{-2\phi}dA)$, extend $P_{a,r}(f)$ to $\mathbb{C}$ by setting

$$P_{a,r}(f)_{|\mathbb{C}\setminus D^r(a)} = 0. \tag{A.4.8}$$

One can check that for $f, g \in L^2_\phi$,

$$P^2_{a,r}(f) = P_{a,r}(f), \quad \text{and} \quad \langle f, P_{a,r}(g)\rangle = \langle P_{a,r}(f), g\rangle. \tag{A.4.9}$$

Moreover, for $h \in F^2_\phi$,

$$P_{a,r}(h) = \chi_{D^r(a)}h, \quad \text{and} \quad \langle h, \chi_{D^r(a)}f - P_{a,r}(f)\rangle = 0. \tag{A.4.10}$$

Proof of Theorem A.1.2. Here we borrow an idea from the proof of Proposition 6.8 in [39] and the proof of Theorem 1.1 in [51]. First we show that (2) $\Longrightarrow$ (1). Let $f \in \mathrm{IDA}^{p,2,-2/p}_r$. Then by Theorem A.1.1, $f = f_1 + f_2$ with

$$\rho^{1-2/p}|\bar{\partial} f_1| + \rho^{1-2/p}(\widehat{|\bar{\partial} f_1|^2}_r)^{1/2} + \rho^{-2/p}(\widehat{|f_2|^2}_r)^{1/2} \in L^p \tag{A.4.11}$$

Applying the definition,

$$\rho^{1-2/p}(z)(\widehat{|\bar{\partial} f_1|^2}_r(z))^{1/2} = \rho^{1-2/p}(z)\Big\{\frac{1}{|D^r(z)|}\int_{D^r(z)} |\bar{\partial} f_1|^2 dA\Big\}^{1/2}, \tag{A.4.12}$$

and

$$\rho^{-2/p}(z)(\widehat{|f_2|^2}_r(z))^{1/2} = \rho^{-2/p}(z)\Big\{\frac{1}{|D^r(z)|}\int_{D^r(z)} |f_2|^2 dA\Big\}^{1/2}, \tag{A.4.13}$$

Set $\Phi := \rho|\bar{\partial} f_1|$ or $\Phi = |f_2|$, and $\mu := |\Phi|^2$. First, if $\Phi = \rho|\bar{\partial} f_1|$,

$$\hat{\mu}_r(z) := \frac{\mu(D^r(z))}{|D^r(z)|} = \frac{1}{|D^r(z)|}\int_{D^r(z)} |\Phi|^2 dA = \frac{1}{|D^r(z)|}\int_{D^r(z)} \rho^2|\bar{\partial} f_1|^2 dA. \tag{A.4.14}$$

We claim that for $f \in \mathrm{IDA}^{p,2,-2/p}_r$, $\hat{\mu}_r \in L^{p/2}(\mathbb{C}, d\sigma)$. Note that

$$\begin{aligned}\|\hat{\mu}_r\|^{p/2}_{L^{p/2}(\mathbb{C},d\sigma)} &= \int_{\mathbb{C}} |\hat{\mu}_r|^{p/2} dA/\rho^2 \\ &= \int_{\mathbb{C}} \frac{1}{|D^r(z)|^{p/2}}\Big[\int_{D^r(z)} \rho^2(w)|\bar{\partial} f_1(w)|^2 dA(w)\Big]^{p/2} \frac{dA(z)}{\rho(z)^2}.\end{aligned} \tag{A.4.15}$$

Since $f \in \mathrm{IDA}_r^{p,2,-2/p}$, we have $\rho^{1-2/p}(\widehat{|\bar{\partial} f_1|^2}_r)^{1/2} \in L^p$ and thus

$$\int_{\mathbb{C}} \rho^{p-2} \Big\{ \frac{1}{|D^r(z)|} \int_{D^r(z)} |\bar{\partial} f_1|^2 dA \Big\}^{p/2} dA(z) < \infty. \tag{A.4.16}$$

Recall that in (A.4.15), $w \in D^r(z)$, and therefore there is a constant C such that $\rho(w) \leq C\rho(z)$. Hence,

$$\|\hat{\mu}_r\|_{L^{p/2}(\mathbb{C},d\sigma)}^{p/2} \leq \int_{\mathbb{C}} \frac{C\rho(z)^{p-2}}{|D^r(z)|^{p/2}} \Big\{ \int_{D^r(z)} |\bar{\partial} f_1|^2 dA \Big\}^{p/2} dA(z) \asymp \text{LHS of (A.4.16)} < \infty. \tag{A.4.17}$$

Thus, we can conclude that $\hat{\mu}_r \in L^{p/2}(\mathbb{C}, d\sigma)$, for $\mu = \rho^2 |\bar{\partial} f_1|^2$. Now, using Theorem A.4.3, $T_\mu \in S_{p/2}(F_\phi^2)$.

Consider the multiplication $M_\Phi : F_\phi^2 \to L_\phi^2$ defined by $M_\Phi f := \Phi f$. Then M_Φ is bounded for $\Phi = \rho|\bar{\partial} f_1|$ or $\Phi = |f_2|$. For $h, g \in F_\phi^2$,

$$\langle M_\Phi^* M_\Phi g, h\rangle_{2,\phi} = \langle M_\Phi g, M_\Phi h\rangle_{2,\phi} = \int_{\mathbb{C}} (\Phi g)\overline{\Phi h} e^{-2\phi} dA = \langle T_{|\Phi|^2} g, h\rangle_{2,\phi}. \tag{A.4.18}$$

so, $M_\Phi^* M_\Phi = T_{|\Phi|^2} \in S_{p/2}$, and thus $M_\Phi \in S_p$. Moreover,

$$\|M_\Phi\|_{S_p} \simeq \|M_\Phi^* M_\Phi\|_{S_{p/2}} \simeq \|T_\mu\|_{S_{p/2}} \simeq \|\hat{\mu}_r\|_{L^{p/2}(\mathbb{C},d\sigma)}. \tag{A.4.19}$$

By equations (3.13) and (3.17) in [63], and using Fock-Carleson measures for F_ϕ^2, we can see that

$$\|H_{f_1} g\|_{2,\phi} \leq \|\rho g \bar{\partial} f_1\|_{2,\phi}, \quad \text{and} \quad \|H_{f_2} g\|_{2,\phi} \leq \|g f_2\|_{2,\phi}. \tag{A.4.20}$$

Therefore,

$$\|H_{f_1}\|_{S_p} \lesssim \|M_\Phi\|_{S_p} \simeq \|\hat{\mu}_r\|_{L^{p/2}(\mathbb{C},d\sigma)} \lesssim \|\rho^{1-2/p}(\widehat{|\bar{\partial} f_1|^2}_r)^{1/2}\|_{L^p} \asymp \|f\|_{\mathrm{IDA}_r^{p,2,-2/p}}. \tag{A.4.21}$$

To complete the proof, it remains to note that when $\mu = |f_2|^2$, we have

$$\begin{aligned} \|\hat{\mu}_r\|_{L^{p/2}(\mathbb{C},d\sigma)}^{p/2} &= \int_{\mathbb{C}} \frac{1}{|D^r(z)|^{p/2}} \Big[\int_{D^r(z)} |f_2|^2 dA \Big]^{p/2} \frac{dA(z)}{\rho(z)^2} \\ &= \int_{\mathbb{C}} \Big[\frac{\rho(z)^{-2/p}}{|D^r(z)|^{1/2}} \Big\{ \int_{D^r(z)} |f_2|^2 dA \Big\}^{1/2} \Big]^p dA(z) \\ &= \|\rho^{-2/p}(\widehat{|f_2|^2}_r)^{1/2}\|_{L^p}, \end{aligned} \tag{A.4.22}$$

so that

$$\|H_{f_2}\|_{S_p} \lesssim \|f\|_{\mathrm{IDA}_r^{p,2,-2/p}}.$$

Consequently, $\|H_f\|_{S_p} \lesssim \|H_{f_1}\|_{S_p} + \|H_{f_2}\|_{S_p} \lesssim \|f\|_{\mathrm{IDA}_r^{p,2,-2/p}}$, and so $H_f \in S_p(F_\phi^2, L_\phi^2)$.

To show (1) $\implies$ (2) for $p \geq 1$, we proceed as follows. Recall that $\{a_j\}_{j=1}^\infty$ is an r-lattice if $\{D^r(a_j)\}_{j=1}^\infty$ covers $\mathbb{C}$ and $D^{r/5}(a_j) \cap D^{r/5}(a_k) = \emptyset$ for $j \neq k$. Let Γ be an r-lattice, and let $\{e_a : a \in \Gamma\}$ be an orthonormal basis of L_ϕ^2. Define linear operators T and B by

$$T = \sum_{a\in\Gamma} k_{2,a} \otimes e_a, \quad \text{and} \quad B = \sum_{a\in\Gamma} g_a \otimes e_a, \tag{A.4.23}$$

where

$$g_a = \begin{cases} \frac{\chi_{D^r(a)} H_f(k_{2,a})}{\|\chi_{D^r(a)} H_f(k_{2,a})\|} & \text{if } \|\chi_{D^r(a)} H_f(k_{2,a})\| \neq 0, \\ 0 & \text{if } \|\chi_{D^r(a)} H_f(k_{2,a})\| = 0. \end{cases} \tag{A.4.24}$$

Since $\|g_a\| \leq 1$ and $\langle g_a, g_b\rangle = 0$ when $a \neq b$, $\|B\|_{L^2_\phi \to L^2_\phi} \lesssim 1$, where the bounding constant depends on (A.2.9). Moreover, by Lemma A.2.4, we can see that $\|T\| \leq C$ for some constant C. Let $H_f \in S_p$. So in particular, H_f is compact. We know from Lemma 2.3 that $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z \to \infty$, where $k_{p,z} = K_z/\|K_z\|_{p,\phi}$ is the normalized Bergman kernel for F^p_ϕ. By compactness of H_f we obtain that

$$\lim_{z\to\infty} \|\chi_{D^r(z)} H_f(k_{2,z})\|_{L^2_\phi} = 0. \tag{A.4.25}$$

Note that

$$\begin{aligned}\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_a\rangle &= \langle \chi_{D^r(a)} H_f \sum_{b\in\Gamma} k_{2,b} \otimes e_b(e_a), \sum_{d\in\Gamma} g_d \otimes e_d(e_a)\rangle \\ &= \langle \chi_{D^r(a)} H_f(k_{2,a}), g_a\rangle = \|\chi_{D^r(z)} H_f(k_{2,z})\|_{L^2_\phi},\end{aligned} \tag{A.4.26}$$

and

$$\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_b\rangle = 0, \quad a \neq b. \tag{A.4.27}$$

Thus, $B^* M_{\chi_{D^r(a)}} H_f T$ is a compact positive operator on L^2_ϕ. By Theorem A.4.2, and since we are dealing with the case of $p \geq 1$,

$$\sum_{a\in\Gamma} |\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_a\rangle| \leq \|B^* M_{\chi_{D^r(a)}} H_f T\|^p_{S_p} \leq C\|H_f\|^p_{S_p}, \tag{A.4.28}$$

as $\|B\| \leq 1$, $\|M_{\chi_{D^r(a)}}\| \leq 1$, and $\|T\| \leq C$. Recall that

$$G_{2,r}(f)(a) = \inf\left\{\left(\frac{1}{|D^r(a)|}\int_{D^r(a)} |f-h|^2 dA\right)^{1/2} : h \in H(D^r(a))\right\}, \tag{A.4.29}$$

and for $1 \leq p < \infty$, $\|K_z\|_{p,\phi} \asymp e^{\phi(z)}\rho(z)^{2/p-2}$. Moreover, recalling Lemma A.2.3 there exists $r_0 > 0$ such that for $w \in D^{r_0}(z)$,

$$|K(w,z)| \asymp \frac{e^{\phi(w)+\phi(z)}}{\rho(z)^2}. \tag{A.4.30}$$

Thus, for $w \in D^{r_0}(z)$,

$$|k_{p,z}(w)|e^{-\phi(w)} = \frac{|K(w,z)|}{\|K_z\|_{p,\phi}} e^{-\phi(w)} \asymp \frac{e^{\phi(w)+\phi(z)}e^{-\phi(w)}}{\rho(z)^2 e^{\phi(z)}} \rho(z)^{-2/p+2} = \rho(z)^{-2/p} > 0, \tag{A.4.31}$$

and we can conclude that $\frac{P(fk_{2,z})}{k_{2,z}} \in H(D^r(z))$. Hence,

$$G_{2,r}(f)(a) \leq \left[\frac{1}{|D^r(a)|}\int_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 dA\right]^{1/2}. \tag{A.4.32}$$

Moreover,

$$\begin{aligned}\|\chi_{D^r(a)} H_f(k_{2,a})\|_{L^2_\phi} &= \left[\int_{D^r(a)} |fk_{2,a} - P(fk_{2,a})|^2 e^{-2\phi} dA\right]^{1/2} \\ &= \left[\int_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 |k_{2,a}|^2 e^{-2\phi} dA\right]^{1/2} \\ &\overset{(A.4.31)}{\asymp} \left[\int_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 \rho(a)^{-2} dA\right]^{1/2} \\ &\asymp \left[\frac{1}{|D^r(a)|}\int_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 dA\right]^{1/2},\end{aligned} \tag{A.4.33}$$

where in the last line we have used the equivalence $|D^r(z)| \asymp \rho(z)^2$. Hence,

$$G_{2,r}(f)(a) \lesssim \|\chi_{D^r(a)} H_f(k_{2,a})\|_{L^2_\phi}, \tag{A.4.34}$$

and therefore,

$$\begin{aligned}\sum_{a\in\Gamma} G_{2,r}(f)(a)^p &\lesssim \sum_{a\in\Gamma} \|\chi_{D^r(a)} H_f(k_{2,a})\|^p_{L^2_\phi} \\ &= \sum_{a\in\Gamma} |\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_a\rangle|^p \leq C\|H_f\|^p_{S_p}.\end{aligned} \tag{A.4.35}$$

Now note that

$$\begin{aligned}\|f\|^p_{\mathrm{IDA}^{p,2,-2/p}_r} &= \int_{\mathbb{C}} \rho^{-2} G_{2,r}(f)^p \, dA \\ &\leq \sum_{a\in\Gamma} \int_{D^r(a)} \rho(z)^{-2} G_{2,r}(f)(z)^p \, dA(z) \\ &\leq \sum_{a\in\Gamma} \sup_{z\in D^r(a)} \rho(z)^{-2} G_{2,r}(f)(z)^p |D^r(a)| \\ &= C \sum_{a\in\Gamma} \rho(a)^{-2} G_{2,r}(f)(a)^p \rho(a)^2 \\ &= C \sum_{a\in\Gamma} G_{2,r}(f)(a)^p \\ &\leq C\|H_f\|^p_{S_p}.\end{aligned} \tag{A.4.36}$$

Now since if Theorem A.1.1 holds for some $r > 0$, it holds for any r, we are done with the proof for $p \geq 1$.

Now we finish the proof of Theorem A.1.2 by showing that (1) $\implies$ (2) for $0 < p < 1$. Since $H_f \in S_p(F^2_\phi, L^2_\phi)$, it is in particular bounded. For $a \in \Gamma$ set

$$g_a = \begin{cases} \frac{\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})}{\|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\|} & \text{if } \|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\| \neq 0, \\ 0 & \text{if } \|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\| = 0. \end{cases} \tag{A.4.37}$$

Then similar as before, $\|g_a\| \leq 1$, and $\langle g_a, g_b\rangle = 0$ for $a \neq b$. Let J be any finite subcollection of Γ, and $\{e_a\}_{a\in J}$ be an orthonormal set of L^2_ϕ. Define

$$A = \sum_{a\in J} e_a \otimes g_a : L^2_\phi \to L^2_\phi. \tag{A.4.38}$$

Then A is of finite rank and $\|A\| \leq 1$. Similarly define

$$T = \sum_{a\in J} k_{2,a} \otimes e_a : L^2_\phi \to F^2_\phi. \tag{A.4.39}$$

Then as before, since Γ is an r-lattice and thus separated, there is a constant C such that $\|T\| \leq C$. Then,

$$A H_f T = \sum_{a,\tau\in J} \langle H_f k_{2,\tau}, g_a\rangle e_a \otimes e_\tau = Y + Z, \tag{A.4.40}$$

where

$$Y = \sum_{a\in J} \langle H_f k_{2,a}, g_a\rangle e_a \otimes e_a, \quad , \quad Z = \sum_{a,\tau\in J, a\neq\tau} \langle H_f k_{2,\tau}, g_a\rangle e_a \otimes e_\tau. \tag{A.4.41}$$

Note that

$$\begin{aligned}
\langle H_f k_{2,a}, g_a\rangle_{2,\phi} &= \langle f k_{2,a} - P(f k_{2,a}), g_a\rangle_{2,\phi} = \langle \chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a}), g_a\rangle_{2,\phi} \\
&= \|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\|_{2,\phi} \\
&= \left[\int_{\mathbb{C}} |\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})|^2 e^{-2\phi} dA\right]^{1/2} \\
&= \left[\int_{D^r(a)} |f k_{2,a} - P_{a,r}(f k_{2,a})|^2 e^{-2\phi} dA\right]^{1/2} \\
&= \left[\int_{D^r(a)} |f - \frac{P_{a,r}(f k_{2,a})}{k_{2,a}}|^2 |k_{2,a}|^2 e^{-2\phi} dA\right]^{1/2} \\
&\asymp \left[\frac{1}{|D^r(a)|}\int_{D^r(a)} |f - \frac{P_{a,r}(f k_{2,a})}{k_{2,a}}|^2 dA\right]^{1/2} \\
&\geq G_{2,r}(f)(a). \qquad \text{(A.4.42)}
\end{aligned}$$

where in the line before the last line we have used (A.4.31) and $|D^r(a)| \asymp \rho(a)^2$. Thus,

$$\langle H_f k_{2,a}, g_a\rangle_{2,\phi} \geq C G_{2,r}(f)(a). \qquad \text{(A.4.43)}$$

Therefore, there exists some N, independent of f and J such that

$$\|Y\|_{S_p}^p = \sum_{a\in J} \langle H_f k_{2,a}, g_a\rangle_{2,\phi}^p \geq N \sum_{a\in J} G_{2,r}(f)(a)^p. \qquad \text{(A.4.44)}$$

On the other hand for $0 < p < 1$,

$$\|Z\|_{S_p}^p \leq \sum_{a,\tau\in J, a\neq\tau} \langle H_f k_{2,\tau}, g_a\rangle_{2,\phi}^p. \qquad \text{(A.4.45)}$$

Let $Q_{a,r} : L^2(D^r(a), dA) \to A^2(D^r(a), dA)$ be the Bergman projection. Then $f k_{2,\tau} - P_{a,r}(f k_{2,\tau})$ and $P_{a,r}(f k_{2,\tau}) - k_{2,\tau} Q_{a,r} f$ are orthogonal, and by Parseval's identity,

$$\|f k_{2,\tau} - P_{a,r}(f k_{2,\tau})\|_{L^2(D^r(a), e^{-2\phi} dA)} \leq \|f k_{2,\tau} - k_{2,\tau} Q_{a,r}(f)\|_{L^2(D^r(a), e^{-2\phi} dA)}. \qquad \text{(A.4.46)}$$

Note that by Lemma 2.3, there exist $C, \epsilon > 0$ such that

$$|K(w,z)| \leq C \frac{e^{\phi(w)+\phi(z)}}{\rho(w)\rho(z)} e^{-\left(\frac{|z-w|}{\rho(z)}\right)^{\epsilon}}. \qquad \text{(A.4.47)}$$

Besides, by Lemma 6.8 in [68], we can see that given $R > 0$ and any finite sequence $\{a_j\}_{j=1}^n$ of different points in $\mathbb{C}$, it can be partitioned into subsequences such that any different points a_j and a_k in the same subsequence satisfy

$$|a_j - a_k| \geq R \min(\rho(a_j), \rho(a_k)). \qquad \text{(A.4.48)}$$

So taking J to be a finite collection of Γ, we can choose an appropriately large $R > 0$ such that

$$|a - b| \geq R \min(\rho(a), \rho(b)), \quad \text{when } a, b \in J, a \neq b. \qquad \text{(A.4.49)}$$

Putting everything together,

$$\begin{aligned}
|\langle H_f k_{2,\tau}, g_a\rangle| &= |\langle f k_{2,\tau} - P(f k_{2,\tau}), g_a\rangle| \\
&= |\langle f k_{2,\tau} - P(f k_{2,\tau}), \frac{\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})}{\|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\|}\rangle| \\
&= \frac{|\langle \chi_{D^r(a)} f k_{2,\tau} - P_{a,r}(f k_{2,\tau}), \chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\rangle|}{\|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\|} \\
&\le \|f k_{2,\tau} - P_{a,r}(f k_{2,\tau})\|_{L^2(D^r(a), e^{-2\phi} dA)} \\
&\overset{(A.4.46)}{\le} \|f k_{2,\tau} - k_{2,\tau} Q_{a,r}(f)\|_{L^2(D^r(a), e^{-2\phi} dA)} \\
&\le \sup_{\xi \in D^r(a)} |k_{2,\tau}(\xi) e^{-\phi}| \|f - Q_{a,r}(f)\|_{L^2(D^r(a), dA)} \\
&\overset{(A.4.47)}{\le} \sup_{\xi \in D^r(a)} \frac{C}{\rho(\xi)} e^{-\left(\frac{|\tau-\xi|}{\rho(\tau)}\right)^\epsilon} \|f - Q_{a,r}(f)\|_{L^2(D^r(a), dA)} \\
&\simeq \frac{C}{\rho(a)} e^{-\left(\frac{|\tau-a|}{\rho(\tau)}\right)^\epsilon} \|f - Q_{a,r}(f)\|_{L^2(D^r(a), dA)} \\
&\simeq \frac{C}{|D^r(a)|^{1/2}} \left[\int_{D^r(a)} |f - Q_{a,r}(f)|^2 dA\right]^{1/2} e^{-\left(\frac{|\tau-a|}{\rho(\tau)}\right)^\epsilon} \\
&= C G_{2,r}(f)(a) e^{-\left(\frac{|\tau-a|}{\rho(\tau)}\right)^\epsilon}, \qquad (A.4.50)
\end{aligned}$$

where in the last line we used that $Q_{a,r}(f)$ is the orthogonal projection of f onto $H(D^r(a))$ in $L^2(D^r(a), dA)$. Hence, by the Hilbert projection theorem,

$$\|f - Q_{a,r}(f)\|_{L^2(D^r(a))} = \inf_{h \in H(D^r(a))} \|f - h\|_{L^2(D^r(a))},$$

which gives the definition of $G_{2,r}(f)(a)$. Therefore,

$$\begin{aligned}
\|Z\|_{S_p}^p &\overset{(A.4.45)}{\le} \sum_{a,\tau \in J, a \neq \tau} G_{2,r}(f)(a)^p e^{-\left(\frac{|\tau-a|}{\rho(\tau)}\right)^{p\epsilon}} \\
&\overset{(A.4.48)}{\le} \sum_{a \in J} G_{2,r}(f)(a)^p \sum_{\substack{\tau \in J \\ \tau \neq a}} e^{-\left(\frac{R \min(\rho(a), \rho(\tau))}{\rho(\tau)}\right)^{p\epsilon}} \\
&\simeq \sum_{a \in J} G_{2,r}(f)(a)^p e^{-R^{p\epsilon}}. \qquad (A.4.51)
\end{aligned}$$

Now we can pick some R large enough such that

$$\|Z\|_{S_p}^p \le \frac{N}{4} \sum_{a \in J} G_{2,r}(f)(a)^p. \qquad (A.4.52)$$

Using

$$\|Y\|_{S_p}^p \le 2\|A H_f T\|_{S_p}^p + 2\|Z\|_{S_p}^p, \qquad (A.4.53)$$

we have

$$N \sum_{a \in J} G_{2,r}(f)(a)^p \le 2\|A H_f T\|_{S_p}^p + \frac{N}{2} \sum_{a \in J} G_{2,r}(f)(a)^p, \qquad (A.4.54)$$

and since J is finite,

$$
\begin{aligned}
N\sum_{a\in J} G_{2,r}(f)(a)^p &\le 4\|AH_f T\|_{S_p}^p \\
&\le 4\|A\|_{L^2_\phi\to L^2_\phi}^p \|H_f\|_{S_p}^p \|T\|_{L^2_\phi\to L^2_\phi}^p \\
&\le C\|H_f\|_{S_p}^p. \qquad \text{(A.4.55)}
\end{aligned}
$$

Since C is independent of f and J,

$$\sum_{a\in\Gamma} G_{2,r}(f)(a)^p \le C\|H_f\|_{S_p}^p. \qquad \text{(A.4.56)}$$

The remaining of the proof is similar to (A.4.36) and we can conclude that for $0 < p < 1$,

$$\|f\|_{\mathrm{IDA}_r^{p,2,-2/p}} \le C\|H_f\|_{S_p}^p. \qquad \text{(A.4.57)}$$

□

A.5 Simultaneous membership of H_f and $H_{\bar{f}}$ in S_p

In this section, we first define the space of functions of integral mean oscillation IMO and prove some of its basic properties. In particular, we prove that H_f and $H_{\bar{f}}$ are simultaneously in $S_p(F^2_\phi, L^2_\phi)$ with $0 < p < \infty$ if and only if the symbol f satisfies a suitable IMO condition (see Theorem A.1.4).

Lemma A.5.1. *Let $0 < p < \infty$ and $r > 0$. Then for $f \in L^2_{loc}$, $f \in \mathrm{IMO}_r^{p,2,\alpha}$ if and only if there exists a continuous function $c(z)$ on $\mathbb{C}$ such that*

$$\rho^\alpha\left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f(w) - c(z)|^2 dA(w)\right)^{1/2} \in L^p \qquad \text{(A.5.1)}$$

Proof. This proof is similar to the proof of Proposition 2.4 in [52]. We can similarly extend the proposition to the case $0 < p < 1$, and the doubling weights by introducing ρ as the following. First note that if $f \in \mathrm{IMO}_r^{p,2,\alpha}$, then (A.5.1) holds with $c(z) = \hat{f}_r(z)$ which is continuous for $z \in \mathbb{C}$. Conversely, assume that (A.5.1) holds. By Minkowski inequality,

$$\rho^\alpha(z) MO_{2,r}(f)(z) \le \rho^\alpha\Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f - c(z)|^2 dA\Big)^{1/2} + \rho^\alpha|\hat{f}_r(z) - c(z)|. \qquad \text{(A.5.2)}$$

By Hölder's inequality,

$$\rho^\alpha|\hat{f}_r(z) - c(z)| \le \rho^\alpha\Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f - c(z)|^2 dA\Big)^{1/2} \in L^p \quad \text{by (A.5.1).} \qquad \text{(A.5.3)}$$

Hence, using (A.5.2) and (A.5.3) we can see that $f \in \mathrm{IMO}_r^{p,2,\alpha}$. □

Proposition A.5.2. *Let $0 < p \le \infty$, $r > 0$, and $f \in L^2_{loc}$. If for each $z \in \mathbb{C}$, there exist $h_1, h_2 \in H(D^r(z))$ such that*

$$\rho^\alpha(z)\Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f - h_1|^2 dA\Big)^{1/2} \in L^p,$$

and

$$\rho^\alpha(z)\Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |\bar{f} - h_2|^2 dA\Big)^{1/2} \in L^p, \qquad \text{(A.5.4)}$$

then $f \in \mathrm{IMO}_r^{p,2,\alpha}$.

Proof. The proof is a more detailed version of the proof of Proposition 2.5 in [52], extended to the case of doubling Fock spaces. For $f \in L^2_{loc}$, recall that

$$\Big(\widehat{|f|^2}_r(z)\Big)^{1/2} = \Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f|^2 dA\Big)^{1/2}. \tag{A.5.5}$$

By the triangle inequality and using (A.5.4),

$$\rho^\alpha\Big(\widehat{|\tfrac{f+\bar f}{2} - \tfrac{h_1+h_2}{2}|^2}_r(z)\Big)^{1/2} \le \rho^\alpha\Big(\widehat{|\tfrac{f-h_1}{2}|^2}_r(z)\Big)^{1/2} + \rho^\alpha\Big(\widehat{|\tfrac{\bar f-h_2}{2}|^2}_r(z)\Big)^{1/2} \in L^p. \tag{A.5.6}$$

Since $f+\bar f$ and ρ^α are real-valued, we can conclude that

$$\rho^\alpha\Big(\widehat{|\operatorname{Im}\tfrac{h_1+h_2}{2}|^2}_r(z)\Big)^{1/2} \in L^p. \tag{A.5.7}$$

As in the proof of the Proposition 2.5 in [52], we know that if $v : D^r(z) \to \mathbb{R}$ is harmonic, there exists a harmonic function u such that $u + iv \in H(D^r(z))$ and

$$\|u - u(z)\|_{L^q(D^r(z),dA)} \le C\|v\|_{L^q(D^r(z),dA)}, \tag{A.5.8}$$

for all $0 < q < \infty$.

Taking $q = 2$ in (A.5.8), and since $h_1 + h_2 \in H(D^r(z))$,

$$\Big(\widehat{|\operatorname{Re}\tfrac{h_1+h_2}{2} - \operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\Big)^{1/2} \le C\Big(\widehat{|\operatorname{Im}\tfrac{h_1+h_2}{2}|^2}_r(z)\Big)^{1/2}. \tag{A.5.9}$$

Thus,

$$\begin{aligned}\rho^\alpha\Big(\widehat{|\tfrac{f+\bar f}{2} - \operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\Big)^{1/2} &\le \rho^\alpha\Big(\widehat{|\tfrac{f+\bar f}{2} - \operatorname{Re}\tfrac{h_1+h_2}{2}|^2}_r(z)\Big)^{1/2} \\ &\qquad + \rho^\alpha\Big(\widehat{|\operatorname{Re}\tfrac{h_1+h_2}{2} - \operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\Big)^{1/2} \\ &\le \rho^\alpha\Big(\widehat{|\tfrac{f+\bar f}{2} - \tfrac{h_1+h_2}{2}|^2}_r(z)\Big)^{1/2} \\ &\qquad + C\rho^\alpha\Big(\widehat{|\operatorname{Im}\tfrac{h_1+h_2}{2}|^2}_r(z)\Big)^{1/2} \in L^p,\end{aligned} \tag{A.5.10}$$

where the first term in the last line is in L^p by (A.5.6), and the second term is in L^p by (A.5.7). Hence,

$$\rho^\alpha\Big(\widehat{|\tfrac{f+\bar f}{2} - \operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\Big)^{1/2} \in L^p. \tag{A.5.11}$$

Similar to (A.5.6), (A.5.7), and (A.5.8), and applying (A.5.4), we have

$$\rho^\alpha\Big(\widehat{|\tfrac{f-\bar f}{2} - \tfrac{h_1-h_2}{2}|^2}_r(z)\Big)^{1/2} \le \rho^\alpha\Big(\widehat{|\tfrac{f-h_1}{2}|^2}_r(z)\Big)^{1/2} + \rho^\alpha\Big(\widehat{|\tfrac{\bar f-h_2}{2}|^2}_r(z)\Big)^{1/2} \in L^p \tag{A.5.12}$$

Since $\frac{f-\bar f}{2}$ is completely imaginary, we can conclude that

$$\rho^\alpha\Big(\widehat{|\operatorname{Re}\tfrac{h_1-h_2}{2}|^2}_r(z)\Big)^{1/2} \in L^p. \tag{A.5.13}$$

We can exchange u and v in (A.5.8), and therefore,

$$\Big(\widehat{|\operatorname{Im}\tfrac{h_1-h_2}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\Big)^{1/2} \le C\Big(\widehat{|\operatorname{Re}\tfrac{h_1-h_2}{2}|^2}_r(z)\Big)^{1/2}. \tag{A.5.14}$$

Thus by (A.5.12) and (A.5.13),

$$\begin{aligned}\rho^\alpha\Big(\widehat{|\tfrac{f-\bar f}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\Big)^{1/2} &\le \rho^\alpha\Big(\widehat{|\tfrac{f-\bar f}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}|^2}_r(z)\Big)^{1/2}\\ &\qquad + \rho^\alpha\Big(\widehat{|\operatorname{Im}\tfrac{h_1-h_2}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\Big)^{1/2}\\ &\le \rho^\alpha\Big(\widehat{|\tfrac{f-\bar f}{2} - \tfrac{h_1-h_2}{2}|^2}_r(z)\Big)^{1/2}\\ &\qquad + C\rho^\alpha\Big(\widehat{|\operatorname{Re}\tfrac{h_1-h_2}{2}|^2}_r(z)\Big)^{1/2} \in L^p. \end{aligned} \tag{A.5.15}$$

Hence, analogous to (A.5.11),

$$\rho^\alpha\Big(\widehat{|\tfrac{f-\bar f}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\Big)^{1/2} \in L^p. \tag{A.5.16}$$

Choose $c(z) = \operatorname{Re}\frac{h_1+h_2}{2}(z) + i\operatorname{Im}\frac{h_1-h_2}{2}(z)$. Then by (A.5.11) and (A.5.16),

$$\rho^\alpha\Big(\widehat{|f - c(z)|^2}_r(z)\Big)^{1/2} \in L^p, \tag{A.5.17}$$

which is equivalent to

$$\rho^\alpha\Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f - c(z)|^2 dA\Big)^{1/2} \in L^p. \tag{A.5.18}$$

Thus by Lemma A.5.1 we can conclude that $f \in \mathrm{IMO}_r^{p,2,\alpha}$. □

Lemma A.5.3. *Let $0 < p \le \infty$. Then for $f \in L^2_{loc}$, $f \in \mathrm{IDA}_r^{p,2,\alpha}$ and $\bar f \in \mathrm{IDA}_r^{p,2,\alpha}$ if and only if $f \in \mathrm{IMO}_r^{p,2,\alpha}$.*

Proof. First assume that $f \in \mathrm{IDA}_r^{p,2,\alpha}$ and $\bar f \in \mathrm{IDA}_r^{p,2,\alpha}$. By the definition of $\mathrm{IDA}_r^{p,2,\alpha}$, this means that $\rho^\alpha G_{2,r}(f) \in L^p$ and $\rho^\alpha G_{2,r}(\bar f) \in L^p$. Moreover, by Lemma A.3.1, for each $z \in \mathbb{C}$, there exist functions $h_1, h_2 \in H(D^r(z))$ such that

$$G_{2,r}(f)(z) = \left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f - h_1|^2\, dA\right)^{1/2}$$

and

$$G_{2,r}(\bar f)(z) = \left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |\bar f - h_2|^2\, dA\right)^{1/2}.$$

Hence the hypotheses of Proposition A.5.2 are satisfied. Therefore $f \in \mathrm{IMO}_r^{p,2,\alpha}$.

Conversely, assume that $f \in \mathrm{IMO}_r^{p,2,\alpha}$. Since $\widehat{\bar f}_r(z) = \overline{\hat f_r(z)}$, we have $MO_{2,r}(\bar f)(z) = MO_{2,r}(f)(z)$. Thus $\bar f \in \mathrm{IMO}_r^{p,2,\alpha}$. Now, for each $z \in \mathbb{C}$, the function $w \mapsto \hat f_r(z)$ is constant on $D^r(z)$, and hence is holomorphic. Therefore, by the definition of $G_{2,r}$,

$$G_{2,r}(f)(z) \le \left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f - \hat f_r(z)|^2\, dA\right)^{1/2} = MO_{2,r}(f)(z).$$

Multiplying by ρ^α and taking the L^p-norm gives $\|f\|_{\mathrm{IDA}_r^{p,2,\alpha}} \le \|f\|_{\mathrm{IMO}_r^{p,2,\alpha}}$. The same argument applied to $\bar f$ gives $\|\bar f\|_{\mathrm{IDA}_r^{p,2,\alpha}} \le \|\bar f\|_{\mathrm{IMO}_r^{p,2,\alpha}} = \|f\|_{\mathrm{IMO}_r^{p,2,\alpha}}$. Hence

$$f, \bar f \in \mathrm{IDA}_r^{p,2,\alpha}.$$

This proves the equivalence. □

We can now give the proof of Theorem A.1.4, which shows that both H_f and $H_{\bar{f}}$ are in S_p if and only if $f \in \mathrm{IMO}_r^{p,2,-2/p}$, where $1 < p < \infty$.

Proof of Theorem A.1.4. By Theorem A.1.2, $H_f \in S_p$ if and only if $f \in \mathrm{IDA}_r^{p,2,-2/p}$ for some (equivalent any) $r > 0$. Similarly, $H_{\bar{f}} \in S_p$ if and only if $\bar{f} \in \mathrm{IDA}_r^{p,2,-2/p}$. An application of Lemma A.5.3 shows that this is equivalent to $f \in \mathrm{IMO}_r^{p,2,-2/p}$, for some (equivalent any) $r > 0$. Further, the norm estimates in (A.1.15) follow from (A.1.12) and (**??**). □

As mentioned in the introduction, we obtain the following result as a consequence of Theorem A.1.4.

Theorem A.5.4. *Let f be a non-constant entire function and F^2_ϕ be a doubling Fock space. Then $H_{\bar{f}}$ is not in $S_2(F^2_\phi, L^2_\phi)$.*

Proof. Since f is holomorphic, $H_f = 0$, and thus belongs to the Hilbert-Schmidt class. Applying Theorem A.1.4, it is enough to show that $f \notin \mathrm{IMO}_1^{2,2,-1}$. First note that $\bar{f}$ is harmonic on $D^1(z)$ and by the mean-value property of harmonic functions,

$$\widehat{f}_1(z) = \frac{1}{|D^1(z)|}\int_{D^1(z)} f\,dA = f(z).$$

By the Cauchy estimate,

$$\begin{aligned} MO_{2,1}(f)(z) &= \left(\frac{1}{|D^1(z)|}\int_{D^1(z)} |f(w) - f(z)|^2 dA(w)\right)^{1/2} \\ &\geq C|\partial f(z)|\rho(z). \end{aligned}$$

Hence,

$$\begin{aligned} \|f\|_{\mathrm{IMO}_1^{2,2,-1}} &= \int_{\mathbb{C}} \rho(z)^{-2} MO_{2,1}(f)(z)^2 dA(z) \\ &\geq C\int_{\mathbb{C}} \rho(z)^{-2}|\partial f(z)|^2\rho(z)^2 dA(z). \end{aligned}$$

So, since f is entire and non-constant, it follows that $f \notin \mathrm{IMO}_1^{2,2,-1}$, and thus $H_{\bar{f}}$ is not Hilbert-Schmidt.

□

A.6 Berger-Coburn phenomenon for doubling Fock spaces

This section contains the proofs of Theorems A.1.5 and A.1.7. We start with the proof of the Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators, that is, we show that for $f \in L^\infty$, H_f is Hilbert-Schmidt if and only if $H_{\bar{f}}$ is Hilbert-Schmidt.

Proof of Theorem A.1.5. Let $H_f \in S_2$. By the assumption, $f \in L^\infty$, and in particular $f \in L^2_{loc}$. Then by Theorem A.1.2, $f \in \mathrm{IDA}_r^{2,2,-1}$ for some (equivalent any) $r > 0$, and

$$\|f\|_{\mathrm{IDA}_r^{2,2,-1}} \simeq \|H_f\|_{S_2} < \infty. \tag{A.6.1}$$

Decompose $f = f_1 + f_2$ as in (A.1.10). Thus $f_1 \in \mathcal{C}^2(\mathbb{C})$ and

$$|\bar\partial f_1| + (\widehat{|\bar\partial f_1|^2}_r)^{1/2} + \rho^{-1}(\widehat{|f_2|^2}_r)^{1/2} \in L^2. \tag{A.6.2}$$

Then the definition

$$\rho^{-1}(z)(\widehat{|f_2|^2}_r(z))^{1/2} = \rho^{-1}(z)\Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f_2|^2 dA\Big)^{1/2} \tag{A.6.3}$$

implies that

$$\rho^{-1}(\widehat{|f_2|^2}_r)^{1/2} = \rho^{-1}(\widehat{|\bar f_2|^2}_r)^{1/2} \in L^2. \tag{A.6.4}$$

By (A.1.11) and (A.1.12), $H_{\bar f_2} \in S_2$. Indeed,

$$\begin{aligned}\|H_{\bar f_2}\|_{S_2} &\overset{(A.1.12)}{=} \|\bar f_2\|_{\mathrm{IDA}_r^{2,2,-1}} \overset{(A.1.11)}{\lesssim} \|\rho^{-1}(\widehat{|\bar f_2|^2}_r)^{1/2}\|_{L^2} \\ &\overset{(A.6.4)}{=} \|\rho^{-1}(\widehat{|f_2|^2}_r)^{1/2}\|_{L^2} \overset{(A.1.11)}{\lesssim} \|f\|_{\mathrm{IDA}_r^{2,2,-1}}.\end{aligned} \tag{A.6.5}$$

To show that $\|H_{\bar f_1}\|_{S_2} \lesssim \|f\|_{\mathrm{IDA}_r^{2,2,-1}}$, we need to follow a more complicated argument, inspired by the proof of Theorem 1.2 in [47]. Let $\{a_j\}_{j=1}^{\infty}$ be a fixed $m_1 r$-lattice for some $m_1 \in (0,1)$ and $r > 0$. Choose a partition of unity $\{\psi_j\}_{j=1}^{\infty}$ subordinate to $\{D^{m_1 r}(a_j)\}$ as in (A.3.9). By Lemma A.3.1 there exists $h_j \in H(D^r(a_j))$ such that

$$\Big(\widehat{|f - h_j|^2}_r(a_j)\Big)^{1/2} = G_{2,r}(f)(a_j), \quad \text{and} \sup_{z \in D^{m_1 r}(a_j)} |h_j(z)| \lesssim \|f\|_{L^\infty}. \tag{A.6.6}$$

Now we get back to the decomposition $f = f_1 + f_2$ as in (A.1.10) with $f_1 = \sum_{j=1}^{\infty} h_j \psi_j$. Without loss of generality we can assume $\psi_j = \bar\psi_j$ for all $j \geq 1$. Since we assumed that f is bounded, $f_1 \in L^\infty$ and moreover

$$\bar\partial \bar f_1 = \sum_{j=1}^{\infty} \bar h_j \bar\partial \psi_j + \sum_{j=1}^{\infty} \psi_j \bar\partial \bar h_j = F + H, \tag{A.6.7}$$

for $F = \sum_{j=1}^{\infty} \bar h_j \bar\partial \psi_j$ and $H = \sum_{j=1}^{\infty} \psi_j \bar\partial \bar h_j$. Similar to (A.3.13) one has

$$\begin{aligned}|F(z)| &= \rho^{-1}(z)\rho(z)|\sum_{j=1}^{\infty} \bar h_j \bar\partial \psi_j| = \rho^{-1}(z)\rho(z)|\sum_{j=1}^{\infty} \bar h_j \bar\partial \psi_j - \sum_{j=1}^{\infty} \bar h_1 \bar\partial \psi_j| \\ &\leq \rho^{-1}(z)\rho(z)\sum_{j=1}^{\infty} |\bar h_j(z) - \bar h_1(z)||\bar\partial \psi_j(z)| \leq C\rho^{-1}(z)G_{2,r}(f)(z).\end{aligned} \tag{A.6.8}$$

Besides,

$$\|H\|_{L^2} \leq \|\bar\partial \bar f_1\|_{L^2} + \|F\|_{L^2}. \tag{A.6.9}$$

By (A.6.8),

$$\|F\|_{L^2} \leq \|f\|_{\mathrm{IDA}_r^{2,2,-1}}. \tag{A.6.10}$$

Lemma 7.1 in [51] implies that

$$\|\bar\partial \bar f_1\|_{L^2} = \|\partial f_1\|_{L^2} \leq C\|\bar\partial f_1\|_{L^2} \leq C\|f\|_{\mathrm{IDA}_r^{2,2,-1}}, \tag{A.6.11}$$

where the last inequality is obtained by multiplying both sides of (A.3.12) with ρ^{-1}. Hence, we can conclude that

$$\|H\|_{L^2} \lesssim \|f\|_{\mathrm{IDA}_r^{2,2,-1}}. \tag{A.6.12}$$

Note that for $m_1, m_2 \in (0,1)$,

$$\|H_{\bar{f}_1}\|_{S_2}^2 \simeq \|\bar{f}_1\|^2_{\mathrm{IDA}_r^{2,2,-1}} \overset{(A.1.10)}{\leq} C\int_{\mathbb{C}} \Big[\big(\widehat{|\bar{\partial}\bar{f}_1|^2}_{mm_2r}\big)^{1/2}\Big]^2 dA$$
$$\lesssim \int_{\mathbb{C}} \Big[\big(\widehat{|F|^2}_{m_1m_2r}\big)^{1/2}\Big]^2 dA + \int_{\mathbb{C}} \Big[\big(\widehat{|H|^2}_{m_1m_2r}\big)^{1/2}\Big]^2 dA, \tag{A.6.13}$$

where for the last inequality we used the equivalence $\rho(w) \simeq \rho(z)$ for $w \in D^{m_1m_2r}(z)$ and (A.6.7). Note that using (A.6.8) one has

$$\int_{\mathbb{C}} \Big[\big(\widehat{|F|^2}_{m_1m_2r}\big)^{1/2}\Big]^2 dA \lesssim \|f\|^2_{\mathrm{IDA}_r^{2,2,-1}}, \tag{A.6.14}$$

and thus we are left to compute $\int_{\mathbb{C}} \Big[\big(\widehat{|H|^2}_{m_1m_2r}\big)^{1/2}\Big]^2 dA$. Let $z \in D^r(a_j) \cap D^r(a_k)$. Since $|\bar{\partial}(\bar{h}_k - \bar{h}_j)| = |\partial(h_k - h_j)|$, applying the Cauchy estimate for the boundary of the disk $D^{m_1m_2r}(z)$ of radius $m_1m_2r\rho(z)$ and Hölder's inequality, we obtain the following.

$$|\bar{\partial}(\bar{h}_k(z) - \bar{h}_j(z))| \leq \frac{C}{\rho(z)}\Big\{\int_{D^{m_1m_2r}(z)} |\bar{h}_k(w) - \bar{h}_j(w)|^2 dA\Big\}^{1/2}. \tag{A.6.15}$$

Using $|\bar{h}_k - \bar{h}_j|^2 = |(f - \bar{h}_k) - (f - \bar{h}_j)|^2 \leq |f - \bar{h}_k|^2 + |f - \bar{h}_j|^2$, and the fact that h_k and h_j are holomorphic, we get

$$|\bar{\partial}(\bar{h}_k(z) - \bar{h}_j(z))| \leq \frac{C}{\rho(z)}\Big(G_{2,m_1m_2r}(f)(a_k) + G_{2,m_1m_2r}(f)(a_j)\Big)$$
$$\leq \frac{C}{\rho(z)} G_{2,R}(f)(z), \tag{A.6.16}$$

for some $R > m_1m_2r$. Recalling H as in (A.6.7),

$$H + \sum_{j=1}^{\infty} \psi_j \bar{\partial}(\bar{h}_k - \bar{h}_j) = H + \sum_{j=1}^{\infty} \psi_j \bar{\partial}\bar{h}_k - H. \tag{A.6.17}$$

since $\{\psi_j\}_{j=1}^{\infty}$ is a partition of unity and therefore $\sum_{j=1}^{\infty} \psi_j = 1$,

$$\bar{\partial}\bar{h}_k = \sum_{j=1}^{\infty} \psi_j \bar{\partial}(\bar{h}_k - \bar{h}_j) + H. \tag{A.6.18}$$

Hence,

$$|\bar{\partial}\bar{h}_k(z)|^2 \lesssim \Big|\sum_{j=1}^{\infty} \psi_j(z)\bar{\partial}(\bar{h}_k(z) - \bar{h}_j(z))\Big|^2 + \big|H(z)\big|^2$$
$$\lesssim \sum_{j \in D^{m_1r}(a_j)} \psi_j(z)|\bar{\partial}(\bar{h}_k(z) - \bar{h}_j(z))|^2 + |H(z)|^2$$
$$\lesssim \Big(\rho^{-1}(z)G_{2,R}(f)(z)\Big)^2 + |H(z)|^2, \tag{A.6.19}$$

where the last inequality follows from (A.6.16). For $z \in D^{m_1r}(a_k)$, notice that $D^{m_1m_2r}(z) \subset D^{m_1r}(a_k)$ for some $m_2 \in (0,1)$. Then by subharmonicity,

$$|\bar{\partial}\bar{h}_k(z)|^2 \leq \frac{1}{|D^{m_1m_2r}(z)|}\int_{D^{m_1m_2r}(z)} |\bar{\partial}\bar{h}_k(w)|^2 dA(w)$$
$$\overset{(A.6.19)}{\lesssim} \frac{1}{|D^{m_1m_2r}(z)|}\int_{D^{m_1m_2r}(z)} \Big[\big|\rho^{-1}(w)G_{2,R}(f)(w)\big|^2 + \big|H(w)\big|^2\Big] dA(w)$$
$$\lesssim (\rho^{-1}(z))^2 G_{2,\tilde{R}}(f)(z)^2 + \widehat{|H|^2}_{m_1m_2r}(z), \tag{A.6.20}$$

for some $\tilde{R} > R$.

Now for $z \in \mathbb{C}$, there exists $w' \in \overline{D^{m_1 m_2 r}(z)}$ such that

$$\begin{aligned}\Big[(\widehat{|H|^2}_{m_1 m_2 r}(z))^{1/2}\Big]^2 &\leq \max\{|H(w)|^2 : w \in \overline{D^{m_1 m_2 r}(z)}\}\\ &= \Big|\sum_{k=1}^{\infty} \psi_k(w')\bar{\partial}\bar{h}_k(w')\Big|^2, \end{aligned} \tag{A.6.21}$$

where the first inequality comes from integration on a bounded domain. Note that $G_{2,\tilde{R}}(f)(w')^2 \lesssim G_{2,s}(f)(z)^2$ for some $s > \tilde{R}$, and

$$\Big[(\widehat{|H|^2}_{m_1 m_2 r}(w'))^{1/2}\Big]^2 \leq \Big[(\widehat{|H|^2}_{m_1 r}(z))^{1/2}\Big]^2, \tag{A.6.22}$$

and we can conclude that

$$\begin{aligned}\Big[(\widehat{|H|^2}_{m_1 m_2 r}(z))^{1/2}\Big]^2 &\overset{(A.6.21)}{\leq} \Big|\sum_{k=1}^{\infty} \psi_k(w')\bar{\partial}\bar{h}_k(w')\Big|^2\\ &\overset{(A.6.20)}{\lesssim} \sum_{k, \psi_k(w') \neq 0} \psi_k(w')\Big\{(\rho^{-1}(w'))^2 G_{2,\tilde{R}}(f)(w')^2\\ &\qquad\qquad + \widehat{|H|^2}_{m_1 m_2 r}(w')\Big\}\\ &\overset{(A.6.22)}{\lesssim} \Big(\rho^{-1}(z))G_{2,s}(f)(z)\Big)^2 + \widehat{|H|^2}_{m_1 r}(z). \end{aligned} \tag{A.6.23}$$

Hence as mentioned in (A.6.13), and applying Theorem A.1.1,

$$\begin{aligned}\|H_{\bar{f}_1}\|_{S_2}^2 &\lesssim \|f\|^2_{\mathrm{IDA}_s^{2,2,-1}} + \int_{\mathbb{C}} \Big[(\widehat{|H|^2}_{m_1 m_2 r}(z))^{1/2}\Big]^2 dA(z)\\ &\lesssim \|f\|^2_{\mathrm{IDA}_s^{2,2,-1}} + \int_{\mathbb{C}} \Big(\rho^{-1}(z))G_{2,s}(f)(z)\Big)^2 dA(z) + \int_{\mathbb{C}} \widehat{|H|^2}_{m_1 r}(z) dA(z)\\ &\lesssim \|f\|^2_{\mathrm{IDA}_s^{2,2,-1}} + \int_{\mathbb{C}} |H|^2 dA\\ &\lesssim \|f\|^2_{\mathrm{IDA}_s^{2,2,-1}}, \end{aligned} \tag{A.6.24}$$

where in the last line we have used (A.6.12).
This together with (A.6.5) implies that

$$\|H_{\bar{f}}\|_{S_2} \lesssim \|H_f\|_{S_2}. \tag{A.6.25}$$

We are done since the proof is symmetric for f and $\bar{f}$. □

We make the following remark related to the Berger-Coburn phenomenon for other values of p.

Remark A.6.1. For $1 < p < \infty$ we say that ω is a Muckenhoupt weight and write $\omega \in A_p$ if there is a constant $C > 0$ such that for any disk $B \subset \mathbb{C}$, we have

$$\left(\frac{1}{|B|}\int_B \omega dA\right)\left(\frac{1}{|B|}\int_B \omega^{-q/p} dA\right)^{p/q} \leq C < \infty, \tag{A.6.26}$$

where q is the Hölder conjugate of p and $|B|$ is the Lebesgue measure of B. As shown in [34], if $\omega \in A_p$ and $1 < p < \infty$, then the Ahlfors-Beurling operator

$$\mathcal{I}(f)(z) = p.v. -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2} dA(z) \tag{A.6.27}$$

is bounded on $L^p(\omega)$. Hence, similarly to the proof of Lemma 7.1 in [51], we can show that when f is bounded,

$$\|\partial f\|_{L^p(\omega)} \le C\|\bar{\partial} f\|_{L^p(\omega)}, \tag{A.6.28}$$

where C is a constant depending only on p.

To generalize Theorem A.1.5 to the other values of $1 < p < \infty$, our approach would require only one additional ingredient that $\omega = \rho^{p-2}$ is a Muckenhoupt weight (see (A.6.11)). However, we have not been able to prove this condition and also note that Lemma A.2.1 does not seem to help because the constants c_r in (A.2.2) are not bounded in general.

Next, we consider the case $0 < p \le 1$. Recently Xia [76] defined the following simple function

$$f(z) := \begin{cases} \frac{1}{z} & \text{if } |z| \ge 1, \\ 0 & \text{if } |z| < 1. \end{cases} \tag{A.6.29}$$

and used it to show that the Berger-Coburn phenomenon does not hold for trace class Hankel operators on the classical Fock space. Hu and Virtanen [50] noticed that when $0 < p \le 1$ the same example shows that there is no Berger-Coburn for Schatten class Hankel operators on generalized Fock spaces. Here we use Xia's example again to show that the Berger-Coburn phenomenon fails for certain Schatten classes $S_p(F^2_\phi, L^2_\phi)$ on doubling Fock spaces. As will be shown below, this failure holds for a large class of doubling weights, including the canonical weights $\phi(z) = |z|^m$ with $m \ge 2$. For general doubling weights not covered by our results, it remains open whether the Berger-Coburn phenomenon fails in the range $0 < p \le 1$.

Proof of Theorem A.1.7. To prove the theorem, we use Theorems A.1.2 and A.1.4. The idea is to find a bounded function f with $f \in \mathrm{IDA}_r^{p,2,-2/p}$ such that $f \notin \mathrm{IMO}_r^{p,2,-2/p}$ for some (equivalent any) $r > 0$. Note that by remark 1 in [64], there are constants $C, \eta > 0$, and $0 \le \beta < 1$ such that for $|z| > 1$,

$$C^{-1}|z|^{-\eta} \le \rho(z) \le C|z|^{\beta}. \tag{A.6.30}$$

Let f be as in (A.6.29). By Theorem A.1.1, the definition of $\mathrm{IDA}_r^{p,2,-2/p}$ is independent of r. So for simplicity, we set $r = 1$. It is easy to see that for a large enough $R > 0$, and $|z| \ge R$, f is holomorphic in $D^1(z) = D(z, \rho(z))$, and hence trivially $G_{2,1}(f_\beta)(z) = 0$. Indeed, one can see that for $|z| \ge R$, $D^1(z) \cap D(0,1) = \emptyset$. Moreover, for all $|z| < R$, there is a constant C such that

$$G_{2,1}(f)(z) < C, \tag{A.6.31}$$

as f is bounded in the bounded domain $D^1(z)$. Thus,

$$\begin{aligned} \|f\|^p_{\mathrm{IDA}_1^{p,2,-2/p}} &= \|\rho^{-2/p} G_{2,1}(f)\|^p_{L^p} = \int_{\mathbb{C}} \rho^{-2} G_{2,1}(f)^p dA \\ &\le C \int_{|z|<R} \rho^{-2} dA < \infty. \end{aligned} \tag{A.6.32}$$

Indeed, by Theorem 14 in [64], there is a smooth function ψ, where $\Delta\psi dA$ is doubling and $\Delta\psi \simeq \rho_{\psi}^{-2} \simeq \rho^{-2}$. Hence,

$$\int_{|z|<R} \rho^{-2} dA \simeq \int_{|z|<R} \Delta\psi dA < \infty, \tag{A.6.33}$$

as the doubling measures are locally finite. So by (A.6.32), $f \in \mathrm{IDA}_1^{p,2,-2/p}$, and Theorem A.1.2 implies that $H_f \in S_p$.

To show that $H_{\bar{f}} \notin S_p$, note that if $|z| \geq R$, $\bar{f}$ is harmonic on $D^1(z)$ and by the mean-value property of harmonic functions,

$$\widehat{\bar{f}}_1(z) = \frac{1}{|D^1(z)|}\int_{D^1(z)} \bar{f} dA = \bar{f}(z). \tag{A.6.34}$$

Moreover, by definition, $MO_{2,r}(f)(z) = MO_{2,r}(\bar{f})(z)$, and thus for $|z| \geq R$,

$$\begin{aligned} MO_{2,1}(f)(z) &= \left(\frac{1}{|D^1(z)|}\int_{D^1(z)} |\bar{f}(w) - \bar{f}(z)|^2 dA(w)\right)^{1/2} \\ &= \left(\frac{1}{|D^1(z)|}\int_{D^1(z)} |\frac{1}{\bar{w}} - \frac{1}{\bar{z}}|^2 dA(w)\right)^{1/2} \\ &= \left(\frac{1}{|D^1(z)|}\int_{D^1(z)} \frac{|w-z|^2}{|zw|^2} dA(w)\right)^{1/2}. \end{aligned} \tag{A.6.35}$$

For $w \in D^1(z)$, we can write $w = z + re^{i\theta}$ where $0 \leq r < \rho(z)$ and $0 \leq \theta < 2\pi$. Therefore,

$$\int_{D^1(z)} \frac{|w-z|^2}{|zw|^2} dA(w) = \frac{1}{|z|^2}\int_0^{\rho(z)} r^3 \int_0^{2\pi} \frac{d\theta dr}{|z+re^{i\theta}|^2} \tag{A.6.36}$$

Let $z = |z|e^{i\psi}$. Then

$$\int_0^{2\pi} \frac{d\theta}{|z+re^{i\theta}|^2} = \int_0^{2\pi} \frac{d\theta}{\left||z|+re^{i\theta}\right|^2} = \int_0^{2\pi} \frac{d\theta}{|z|^2 + r^2 + 2|z|r\cos\theta}. \tag{A.6.37}$$

Defining $y = \frac{r}{|z|}$,

$$\begin{aligned} \frac{1}{|z|^2}\int_0^{\rho(z)}\int_0^{2\pi} \frac{r^3 d\theta dr}{|z|^2 + r^2 + 2|z|r\cos\theta} &= \frac{1}{|z|^2}\int_0^{\frac{\rho(z)}{|z|}}\int_0^{2\pi} \frac{y^3|z|^4 d\theta dy}{|z|^2 + y^2|z|^2 + 2|z|^2 y\cos\theta} \\ &= \int_0^{\frac{\rho(z)}{|z|}} \frac{y^3}{2y}\int_0^{2\pi} \frac{d\theta dy}{\frac{1+y^2}{2y} + \cos\theta}. \end{aligned}$$

Let $x = \frac{1+y^2}{2y}$. Then

$$\int_0^{2\pi} \frac{d\theta}{\frac{1+y^2}{2y} + \cos\theta} = \int_0^{2\pi} \frac{d\theta}{x + \cos\theta}. \tag{A.6.38}$$

Taking $t = \tan\frac{\theta}{2}$, we have $\theta = 2\tan^{-1}(t)$, $d\theta = \frac{2dt}{1+t^2}$, and $\cos\theta = \frac{1-t^2}{1+t^2}$. Since the cosine function is even, one has

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{x+\cos\theta} &= 2\int_0^{\pi} \frac{d\theta}{x+\cos\theta} = 2\int_0^{\infty} \frac{2dt}{x(1+t^2)+1-t^2} \\ &= 2\int_0^{\infty} \frac{2dt}{t^2(x-1)+(x+1)} = \frac{4}{x+1}\int_0^{\infty} \frac{dt}{1+(\frac{x-1}{x+1})t^2}. \end{aligned} \tag{A.6.39}$$

Taking $u = \sqrt{\frac{x-1}{x+1}}t$, we obtain

$$\begin{aligned}\frac{4}{x+1}\int_0^\infty \frac{dt}{1+(\frac{x-1}{x+1})t^2} &= \frac{2}{x+1}\int_0^\infty \frac{2\sqrt{\frac{x+1}{x-1}}du}{u^2+1} = \frac{2}{x+1}\sqrt{\frac{x+1}{x-1}}\int_0^\infty \frac{2du}{u^2+1}\\ &= \frac{2}{x+1}\sqrt{\frac{x+1}{x-1}}\int_0^\pi d\theta = \frac{2\pi}{\sqrt{(x-1)(x+1)}}\\ &= \frac{2\pi}{\sqrt{(\frac{1+y^2}{2y}-1)(\frac{1+y^2}{2y}+1)}} = \frac{4\pi y}{(1-y)(1+y)}. \end{aligned}\tag{A.6.40}$$

Thus,

$$\int_0^{\rho(z)/|z|}\frac{y^3}{2y}\int_0^{2\pi}\frac{d\theta dy}{\frac{1+y^2}{2y}+\cos\theta} = \int_0^{\rho(z)/|z|}\frac{y^2}{2}\frac{4\pi y dy}{(1-y^2)}. \tag{A.6.41}$$

Let $v = y^2$, then

$$\begin{aligned}\int_0^{\rho(z)/|z|}\frac{y^2}{2}\frac{4\pi y dy}{(1-y^2)} &= \int_0^{(\rho(z)/|z|)^2}\frac{v}{2}\frac{4\pi\sqrt{v}dv}{(1-v)}\frac{dv}{2\sqrt{v}}\\ &= \pi\int_0^{(\rho(z)/|z|)^2}\frac{v-1+1}{1-v}dv = \pi\int_0^{(\rho(z)/|z|)^2}\left(-1+\frac{1}{1-v}\right)dv\\ &= \pi\left[-(\frac{\rho(z)}{|z|})^2 - \ln(1-(\frac{\rho(z)}{|z|})^2)\right]. \end{aligned}\tag{A.6.42}$$

Hence,

$$MO_{2,1}(f)(z) = \frac{\pi}{\rho(z)}\left[-(\frac{\rho(z)}{|z|})^2 - \ln(1-(\frac{\rho(z)}{|z|})^2)\right]^{1/2}. \tag{A.6.43}$$

Therefore,

$$\begin{aligned}\|f\|^p_{\mathrm{IMO}_1^{p,2,-2/p}} &= \int_{\mathbb{C}}\rho(z)^{-2}\mathrm{MO}_{2,1}(f)(z)^p dA(z)\\ &\simeq \int_{\mathbb{C}}\frac{1}{\rho(z)^2}\frac{1}{\rho(z)^p}\left[-(\frac{\rho(z)}{|z|})^2 - \ln(1-(\frac{\rho(z)}{|z|})^2)\right]^{p/2} dA(z). \end{aligned}\tag{A.6.44}$$

Note that taking $x = -(\rho(z)/|z|)^2$, the term in the bracket is $x - \ln(1+x) = x - x + x^2/2 - x^3/3 + \cdots$, and hence the most contribution comes from the term $x^2/2$. Thus,

$$\begin{aligned}\|f\|^p_{\mathrm{IMO}_1^{p,2,-2/p}} &\simeq \int_{\mathbb{C}}\frac{1}{\rho(z)^{p+2}}\frac{\rho(z)^{2p}}{|z|^{2p}}dA(z) = \int_{\mathbb{C}}\frac{1}{\rho(z)^{2-p}}\frac{1}{|z|^{2p}}dA(z)\\ &\geq \int_{|z|\geq R}\frac{1}{|z|^{\beta(2-p)}}\frac{1}{|z|^{2p}}dA(z) \simeq \int_R^\infty \frac{rdr}{r^{2p+\beta(2-p)}}\\ &= \int_R^\infty r^{1-2p-\beta(2-p)}dr \simeq r^{2-2p-\beta(2-p)}\Big|_{r=R}^{\infty}. \end{aligned}\tag{A.6.45}$$

Note that $2-2p-\beta(2-p) = (2-p)\Big(\frac{2(1-p)}{2-p}-\beta\Big)$, and since $0 < p \leq 1$, the integral diverges when $\beta \leq \frac{2(1-p)}{2-p}$. So, when $p = 1$, β must be zero. When p is very close to zero, β can get very close to 1, implying that Xia's example is a counterexample for any doubling measure.

□

Remark A.6.2. One could also hope to modify (A.6.29) so that it takes into account the growth condition of ρ; see (A.1.17). However, there are no holomorphic functions that behave like $|z|^c$ at infinity unless c is an integer. Indeed, suppose that f is holomorphic in the complement of a disk centered at the origin, and assume that $\sup_\theta |f(re^{i\theta})| \simeq r^c$ as $r \to \infty$. Then $c \in \mathbb{Z}$. To see this, for such a function f, set $g(z) = z^k f(1/z)$, where $k \geq c$ is an integer. Then g has a removable singularity at the origin since $|g(re^{i\theta})| = \mathcal{O}(r^{k-c})$ as $r \to 0$. So g is bounded at zero, and hence g has a power series $\sum a_k z^k$ near the origin, which implies that $c \in \mathbb{Z}$.

Finally, notice that substituting (A.6.29) in the proof of Theorem A.1.7 by the functions $f(z) = 1/z^n$ for $|z| > 1$ and $f(z) = 0$ elsewhere actually works worse when the integer n is larger than 1.

Proof of Corollary A.1.8. We apply Theorem A.1.7 to the canonical doubling weights $\phi(z) = |z|^m$ with $m > 0$. Recall that by Lemma A.2.5, there is some $R > 0$ such that $\rho(z) \leq |z|^{1-m/2}$ for $|z| \geq R$. Therefore, $\beta_\phi = 1 - m/2$. We can conclude that the Berger-Coburn phenomenon fails for $S_p(F^2_{|z|^m}, F^2_{|z|^m})$ if $1 - m/2 \leq \frac{1-p}{1-p/2}$, which is equivalent to $m \geq \frac{p}{1-p/2}$. In particular, if $m \geq 2$, then the phenomenon fails for all Schatten classes S_p with $0 < p \leq 1$. $\square$

Appendix B

Toeplitz operators on large vector-valued Fock spaces[1]

Abstract

We characterize boundedness and compactness of Toeplitz operators on large vector-valued Fock spaces with weights introduced by Dall'Ara's [Adv. Math., 285 (2015) 1706–1740] in terms of generalized Berezin transforms, averaging functions, and Carleson measures. We also introduce the operator-valued Berezin transform and averaging functions to describe the Schatten class properties of Toeplitz operators. This class of weights extends doubling weights on the complex plane to the setting of $\mathbb{C}^n$.

B.1 Introduction and main results

The classical Fock space of square-integrable entire functions with respect to a Gaussian measure, originally introduced in quantum mechanics, has long served as an important object of study in functional analysis, complex analysis, and mathematical physics. From the complex analysis point of view, it provides a canonical example of a reproducing kernel Hilbert space, which is a central setting to the study of Toeplitz and Hankel operators. From a geometric perspective, the Fock space can be interpreted as the space of holomorphic sections of a Hermitian line bundle over $\mathbb{C}^n$, where the Gaussian weight naturally induces a Kähler metric. This interpretation establishes deep connections with complex differential geometry and the framework of geometric quantization.

Motivated by these and other applications (such as sampling and interpolation), considerable effort has been devoted to extending classical Fock spaces to those defined via more general scalar weights, which typically satisfy certain growth or curvature conditions, allowing for a broader and more nuanced geometric and analytic framework (see, e.g., [7, 46, 48, 81]). In his seminal work, Christ [24] studied doubling Fock spaces over the complex plane $\mathbb{C}$, associated with suitable subharmonic weight functions. His analysis established a profound link between the study of Bergman kernels and partial differential equations through geometric and analytic techniques. This framework was subsequently extended by Dall'Ara [30] to

---

[1] This appendix reproduces the paper *"Toeplitz operators on large vector-valued Fock spaces"* by H. Arroussi, G. Asghari, and J. A. Virtanen, available as an arXiv preprint (arXiv:2504.15239, 2025).

Apart from formatting adjustments, this appendix coincides with the current version of the arXiv preprint.

higher-dimensional complex spaces $\mathbb{C}^n$, which is the underlying space for the operators that we consider in our work.

Compared to the long-term interest in the scalar-valued Fock spaces, less attention has been paid to the vector-valued case, where functions take values in finite- or infinite-dimensional Hilbert spaces, where additional difficulties are caused by the complexity of kernel functions taking values in operator algebras and the interplay between geometry and operator theory in infinite dimensions.

Let $\mathcal{H}$ be a separable Hilbert space. We denote by $L^2_\phi(\mathbb{C}^n, \mathcal{H})$ the space of all measurable $\mathcal{H}$-valued functions on $\mathbb{C}^n$ for which

$$\|f\|_{2,\phi}^2 = \int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}}^2 e^{-2\phi(z)}\, dA(z) < \infty, \tag{B.1.1}$$

where dA is the Lebesgue measure on $\mathbb{C}^n$ and ϕ is an *admissible weight* introduced by Dall'Ara [30] (see §B.2 below). When equipped with the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} \langle f(z), g(z)\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z),$$

$L^2_\phi(\mathbb{C}^n, \mathcal{H})$ becomes a Hilbert space. We say that $f : \mathbb{C}^n \to \mathcal{H}$ is holomorphic if for every continuous linear functional $\phi \in \mathcal{H}^*$, the scalar-valued function $\phi \circ f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic in the usual sense (see, e.g., §3.10 in [42]). The *large vector-valued Fock space* $F^2_\phi(\mathcal{H})$ is defined by

$$F^2_\phi(\mathbb{C}^n, \mathcal{H}) = L^2_\phi(\mathbb{C}^n, \mathcal{H}) \cap H(\mathbb{C}^n, \mathcal{H}),$$

where $H(\mathbb{C}^n, \mathcal{H})$ stands for the space of all $\mathcal{H}$-valued holomorphic functions on $\mathbb{C}^n$.

Our large Fock spaces $F^2_\phi(\mathcal{H})$ generalize the concept of doubling Fock spaces on $\mathbb{C}$ to higher dimensions and allow for vector-valued functions. Note that the class of admissible weights contains all weights ϕ for which there are constants $0 < m < M$ such that

$$m\omega_o \leq dd^c\phi \leq M\omega_o, \tag{B.1.2}$$

where $\omega_o = \frac{1}{2} dd^c|z|^2$ is the Euclidean Kähler form in $\mathbb{C}^n$, $d = \partial + \bar{\partial}$ is the exterior derivative, and $d^c = \frac{i}{2}(\bar{\partial} - \partial)$; see, e.g., [73] for further details of the weights satisfying (B.1.2). When $n = 1$, the condition in (B.1.2) is equivalent to $m \leq \Delta\phi \leq M$, where Δ is the Laplacian.

It is easy to see that $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is a closed subspace of $L^2_\phi(\mathbb{C}^n, \mathcal{H})$ and hence a Hilbert space. Indeed, given $z \in \mathbb{C}^n$, by Lemma B.2.8, there is a constant $C(z)$ such that

$$\|f(z)\|_{\mathcal{H}} \leq C(z)\|f\|_{2,\phi}, \quad \text{for } f \in F^2_\phi(\mathbb{C}^n, \mathcal{H})$$

(see Remark B.2.9), which implies that the point evaluation map $f \mapsto f(z)$ is a bounded linear homomorphism from $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ to $\mathcal{H}$ and uniformly bounded in bounded domains of $\mathbb{C}^n$. Since locally uniform limits of holomorphic functions are holomorphic, we conclude that $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is a closed subspace of $L^2_\phi(\mathbb{C}^n, \mathcal{H})$.

Reproducing kernel Hilbert spaces, such as the classical Fock space of square-integrable complex-valued holomorphic functions, have been an exciting area of research in analysis and operator theory. One of the basic properties of the reproducing kernel in the scalar setting, i.e, spaces of complex-valued functions, is that the reproducing kernel itself is holomorphic and belongs to the space. In the previous work on vector-valued Fock spaces, such as [19], reproducing kernels were not always explicitly defined and we introduce them here for the first time in Definition B.1.1. However, a notion of *operator-valued positive definite kernel*, introduced by Aronszajn in 1950 [5], has been used in the finite-dimensional Euclidean setting in machine learning [66, 67]. Our definition agrees with this and also with the definition of *matrix-valued reproducing kernel Hilbert spaces*; see e.g., [35]. What will be different, though, from the scalar case is that, although the reproducing kernel reproduces the elements of the Hilbert space in the sense of the integral equation (B.1.3), it is not an element of the space itself.

Definition B.1.1. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{H}^*$ be its dual, and let $\mathcal{F}$ be a Hilbert space of functions $f : \mathbb{C}^n \to \mathcal{H}$. We say that $\mathcal{F}$ is a *vector-valued reproducing kernel Hilbert* space if there is a map $K^{\mathcal{H}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{H} \otimes \mathcal{H}^*$ with $K^{\mathcal{H}}(z,w)^* \cong K^{\mathcal{H}}(w,z)$, and

$$f(z) = \int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w) f(w)\, dV(w) \quad \text{for } f \in \mathcal{F}, \tag{B.1.3}$$

where dV is a measure on $\mathbb{C}^n$.

Note that here $\cong$ stands for the natural isomorphism $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{H}^* \otimes \mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Then there is a natural isomorphism $\mathcal{L}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$. In fact, using the map $B : \mathcal{H}^* \times \mathcal{H} \to \mathcal{L}(\mathcal{H})$ defined by $B(\lambda, w)(v) = \lambda(v)w$, and the universal property of the tensor products, we obtain a linear map $T_B : \mathcal{H}^* \otimes \mathcal{H} \to \mathcal{L}(\mathcal{H})$. This map is an isomorphism with inverse $S(L) = \sum_{i=1}^{\infty} e^i \otimes Le_i$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ and $\{e^i\}_{i=1}^{\infty}$ is the dual basis of $\mathcal{H}^*$. Therefore, the vector-valued reproducing kernel $K^{\mathcal{H}}$ can be viewed as a map $K^{\mathcal{H}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$.

Write $K_z^{\mathcal{H}}(\cdot) = K^{\mathcal{H}}(\cdot, z)$. In Definition B.1.1, consider $K^{\mathcal{H}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$ and $K^{\mathcal{H}}(z,w)^* = K^{\mathcal{H}}(w,z)$. Let us consider the inner product of $\mathcal{F}$ as

$$\langle f, g \rangle_{\mathcal{F}} = \int_{\mathbb{C}^n} \langle f(z), g(z) \rangle_{\mathcal{H}}\, dV(z).$$

It follows that $\langle f(z), h \rangle_{\mathcal{H}} = \langle f, K_z^{\mathcal{H}} h \rangle_{\mathcal{F}}$ for every $h \in \mathcal{H}$ and $z \in \mathbb{C}^n$. This can be seen as an analog to $f(z) = \langle f, K_z \rangle$ in the scalar setting.

For the rest of the paper, we assume that $dV = e^{-2\phi} dA$, which implies that $F_\phi^2(\mathbb{C}^n, \mathcal{H})$ is a vector-valued reproducing kernel Hilbert space and its reproducing kernel $K_z^{\mathcal{H}}$ is a map from $\mathbb{C}^n$ to $\mathcal{H} \otimes \mathcal{H}^*$. The reproducing kernel property takes the form

$$f(z) = \int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w) f(w) e^{-2\phi(w)}\, dA(w).$$

When $\mathcal{H} = \mathbb{C}$, we denote the scalar-valued weighted Fock space on $\mathbb{C}^n$ by $F_\phi^2(\mathbb{C}^n)$, which consists of all complex-valued holomorphic functions on $\mathbb{C}^n$ such that the norm defined in (B.1.1) is finite. Notice that the above integral is equivalent to the scalar reproducing kernel property, where the action of the reproducing kernel in the scalar case $F_\phi^2(\mathbb{C}^n)$ is given by the usual multiplication. Being an element of $\mathcal{H} \otimes \mathcal{H}^*$, the most general $K^{\mathcal{H}}(z,w)$ is of the form $\sum_{m,n=1}^{\infty} K_{mn}(z,w) e_m \otimes e^n$, where $K_{mn}(z,w)$ are some complex scalars. In fact, we will see in §B.2 that the reproducing kernel for $F_\phi^2(\mathbb{C}^n, \mathcal{H})$ is obtained by taking $K_{mn}(z,w) = \delta_{mn} K(z,w)$, where $K(z,w)$ is the reproducing kernel of $F_\phi^2(\mathbb{C}^n)$; that is,

$$K_w^{\mathcal{H}}(z) = K^{\mathcal{H}}(z,w) = \sum_{n=1}^{\infty} K(z,w) e_n \otimes e^n.$$

Define an integral operator $P : L^2_\phi(\mathbb{C}^n, \mathcal{H}) \to F^2_\phi(\mathbb{C}^n, \mathcal{H})$ by

$$P(f)(z) = \int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w) f(w) e^{-2\phi(w)} dA(w) = \int_{\mathbb{C}^n} f(w) K(z,w) e^{-2\phi(w)} dA(w), \tag{B.1.4}$$

which is shown to be the orthogonal projection of $L^2_\phi(\mathbb{C}^n, \mathcal{H})$ onto $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ in Lemma B.2.12. To define vectorial Toeplitz operators, we denote by $T_\phi(\mathcal{L}(\mathcal{H}))$ the space of holomorphic operator-valued functions $G : \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$ such that each $G(z)$ is positive and satisfies

$$K_z(\cdot) \|G(\cdot)\|_{\mathcal{L}(\mathcal{H})} \in L^2_\phi(\mathbb{C}^n), \quad z \in \mathbb{C}^n. \tag{B.1.5}$$

For $G \in T_\phi(\mathcal{L}(\mathcal{H}))$, the *vectorial Toeplitz operator* T_G is densely defined by

$$T_G f(z) = P(Gf)(z) = \int_{\mathbb{C}^n} G(w) f(w) K(z,w) e^{-2\phi(w)} dA(w),$$

for $f \in F^2_\phi(\mathbb{C}^n, \mathcal{H})$. For more details see Section B.2.

To characterize the boundedness and compactness of T_G, we define the Berezin transform $\tilde{G}$ by

$$\tilde{G}(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w), \quad z \in \mathbb{C}^n,$$

where

$$k_z = \frac{K_z}{\|K_z\|_{F^2_\phi(\mathbb{C}^n)}}, \quad z \in \mathbb{C}^n,$$

is the normalized Bergman kernel of $F^2_\phi(\mathbb{C}^n)$. For $r > 0$, the corresponding averaging function $\widehat{G}_r$ is defined by

$$\widehat{G}_r(z) = \frac{\int_{D^r(z)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w)}{|D^r(z)|} \simeq \frac{\int_{D^r(z)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w)}{\rho(z)^{2n}},$$

where $|D^r(z)|$ is the Lebesgue measure of the disk $D^r(z) = D(z, r\rho(z))$ and $\simeq$ is defined below in §B.1.

We say that G satisfies the Carleson condition if the inclusion map $I_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to L^2_\phi(\mathbb{C}^n, \mathcal{H}, \|G\|_{\mathcal{L}(\mathcal{H})} dA)$ is bounded, that is, there is a constant C such that

$$\left(\int_{\mathbb{C}^n} \|f(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) \right)^{1/2} \le C \|f\|_{2,\phi}, \quad \text{for } f \in F^2_\phi(\mathbb{C}^n, \mathcal{H}). \tag{B.1.6}$$

We say that G satisfies the vanishing Carleson condition if the embedding operator $I_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to L^2_\phi(\mathbb{C}^n, \mathcal{H}, \|G\|_{\mathcal{L}(\mathcal{H})} dA)$ is compact, that is, for any bounded sequence $\{f_j\}_{j=1}^\infty$ in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ that converges to zero uniformly on any compact subset of $\mathbb{C}^n$ as $j \to \infty$,

$$\lim_{j \to \infty} \int_{\mathbb{C}^n} \|f_j(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z) = 0. \tag{B.1.7}$$

In the scalar setting, the basic properties of Toeplitz operators are relatively well understood. Indeed, in the scalar case, boundedness, compactness, and Schatten class properties for Dall'Ara weights were described in [8], while their Fredholm properties are understood up to doubling weights (see [48]). In the vectorial case, see [33] for boundedness and compactness when the weights satisfy (B.1.2) and see [78] for logarithmic growth weights introduced by

Seip and Youssfi [74]. Regarding Schatten class properties, there are no characterizations for vectorial Toeplitz operators but there is a characterization for vectorial Hankel operators on Fock spaces with logarithmic growth weights (see [19]). The present work provides the first Schatten class characterization for vectorial Toeplitz operators on weighted Fock spaces and also provides descriptions of their boundedness and compactness.

Main results

The following three results describe boundedness, compactness, and Schatten class properties of Toeplitz operators acting on large vector-valued Fock spaces.

Theorem B.1.2. *Let $G \in T_\phi(\mathcal{L}(\mathcal{H}))$ and α be as in* (B.2.11). *Then the following conditions are equivalent:*

1. $T_G : F^2_\phi(\mathcal{H}) \to F^2_\phi(\mathcal{H})$ *is bounded;*
2. $\widetilde{G} \in L^\infty(\mathbb{C}^n, dA)$*;*
3. $\widehat{G}_\delta \in L^\infty(\mathbb{C}^n, dA)$ *for some (or any)* $0 < \delta \le \alpha$*;*
4. $\{\widehat{G}_\delta(z_k)\}_k$ *is a bounded sequence for some (or any) δ-lattice $\{z_k\}_k$ with* $0 < \delta \le \alpha$*;*
5. *G satisfies a Carleson condition.*

Moreover,

$$\|T_G\| \simeq \|\widetilde{G}\|_{L^\infty(\mathbb{C}^n,dA)} \simeq \|\widehat{G}_\delta\|_{L^\infty(\mathbb{C}^n,dA)} \simeq \|\{\widehat{G}_\delta(z_k)\}_k\|_{l^\infty}. \tag{B.1.8}$$

Theorem B.1.3. *Let $G \in T_\phi(\mathcal{L}(\mathcal{H}))$ and α be as in* (B.2.11). *Then the following conditions are equivalent:*

1. $T_G : F^2_\phi(\mathcal{H}) \to F^2_\phi(\mathcal{H})$ *is compact;*
2. $\widetilde{G}(z) \to 0$ *as* $|z| \to \infty$*;*
3. $\widehat{G}_\delta(z) \to 0$ *as* $|z| \to \infty$ *for some (or any)* $0 < \delta \le \alpha$*;*
4. $\widehat{G}_\delta(z_k) \to 0$ *as* $k \to \infty$ *for some (or any) δ-lattice $\{z_k\}_k$ with* $0 < \delta \le \alpha$*;*
5. *G satisfies a vanishing Carleson condition.*

To characterize the Schatten class membership of the vectorial Toeplitz operator T_G, we define the operator-valued Berezin transform of G by

$$\tilde{G}^{op}(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)} G(w)\, dA(w), \quad z \in \mathbb{C}^n,$$

and the corresponding averaging operator by

$$\hat{G}^{op}_r(z) = \frac{\int_{D^r(z)} G(w)\, dA(w)}{|D^r(z)|} \simeq \frac{\int_{D^r(z)} G(w) dA(w)}{\rho(z)^{2n}}, \quad z \in \mathbb{C}^n.$$

These operator-valued versions of the Berezin transform and the averaging operator will likely be useful for the study of various classes of concrete operators. In our present work we use them to characterize the Schatten class membership of the vectorial Toeplitz operators.

Theorem B.1.4. *Let $1 \leq p < \infty$, and $0 < \delta < \alpha$, where α is as in (B.2.11). Then for any orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, the following statements are equivalent:*

1. *The operator T_G belongs to $S_p(F^2_\phi(\mathbb{C}^n,\mathcal{H}))$;*
2. $$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z)\, e_m, e_m\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$
3. $$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \hat{G}^{op}_\delta(z)\, e_m, e_m\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$
4. *Let $\{z_j\}_{j\geq 1}$ be a δ-lattice. Then*
$$\sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_\delta(z_j)\, e_m, e_m\rangle_{\mathcal{H}}\Big)^p < \infty.$$

The characterization of Schatten class membership of Toeplitz operators T_G, for $0 < p < 1$, is more complicated. As one can see in Theorem B.1.7, to get a full characterization, we need to add an extra condition about the symbol $G(z)$, that is, $G(z)$ is a compact operator on $\mathcal{H}$, for every $z \in \mathbb{C}^n$. However, sufficient conditions for $T_G \in S_p(F^2_\phi(\mathbb{C}^n,\mathcal{H}))$ is exactly as those when $1 \leq p < \infty$, as explained in Proposition B.1.6 below.

Proposition B.1.5. *Let $0 < p < 1$, and $0 < \delta < \alpha$, where α is as in (B.2.11). Then for any orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, the following statements are equivalent:*

1. $$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z)\, e_m, e_m\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$
2. $$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \hat{G}^{op}_\delta(z)\, e_m, e_m\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty;$$
3. *Let $\{z_j\}_{j\geq 1}$ be a δ-lattice. Then*
$$\sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_\delta(z_j)\, e_m, e_m\rangle_{\mathcal{H}}\Big)^p < \infty.$$

Proposition B.1.6. *Let $0 < p < 1$, and $0 < \delta < \alpha$, where α is as in (B.2.11). If there is an orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, such that*

$$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z)\, e_m, e_m\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty,$$

then the operator T_G belongs to $S_p(F^2_\phi(\mathbb{C}^n,\mathcal{H}))$.

The following theorem gives the necessary condition for the Schatten class membership of T_G by assuming that $G(z)$ is a compact operator on $\mathcal{H}$.

Theorem B.1.7. *Let* $0 < p < 1$*,* $\delta < \min(1/2, \alpha)$*, where* α *is as in (B.2.11), and* $\{z_j\}_{j\geq 1}$ *be a* δ*-lattice. Assume that* $G(z)$ *is compact for every* $z \in \mathbb{C}^n$*, and* $T_G \in S_p(F^2_\phi(\mathbb{C}^n, \mathcal{H}))$*. Then there is a family of orthonormal bases* $\{e^j_m\}_{m\geq 1}$ *of* $\mathcal{H}$*, possibly depending on* $z_j \in \mathbb{C}^n$*, such that*

$$\sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_\delta(z_j)\, e^j_m, e^j_m \rangle_{\mathcal{H}} \Big)^p < \infty,$$

where $\{e^j_m\}_{m\geq 1}$ *is the basis of* $\mathcal{H}$*, obtained by eigenvectors of* $\hat{G}^{op}_\delta(z_j)$*, for each* $j \geq 1$*.*

Note that the integrals in the preceding theorems and propositions are taken with respect to the volume form associated with the Riemannian metric tensor $g = \sum_{j=1}^n \rho(z)^{-2} dz_j \otimes d\bar{z}_j$ over $\mathbb{C}^n$, taking into account the underlying geometry of the space. One can see that the associated Riemannian metric is conformal, with the conformal factor $\rho(z)^{-1}$. For more details on function ρ see §B.2. In the case of the classical Fock space, $\rho = 1$, and hence the study of Schatten class membership of the vectorial Toeplitz operator is much easier, as one is dealing with the usual Riemannian metric.

In general, when dealing with Toeplitz operators on Fock spaces with doubling or Dall'Ara weights, the difficulty is caused by the geometry induced by the weight ϕ, that effects both the kernel estimates and the Riemannian metric over $\mathbb{C}^n$. Accordingly, the proofs typically require new techniques adapted to such weights, such as generalized criteria of Carleson measures and decompositions of the complex plane by r-lattices, which differ from the conventional decompositions into subsets with essentially constant radius. Moreover, because of the lack of an explicit expression for the reproducing kernel, which makes our goal more difficult, we use some of its pointwise and norm estimates.

More precisely, Theorems B.1.2 and B.1.3 extend classical results for scalar symbols, the main difference from the scalar case in our approach is that we introduce a Carleson condition for the vector-valued case, while for our Schatten class result in Theorem B.1.4, we have to introduce operator-valued versions of the Berezin transform and the averaging function. This is in contrast, for example, to the case in which the symbol of the Toeplitz operator is a positive measure. Another useful observation for our approach is that the compactness of the vectorial Toeplitz operator implies the compactness of the operator-valued averaging function, as shown in Lemma B.2.16. This enables us to apply the spectral theorem introducing bases e^j_m in the proof of Theorem B.1.4.

The paper is organized as follows. In Section B.2, we give some background on the radius function ρ and some useful estimates of the reproducing kernel. Further, we elaborate more on the relationship between the reproducing kernel $K^{\mathcal{H}}(z,w)$ of $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ and that of $F^2_\phi(\mathbb{C}^n)$, and discuss the properties of the orthogonal projection. Furthermore, we provide some lemmas on the Schatten class properties of Toeplitz operators that turn out to be very useful in the proof of Theorem B.1.4. Section B.3 is mostly devoted to the proof of Theorem B.1.2 and Theorem B.1.3. Finally, the proof of Theorem B.1.4 is given in Section B.4.

Notation

We use C to denote positive constants whose value may change from line to line but does not depend on the functions being considered. We say that $A \simeq B$ if there exists a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$. Moreover, $A \lesssim B$ if $A \leq CB$ for some positive constant C.

B.2 Preliminaries

In this section, we state the definition of Dall'Ara's weights, prove some key lemmas on the radius function ρ, and deal with the reproducing kernels of $F^2_\phi(\mathbb{C}^n)$ and $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. We finish this section by providing some auxiliary results on the Schatten class membership of vectorial Toeplitz operators.

Dall'Ara's weights and the corresponding weighted Fock spaces

Let $\mathcal{H}$ be a separable Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and $\phi : \mathbb{C}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ plurisubharmonic function.

Definition B.2.1. We say that ϕ belongs to the weight class $\mathcal{W}$ if ϕ satisfies the following statements:

(I) There exists $c > 0$ such that

$$\inf_{z\in\mathbb{C}^n} \sup_{\xi\in D(z,c)} \Delta\phi(\xi) > 0, \tag{B.2.1}$$

where $D(z,c)$ is the Euclidean disk centered at z with radius c,

(II) $\Delta\phi$ satisfies the reverse-Hölder inequality. That is, there exists a positive real number C such that

$$\|\Delta\phi\|_{L^\infty(D(z,r))} \le Cr^{-2n} \int_{D(z,r)} \Delta\phi(\xi)\,dA(\xi), \quad \text{for any } z \in \mathbb{C}^n \text{ and } r > 0,$$

(III) the eigenvalues of H_ϕ are comparable, i.e., there exists a $\delta_0 > 0$ such that

$$\langle H_\phi(z)u, u\rangle \ge \delta_0 \Delta\phi(z)|u|^2, \quad \text{for any } u, z \in \mathbb{C}^n,$$

where The Hessian matrix of ϕ is given by

$$H_\phi = \left(\frac{\partial^2\phi}{\partial z_j \partial \overline{z_k}}\right)_{j,k\ge 1}.$$

Suppose $0 < p < \infty$ and $\phi \in \mathcal{W}$. The space $L^p_\phi(\mathbb{C}^n)$ is the space of all measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{L^p_\phi(\mathbb{C}^n)} = \left(\int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)}\,dA(z)\right)^{1/p} < \infty,$$

and the space $L^\infty_\phi(\mathbb{C}^n)$ consists of measurable functions endowed with the norm

$$\|f\|_{L^\infty_\phi(\mathbb{C}^n)} = \sup_{z\in\mathbb{C}^n} |f(z)|e^{-\phi(z)} < \infty.$$

Denote by $H(\mathbb{C}^n)$ the space of all holomorphic functions on $\mathbb{C}^n$. Then the scalar weighted Fock space is defined as

$$F^p_\phi(\mathbb{C}^n) = L^p_\phi(\mathbb{C}^n) \cap H(\mathbb{C}^n).$$

with the same norm which was defined above. It is easy to check that $F^p_\phi(\mathbb{C}^n)$ is a Banach space under the above norm for $1 \le p < \infty$, and a complete metrizable topological vector space with the metric

$$\varrho(f,g) = \|f - g\|_{F^p_\phi(\mathbb{C}^n)}, \quad \text{for } 0 < p < 1.$$

For $z \in \mathbb{C}^n$, we define the associated function ρ to ϕ as

$$\rho(z) = \sup\{r > 0 : \sup_{w \in D(z,r)} \Delta\phi(w) \leq r^{-2}\}. \tag{B.2.2}$$

This function satisfies many different properties, as presented in Lemma B.2.2. Let μ be a positive Borel measure defined by

$$\mu(D(z,r)) = r^2 \|\Delta\phi\|_{L^\infty(D(z,r))}.$$

One can see that μ is doubling, and $\mu(D(z,\rho(z))) = 1$ using the reverse Hölder inequality, as was shown in [62].

The orthogonal projection of $L^2_\phi(\mathbb{C}^n)$ onto $F^2_\Phi(\mathbb{C}^n)$ is denoted by $P_\mathbb{C}$ and the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$ by $K_w(z) = K(z,w)$. It is well known that $P_\mathbb{C}$ can be represented as an integral operator

$$P_\mathbb{C}(f)(z) = \int_{\mathbb{C}^n} f(w)K(z,w)e^{-2\phi(w)}dA(w), \quad z \in \mathbb{C}^n,$$

which extends to a bounded projection from $L^p_\varphi(\mathbb{C}^n)$ to $F^p_\phi(\mathbb{C}^n)$ if $1 < p < \infty$. In particular, Theorem 20 of [30] proves that there are constants $C, \varepsilon > 0$ such that

$$|K(z,w)| \leq C \frac{e^{\phi(z)}}{\rho(z)^n} \frac{e^{\phi(w)}}{\rho(w)^n} e^{-\varepsilon d_\rho(z,w)}, \quad z,w \in \mathbb{C}^n, \tag{B.2.3}$$

where if $\gamma : [0,1] \to \mathbb{C}^n$ is a piecewise C^1 curve, we define

$$L_\rho(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))} dt,$$

and

$$d_\rho(z,w) = \inf_\gamma L_\rho(\gamma),$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : I \to \mathbb{C}^n$ with $\gamma(0) = z$ and $\gamma(1) = w$. Moreover, $d_\rho(z,w) \simeq \frac{|z-w|}{\rho(z)}$, for $z,w \in \mathbb{C}^n$. For more details, please see proposition 5 in [30].

Some useful estimates

The first lemma shows some properties of the associated function ρ and construction of the r-lattice as defined below.

Lemma B.2.2 (See [7], Lemma A)**.** *Let ϕ be defined as in Definition B.2.1. Then the radius function ρ satisfies the following properties.*

(1) There exists $M > 0$ such that

$$\sup_{z \in \mathbb{C}^n} \rho(z) \leq M, \tag{B.2.4}$$

(2) The function ρ is Lipschitz. That is, for every $z,w \in \mathbb{C}^n$,

$$|\rho(z) - \rho(w)| \leq |z - w|, \tag{B.2.5}$$

(3) For $r \in (0,1)$ and $w \in D^r(z)$,

$$(1-r)\rho(z) \leq \rho(w) \leq (1+r)\rho(z), \tag{B.2.6}$$

(4) There exist $A, B > 0$ such that

$$|z|^{-A} \lesssim \rho(z) \lesssim |z|^{B}, \quad \textit{for } |z| > 1. \tag{B.2.7}$$

By (B.2.6) and the triangle inequality, for any $r \in (0,1)$, there are $m_1 = m_1(r) > 1$ and $m_2 = m_2(r) > 1$ such that

$$D^r(z) \subset D^{m_1 r}(w), \quad \text{and } D^r(w) \subset D^{m_2 r}(z), \quad \text{for every } w \in D^r(z). \tag{B.2.8}$$

It is easy to see that

$$\beta = \sup_{0<r<1} [m_1(r) + m_2(r)] < \infty. \tag{B.2.9}$$

Given a sequence $\{z_k\}_{k\geq 1} \subset \mathbb{C}^n$ and $r > 0$, we call $\{z_k\}_{k\geq 1}$ an r-lattice if $\{D^r(z_k)\}_{k=1}^{\infty}$ covers $\mathbb{C}^n$, and the balls of the form $\{D^{r/5}(z_k)\}_{k=1}^{\infty}$ are pairwise disjoint. Moreover, for an r-lattice $\{z_k\}_{k\geq 1}$ and a real number $m \geq 1$, there exists some integer N, only depending on m and r, such that each $z \in \mathbb{C}^n$ can be in at most N balls of the form $D^{mr}(z_k)$. That is,

$$\sum_{k=1}^{\infty} \chi_{D^{mr}(z_k)}(z) \leq N, \quad \text{for } z \in \mathbb{C}^n, \tag{B.2.10}$$

where χ_E is a characteristic function of a subset E of $\mathbb{C}^n$.

The second lemma presents some estimates of the reproducing kernels and their norms.

Lemma B.2.3 (See [8], Lemma 2.3). *Let $K_z = K(\cdot, z)$ be the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$. The following assertions are true.*

(a) There exists $\alpha \in (0,1]$ such that

$$|K_z(w)| \simeq \|K_z\|_{F^2_\phi(\mathbb{C}^n)} \|K_w\|_{F^2_\phi(\mathbb{C}^n)}, \quad w \in D^\alpha(z), \tag{B.2.11}$$

(b) For $0 < p \leq \infty$,

$$\|K_z\|_{F^p_\phi(\mathbb{C}^n)} \simeq e^{\phi(z)} \rho(z)^{2n(1-p)/p}, \quad z \in \mathbb{C}^n, \tag{B.2.12}$$

(c) Let α be as defined in (B.2.11). Then

$$|k_z(w)|^2 e^{-2\phi(w)} \simeq \rho(z)^{-2n}, \quad w \in D^\alpha(z), \tag{B.2.13}$$

(d) For each $z \in \mathbb{C}^n$, $0 < p \leq \infty$ and $\beta \in \mathbb{R}$,

$$\int_{\mathbb{C}^n} |K_z(w)|^p e^{-p\phi(w)} \rho(w)^\beta \, dA(w) \simeq e^{p\phi(z)} \rho(z)^{2n(1-p)+\beta}, \tag{B.2.14}$$

(e) The set $\{k_z : z \in \mathbb{C}^n\}$ is bounded in $F^2_\phi(\mathbb{C}^n)$ and $k_z \to 0$ uniformly on any compact subsets of $\mathbb{C}^n$ as $|z| \to \infty$.

The third lemma shows how we transform any function f in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ into another one in $F^2_\phi(\mathbb{C}^n)$.

Lemma B.2.4. *If $f \in F^2_\phi(\mathbb{C}^n, \mathcal{H})$, then $z \mapsto \langle f(z), e \rangle_{\mathcal{H}} \in F^2_\phi(\mathbb{C}^n)$, for any unit element $e \in \mathcal{H}$.*

Proof. By Cauchy-Schwarz inequality,

$$\int_{\mathbb{C}^n} |\langle f(z), e\rangle_{\mathcal{H}}|^2 e^{-2\phi(z)} dA(z) \leq \int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}}^2 \|e\|_{\mathcal{H}}^2 e^{-2\phi(z)} dA(z) < \infty,$$

which finishes the proof. □

Remark B.2.5. Lemma B.2.4 can be generalized for any $h \in H$. Indeed, similarly, one can observe that $z \mapsto \langle f(z), h\rangle_{\mathcal{H}} \in F_\phi^2(\mathbb{C}^n)$, for any element $h \in \mathcal{H}$.

Lemma B.2.6. *Let $e \in \mathcal{H}$ be a unit element. The set $\{k_z(\cdot)e : z \in \mathbb{C}^n\}$ is bounded in $F_\phi^2(\mathbb{C}^n, \mathcal{H})$ and $k_z(\cdot)e \to 0$ uniformly on any compact subsets of $\mathbb{C}^n$ as $|z| \to \infty$.*

Proof. It is similar to the proof of the statement (e) of Lemma B.2.3. □

Lemma B.2.7 (See [7], Lemma B)**.** *Let $0 < p < \infty$ and define ϕ as in (B.2.1). For any $\delta \in (0,1]$, there exists $C > 0$ such that for any $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$,*

$$|f(z)|^p e^{-p\phi(z)} \leq \frac{C}{\delta^{2n}\rho(z)^{2n}} \int_{D^\delta(z)} |f(w)|^p e^{-p\phi(w)} dA(w). \tag{B.2.15}$$

Lemma B.2.8. *For any $\delta \in (0,1]$, there exists $C > 0$ such that for any $f \in F_\phi^2(\mathbb{C}^n, \mathcal{H})$ and $z \in \mathbb{C}^n$,*

$$\|f(z)\|_{\mathcal{H}}^2 e^{-2\phi(z)} \leq \frac{C}{\delta^{2n}\rho(z)^{2n}} \int_{D^\delta(z)} \|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)} dA(w). \tag{B.2.16}$$

Proof. Let $f \in F_\phi^2(\mathbb{C}^n, \mathcal{H})$. By Lemma B.2.4, $\langle f(z), e\rangle_{\mathcal{H}}$ belongs to $F_\phi^2(\mathbb{C}^n)$ and hence holomorphic, for any unit vector $e \in \mathcal{H}$. Hence by Lemma B.2.7, and applying the Cauchy-Schwarz inequality

$$\begin{aligned} |\langle f(z), e\rangle_{\mathcal{H}}|^2 e^{-2\phi(z)} &\leq \frac{C}{\delta^{2n}\rho(z)^{2n}} \int_{D^\delta(z)} |\langle f(w), e\rangle_{\mathcal{H}}|^2 e^{-2\phi(w)} dA(w) \\ &\leq \frac{C}{\delta^{2n}\rho(z)^{2n}} \int_{D^\delta(z)} \|f(w)\|_{\mathcal{H}}^2 \|e\|_{\mathcal{H}}^2 e^{-2\phi(w)} dA(w). \end{aligned}$$

Since $\|e\|_{\mathcal{H}} = 1$, we obtain (B.2.16) and the proof is complete. □

Remark B.2.9. Let $z \in \mathbb{C}^n$. Then by Lemma B.2.8, $\|f(z)\|_{\mathcal{H}} \leq C\frac{e^{\phi(z)}}{\rho(z)^n}\|f\|_{2,\phi}$, and hence the point evaluation map $f \mapsto f(z)$ is a bounded linear homomorphism from $F_\phi^2(\mathbb{C}^n, \mathcal{H})$ to $\mathcal{H}$. Let $C(z)$ be the bounding constant, depending only on z, ϕ, and n. One can see that for any compact subset $K \subset \mathbb{C}^n$, and any $z \in K$, $C(z)$ is bounded. To see this, first take K not overlapping the unit disk centered at the origin. Then (B.2.7) implies that $C(z) \simeq e^{\phi(z)}|z|^{nA}$ for some $A > 0$, and thus bounded. Now, assume that K overlaps the unit disk $D(0,1)$. By (B.2.5), ρ is continuous, and since ϕ is C^2, it is enough to show that ρ never vanishes on the unit disk, to conclude that $C(z)$ is continuous and thus bounded on K. Let $z \in D(0,1)$. By continuity of $\Delta\phi$, and since ϕ is plurisubharmonic, there is some constant $M > 0$ such that $\sup_{w\in D(0,2)} \Delta\phi(w) = M$. Let $N = \max\{M, 1\}$. Then $\frac{1}{N} \leq 1$, and thus $D(z, \frac{1}{N}) \subset D(0,2)$, for every $z \in D(0,1)$. Hence, $\sup_{w\in D(z,\frac{1}{N})} \Delta\phi(w) \leq N \leq N^2$. Using (B.2.2), we can conclude that $\rho(z) \geq \frac{1}{N}$ for every $z \in D(0,1)$, and in particular $\rho(z) \neq 0$.

Reproducing kernel of $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ and the orthogonal projection

Lemma B.2.10. *Let ϕ be as in Definition B.2.1, and $\mathcal{H}$ be a separable Hilbert space. The reproducing kernel of $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is of the form*

$$K^{\mathcal{H}}_w(z) = K^{\mathcal{H}}(z,w) = \sum_{n=1}^{\infty} K(z,w) e_n \otimes e^n,$$

where $K(z,w)$ is the reproducing kernel of $F^2_\phi(\mathbb{C}^n)$.

Proof. Applying Lemma B.2.4, we can write

$$\begin{aligned}
&\int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w) f(w) e^{-2\phi(w)} dA(w) = \\
&\int_{\mathbb{C}^n} \sum_{n=1}^{\infty} \langle f(w), e_n \rangle_{\mathcal{H}} e_n K(z,w) e^{-2\phi(w)} dA(w) \\
&= \sum_{n=1}^{\infty} \langle f(z), e_n \rangle_{\mathcal{H}} e_n \\
&= f(z),
\end{aligned}$$

showing that the choice we made for the reproducing kernel does make sense. Moreover, since $K(z,w)$ is conjugate symmetric, and by the natural isomorphism $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{H}^* \otimes \mathcal{H}$,

$$K^{\mathcal{H}}(z,w)^* = \sum_{n=1}^{\infty} \overline{K(z,w)} e^n \otimes e_n \cong \sum_{n=1}^{\infty} K(w,z) e_n \otimes e^n = K^{\mathcal{H}}(w,z).$$

□

Remark B.2.11. The reproducing kernel in Lemma B.2.10 is unique when viewed as an $\mathcal{L}(\mathcal{H})$-valued kernel. More precisely, if $K^{\mathcal{H}}_1$, $K^{\mathcal{H}}_2 : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$ both satisfy the reproducing kernel property

$$\langle f(z), h \rangle_{\mathcal{H}} = \langle f, K^{\mathcal{H}}_{j,z} h \rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})}, \qquad f \in F^2_\phi(\mathbb{C}^n, \mathcal{H}),\ h \in \mathcal{H},\ z \in \mathbb{C}^n,$$

for $j = 1, 2$, then $K^{\mathcal{H}}_1 = K^{\mathcal{H}}_2$. Indeed, for fixed $z \in \mathbb{C}^n$ and $h \in \mathcal{H}$, the point evaluation map $ev_z : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to \mathcal{H}$, with $f \mapsto f(z)$, is bounded (see the discussion preceding Definition B.1.1). Hence, the scalar-valued map

$$L_{z,h} : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to \mathbb{C}, \qquad L_{z,h}(f) = \langle f(z), h \rangle_{\mathcal{H}},$$

is a bounded linear functional. Since $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is a Hilbert space, the Riesz representation theorem yields a unique vector $g_{z,h} \in F^2_\phi(\mathbb{C}^n, \mathcal{H})$ such that

$$\langle f(z), h \rangle_{\mathcal{H}} = \langle f, g_{z,h} \rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} \qquad \text{for all } f \in F^2_\phi(\mathbb{C}^n, \mathcal{H}).$$

Thus, for each reproducing kernel $K^{\mathcal{H}}_j$, the vector $K^{\mathcal{H}}_{j,z} h$ is precisely this unique Riesz representer, and in particular

$$K^{\mathcal{H}}_{j,z} h \in F^2_\phi(\mathbb{C}^n, \mathcal{H}).$$

Therefore,

$$\langle f, (K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z}) h \rangle_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} = 0 \qquad \text{for all } f \in F^2_\phi(\mathbb{C}^n, \mathcal{H}).$$

Choosing

$$f = (K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z})h \in F^2_\phi(\mathbb{C}^n, \mathcal{H}),$$

we obtain

$$\|(K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z})h\|^2_{F^2_\phi(\mathbb{C}^n,\mathcal{H})} = 0.$$

Hence

$$(K^{\mathcal{H}}_{1,z} - K^{\mathcal{H}}_{2,z})h = 0 \qquad \text{for every } h \in \mathcal{H},$$

which implies

$$K^{\mathcal{H}}_{1,z} = K^{\mathcal{H}}_{2,z}.$$

Since z was arbitrary, it follows that

$$K^{\mathcal{H}}_1 = K^{\mathcal{H}}_2.$$

Consequently, the reproducing kernel is genuinely unique as an $\mathcal{L}(\mathcal{H})$-valued kernel. The formula in Lemma 2.4.6,

$$K^{\mathcal{H}}(z,w) = \sum_{n=1}^{\infty} K(z,w) e_n \otimes e^n,$$

depends on the chosen orthonormal basis $\{e_n\}_{n\geq 1}$ only at the level of coordinates. Under the natural identification $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{L}(\mathcal{H})$, one has

$$\sum_{n=1}^{\infty} e_n \otimes e^n = I_{\mathcal{H}},$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. So the above expression is simply

$$K^{\mathcal{H}}(z,w) = K(z,w) I_{\mathcal{H}}.$$

Thus the kernel is not merely unique up to isomorphism of $\mathcal{H} \otimes \mathcal{H}^*$, but it is uniquely determined as the operator-valued kernel $K(z,w)I_{\mathcal{H}}$. Different orthonormal bases only give different tensor-coordinate expressions for the same operator.

We now show that the integral operator defined in (B.1.4) is the orthogonal projection onto $F^2_\phi(\mathcal{H})$.

Lemma B.2.12. *Let ϕ be as in Definition B.2.1, and $\mathcal{H}$ be a separable Hilbert space. The integral operator*

$$P(f)(z) = \int_{\mathbb{C}^n} K^{\mathcal{H}}(z,w) f(w) e^{-2\phi(w)} dA(w) = \int_{\mathbb{C}^n} f(w) K(z,w) e^{-2\phi(w)} dA(w), \quad z \in \mathbb{C}^n,$$

is the orthogonal projection of $L^2_\phi(\mathbb{C}^n, \mathcal{H})$ onto $F^2_\phi(\mathbb{C}^n, \mathcal{H})$.

Proof. Let $\{e_m\}_{m=1}^\infty$ be an orthonormal basis of $\mathcal{H}$. For $f \in L^2_\phi(\mathbb{C}^n, \mathcal{H})$, define the scalar-valued functions $f^*_m(z) := \langle f(z), e_m \rangle_{\mathcal{H}}$, for $z \in \mathbb{C}^n$. Then

$$f(z) = \sum_{m=1}^{\infty} f^*_m(z) e_m$$

in $\mathcal{H}$ for a.e. z, and

$$\|f\|^2_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} = \sum_{m=1}^{\infty} \|f^*_m\|^2_{L^2_\phi(\mathbb{C}^n)}.$$

By Lemma 2.4.6, for each $z \in \mathbb{C}^n$,

$$P(f)(z) = \sum_{m=1}^{\infty} e_m \, P_{\mathbb{C}}(f_m^*)(z),$$

where $P_{\mathbb{C}} : L^2_\phi(\mathbb{C}^n) \to F^2_\phi(\mathbb{C}^n)$ is the scalar Bergman projection. Since $P_{\mathbb{C}}(f_m^*) \in F^2_\phi(\mathbb{C}^n)$ for every m, it follows that $P(f)$ is $\mathcal{H}$-valued holomorphic. Moreover,

$$\|P(f)\|^2_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} = \sum_{m=1}^{\infty} \|P_{\mathbb{C}}(f_m^*)\|^2_{L^2_\phi(\mathbb{C}^n)} \le \sum_{m=1}^{\infty} \|f_m^*\|^2_{L^2_\phi(\mathbb{C}^n)} = \|f\|^2_{L^2_\phi(\mathbb{C}^n,\mathcal{H})},$$

so $P(f) \in F^2_\phi(\mathcal{H})$. Next, for $f \in L^2_\phi(\mathbb{C}^n, \mathcal{H})$,

$$P(Pf)(z) = \sum_{m=1}^{\infty} e_m \, P_{\mathbb{C}}\Big((Pf)_m^*\Big)(z).$$

But

$$(Pf)_m^*(z) = \langle Pf(z), e_m \rangle_{\mathcal{H}} = P_{\mathbb{C}}(f_m^*)(z),$$

hence, since $P_{\mathbb{C}}$ is a projection,

$$P(Pf)(z) = \sum_{m=1}^{\infty} e_m \, P_{\mathbb{C}}(P_{\mathbb{C}}(f_m^*))(z) = \sum_{m=1}^{\infty} e_m \, P_{\mathbb{C}}(f_m^*)(z) = Pf(z).$$

Therefore $P^2 = P$, so P is a projection onto $F^2_\phi(\mathcal{H})$. It remains to show that P is orthogonal. Let $f \in L^2_\phi(\mathbb{C}^n, \mathcal{H})$. Then $f = (f - Pf) + Pf$. We claim that $\langle f - Pf, Pf \rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} = 0$. Indeed, using the orthonormal basis expansion,

$$\begin{aligned}
\langle f, Pf \rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} &= \int_{\mathbb{C}^n} \langle f(z), Pf(z) \rangle_{\mathcal{H}} e^{-2\phi(z)} \, dA(z) \\
&= \int_{\mathbb{C}^n} \left\langle \sum_{m=1}^{\infty} f_m^*(z) e_m, \sum_{k=1}^{\infty} P_{\mathbb{C}}(f_k^*)(z) e_k \right\rangle_{\mathcal{H}} e^{-2\phi(z)} \, dA(z) \\
&= \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} f_m^*(z) \, \overline{P_{\mathbb{C}}(f_m^*)(z)} \, e^{-2\phi(z)} \, dA(z) \\
&= \sum_{m=1}^{\infty} \langle f_m^*, P_{\mathbb{C}}(f_m^*) \rangle_{L^2_\phi(\mathbb{C}^n)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle Pf, Pf \rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} &= \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} P_{\mathbb{C}}(f_m^*)(z) \, \overline{P_{\mathbb{C}}(f_m^*)(z)} \, e^{-2\phi(z)} \, dA(z) \\
&= \sum_{m=1}^{\infty} \langle P_{\mathbb{C}}(f_m^*), P_{\mathbb{C}}(f_m^*) \rangle_{L^2_\phi(\mathbb{C}^n)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle f - Pf, Pf \rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} &= \langle f, Pf \rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} - \langle Pf, Pf \rangle_{L^2_\phi(\mathbb{C}^n,\mathcal{H})} \\
&= \sum_{m=1}^{\infty} \Big(\langle f_m^*, P_{\mathbb{C}}(f_m^*) \rangle_{L^2_\phi(\mathbb{C}^n)} - \langle P_{\mathbb{C}}(f_m^*), P_{\mathbb{C}}(f_m^*) \rangle_{L^2_\phi(\mathbb{C}^n)} \Big) = 0,
\end{aligned}$$

because $P_{\mathbb{C}}$ is the orthogonal projection of $L^2_\phi(\mathbb{C}^n)$ onto $F^2_\phi(\mathbb{C}^n)$. Thus we can conclude that P is the orthogonal projection of $L^2_\phi(\mathbb{C}^n, \mathcal{H})$ onto $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. □

Vectorial Toeplitz operator

Lemma B.2.13. *Let* $\{e_i\}_{i=1}^{\infty}$ *be an orthonormal basis of* $\mathcal{H}$*, and* $K_z^{\mathcal{H}} = \sum_{j=1}^{\infty} K_z e_j \otimes e^j$ *be the reproducing kernel of* $F^2_\phi(\mathbb{C}^n, \mathcal{H})$*, with* K_z *the reproducing kernel of* $F^2_\phi(\mathbb{C}^n)$*. Then the linear span of* $\{K_z^{\mathcal{H}} e_i : z \in \mathbb{C}^n, i \geq 1\}$ *is dense in* $F^2_\phi(\mathbb{C}^n, \mathcal{H})$*.*

Proof. First, notice that for any $z, w \in \mathbb{C}^n$ and $i \geq 1$, $K_z^{\mathcal{H}}(w)e_i = K_z(w)e_i \in \mathcal{H}$. Moreover, it is easy to see that for any $z \in \mathbb{C}^n$ and $i \geq 1$, $K_z e_i \in F^2_\phi(\mathbb{C}^n, \mathcal{H})$. Let $\Gamma = \operatorname{span}\{K_z^{\mathcal{H}} e_i : z \in \mathbb{C}^n, i \geq 1\}$. Since $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ is a Hilbert space, the density of Γ is equivalent to $\Gamma^\perp = \{0\}$, where $\perp$ represents the orthogonal complement. So, let $f \in \Gamma^\perp$. Then by definition, for any $z \in \mathbb{C}^n$ and any $i \geq 1$, $\langle f, K_z^{\mathcal{H}} e_i \rangle_{F^2_\phi(\mathbb{C}^n, \mathcal{H})} = 0$. Consider the natural isomorphism $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{L}(\mathcal{H})$. By the reproducing kernel property,

$$0 = \langle f, K_z^{\mathcal{H}} e_i \rangle_{F^2_\phi(\mathbb{C}^n, \mathcal{H})} = \langle f(z), e_i \rangle_{\mathcal{H}}.$$

Hence, $\langle f(z), h \rangle_{\mathcal{H}} = 0$ for every h is the linear span of $\{e_i\}_{i \geq 1}$. Because this span is dense in $\mathcal{H}$, we get $\langle f(z), h \rangle_{\mathcal{H}} = 0$, for every $h \in \mathcal{H}$. Thus, $f(z) = 0$ for all $z \in \mathbb{C}^n$. Hence, f is identically zero, and we can conclude that $\Gamma^\perp = \{0\}$. Therefore, Γ is a dense subset of $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. ☐

Let $\Gamma = \operatorname{span}\{K_z e_i : z \in \mathbb{C}^n, i \geq 1\}$. By Lemma B.2.13, Γ is a dense subset of $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. Let $f \in \Gamma$. Using $\|G(w)f(w)\|^2_{\mathcal{H}} \leq \|G(w)\|^2_{\mathcal{L}(\mathcal{H})} \|f(w)\|^2_{\mathcal{H}}$ and (B.1.5), one can conclude that $Gf \in L^2_\phi(\mathbb{C}^n, \mathcal{H})$. Therefore $P(Gf)$ is well-defined for $f \in \Gamma$. Hence, for $G \in T_\phi(\mathcal{L}(\mathcal{H}))$, the *vectorial Toeplitz operator* T_G is densely defined by

$$T_G f(z) = P(Gf)(z) = \int_{\mathbb{C}^n} G(w) f(w) K(z,w) e^{-2\phi(w)} dA(w),$$

for $f \in F^2_\phi(\mathbb{C}^n, \mathcal{H})$.

Remark B.2.14. Let G be a map from $\mathbb{C}^n$ to the Banach space of bounded linear operators on $\mathcal{H}$. Definition 3.10.1 in [42] states that G is a holomorphic operator-valued function if for every $u, v \in \mathcal{H}$, $z \mapsto \langle G(z)u, v \rangle_{\mathcal{H}}$ is holomorphic.

Schatten classes

In this subsection, we prove some lemmas that will be useful in the proof of Theorem B.1.4.

Lemma B.2.15. *Let* $\{e_k^z\}_{k \geq 1}$ *be an orthonormal basis of* $\mathcal{H}$*, possibly depending on* $z \in \mathbb{C}^n$*, and assume that* $1 \leq p \leq \infty$*. Then,* $T_G \in S_p(F^2_\phi(\mathbb{C}^n, \mathcal{H}))$ *if*

$$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} (\langle G(z) e_m^z, e_m^z \rangle)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty.$$

Proof. Let $\{f_k\}_{k \geq 1}$ be an orthonormal basis of $F^2_\phi(\mathbb{C}^n)$. Consider $\{B_{k,m}(z) = f_k(z) e_m^z\}_{m,k \geq 1}$, which is an orthonormal basis of $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. First, by the reproducing kernel property, it is easy to see that for any $f, g \in F^2_\phi(\mathbb{C}^n, \mathcal{H})$,

$$\begin{aligned}
\langle T_G f, g \rangle &= \int_{\mathbb{C}^n} \langle T_G f(z), g(z) \rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z) \\
&= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle G(w) f(w), g(z) \rangle_{\mathcal{H}} K(z,w) e^{-2\phi(w)} e^{-2\phi(z)} dA(w) dA(z) \\
&= \int_{\mathbb{C}^n} \langle G(w) f(w), g(w) \rangle_{\mathcal{H}} e^{-2\phi(w)} dA(w).
\end{aligned}$$

Hence,

$$\begin{aligned}\langle T_G B_{k,m}, B_{k,m}\rangle &= \int_{\mathbb{C}^n} \langle T_G f_k(z) e_m^z, f_k(z) e_m^z\rangle_{\mathcal{H}}\, e^{-2\phi(z)}\, dA(z)\\ &= \int_{\mathbb{C}^n} \langle G(z) e_m^z, e_m^z\rangle_{\mathcal{H}} |f_k(z)|^2\, e^{-2\phi(z)}\, dA(z).\end{aligned}$$

Hence, for $p = \infty$,

$$\|T_G\|_{S_\infty(F^2_\phi(\mathbb{C}^n,\mathcal{H}))} \lesssim \sup_{z\in\mathbb{C}^n} \sum_{m=1}^{\infty} \langle G(z) e_m^z, e_m^z\rangle_{\mathcal{H}}.$$

On the other hand, for $p = 1$, applying (B.2.12), and $K_z(z) = \sum_{k=1}^{\infty} |f_k(z)|^2$, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \langle T_G B_{k,m}, B_{k,m}\rangle &= \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \langle G(z) e_m^z, e_m^z\rangle_{\mathcal{H}} \left(\sum_{k=1}^{\infty} |f_k(z)|^2\right) e^{-2\phi(z)}\, dA(z)\\ &= \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \langle G(z) e_m^z, e_m^z\rangle_{\mathcal{H}} K_z(z)\, e^{-2\phi(z)}\, dA(z)\\ &\simeq \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \langle G(z) e_m^z, e_m^z\rangle_{\mathcal{H}} \frac{dA(z)}{\rho(z)^{2n}}.\end{aligned}$$

Then by Proposition 1.29 in [80],

$$\|T_G\|_{S_1(F^2_\phi(\mathbb{C}^n,\mathcal{H}))} \lesssim \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \langle G(z) e_m, e_m\rangle_{\mathcal{H}} \frac{dA(z)}{\rho(z)^{2n}}.$$

By the interpolation, we can obtain the desired result, which completes the proof. □

Lemma B.2.16. *Assume that the vectorial Toeplitz operator $T_G : F^2_\phi(\mathbb{C}^n,\mathcal{H}) \to F^2_\phi(\mathbb{C}^n,\mathcal{H})$ is compact. Moreover, take $G(w) : \mathcal{H} \to \mathcal{H}$ to be compact for every $w \in \mathbb{C}^n$ and let $\delta > 0$. Then for every fixed $z \in \mathbb{C}^n$, the average operator*

$$\widehat{G}^{op}_\delta(z) \simeq \int_{D^\delta(z)} G(w) \frac{dA(w)}{\rho(w)^{2n}} : \mathcal{H} \to \mathcal{H}$$

is compact.

Proof. Fix $z \in \mathbb{C}^n$. Using (B.2.6), (B.2.12), and (B.2.13) we can write

$$\hat{G}^{op}_\delta(z) = C(z) \int_{D^\delta(z)} G(w) K(z,w) K(w,z) e^{-2\phi(w)} dA(w),$$

where $C(z) = \frac{\rho(z)^{2n}}{e^{2\phi(z)}}$. Let $h \in \mathcal{H}$ be arbitrary and $f = K_z(\cdot)h$ be an element of $F^2_\phi(\mathbb{C}^n,\mathcal{H})$. Then

$$\begin{aligned}C(z) T_G(K_z(\cdot)h)(z) &= C(z) \int_{\mathbb{C}^n} G(w)[hK(w,z)] K(z,w) e^{-2\phi(w)} dA(w)\\ &= \hat{G}^{op}_\delta(z)h + C(z) \int_{\mathbb{C}^n\setminus D^\delta(z)} G(w)[hK(z,w)] K(w,z) e^{-2\phi(w)} dA(w).\end{aligned}$$

Denote the “tail” term by

$$B(h) := C(z) \int_{\mathbb{C}^n\setminus D^\delta(z)} G(w)\big[hK(z,w)\big] K(w,z) e^{-2\phi(w)}\, dA(w).$$

Hence

$$\widehat{G}_\delta^{op}(z)h = C(z)\, T_G(K_z(\cdot)h)(z) - B(h).$$

Let $\{x_m\}_{m\geq 1}$ be a sequence in $\mathcal{H}$, converging weakly to zero. Then $\{K_z(\cdot)x_m\}_{m\geq 1}$ is a sequence in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$, that converges weakly to zero. Since T_G is compact, we have

$$\|T_G(K_z(\cdot)x_m)\|_{2,\phi} \to 0.$$

By Lemma B.2.8, it follows that $\|T_G(K_z(\cdot)x_n)(z)\|_{\mathcal{H}} \to 0$. To prove compactness of $\widehat{G}_\delta^{op}(z)$, it suffices to show that $\|\widehat{G}_\delta^{op}(z)x_m\|_{\mathcal{H}} \to 0$. From the above identity this follows if we prove that $\|B(x_m)\|_{\mathcal{H}} \to 0$.

For each fixed $w \in \mathbb{C}^n$, the operator $G(w)$ is compact. Hence, since $x_m \rightharpoonup 0$, we have

$$\|G(w)x_m\|_{\mathcal{H}} \to 0 \quad \text{for each fixed } w.$$

Computing the norm of the tail, we obtain

$$\|B(x_m)\|_{\mathcal{H}} \leq C(z) \int_{\mathbb{C}^n \setminus D^\delta(z)} \|G(w)x_m\|_{\mathcal{H}} |K(z,w)K(w,z)|\, e^{-2\phi(w)}\, dA(w).$$

Let $M := \sup_m \|x_m\|_{\mathcal{H}} < \infty$. Then $\|G(w)x_m\|_{\mathcal{H}} \leq M\|G(w)\|_{\mathcal{L}(\mathcal{H})}$, so the integrand is dominated by

$$\Phi(w) := M\, \|G(w)\|_{\mathcal{L}(\mathcal{H})}\, |K(z,w)K(w,z)|\, e^{-2\phi(w)}.$$

To see that $\Phi \in L^1(\mathbb{C}^n)$, use Cauchy–Schwarz together with assumption (B.1.5) and the kernel estimates (B.2.12),

$$\begin{aligned}
&\int_{\mathbb{C}^n \setminus D^\delta(z)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} |K(z,w)K(w,z)| e^{-2\phi(w)}\, dA(w) \\
&\leq \left(\int_{\mathbb{C}^n} \|G(w)\|^2_{\mathcal{L}(\mathcal{H})} |K(z,w)|^2 e^{-2\phi(w)}\, dA(w) \right)^{1/2} \|K_z\|_{2,\phi} < \infty.
\end{aligned}$$

Thus, Φ is integrable and independent of m. Since $\|G(w)x_m\|_{\mathcal{H}} \to 0$ for each w, the dominated convergence theorem yields $\|B(x_m)\|_{\mathcal{H}} \to 0$. Consequently,

$$\|\widehat{G}_\delta^{op}(z)x_m\|_{\mathcal{H}} \leq \|C(z)\, T_G(K_z(\cdot)x_m)(z)\|_{\mathcal{H}} + \|B(x_m)\|_{\mathcal{H}} \xrightarrow[m\to\infty]{} 0.$$

Therefore $\widehat{G}_\delta^{op}(z)$ sends weakly null sequences to norm-null sequences, and hence it is compact.

□

Remark B.2.17. The corollary after Theorem 3.18 in [70] states that if X is a normed space, $E \subset X$, and if $\sup_{x\in E} |\Lambda x| < \infty$ for all $\Lambda \in X^*$, then there is a constant $\gamma < \infty$ such that $\|x\| < \gamma$ for all $x \in E$. Applying this to the Hilbert space $\mathcal{H}$, we can see that any weakly convergent sequence is bounded, and therefore, $M := \sup_m \|x_m\| < \infty$.

B.3 Boundedness and compactness of vectorial Toeplitz operators

In this section, we prove Theorem B.1.2 and Theorem B.1.3, to characterize boundedness and compactness of the vectorial Toeplitz operator $T_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to F^2_\phi(\mathbb{C}^n, \mathcal{H})$.

Proof of Theorem B.1.2. First, we prove that (2) implies (3). By using (B.2.13), we obtain

$$\begin{aligned}
\hat{G}_\delta(z) &\simeq \rho(z)^{-2n}\int_{D^\delta(z)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dA(w)\\
&\simeq \int_{D^\delta(z)}|k_z(w)|^2e^{-2\phi(w)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dA(w)\\
&\leq \tilde{G}(z),
\end{aligned}$$

which gives (3). Moreover,

$$\|\hat{G}_\delta\|_{L^\infty(\mathbb{C}^n,dA)} \lesssim \|\tilde{G}\|_{L^\infty(\mathbb{C}^n,dA)} \tag{B.3.1}$$

It is obvious that (3) implies (4) and

$$\|\{\hat{G}_\delta(z_k)\}_k\|_{\ell^\infty} \lesssim \|\hat{G}_\delta\|_{L^\infty(\mathbb{C}^n,dA)}. \tag{B.3.2}$$

Now, let us show that (4) implies (2). Note that

$$\begin{aligned}
\tilde{G}(z) &= \int_{\mathbb{C}^n}|k_z(w)|^2e^{-2\phi(w)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dA(w)\\
&\leq \sum_{k=1}^{\infty}\int_{D^\delta(z_k)}|k_z(w)|^2e^{-2\phi(w)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dA(w)\\
&\lesssim \sum_{k=1}^{\infty}\hat{G}_\delta(z_k)\rho(z_k)^{2n}\sup_{w\in D^\delta(z_k)}|k_z(w)|^2e^{-2\phi(w)}.
\end{aligned} \tag{B.3.3}$$

By Lemma B.2.7 and taking a real number $m > 1$, as well as (B.2.10), we obtain

$$\begin{aligned}
\tilde{G}(z) &\lesssim \sum_{k=1}^{\infty}\hat{G}_\delta(z_k)\int_{D^{m\delta}(z_k)}|k_z(w)|^2e^{-2\phi(w)}dA(w)\\
&\leq \sup_k \hat{G}_\delta(z_k)\sum_{k=1}^{\infty}\int_{D^{m\delta}(z_k)}|k_z(w)|^2e^{-2\phi(w)}dA(w)\\
&\leq N\sup_k \hat{G}_\delta(z_k)\|k_z\|^2_{F^2_\phi(\mathbb{C}^n)}\\
&\lesssim \sup_k \hat{G}_\delta(z_k).
\end{aligned}$$

Therefore,

$$\|\tilde{G}\|_{L^\infty(\mathbb{C}^n,dA)} \lesssim \|\{\hat{G}_\delta(z_k)\}_k\|_{l^\infty}. \tag{B.3.4}$$

Thus, (2), (3) and (4) are all equivalent.

Next we show that (1) implies (2). Let $T_G : F^2_\phi(\mathbb{C}^n,\mathcal{H}) \to F^2_\phi(\mathbb{C}^n,\mathcal{H})$ be bounded. Since $G(w)$ is positive for every $w \in \mathbb{C}^n$, $\|G(w)\|_{\mathcal{L}(\mathcal{H})} = \sup_{\|e\|=1}\langle G(w)e,e\rangle_{\mathcal{H}}$. Applying (B.2.12), Lemma B.2.6, and Lemma B.2.8, we obtain

$$\begin{aligned}
\tilde{G}(z) &= \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) \\
&\simeq \rho(z)^n e^{-\phi(z)} \int_{\mathbb{C}^n} k_z(w) K(z,w) e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) \\
&\simeq \rho(z)^n e^{-\phi(z)} \int_{\mathbb{C}^n} \sup_{\|e\|=1} \langle G(w) k_z(w) e, e \rangle_{\mathcal{H}} K(z,w) e^{-2\phi(w)} dA(w) \\
&\simeq \rho(z)^n e^{-\phi(z)} \sup_{\|e\|=1} \langle T_G k_z(z) e, e \rangle_{\mathcal{H}} \\
&\leq \rho(z)^n e^{-\phi(z)} \|T_G(k_z(z)e)\|_{\mathcal{H}} \\
&\lesssim \rho(z)^n \left(\frac{1}{\rho(z)^{2n}} \int_{\mathbb{C}^n} \|T_G k_z(\zeta) e\|_{\mathcal{H}}^2 e^{-2\phi(\zeta)} dA(\zeta) \right)^{1/2} \\
&\lesssim \|T_G k_z(\cdot) e\|_{2,\varphi} \lesssim \|T_G\|.
\end{aligned} \tag{B.3.5}$$

To show that (3) implies (1), we claim that there is a constant $C > 0$ such that

$$\|T_G f\|_{2,\phi}^2 \leq C \int_{\mathbb{C}^n} \|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)} \hat{G}_\delta(w)^2 dA(w), \tag{B.3.6}$$

for any $f \in F_\phi^2(\mathbb{C}^n, \mathcal{H})$ and any $\delta > 0$. Then (B.3.6) and (B.2.16) imply that

$$\begin{aligned}
\|T_G f\|_{2,\phi}^2 &\lesssim \int_{\mathbb{C}^n} \hat{G}_\delta(w)^2 \|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)} dA(w) \\
&\leq \|\hat{G}_\delta\|_{L^\infty(\mathbb{C}^n, dA)}^2 \|f\|_{2,\phi}^2.
\end{aligned}$$

Hence, $T_G : F_\phi^2(\mathbb{C}^n, \mathcal{H}) \to F_\phi^2(\mathbb{C}^n, \mathcal{H})$ is bounded and

$$\|T_G\| \lesssim \|\hat{G}_\delta\|_{L^\infty(\mathbb{C}^n, dA)}. \tag{B.3.7}$$

To finish the proof, we should justify the inequality (B.3.6). Let $f \in F_\phi^2(\mathbb{C}^n, \mathcal{H})$ and $z \in \mathbb{C}^n$. Take $r > 0$ such that $\beta^2 r \leq \delta$, where β is as in (B.2.9), and let $\{z_k\}_k$ be an r-lattice. Then

$$\begin{aligned}
\|T_G(f)(z)\|_{\mathcal{H}} &= \left\| \int_{\mathbb{C}^n} G(w) f(w) K(z,w) e^{-2\phi(w)} dA(w) \right\|_{\mathcal{H}} \\
&\leq \int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} \|f(w)\|_{\mathcal{H}} |K(z,w)| e^{-2\phi(w)} dA(w) \\
&\leq \sum_{k=1}^{\infty} \int_{D^r(z_k)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} \|f(w)\|_{\mathcal{H}} |K(z,w)| e^{-2\phi(w)} dA(w) \\
&\lesssim \sum_{k=1}^{\infty} \hat{G}_r(z_k) \rho(z_k)^{2n} \left(\sup_{w \in D^r(z_k)} \|f(w)\|_{\mathcal{H}} |K(z,w)| e^{-2\phi(w)} \right).
\end{aligned}$$

Applying (B.2.16), since the function $f K_z \in H(\mathbb{C}^n, \mathcal{H})$, using $\hat{G}_r(z_k) \lesssim \hat{G}_{\beta^2 r}(w)$, for any $w \in D^{\beta r}(z_k)$, and (B.2.6), we obtain

$$\begin{aligned}
\|T_G(f)(z)\|_{\mathcal{H}} &\lesssim \sum_{k=1}^{\infty} \hat{G}_r(z_k) \int_{D^{\beta r}(z_k)} \|f(w)\|_{\mathcal{H}} |K(z,w)| e^{-2\phi(w)} dA(w) \\
&\lesssim N \int_{\mathbb{C}^n} \hat{G}_{\beta^2 r}(w) \|f(w)\|_{\mathcal{H}} |K(z,w)| e^{-2\phi(w)} dA(w) \\
&\lesssim \int_{\mathbb{C}^n} \hat{G}_\delta(w) \|f(w)\|_{\mathcal{H}} |K(z,w)| e^{-2\phi(w)} dA(w).
\end{aligned} \tag{B.3.8}$$

Now, applying Hölder's inequality and (B.2.12), we get

$$\begin{aligned}
\|T_G f(z)\|_{\mathcal{H}}^2 e^{-2\phi(z)} &\lesssim \left(\int_{\mathbb{C}^n} \hat{G}_\delta(w)\|f(w)\|_{\mathcal{H}}|K(z,w)|e^{-2\phi(w)}e^{-\phi(z)}dA(w)\right)^2 \\
&\lesssim \Bigg[\left(\int_{\mathbb{C}^n} \hat{G}_\delta(w)\|f(w)\|_{\mathcal{H}}|K(z,w)|^{1/2}e^{-\phi(w)}e^{-\phi(w)/2}e^{-\phi(z)/2}\right) \\
&\qquad \left(|K(z,w)|^{1/2}e^{-\phi(w)/2}e^{-\phi(z)/2}\right)dA(w)\Bigg]^2 \\
&\leq \left(\int_{\mathbb{C}^n} \hat{G}_\delta^2(w)\|f(w)\|_{\mathcal{H}}^2|K(z,w)|e^{-2\phi(w)}e^{-\phi(w)}e^{-\phi(z)}dA(w)\right) \\
&\qquad \left(\int_{\mathbb{C}^n} |K(z,w)|e^{-\phi(w)}e^{-\phi(z)}dA(w)\right) \\
&\lesssim \int_{\mathbb{C}^n} \hat{G}_\delta^2(w)\|f(w)\|_{\mathcal{H}}^2|K(z,w)|e^{-2\phi(w)}e^{-\phi(w)}e^{-\phi(z)}dA(w).
\end{aligned}$$

Then, by Fubini's theorem and using again (B.2.12), we have

$$\begin{aligned}
\|T_G f\|_{2,\phi}^2 &= \int_{\mathbb{C}^n} \|T_G f(z)\|_{\mathcal{H}}^2 e^{-2\phi(z)}dA(z) \\
&\lesssim \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \hat{G}_\delta^2(w)\|f(w)\|_{\mathcal{H}}^2|K(z,w)|e^{-2\phi(w)}e^{-\phi(w)}e^{-\phi(z)}dA(w)dA(z) \\
&= \int_{\mathbb{C}^n} \hat{G}_\delta^2(w)\|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)}e^{-\phi(w)} \int_{\mathbb{C}^n} |K(z,w)|e^{-\phi(z)}dA(z)dA(w) \\
&\simeq \int_{\mathbb{C}^n} \hat{G}_\delta^2(w)\|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)}dA(w),
\end{aligned}$$

and we are done. Furthermore, by (B.3.1), (B.3.2), (B.3.4), and (B.3.7), one has (B.1.8).

To show that (4) implies (5), note that using Lemma B.2.8, (B.2.6), (B.2.8), and (B.2.10), there is $m > 1$ such that

$$\begin{aligned}
&\int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}}^2 e^{-2\phi(z)}\|G(z)\|_{\mathcal{L}(\mathcal{H})}dA(z) \\
&\lesssim \int_{\mathbb{C}^n} \left(\frac{1}{\rho(z)^{2n}} \int_{D^{\frac{\delta}{1+\delta}}(z)} \|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)}dA(w)\right)\|G(z)\|_{\mathcal{L}(\mathcal{H})}dA(z) \\
&\leq \sum_{k=1}^{\infty} \int_{D^\delta(z_k)} \frac{1}{\rho(z)^{2n}}\left(\int_{D^{\frac{\delta}{1+\delta}}(z)} \|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)}dA(w)\right)\|G(z)\|_{\mathcal{L}(\mathcal{H})}dA(z) \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{\rho(z_k)^{2n}} \int_{D^\delta(z_k)}\left(\int_{D^{m\delta}(z_k)} \|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)}dA(w)\right)\|G(z)\|_{\mathcal{L}(\mathcal{H})}dA(z) \\
&\simeq \sum_{k=1}^{\infty}\left(\int_{D^{m\delta}(z_k)} \|f(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)}dA(w)\right)\hat{G}_\delta(z_k) \\
&\lesssim \|\{\hat{G}_\delta(z_k)\}_k\|_{l^\infty}\|f\|_{2,\phi}^2.
\end{aligned} \tag{B.3.9}$$

We finish the proof by showing that (5) implies (2). Take $f = k_z e$ in (B.1.6). Then $\tilde{G}(z) \lesssim \|k_z\|_{F_\phi^2(\mathbb{C}^n)}^2 = 1$, implying that $\tilde{G}(z) \in L^\infty(\mathbb{C}^n, dA)$, and we are done with the proof. ☐

Proof of Theorem B.1.3. One can show that (2) implies (3), and (3) implies (4) similarly as in the proof of Theorem B.1.2.

Now, let us show that (4) implies (2). Let $\{z_k\}_k$ be a δ-lattice. Assuming (4) holds, for every $\epsilon > 0$, there is $K \in \mathbb{N}$ such that whenever $k > K$,

$$\hat{G}_\delta(z_k) < \epsilon. \tag{B.3.10}$$

Let m be defined as in (B.2.8). Then

$$\begin{aligned}\tilde{G}(z) &= \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) \\ &\lesssim \int_{\cup_{k=1}^K \overline{D^{m\delta}(z_k)}} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) \\ &\qquad + \sum_{k=K+1}^{\infty} \rho(z_k)^{2n} \hat{G}_\delta(z_k) \left(\sup_{w \in D^\delta(z_k)} |k_z(w)|^2 e^{-2\phi(w)} \right).\end{aligned}$$

On one hand, by Lemma B.2.7, (B.3.10) and (B.2.10), we obtain

$$\begin{aligned}&\sum_{k=K+1}^{\infty} \rho(z_k)^{2n} \hat{G}_\delta(z_k) \left(\sup_{w \in D^\delta(z_k)} |k_z(w)|^2 e^{-2\phi(w)} \right) \\ &\lesssim \sum_{k=K+1}^{\infty} \hat{G}_\delta(z_k) \left(\int_{D^{m\delta}(z_k)} |k_z(w)|^2 e^{-2\phi(w)} dA(w) \right) \\ &\le \sup_{k \ge K+1} \hat{G}_\delta(z_k) \left(\sum_{k=K+1}^{\infty} \int_{D^{m\delta}(z_k)} |k_z(w)|^2 e^{-2\phi(w)} dA(w) \right) \\ &\lesssim \epsilon N \|k_z\|^2_{F^2_\phi(\mathbb{C}^n)} \lesssim \epsilon.\end{aligned} \tag{B.3.11}$$

On the other hand, Lemma B.2.3 implies that $k_z \to 0$ uniformly on $\cup_{k=1}^K \overline{D^{m\delta}(z_k)}$ as $|z| \to \infty$. Therefore, as $|z| \to \infty$,

$$\int_{\cup_{k=1}^K \overline{D^{m\delta}(z_k)}} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) < \epsilon. \tag{B.3.12}$$

This, together with (B.3.11), proves (2).

Next, we show that (1) implies (2). Since T_G is compact, Lemma B.2.6 implies that $\|T_G k_z e\|_{2,\phi}$ converges to zero as $|z| \to \infty$. By (B.3.5), we have

$$\tilde{G}(z) \lesssim \|T_G k_z e\|_{2,\phi} \to 0,$$

as $|z| \to \infty$.

To show that (3) implies (1), let $\epsilon > 0$ be arbitrary and $\{f_j\}_{j=1}^\infty$ be a sequence in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ that converges to zero uniformly on any compact subset of $\mathbb{C}^n$. We want to show that for big enough $j \in \mathbb{N}$,

$$\|T_G f_j\|_{2,\phi} \lesssim \epsilon. \tag{B.3.13}$$

By our assumption, there is some $R > 0$ such that

$$\hat{G}_\delta(z) < \sqrt{\epsilon}, \tag{B.3.14}$$

whenever $|z| > R$. This together with (B.3.6), for big enough j we have

$$\begin{aligned}\|T_G f_j\|^2_{2,\phi} &\lesssim \int_{|z| \le R} \hat{G}_\delta(w)^2 \|f_j(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} dA(w) \\ &\qquad + \int_{|z| > R} \hat{G}_\delta(w)^2 \|f_j(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} dA(w) \\ &\lesssim \epsilon + \epsilon \|f_j\|^2_{2,\phi} \lesssim \epsilon,\end{aligned} \tag{B.3.15}$$

where to bound the first integral we used the continuity of $\hat{G}_\delta(w)$ and the fact that $\{f_j\}_{j=1}^\infty$ converges to zero uniformly on any compact $\{|z| \le R\}$.

To show that (4) implies (5), let $\{f_k\}_k$ be a bounded sequence in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ that converges to zero uniformly on compact subsets of $\mathbb{C}^n$. By our assumption, letting $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$\sup_{|z_k|>r_o} \hat{G}_\delta(z_k) < \varepsilon. \tag{B.3.16}$$

Observe that there is $r_0 \le r_1$ such that if a point z_k of the sequence $\{z_k\}_k$ belongs to $\{|z| \le r_0\}$, then $D^\delta(z_k) \subset \{|z| \le r_1\}$. So take k big enough such that

$$\sup_{\{|z|\le r_1\}} \|f_k(z)\|_{\mathcal{H}} < \varepsilon.$$

This together with (B.2.10), and similarly to (B.3.9), we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n} \|f_k(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z)\\
&\lesssim \int_{\{|\zeta|\le r_1\}} \|f_k(z)\|^2_{\mathcal{H}} e^{-2\phi(z)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dA(z)\\
&+ \sup_{|z_k|>r_o} \hat{G}_\delta(z_k) \sum_{|z_k|>r_0} \int_{D^{m\delta}(z_k)} \|f_k(w)\|^2_{\mathcal{H}} e^{-2\phi(w)} dA(w)\\
&\lesssim \varepsilon + \varepsilon \|f_k\|^2_{2,\phi} \lesssim \varepsilon,
\end{aligned}$$

because $\{f_k\}_k$ belongs to $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ and is bounded. This implies that G satisfies a vanishing Carleson condition.

To finish the proof, we show that (5) implies (2). Assuming (v), $I_G : F^2_\phi(\mathbb{C}^n, \mathcal{H}) \to L^2_\phi(\mathbb{C}^n, \mathcal{H}, \|G\|_{\mathcal{L}(\mathcal{H})} dA)$ is compact. Since $\{k_z e : z \in \mathbb{C}^n\}$ is bounded in $F^2_\phi(\mathbb{C}^n, \mathcal{H})$ and $k_z e \to 0$ uniformly on compact subsets of $\mathbb{C}^n$, using (B.1.7), we have

$$\int_{\mathbb{C}^n} |k_z(w)|^2 e^{-2\phi(w)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dA(w) \to 0, \quad \text{as } |z| \to \infty,$$

and therefore (2) holds. This proves the desired result and completes the proof. ☐

B.4 Schatten class membership of vectorial Toeplitz operators

Here we give proofs of Theorem B.1.4, Proposition B.1.5, Proposition B.1.6, and Theorem B.1.7, characterizing the p-Schatten class membership of the vectorial Toeplitz operator T_G acting on the Hilbert space $F^2_\phi(\mathbb{C}^n, \mathcal{H})$. Before going through the proof, notice the following lemmas regarding an arbitrary δ-lattice that are going to be useful in the proof of Theorem B.1.7.

Lemma B.4.1. *For $R > 0$ and any finite sequence $\{z_j\}_{j=1}^m$ of different points in $\mathbb{C}^n$, let*

$$M_R(\{z_j\}_{j=1}^m) := \max_{1\le j\le m} \#\{k \in \{1, \cdots, m\} : |z_j - z_k| < R \min(\rho(z_j), \rho(z_k))\}.$$

Then $\{z_j\}_{j=1}^m$ can be partitioned into at most $M_R(\{z_j\}_{j=1}^m)$ subsequences such that any two different points z_j and z_k in the same subsequence satisfy either $z_j \notin D^R(z_k)$, or $z_k \notin D^R(z_j)$. That is, $|z_j - z_k| \ge R \min(\rho(z_j), \rho(z_k))$.

Proof. The proof is identical to the doubling Fock spaces of the complex plane, as done in Lemma 6.8 in [68]. ☐

Lemma B.4.2. *Let* $\delta > 0$, $R > 1$, *and* $\{z_j\}_{j\geq 1}$ *be a* δ*-lattice. Then* $M_R(\{z_j\}_{j=1}^m) \leq 6^{2n}R^{4n}\delta^{-2n}N_\delta$, *for every finite sublattice* $\{z_j\}_{j=1}^m$, *where* $N_\delta = \sup_{z\in\mathbb{C}^n}\sum_{j=1}^\infty \chi_{D^\delta(z_j)}(z)$, *as in (B.2.10).*

Proof. The proof can be done similarly as in Lemma 6.9 in [68]. ☐

Proof of Theorem B.1.4. Let $\{e_m\}_{m\geq 1}$ be any orthonormal basis of $\mathcal{H}$. Notice that since $G(z)$ is a positive operator on $\mathcal{H}$, so are $\hat{G}_\delta^{op}(z)$ and $\tilde{G}^{op}(z)$. Moreover, T_G is also a positive operator acting on $F_\phi^2(\mathbb{C}^n,\mathcal{H})$. First, we show that (1) implies (2). By the definition of the operator-valued Berezin transform, applying Proposition 1.31 in [80], Lemma B.2.3, and Lemma 5.1 in [19], we obtain

$$
\begin{aligned}
&\int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\langle \widetilde{G}^{op}(z)e_m, e_m\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&= \int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\int_{\mathbb{C}^n}\langle G(\zeta)e_m, e_m\rangle_{\mathcal{H}}|k_z(\zeta)|^2\, e^{-2\phi(\zeta)}dA(\zeta)\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&= \int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\int_{\mathbb{C}^n}\langle G(\zeta)k_z(\zeta)e_m, k_z(\zeta)e_m\rangle_{\mathcal{H}}\, e^{-2\phi(\zeta)}dA(\zeta)\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&= \int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\int_{\mathbb{C}^n}\langle T_G k_z(\zeta)e_m, k_z(\zeta)e_m\rangle_{\mathcal{H}}\, e^{-2\phi(\zeta)}dA(\zeta)\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&= \int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\langle T_G k_z e_m, k_z e_m\rangle\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&\leq \int_{\mathbb{C}^n}\sum_{m=1}^\infty \langle T_G^p k_z e_m, k_z e_m\rangle \frac{dA(z)}{\rho(z)^{2n}}\\
&\simeq \int_{\mathbb{C}^n}\sum_{m=1}^\infty \langle T_G^p K_z e_m, K_z e_m\rangle e^{-2\phi(z)}dA(z)\\
&= \mathrm{Tr}(T_G^p) < \infty.
\end{aligned}
\tag{B.4.1}
$$

Now we show that (2) implies (3). Indeed, using (B.2.13) and (B.2.6),

$$
\begin{aligned}
&\int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\Big\langle \hat{G}_\delta^{op}(z)e_m, e_m\Big\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&\simeq \int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\Big\langle \int_{D^\delta(z)} G(\zeta)\, e_m \frac{dA(\zeta)}{\rho(\zeta)^{2n}}, e_m\Big\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&\simeq \int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\Big\langle \int_{\mathbb{C}^n} G(\zeta)\, e_m |k_z(\zeta)|^2\, e^{-2\phi(\zeta)}dA(\zeta), e_m\Big\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}}\\
&= \int_{\mathbb{C}^n}\sum_{m=1}^\infty \Big(\Big\langle \tilde{G}^{op}(\zeta)\, e_m, e_m\Big\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}},
\end{aligned}
$$

and we can see that (2) and (3) are equivalent. Moreover, the statement (3) implies (1). Indeed, let $\{f_k\}_{k\geq 1}$ be an orthonomal basis of $F_\phi^2(\mathbb{C}^n)$. Then $\{f_k e_m\}_{k,m\geq 1}$ is an orthonomal basis of $F_\phi^2(\mathbb{C}^n,\mathcal{H})$. By Lemma B.2.15 we see that $T_{\hat{G}_\delta^{op}} \in S_p(F_\phi^2(\mathbb{C}^n,\mathcal{H}))$. Recall that for any $\zeta \in D^\delta(z)$, there exists some small enough $r > 0$ such that $D^r(\zeta) \subset D^\delta(z)$, coming from the Hausdorff property of $\mathbb{C}^n$. Moreover, for any $m, k \geq 1$, applying Fubini's Theorem and Lemma B.2.7, we get

$$\begin{aligned}
\langle T_{\hat{G}_\delta^{op}} f_k e_m, f_k e_m \rangle &= \int_{\mathbb{C}^n} \langle T_{\hat{G}_\delta^{op}} f_k(z) e_m, f_k(z) e_m \rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z) \\
&= \int_{\mathbb{C}^n} \langle \hat{G}_\delta^{op}(z) f_k(z) e_m, f_k(z) e_m \rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z) \\
&\simeq \int_{\mathbb{C}^n} \frac{1}{\rho(z)^{2n}} \int_{D^\delta(z)} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} |f_k(z)|^2 e^{-2\phi(z)} dA(\zeta) dA(z) \\
&\gtrsim \int_{\mathbb{C}^n} \frac{1}{\rho(\zeta)^{2n}} \int_{D^r(\zeta)} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} |f_k(z)|^2 e^{-2\phi(z)} dA(z) dA(\zeta) \\
&\gtrsim \int_{\mathbb{C}^n} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} |f_k(\zeta)|^2 e^{-2\phi(\zeta)} dA(\zeta) \\
&= \int_{\mathbb{C}^n} \langle G(\zeta) f_k(\zeta) e_m, f_k(\zeta) e_m \rangle_{\mathcal{H}} e^{-2\phi(\zeta)} dA(\zeta) \\
&= \langle T_G f_k e_m, f_k e_m \rangle.
\end{aligned}$$

Since $T_{\hat{G}_\delta^{op}} \in S_p(F_\phi^2(\mathbb{C}^n, \mathcal{H}))$, by definition $\sum_{k,m=1}^\infty (\langle T_{\hat{G}_\delta^{op}} f_k e_m, f_k e_m \rangle)^p < \infty$, implying that

$$\sum_{k,m=1}^\infty (\langle T_G f_k e_m, f_k e_m \rangle)^p < \infty.$$

Thus $T_G \in S_p(F_\phi^2(\mathbb{C}^n, \mathcal{H}))$.

To finish the proof, it remains to show that the statements (3) and (4) are equivalent. Suppose that (3) holds and let $\{z_j\}_{j\geq 1}$ be a δ-lattice. Let $z \in D^\delta(z_j)$. Then by (B.2.8), there is some $c > 1$ such that $D^\delta(z_j) \subset D^{c\delta}(z)$. Then by definition it is easy to see that $\hat{G}_\delta^{op}(z_j) \lesssim \hat{G}_{c\delta}^{op}(z)$. That is, for any $h \in \mathcal{H}$, $\langle \hat{G}_\delta^{op}(z_j) h, h \rangle_{\mathcal{H}} \lesssim \langle \hat{G}_{c\delta}^{op}(z) h, h \rangle_{\mathcal{H}}$. Hence, using (B.2.6) and (B.2.10),

$$\begin{aligned}
\sum_{j,m=1}^\infty \left(\langle \hat{G}_\delta^{op}(z_j) e_m, e_m \rangle_{\mathcal{H}} \right)^p &= \sum_{j,m=1}^\infty \frac{\rho(z_j)^{2n}}{\rho(z_j)^{2n}} \left(\langle \hat{G}_\delta^{op}(z_j) e_m, e_m \rangle_{\mathcal{H}} \right)^p \\
&\simeq \sum_{j,m=1}^\infty \int_{D^\delta(z_j)} \left(\langle \hat{G}_\delta^{op}(z_j) e_m, e_m \rangle_{\mathcal{H}} \right)^p \frac{dA(z)}{\rho(z_j)^{2n}} \\
&\lesssim \sum_{j,m=1}^\infty \int_{D^\delta(z_j)} \left(\langle \hat{G}_{c\delta}^{op}(z) e_m, e_m \rangle_{\mathcal{H}} \right)^p \frac{dA(z)}{\rho(z)^{2n}} \\
&= \sum_{m=1}^\infty \int_{\mathbb{C}^n} \sum_{j=1}^\infty \chi_{D^\delta(z_j)}(z) \left(\langle \hat{G}_{c\delta}^{op}(z) e_m, e_m \rangle_{\mathcal{H}} \right)^p \frac{dA(z)}{\rho(z)^{2n}} \\
&\lesssim \sum_{m=1}^\infty \int_{\mathbb{C}^n} \left(\langle \hat{G}_{c\delta}^{op}(z) e_m, e_m \rangle_{\mathcal{H}} \right)^p \frac{dA(z)}{\rho(z)^{2n}}.
\end{aligned}$$

Conversely, since $\rho(z_j) \simeq \rho(z)$ for any $z \in D^{\delta}(z_j)$, and $D^{\delta}(z) \subset D^{c\delta}(z_j)$ for some $c > 1$, we have

$$
\begin{aligned}
&\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \hat{G}^{op}_{\delta}(z) e_m, e_m \rangle_{\mathcal{H}} \Big)^p \frac{dA(z)}{\rho(z)^{2n}} \\
&\simeq \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \bigg(\frac{1}{\rho(z)^{2n}} \int_{D^{\delta}(z)} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} dA(\zeta) \bigg)^p \frac{dA(z)}{\rho(z)^{2n}} \\
&\lesssim \sum_{j=1}^{\infty} \int_{D^{\delta}(z_j)} \sum_{m=1}^{\infty} \bigg(\frac{1}{\rho(z_j)^{2n}} \int_{D^{\delta}(z)} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} dA(\zeta) \bigg)^p \frac{dA(z)}{\rho(z_j)^{2n}} \\
&\leq \sum_{j,m=1}^{\infty} \int_{D^{\delta}(z_j)} \bigg(\frac{1}{\rho(z_j)^{2n}} \int_{D^{c\delta}(z_j)} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} dA(\zeta) \bigg)^p \frac{dA(z)}{\rho(z_j)^{2n}} \\
&\simeq \sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_{c\delta}(z_j) e_m, e_m \rangle_{\mathcal{H}} \Big)^p .
\end{aligned}
$$

This completes the proof for any choice of orthonormal basis $\{e_m\}_{m \geq 1}$. □

Proof of Proposition B.1.5. Let $0 < p \leq 1$. Similar to the previous case, we can see that (1) implies (2) and (2) implies (3). Now, we prove that (3) implies (1). First note that by Lemma B.2.7, (B.2.8), and Fubini's Theorem, there is some $c > 1$ such that

$$
\begin{aligned}
\langle \tilde{G}^{op}(z) e_m, e_m \rangle_{\mathcal{H}} &= \int_{\mathbb{C}^n} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} |k_z(\zeta)|^2 e^{-2\phi(\zeta)} dA(\zeta) \\
&\lesssim \int_{\mathbb{C}^n} \langle G(\zeta) e_m, e_m \rangle_{\mathcal{H}} \bigg(\int_{D^{\delta}(\zeta)} |k_z(\xi)|^2 e^{-2\phi(\xi)} \frac{dA(\xi)}{\rho(\zeta)^{2n}} \bigg) dA(\zeta) \\
&\lesssim \int_{\mathbb{C}^n} \bigg(\langle \int_{D^{c\delta}(\xi)} G(\zeta) e_m \frac{dA(\zeta)}{\rho(\zeta)^{2n}}, e_m \rangle_{\mathcal{H}} \bigg) |k_z(\xi)|^2 e^{-2\phi(\xi)} dA(\xi) \\
&\simeq \int_{\mathbb{C}^n} \langle \hat{G}^{op}_{c\delta}(\xi) e_m, e_m \rangle_{\mathcal{H}} |k_z(\xi)|^2 e^{-2\phi(\xi)} dA(\xi).
\end{aligned}
$$

Since $p \leq 1$, $(x+y)^p \leq x^p + y^p$. Moreover, whenever $\xi \in D^{\delta}(z_j)$, $D^{c\delta}(\xi) \subset D^{r}(z_j)$ for some large enough $r > 0$ and $\rho(\xi) \simeq \rho(z_j)$. Hence, it is easy to see that $\hat{G}^{op}_{c\delta}(\xi) \lesssim \hat{G}^{op}_{r}(z_j)$. Thus, we obtain the following.

$$
\begin{aligned}
&\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z) e_m, e_m \rangle_{\mathcal{H}} \Big)^p dA(z) \\
&\lesssim \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \bigg(\int_{\mathbb{C}^n} \langle \hat{G}^{op}_{c\delta}(\xi) e_m, e_m \rangle_{\mathcal{H}} |k_z(\xi)|^2 e^{-2\phi(\xi)} dA(\xi) \bigg)^p \frac{dA(z)}{\rho(z)^{2n}} \\
&\leq \sum_{j,m=1}^{\infty} \int_{\mathbb{C}^n} \bigg(\int_{D^{\delta}(z_j)} \langle \hat{G}^{op}_{c\delta}(\xi) e_m, e_m \rangle_{\mathcal{H}} |k_z(\xi)|^2 e^{-2\phi(\xi)} dA(\xi) \bigg)^p \frac{dA(z)}{\rho(z)^{2n}} \\
&\leq \sum_{j,m=1}^{\infty} \Big(\langle \hat{G}^{op}_{r}(z_j) e_m, e_m \rangle_{\mathcal{H}} \Big)^p \int_{\mathbb{C}^n} \bigg(\sup_{\xi \in D^{\delta}(z_j)} |k_z(\xi)|^2 e^{-2\phi(\xi)} \rho(z_j)^{2n} \bigg)^p \frac{dA(z)}{\rho(z)^{2n}}.
\end{aligned}
$$

Using the fact that $|k_z(\xi)|^2 e^{-2\phi(\xi)} \rho(\xi)^{2n} \lesssim \exp\Big(-\varepsilon d_{\rho}(z,\xi)\Big)$, for any $\varepsilon > 0$, and Lemma 2 in [62], we get

$$\int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \tilde{G}^{op}(z)e_m, e_m\rangle_{\mathcal{H}}\Big)^p dA(z) \lesssim \sum_{j,m=1}^{\infty} \Big(\langle \hat{G}_r^{op}(z_j)e_m, e_m\rangle_{\mathcal{H}}\Big)^p \int_{\mathbb{C}^n} e^{-p\epsilon d_\rho(z,z_j)} \frac{dA(z)}{\rho(z)^{2n}}$$

$$\lesssim \sum_{j,m=1}^{\infty} \Big(\langle \hat{G}_r^{op}(z_j)e_m, e_m\rangle_{\mathcal{H}}\Big)^p \rho(z_j)^{2n-2n}$$

$$= \sum_{j,m=1}^{\infty} \Big(\langle \hat{G}_r^{op}(z_j)e_m, e_m\rangle_{\mathcal{H}}\Big)^p.$$

We have just proved that (1), (2), and (3) are equivalent. □

Proof of Proposition B.1.6. We know that $T_G \in S_p$ if and only if $T_G^p \in S_1$. By Lemma 5.1 in [19] and (B.4.1), $T_G \in S_p$ is equivalent to

$$\operatorname{Tr}(T_G^p) \simeq \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \langle T_G^p k_z e_m, k_z e_m\rangle \frac{dA(z)}{\rho(z)^{2n}} < +\infty.$$

By Proposition 1.31 in [80] for when $p \leq 1$, and since positivity of $G(z)$ implies positivity of T_G, we get

$$\langle T_G^p k_z e_m, k_z e_m\rangle \lesssim \langle T_G k_z e_m, k_z e_m\rangle^p.$$

This together with (B.4.1) implies that

$$\operatorname{Tr}(T_G^p) \lesssim \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} (\langle T_G k_z e_m, k_z e_m\rangle)^p \frac{dA(z)}{\rho(z)^{2n}} = \int_{\mathbb{C}^n} \sum_{m=1}^{\infty} \Big(\langle \widetilde{G}^{op}(z)e_m, e_m\rangle_{\mathcal{H}}\Big)^p \frac{dA(z)}{\rho(z)^{2n}} < \infty,$$

and we are done. □

We finish this section by giving a proof of Theorem B.1.7. First, notice that in Theorem B.1.4, Proposition B.1.5, and Proposition B.1.6, we have not made any specific assumptions on the basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, as each of the proofs allows for any generic basis of $\mathcal{H}$. But now we will see how we are forced to restrict ourselves to specific bases of $\mathcal{H}$. Note that the idea of the proof originally comes from the work of Luecking [61] in studying the Schatten class Toeplitz operators on Hardy and Bergman spaces. Later, an analogous idea was applied to studying the Schatten class Toeplitz operators with measure symbols on doubling Fock spaces in [68]. This approach has also been used to study the Schatten class Hankel operators acting on generalized Fock spaces in [10, 51].

Proof of Theorem B.1.7. Assume that $0 < \delta < 1/2$, $R > 1$, and fix $M \in \mathbb{N}$. Let $\{z_j\}_{j=1}^M$ be the finite sublattice obtained by considering the first M elements of the δ-lattice $\{z_j\}_{j\geq 1}$. Then Lemma B.4.1 implies that $\{z_j\}_{j=1}^M$ can be partitioned into $M_R(\{z_j\}_{j=1}^M)$ subsequences such that any two different points a_j and a_k in the same subsequence satisfy $|a_j - a_k| \geq R\min(\rho(a_j), \rho(a_k))$. Let $\{a_j\}_{j=1}^s$ be one such subsequence. Let $\{B_{k,m}(z) = f_k(z)e_m\}_{k,m\geq 1}$ be an orthonormal basis of $F_\phi^2(\mathbb{C}^n, \mathcal{H})$ with $\{f_k\}_{k\geq 1}$ being an orthonormal basis of $F_\phi^2(\mathbb{C}^n)$, and $\{e_m\}_{m\geq 1}$ any orthonormal basis of $\mathcal{H}$. We consider a bounded linear operator A on $F_\phi^2(\mathbb{C}^n, \mathcal{H})$ by $Af(z) = \sum_{k,m=1}^s \langle f, B_{k,m}\rangle k_{a_k}(z)e_m$. Indeed, let $f, g \in F_\phi^2(\mathbb{C}^n, \mathcal{H})$. Then by the Cauchy-Schwarz inequality, (B.2.12), Lemma B.2.4, Lemma B.2.7, and (B.2.10), we obtain

$$\begin{aligned}
|\langle Af, g\rangle| &\leq \|f\|_{2,\phi}\left(\sum_{k,m=1}^{s}|\langle k_{a_k}e_m, g\rangle|^2\right)^{1/2}\\
&\simeq \|f\|_{2,\phi}\left(\sum_{k,m=1}^{s}\left|\int_{\mathbb{C}^n}\rho(a_k)^n e^{-\phi(a_k)}\langle e_m, g(z)\rangle_{\mathcal{H}}K(z,a_k)e^{-2\phi(z)}dA(z)\right|^2\right)^{1/2}\\
&= \|f\|_{2,\phi}\left(\sum_{k,m=1}^{s}\rho(a_k)^{2n}e^{-2\phi(a_k)}|\langle e_m, g(a_k)\rangle_{\mathcal{H}}|^2\right)^{1/2}\\
&\lesssim \|f\|_{2,\phi}\left(\sum_{k,m=1}^{s}\int_{D^\delta(a_k)}|\langle e_m, g(w)\rangle_{\mathcal{H}}|^2 e^{-2\phi(w)}dA(w)\right)^{1/2}\\
&\leq \|f\|_{2,\phi}\left(\sum_{k=1}^{s}\int_{D^\delta(a_k)}\|g(w)\|_{\mathcal{H}}^2 e^{-2\phi(w)}dA(w)\right)^{1/2} \leq N^{1/2}\|f\|_{2,\phi}\|g\|_{2,\phi},
\end{aligned}$$

where the constants do not depend on s. Hence, A is bounded. Moreover, let $U(z) = (\sum_{j=1}^{s}\chi_{D^\delta(a_j)}(z))G(z)$. Then $U \leq NG$, where N is as in (B.2.10). That is, $NG(z) - U(z)$ is a positive operator on $\mathcal{H}$ for every $z \in \mathbb{C}^n$. Since $T_G \in S_p$, we can conclude that $T_U \in S_p$, and $\|T_U\|_{S_p} \leq N\|T_G\|_{S_p}$. Set $T = A^*T_UA$ such that $\|T\|_{S_p} \lesssim \|T_U\|_{S_p}$. It is easy to see that when $k, m > s$, $\langle TB_{k,m}, B_{k,m}\rangle = \langle T_UAB_{k,m}, AB_{k,m}\rangle = 0$. We can split T as $T = D_s + M_s$, where D_s is the diagonal operator defined by

$$D_sf = \sum_{k,m=1}^{s}\langle TB_{k,m}, B_{k,m}\rangle\langle f, B_{k,m}\rangle B_{k,m}, \quad \text{where } f \in F_\phi^2(\mathbb{C}^n, \mathcal{H}), \tag{B.4.2}$$

and M_s is the off-diagonal operator defined by

$$\begin{aligned}
M_sf = &\sum_{k,m=1}^{s}\sum_{\substack{r,n=1\\ r\neq k, m\neq n}}^{s}\langle TB_{k,m}, B_{r,n}\rangle\langle f, B_{k,m}\rangle B_{r,n} + \sum_{k,m=1}^{s}\sum_{\substack{r=1\\ r\neq k,}}^{s}\langle TB_{k,m}, B_{r,m}\rangle\langle f, B_{k,m}\rangle B_{r,m}\\
&+\sum_{k,m=1}^{s}\sum_{\substack{n=1\\ m\neq n,}}^{s}\langle TB_{k,m}, B_{k,n}\rangle\langle f, B_{k,m}\rangle B_{k,n}, \quad \text{where } f \in F_\phi^2(\mathbb{C}^n, \mathcal{H}).
\end{aligned} \tag{B.4.3}$$

Recall that $U(z) = 0$ if $z \notin \cup_{j=1}^{s}D^\delta(a_j)$. Then using (B.2.13), and positivity of $G(z)$ and $\hat{G}_\delta^{op}(z)$, there is a constant $C_1 > 0$, only depending on δ such that

$$\begin{aligned}
\|D_s\|_{S_p}^p &= \sum_{i,j=1}^{s} \left|\langle D_s B_{i,j}, B_{i,j}\rangle\right|^p = \sum_{m=1}^{s}\sum_{k=1}^{s} \left|\langle T B_{k,m}, B_{k,m}\rangle\right|^p \\
&= \sum_{m=1}^{s}\sum_{k=1}^{s} \left| \int_{\mathbb{C}^n} \langle T B_{k,m}(z), B_{k,m}(z)\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z)\right|^p \\
&= \sum_{m=1}^{s}\sum_{k=1}^{s} \left| \int_{\mathbb{C}^n} \langle T_U k_{a_k}(z) e_m, k_{a_k}(z) e_m\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z)\right|^p \\
&= \sum_{m=1}^{s}\sum_{k=1}^{s} \left| \int_{\mathbb{C}^n} \langle U(z) k_{a_k}(z) e_m, k_{a_k}(z) e_m\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z)\right|^p \\
&\geq \sum_{m=1}^{s}\sum_{k=1}^{s} \left| \int_{D^{\delta}(a_k)} \langle G(z) e_m, e_m\rangle_{\mathcal{H}} |k_{a_k}(z)|^2 e^{-2\phi(z)} dA(z)\right|^p \\
&\geq C_1 \sum_{m=1}^{s}\sum_{k=1}^{s} \left| \int_{D^{\delta}(a_k)} \langle G(z) e_m, e_m\rangle_{\mathcal{H}} \frac{dA(z)}{\rho(z)^{2n}}\right|^p = C_1 \sum_{m=1}^{s}\sum_{k=1}^{s} \left(\langle \hat{G}_{\delta}^{op}(a_k) e_m, e_m\rangle_{\mathcal{H}}\right)^p.
\end{aligned} \tag{B.4.4}$$

On the other hand, by Proposition 1.29 in [80], and using the fact that $(x+y)^p \leq x^p + y^p$ for $p \leq 1$, we obtain

$$\begin{aligned}
\|M_s\|_{S_p}^p &\leq \sum_{s,q=1}^{s}\sum_{i,j=1}^{s} \left|\langle M_s B_{s,q}, B_{i,j}\rangle\right|^p \\
&= \sum_{\substack{r,k=1 \\ r\neq k}}^{s}\sum_{\substack{m,n=1 \\ m\neq n}}^{s} \left|\langle T B_{k,m}, B_{r,n}\rangle\right|^p + \sum_{\substack{k,r=1 \\ r\neq k}}^{s}\sum_{m=1}^{s} \left|\langle T B_{k,m}, B_{r,m}\rangle\right|^p + \sum_{k=1}^{s}\sum_{\substack{n=1 \\ m\neq n}}^{s} \left|\langle T B_{k,m}, B_{k,n}\rangle\right|^p \\
&= \sum_{\substack{r,k=1 \\ r\neq k}}^{s}\sum_{\substack{m,n=1 \\ m\neq n}}^{s} \left| \int_{\mathbb{C}^n} \langle U(z) k_{a_k}(z) e_m, k_{a_r}(z) e_n\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z)\right|^p \\
&\quad + \sum_{\substack{k,r=1 \\ r\neq k}}^{s}\sum_{m=1}^{s} \left| \int_{\mathbb{C}^n} \langle U(z) k_{a_k}(z) e_m, k_{a_r}(z) e_m\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z)\right|^p \\
&\quad + \sum_{k=1}^{s}\sum_{\substack{n=1 \\ m\neq n}}^{s} \left| \int_{\mathbb{C}^n} \langle U(z) k_{a_k}(z) e_m, k_{a_k}(z) e_n\rangle_{\mathcal{H}} e^{-2\phi(z)} dA(z)\right|^p.
\end{aligned} \tag{B.4.5}$$

Then by the definition of $U(z)$, and since $G(z)$ is a positive operator, we can write (B.4.5) as follows. Note that positivity of $G(z)$ implies that $\langle G(z) e_m, e_m\rangle_{\mathcal{H}} \geq 0$, but $\langle G(z) e_m, e_n\rangle_{\mathcal{H}}$ is a complex number.

$$\begin{aligned}
\|M_s\|_{S_p}^p \leq N^p &\sum_{\substack{r,k=1 \\ r\neq k}}^{s}\sum_{\substack{m,n=1 \\ m\neq n}}^{s} \left| \sum_{j=1}^{s} \int_{D^{\delta}(a_j)} \langle G(z) e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)| |k_{a_r}(z)| e^{-2\phi(z)} dA(z)\right|^p \\
&+ N^p \sum_{\substack{k,r=1 \\ r\neq k}}^{s}\sum_{m=1}^{s} \left(\sum_{j=1}^{s} \int_{D^{\delta}(a_j)} \langle G(z) e_m, e_m\rangle_{\mathcal{H}} |k_{a_k}(z)| |k_{a_r}(z)| e^{-2\phi(z)} dA(z)\right)^p \\
&+ N^p \sum_{k=1}^{s}\sum_{\substack{n=1 \\ m\neq n}}^{s} \left| \sum_{j=1}^{s} \int_{D^{\delta}(a_j)} \langle G(z) e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)|^2 e^{-2\phi(z)} dA(z)\right|^p.
\end{aligned} \tag{B.4.6}$$

Define

$$J_{k,r}^{m,n}(G,s) = \sum_{j=1}^{s} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)||k_{a_r}(z)|e^{-2\phi(z)}dA(z).$$

Since $k \neq r$, then $|a_k - a_r| \geq R\min(\rho(a_k), \rho(a_r))$. Thus for $z \in D^\delta(a_j)$ it is easy to see that either

$$|z - a_k| \geq \tilde{R}\min(\rho(z), \rho(a_k)), \tag{B.4.7}$$

or

$$|z - a_r| \geq \tilde{R}\min(\rho(z), \rho(a_r)), \tag{B.4.8}$$

where $\tilde{R} = \frac{R-1}{3}$. Indeed, since $k \neq r$, then either $j \neq k$ or $j \neq r$. Without loss of generality assume that $j \neq k$, and therefore $|a_j - a_k| \geq R\min(\rho(a_j), \rho(a_k))$. By contradiction, assume that $|z - a_k| < \tilde{R}\min(\rho(z), \rho(a_k))$. Then by (B.2.6),

$$\begin{aligned} |a_j - a_k| &\leq |a_j - z| + |z - a_k| < \delta\rho(a_j) + \tilde{R}\min(\rho(z), \rho(a_k)) \\ &\leq \delta\rho(a_j) + \tilde{R}\min((1+\delta)\rho(a_j), \rho(a_k)) \\ &\leq \delta\rho(a_j) + (1+\delta)\tilde{R}\min(\rho(a_j), \rho(a_k)). \end{aligned}$$

If $\rho(a_j) \leq \rho(a_k)$, then $|a_j - a_k| < (\delta + (1+\delta)\tilde{R})\rho(a_j) < \frac{1+3\tilde{R}}{2}\rho(a_j) = \frac{R}{2}\rho(a_j)$, which is a contradiction. Similarly, if $\rho(a_k) \leq \rho(a_j)$, (B.2.5) implies that

$$\begin{aligned} |a_j - a_k| &< \delta\rho(a_j) + (1+\delta)\tilde{R}\min(\rho(a_j), \rho(a_k)) \\ &\leq \delta\rho(a_k) + \delta|a_j - a_k| + (1+\delta)\tilde{R}\rho(a_k). \end{aligned}$$

Hence, $|a_j - a_k| < \frac{\delta + (1+\delta)\tilde{R}}{1-\delta}\rho(a_k) < (1 + 3\tilde{R})\rho(a_k) = R\rho(a_k)$, resulting in a contradiction. Therefore we can conclude that either (B.4.7) or (B.4.8) holds.

Using (B.2.3) and (B.2.12), we can write

$$|k_{a_k}(z)||k_{a_r}(z)|e^{-2\phi(z)} \lesssim \frac{e^{\frac{-\epsilon}{2}[d_\rho(z,a_k) + d_\rho(z,a_r)]}}{\rho(z)^n}|k_{a_k}(z)|^{1/2}|k_{a_r}(z)|^{1/2}e^{-\phi(z)}.$$

Furthermore, since $z \in D^\delta(a_j)$, and $r \neq k$, we can assume that $j \neq k$, and by the above argument we have $d_\rho(z, a_k) + d_\rho(z, a_r) \geq d_\rho(a_r, a_k) \geq \tilde{R}$. Hence, for $z \in D^\delta(a_j)$, we can conclude that $e^{\frac{-\epsilon}{2}[d_\rho(z,a_k) + d_\rho(z,a_r)]} \leq e^{\frac{-\epsilon}{2}\tilde{R}}$. Hence, using $\rho(z) \simeq \rho(a_j)$, we obtain

$$J_{k,r}^{m,n}(G,s) \lesssim e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \frac{1}{\rho(a_j)^n} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} |k_{a_k}(z)|^{1/2}|k_{a_r}(z)|^{1/2}e^{-\phi(z)}dA(z).$$

By Lemma B.2.7 we have

$$|k_{a_k}(z)|^{1/2}e^{-\phi(z)/2} \lesssim \left(\frac{1}{\rho(z)^{2n}} \int_{D^\delta(z)} |k_{a_k}(\xi)|^{p/2}e^{-\frac{p}{2}\phi(\xi)}dA(\xi)\right)^{1/p}.$$

Since $z \in D^\delta(a_j)$, there exists some $m_1 > 1$ such that $D^\delta(z) \subset D^{m_1\delta}(a_j)$. Therefore,

$$|k_{a_k}(z)|^{1/2}e^{-\phi(z)/2} \lesssim \frac{1}{\rho(z)^{2n/p}}\left(\int_{D^{m_1\delta}(a_j)} |k_{a_k}(\xi)|^{p/2}e^{-\frac{p}{2}\phi(\xi)}dA(\xi)\right)^{1/p} = \frac{1}{\rho(z)^{2n/p}}S_k(a_j)^{1/p},$$

where

$$S_k(z) = \int_{D^{m_1\delta}(z)} |k_{a_k}(\xi)|^{p/2}e^{-\frac{p}{2}\phi(\xi)}dA(\xi).$$

Similarly,

$$|k_{a_r}(z)|^{1/2}e^{-\phi(z)/2} \lesssim \frac{1}{\rho(z)^{2n/p}} S_r(a_j)^{1/p}.$$

Therefore,

$$J_{k,r}^{m,n}(G,s) \lesssim e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \frac{1}{\rho(a_j)^n} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} \frac{1}{\rho(z)^{4n/p}} S_k(a_j)^{1/p} S_r(a_j)^{1/p} dA(z).$$

Since $0 < p < 1$, $4/p - 1 > 1$, and

$$\begin{aligned}
J_{k,r}^{m,n}(G,s) &\lesssim e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \frac{1}{\rho(a_j)^n} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} \frac{1}{\rho(a_j)^{4n/p}} S_k(a_j)^{1/p} S_r(a_j)^{1/p} dA(z) \\
&\simeq e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \frac{1}{\rho(a_j)^{n(\frac{4}{p}-1)}} S_k(a_j)^{1/p} S_r(a_j)^{1/p} \int_{D^\delta(a_j)} \langle G(z)e_m, e_n\rangle_{\mathcal{H}} \frac{dA(z)}{\rho(z)^{2n}} \\
&\simeq e^{\frac{-\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \rho(a_j)^{n(1-\frac{4}{p})} S_k(a_j)^{1/p} S_r(a_j)^{1/p} \langle \hat{G}_\delta^{op}(a_j)e_m, e_n\rangle_{\mathcal{H}}.
\end{aligned}$$

Then since $0 < p < 1$,

$$\left|J_{k,r}^{m,n}(G,s)\right|^p \lesssim e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} |\langle \hat{G}_\delta^{op}(a_j)e_m, e_n\rangle_{\mathcal{H}}|^p \rho(a_j)^{n(p-4)} S_k(a_j) S_r(a_j). \tag{B.4.9}$$

Recall that

$$\begin{aligned}
\sum_{k=1}^{s} S_k(a_j) &= \sum_{k=1}^{s} \int_{D^{m_1\delta}(a_j)} |k_{a_k}(\xi)|^{p/2} e^{-\frac{p}{2}\phi(\xi)} dA(\xi) \\
&= \int_{D^{m_1\delta}(a_j)} \left(\sum_{k=1}^{s} |k_{a_k}(\xi)|^{p/2} \right) e^{-\frac{p}{2}\phi(\xi)} dA(\xi).
\end{aligned}$$

Moreover, using (B.2.10), (B.2.12), (B.2.14), and Lemma B.2.7, we can write

$$\begin{aligned}
\sum_{k=1}^{s} |k_{a_k}(\xi)|^{p/2} &= \sum_{k=1}^{s} \rho(a_k) |K_\xi(a_k)|^{p/2} e^{\frac{-p}{2}\phi(a_k)} \\
&\lesssim \sum_{k=1}^{s} \int_{D^\delta(a_k)} |K_\xi(z)|^{p/2} e^{\frac{-p}{2}\phi(z)} \rho(z)^{n(\frac{p}{2}-2)} dA(z) \\
&\le N \int_{\mathbb{C}^n} |K_\xi(z)|^{p/2} e^{\frac{-p}{2}\phi(z)} \rho(z)^{n(\frac{p}{2}-2)} dA(z) \simeq N e^{\frac{p}{2}\phi(\xi)} \rho(\xi)^{\frac{-np}{2}}.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{s} S_k(a_j) \lesssim N \int_{D^{m_1\delta}(a_j)} \rho(\xi)^{\frac{-np}{2}} dA(\xi) \simeq \rho(a_j)^{2n-\frac{np}{2}},$$

where the constant only depends on δ. Similarly,

$$\sum_{r=1}^{s} S_r(a_j) \lesssim \rho(a_j)^{2n-\frac{np}{2}},$$

and this together with (B.4.9) implies that

$$\sum_{\substack{k,r=1\\ r\neq k}}^{s} \left|J^{m,n}_{k,r}(G,s)\right|^p \lesssim e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e_m, e_n\rangle_{\mathcal{H}}|^p, \tag{B.4.10}$$

where the constant only depends on $0 < \delta < 1/2$.

Since $T_G \in S_p(F^2_{\phi}(\mathbb{C}^n, \mathcal{H}))$, it is in particular compact. Lemma B.2.16 along with compactness of $G(w)$ for every $w \in \mathbb{C}^n$, then implies that $\hat{G}^{op}_{\delta}(a_j)$ is a compact operator on $\mathcal{H}$ for every $a_j \in \mathbb{C}^n$. Since $\hat{G}^{op}_{\delta}(a_j)$ is positive, it is, in particular, self-adjoint. Then the spectral Theorem for self-adjoint and compact operators implies that for each $j \geq 1$ there exists an orthonormal basis $\{e^j_m\}_{m\geq 1}$ of $\mathcal{H}$ consisting of eigenvectors of $\hat{G}^{op}_{\delta}(a_j)$. That is,

$$\mathcal{H} = \overline{\operatorname{span}\{e^j_m\}}, \quad \text{where } m \geq 1.$$

Hence, $\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}} = 0$, for $m \neq n$. Comparing (B.4.6) with (B.4.9), and by the positivity of $\hat{G}^{op}_{\delta}(a_j)$, we can see that

$$\begin{aligned}
\|M_s\|^p_{S_p} &\lesssim \sum_{k,m=1}^{s} \sum_{\substack{r,n=1\\ r\neq k, m\neq n}}^{s} \left|J^{m,n}_{k,r}(G,s)\right|^p + \sum_{k,m=1}^{s} \sum_{\substack{r=1\\ r\neq k}}^{s} \left(J^{m,m}_{k,r}(G,s)\right)^p + \sum_{k,m=1}^{s} \sum_{\substack{n=1\\ m\neq n}}^{s} \left|J^{m,n}_{k,k}(G,s)\right|^p \\
&\lesssim e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{\substack{m,n=1\\ m\neq n}}^{s} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e_m, e_n\rangle_{\mathcal{H}}|^p + e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{m=1}^{s} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e_m, e_m\rangle_{\mathcal{H}}|^p \\
&\qquad + e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{\substack{m,n=1\\ m\neq n}}^{s} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e_m, e_n\rangle_{\mathcal{H}}|^p \\
&= e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \sum_{\substack{m,n=1\\ m\neq n}}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}}|^p + e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{m=1}^{s} \sum_{j=1}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_m\rangle_{\mathcal{H}}|^p \\
&\qquad + e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j=1}^{s} \sum_{\substack{m,n=1\\ m\neq n}}^{s} |\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}}|^p \\
&= e^{\frac{-p\epsilon}{2}\tilde{R}} \sum_{j,m=1}^{s} \left(\langle \hat{G}^{op}_{\delta}(a_j)e^j_m, e^j_n\rangle_{\mathcal{H}}\right)^p,
\end{aligned} \tag{B.4.11}$$

where the first and the third terms vanish because of the compactness argument above. Note that inequality (B.4.10) was established for an *arbitrary* orthonormal basis $\{e_m\}_{m\geq 1}$ of $\mathcal{H}$, and the constant involved does not depend on the chosen basis. Hence, for each fixed index j, we may apply (B.4.10) with a possibly different orthonormal basis $\{e^j_m\}_{m\geq 1}$, for instance, the eigenbasis of $\hat{G}^{op}_{\delta}(a_j)$. Doing so yields a valid inequality for each j, and summing these inequalities over j gives (B.4.11). In other words, the passage from (B.4.10) to (B.4.11) does not require a single global orthonormal basis.

Therefore, by (B.4.4) and (B.4.11), we can conclude that

$$\begin{aligned}
\|T_G\|^p_{S_p} &\gtrsim \|T\|^p_{S_p} \geq \|D_s\|^p_{S_p} - \|M_s\|^p_{S_p} \\
&\geq (C_1 - Q(N)e^{\frac{-p\epsilon}{2}\tilde{R}}) \sum_{m,j=1}^{s} \left(\langle \hat{G}^{op}_{\delta}(z_j)e^j_m, e^j_m\rangle_{\mathcal{H}}\right)^p,
\end{aligned} \tag{B.4.12}$$

where $Q(N)$ is some power of N, not depending on s. Since $e^{\frac{-p\epsilon}{2}\tilde{R}} \to 0$ as $R \to \infty$, there is always a constant $R = R(p,\delta,N)$ such that $C(p,\delta,N) = C_1 - Q(N)e^{\frac{-p\epsilon}{2}\tilde{R}} > 0$.

Fix M to be a positive integer. Then Lemma B.4.2 implies that $\{z_j\}_{j=1}^M$ can be partitioned into at most $6^{2n}R^{4n}\delta^{-2n}N_\delta$ subsequences such that any different points z_j and z_k in the same subsequence satisfy $|z_j - z_k| \geq R\min(\rho(z_j), \rho(z_k))$. Then by (B.4.12), we obtain

$$\sum_{m,j=1}^{M} \left(\langle \hat{G}_\delta^{op}(z_j) e_m^j, e_m^j \rangle_{\mathcal{H}} \right)^p \leq C(p,\delta,N_\delta)^{-1} 6^{2n} R^{4n} \delta^{-2n} N_\delta \|T_G\|_{S_p}^p < \infty.$$

Since the RHS does not depend on M, we are done with the proof of Theorem B.1.7. ☐

Open Problem: Can we prove the result of Theorem B.1.7 without assuming that the symbol $G(z)$ is a compact operator on $\mathcal{H}$, for any $z \in \mathbb{C}^n$?

Acknowledgments

The authors would like to thank Haakan Hedenmalm for fruitful discussions.

Appendix C

Fixed points of the Berezin transform on Fock-type spaces[1]

Abstract

We study the fixed points of the Berezin transform on the Fock-type spaces F_m^2 with the weight $e^{-|z|^m}, m > 0$. It is known that the Berezin transform is well-defined on the polynomials in z and $\bar{z}$. In this paper we focus on the polynomial fixed points and we show that these polynomials must be harmonic, except possibly for countably many $m \in (0, \infty)$. We also show that, in some particular cases, the fixed point polynomials are harmonic for all m.

C.1 Introduction and main results

The study of operators on the Bergman spaces is closely related to the properties of the Berezin transform B. This connection comes in a natural way through the use of the Bergman kernel, which is an essential object in the study of Bergman spaces on different domains. The Berezin transform is also an interesting object in its own right. A lot of work has been done in studying the regularity properties of the Berezin transform (see [4, 25, 37, 29]), as well as studying its range (see [1, 28, 69, 27]).

In this paper, we are interested in finding the fixed points of the Berezin transform. Characterizing functions that satisfy the equality $Bf = f$ is an interesting and deep problem. The known results show the connection of these functions and different notions of harmonicity. One of the first work in this direction was the paper by Axler and Čučković [13] who studied this problem in connection with the problem of commuting Toeplitz operators on the Bergman space on the unit disk. Engliš [36] showed that on the unit disk, bounded fixed points are precisely harmonic functions. Several authors have continued this line of investigation, we mention [2], Lee [58, 57], Arazy and Engliš [4], Jevtić [54], and Casseli [22], among others.

On the Fock space F_2^2, the situation is different. Engliš [38, 36] (see also [79, section 3.3]) showed that there are non harmonic fixed points of the corresponding Berezin transform. For example $f(x+iy) = e^{ax+by}$ is fixed in F_2^2 for any $a, b \in \mathbb{C}$ such that $a^2 + b^2 = 8\pi i$, however f is

---

[1] This appendix reproduces the paper *"Fixed points of the Berezin transform on Fock-type spaces"* by G. Asghari, Ž. Čučković, and S. Şahutoğlu, available as an arXiv preprint (arXiv:2508.13115, 2025), and accepted for publication in *Proceedings of the American Mathematical Society*.

Apart from formatting adjustments, this appendix coincides with the accepted version of the paper.

not harmonic. On the other hand, if f is a bounded fixed point of the Berezin transform on F_2^2, then it has to be a constant. In this paper we are interested in studying the fixed point problem on the family of Fock-type spaces F_m^2.

Let m be a positive real number. Consider the space $L_m^2 = L^2(\mathbb{C}, e^{-|z|^m} dA)$, where $dA = r dr d\theta / 2\pi$ is the Lebesgue measure on $\mathbb{C}$. L_m^2 is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^m} dA(z).$$

The Fock-type space F_m^2 is the closed subspace of entire functions inside L_m^2. It is a reproducing kernel Hilbert space with kernel, called the Bergman kernel, given by

$$K_m(w, z) = m \sum_{k=0}^{\infty} \frac{w^k \bar{z}^k}{\Gamma(\frac{2k+2}{m})}.$$

Let $k_{m,z} = \frac{K_{m,z}}{\|K_{m,z}\|}$ be the normalized Bergman kernel, where $K_{m,z}(w) = K_m(w, z)$. One can define the Berezin transform of a function f as

$$B_m f(z) = \langle f k_{m,z}, k_{m,z} \rangle = \int_{\mathbb{C}} f(w) |k_{m,z}(w)|^2 e^{-|w|^m} dA(w),$$

whenever the above integral exists. We note that the estimate in the proof of [20, Lemma 5.2] implies that the Berezin transform is defined for any polynomial in z and $\bar{z}$. For more information about Fock-type spaces, we refer the reader to [18, 21].

In view of the counterexample above, we will study the polynomials in z and $\bar{z}$ that are fixed points of the Berezin transform on Fock-type spaces. First, we show that if a polynomial with non-negative coefficients is fixed, then it has to be harmonic.

Theorem C.1.1. *Assume that $B_m f = f$ for a polynomial f of z and $\bar{z}$ with nonnegative coefficients and for some $m > 0$. Then f is harmonic.*

Next, we show that on F_2^2, if $B_2 f = f$ and f is a polynomial, then f must be harmonic.

Theorem C.1.2. *Let f be a polynomial of z and $\bar{z}$ such that $B_2 f = f$. Then f is harmonic.*

For $m \neq 2$, the situation is more difficult, since the Bergman kernels do not have a closed form, and the computations involve many gamma functions. Our main result shows that $B_m f = f$ for a polynomial f implies that f is harmonic for all m, except possibly a countably many m. We denote the non-negative integers as $\mathbb{N}_0$.

Theorem C.1.3. *For $n \in \mathbb{N}_0$, there exists a discrete (possibly empty) set $Z_n \subset (0, \infty)$ with no cluster points in $(0, \infty)$ such that if $m \in (0, \infty) \setminus Z_n$ and $B_m f = f$ for a polynomial f of degree at most n, then f is harmonic.*

Since harmonic polynomials are fixed points of Berezin transform (see Lemma C.3.1) we have the following corollary.

Corollary C.1.4. *Let f be a polynomial of z and $\bar{z}$ of degree at most n. Then the following are equivalent.*

i.) $B_m f = f$ for some $m \in (0, \infty) \setminus Z_n$, *ii.)* $B_m f = f$ *for all* $m \in (0,\infty) \setminus Z_n$.

Since countable union of countable sets is countable, we have the following corollary.

Corollary C.1.5. *There exists a countable (possibly empty) set* $Z \subset (0,\infty)$ *such that if* $B_m f = f$ *for a polynomial* f *of* z *and* $\bar{z}$ *and* $m \in (0,\infty) \setminus Z$, *then* f *is harmonic.*

By using a more computational approach, we also show that if $B_m f = f$ for a binomial function f, then f is harmonic.

Theorem C.1.6. *Let* $m > 0$ *and* $f(z) = c_1 z^a \bar{z}^b + c_2 z^c \bar{z}^d$, *where* a, b, c, *and* d *are positive integers. Then* f *is a fixed point of the Berezin transformation* B_m *if and only if* $c_1 = c_2 = 0$.

It would be interesting to know if the set Z in Corollary C.1.5 can be non-empty.

Conjecture C.1.7. Let f be a polynomial of z and $\bar{z}$. If for any $m > 0$ we have $B_m f = f$, then f is harmonic.

C.2 Preliminaries

For m a positive real number, the space F^2_m is defined as a subspace of holomorphic functions in $L^2(\mathbb{C}, e^{-|z|^m} dA)$. We note that, throughout the paper, we will use

$$dA(z) = \frac{r dr d\theta}{2\pi}.$$

Given a function $f \in F^2_m$, its Taylor series $f(z) = \sum_{j=0}^{\infty} f_j z^j$ converges uniformly on compact subsets of $\mathbb{C}$. Furthermore,

$$\begin{aligned}
||f||^2 &= \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} dA(z) \\
&= \int_{\mathbb{C}} \sum_{j,k=0}^{\infty} f_j z^j \overline{f}_k \bar{z}^k e^{-|z|^m} dA(z) \\
&= \int_0^{2\pi} \int_0^{\infty} \sum_{j,k=0}^{\infty} f_j \overline{f}_k r^{j+k} e^{i\theta(j-k)} e^{-r^m} \frac{r dr d\theta}{2\pi} \\
&= \sum_{j=0}^{\infty} |f_j|^2 \int_0^{\infty} r^{2j+1} e^{-r^m} dr \\
&= \sum_{j=0}^{\infty} |f_j|^2 \frac{1}{m} \Gamma\left(\frac{2j+2}{m}\right).
\end{aligned}$$

We note that the scalar product for $f, g \in F_m^2$ is defined by

$$\begin{aligned}
\langle f, g\rangle &= \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^m}\,dA(z)\\
&= \int_{\mathbb{C}} \sum_{j,k=0}^{\infty} f_j z^j \overline{g_k}\bar{z}^k e^{-|z|^m}\,dA(z)\\
&= \int_0^{2\pi}\int_0^{\infty}\sum_{j,k=0}^{\infty} f_j\overline{g}_k r^{j+k+1} e^{i\theta(j-k)} e^{-r^m}\frac{dr\,d\theta}{2\pi}\\
&= \sum_{j=0}^{\infty} f_j\overline{g}_j \frac{1}{m}\Gamma\Big(\frac{2j+2}{m}\Big).
\end{aligned}$$

Hence, the monomials z^n, with $n = 0, 1, 2, \cdots$ form an orthogonal basis for F_m^2. In particular,

$$\Big\{\sqrt{md_j}\,z^j;\quad j = 0, 1, 2, \cdots\Big\}$$

is an orthonormal basis for F_m^2 where

$$d_j = \frac{1}{\Gamma\big(\frac{2j+2}{m}\big)}. \tag{C.2.1}$$

To find the Bergman kernel, we proceed as follows. Let $f \in F_m^2$ and $z \in \mathbb{C}$. Then

$$\begin{aligned}
|f(z)| = |\sum_{j=0}^{\infty} f_j z^j| &\le \sum_{j=0}^{\infty} |f_j||z|^j\\
&= \sum_{j=0}^{\infty} \frac{|f_j|}{\sqrt{md_j}}\sqrt{md_j}|z|^j\\
&\le \left(\sum_{j=0}^{\infty}\frac{|f_j|^2}{md_j}\right)^{1/2}\left(\sum_{j=0}^{\infty} md_j|z|^{2j}\right)^{1/2}.
\end{aligned}$$

The first sum on the right hand side equals $\|f\|$ and the ratio test shows that the second sum above converge uniformly on compact subsets. Thus, the evaluation map $f \mapsto f(z)$ is a bounded linear functional on $\mathbb{C}$ and uniformly bounded for z. Furthermore, F_m^2 is closed inside $L^2(\mathbb{C}, e^{-|z|^m}dA)$, and hence a Hilbert space. Thus, there exists $K_{m,z} \in F_m^2$ such that for any $f \in F_m^2$, $f(z) = \langle f, K_{m,z}\rangle$. Indeed,

$$f(z) = \sum_{j=0}^{\infty} f_j z^j = \sum_{j=0}^{\infty} f_j md_j z^j \frac{1}{md_j} = \langle f, K_{m,z}\rangle,$$

where, $K_{m,z}(w) = m\sum_{j=0}^{\infty} d_j w^j \bar{z}^j$, for any $z, w \in \mathbb{C}$. From now on, we will write

$$K_m(w, z) = K_{m,z}(w) = m\sum_{j=0}^{\infty} d_j w^j \bar{z}^j.$$

We finish this section with the following property about the beta function

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

for $x, y > 0$.

Lemma C.2.1. *The function $x \to \beta(x, k-x)$ is convex on $(0,k)$ and attains its minimum at $x = k/2$.*

Proof. By definition,

$$\beta(x, k-x) = \int_0^1 t^{x-1}(1-t)^{k-x-1}dt = \int_0^1 \left(\frac{t}{1-t}\right)^{x-1}(1-t)^{k-2}dt.$$

Writing $(\frac{t}{1-t})^{x-1} = e^{(x-1)\log(\frac{t}{1-t})}$, we can compute the partial derivative as

$$\frac{\partial}{\partial x}\beta(x, k-x) = \int_0^1 \log\left(\frac{t}{1-t}\right)\left(\frac{t}{1-t}\right)^{x-1}(1-t)^{k-2}dt.$$

It is easy to see that $x = \frac{k}{2}$ is a critical point. Indeed, taking $u = 1-t$ for when $1/2 < t < 1$,

$$\begin{aligned}\frac{\partial}{\partial x}\beta\left(\frac{k}{2}, \frac{k}{2}\right) &= \int_0^{1/2} \log\left(\frac{t}{1-t}\right)t^{k/2-1}(1-t)^{k/2-1}dt \\ &\quad + \int_{1/2}^1 \log\left(\frac{t}{1-t}\right)t^{k/2-1}(1-t)^{k/2-1}dt \\ &= \int_0^{1/2} \log\left(\frac{t}{1-t}\right)t^{k/2-1}(1-t)^{k/2-1}dt \\ &\quad - \int_{1/2}^0 \log\left(\frac{1-u}{u}\right)(1-u)^{k/2-1}u^{k/2-1}du \\ &= \int_0^{1/2} \log\left(\frac{t}{1-t}\right)t^{k/2-1}(1-t)^{k/2-1}dt \\ &\quad - \int_0^{1/2} \log\left(\frac{u}{1-u}\right)(1-u)^{k/2-1}u^{k/2-1}du = 0.\end{aligned}$$

Computing

$$\frac{\partial^2}{\partial x^2}\beta(x, k-x) = \int_0^1 \left(\log\left(\frac{t}{1-t}\right)\right)^2\left(\frac{t}{1-t}\right)^{x-1}(1-t)^{k-2}dt > 0,$$

implies that the beta function is convex on the line $x + y = k$. Hence, we can conclude that $x = y = k/2$ is the minimum of the beta function on the line $x + y = k$. ☐

C.3 Preparatory results

Let f be a polynomial in z and $\bar{z}$ (or a function so that the following integrals make sense). Then

$$\begin{aligned}B_m f(z) &= \int_{\mathbb{C}} f(w)|k_{m,z}(w)|^2 e^{-|w|^m}dA(w) \\ &= \frac{1}{K_m(z,z)}\int_{\mathbb{C}} f(w)|K_m(w,z)|^2 e^{-|w|^m}dA(w) \\ &= \frac{m^2}{K_m(z,z)}\sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \int_{\mathbb{C}} f(w)\bar{w}^k w^l e^{-|w|^m}dA(w) \\ &= \frac{m^2}{K_m(z,z)}\sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \lambda_{k,l}^f,\end{aligned} \tag{C.3.1}$$

where d_j is defined is (C.2.1) and

$$\lambda_{k,l}^f = \int_{\mathbb{C}} f(w)\bar{w}^k w^l e^{-|w|^m}dA(w).$$

The following lemma is very easy to show so we will skip the proof.

Lemma C.3.1. *Harmonic polynomials are fixed points of the Berezin transform.*

Let us define

$$H_{n,\tau} = \left\{ \sum_{j=0}^{n} a_j z^{j+\tau} \bar{z}^j : a_j \in \mathbb{C} \right\}$$

for $n, \tau \in \mathbb{N}_0$, and

$$H_{n,\tau} = \left\{ \sum_{j=0}^{n} a_j z^{j} \bar{z}^{j-\tau} : a_j \in \mathbb{C} \right\}$$

for $\tau \in \mathbb{Z} \backslash \mathbb{N}_0$ and $n \in \mathbb{N}_0$. Let $C^{\omega}(\mathbb{C}, e^{-|z|^m} dA)$ denote the real analytic functions in $L^2(\mathbb{C}, e^{-|z|^m} dA)$ and $A_\tau \subset C^{\omega}(\mathbb{C}, e^{-|z|^m} dA)$ denote the set of real analytic functions whose nth Taylor polynomial belong to $H_{n,\tau}$ for all n. We note that $A_\tau \cap A_s = \{0\}$ for $s \neq \tau$.

Lemma C.3.2. *Let $n \in \mathbb{N}_0, \tau \in \mathbb{Z}$, and $m > 0$. Then $B_m f \in A_\tau$ and $K_m(\cdot,\cdot) B_m f \in A_\tau$ for all $f \in H_{n,\tau}$.*

Proof. Since $K_m(z,z) = m \sum_{k=0}^{\infty} d_k |z|^{2k} \in A_0$, it is enough to show that $K_m(\cdot,\cdot) B_m$ maps a monomial in $H_{n,\tau}$ into A_τ. Since $H_{n,\tau}$ is composed of polynomials, it is enough to prove the lemma for monomials. So, without loss of generality, let $\tau \geq 0$ and $f(z) = z^{\alpha+\tau} \bar{z}^{\alpha}$ for some $\alpha \in \mathbb{N}_0$. Then by (C.3.1) we have

$$K_m(z,z) B_m f(z) = m^2 \sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \lambda_{k,l}^f,$$

where

$$\lambda_{k,l}^f = \int_{\mathbb{C}} w^{\alpha+\tau} \bar{w}^{\alpha} \bar{w}^k w^l e^{-|w|^m} dA(w) = \int_{\mathbb{C}} w^{\alpha+\tau+l} \bar{w}^{\alpha+k} e^{-|w|^m} dA(w).$$

Taking $w = re^{i\theta}$ and the normalized measure $dA(w) = \frac{1}{2\pi} r dr d\theta$, it is easy to see that the above integral is nonzero only if $k = l + \tau$. Then

$$\begin{aligned} K_m(z,z) B_m f(z) &= m^2 \sum_{l=0}^{\infty} d_{l+\tau} d_l z^{l+\tau} \bar{z}^l \int_0^{\infty} r^{2(\alpha+\tau+l)+1} e^{-r^m} dr \\ &= m \sum_{l=0}^{\infty} d_{l+\tau} d_l z^{l+\tau} \bar{z}^l \Gamma\left(\frac{2(\alpha+\tau+l)+2}{m} \right) \\ &= m \sum_{l=0}^{\infty} \frac{d_{l+\tau} d_l}{d_{\alpha+\tau+l}} z^{l+\tau} \bar{z}^l \in A_\tau. \end{aligned} \tag{C.3.2}$$

Therefore, one can use long division to show that $B_m f \in A_\tau$ as $K_m \in A_0$. ☐

Lemma C.3.3. *Let f be a polynomial (of z and $\bar{z}$) of degree n such that $B_m f = f$. Then $B_m f_\tau = f_\tau$ for $-n \leq \tau \leq n$ where $f = \sum_{\tau=-n}^{n} f_\tau$ and $f_\tau \in H_{n,\tau}$.*

Proof. Since f is a polynomial (of z and $\bar{z}$) of degree n, we have a unique decomposition

$$f = \sum_{\tau=-n}^{n} f_\tau,$$

where $f_\tau \in H_{n,\tau}$. We assume that $B_m f = f$. Then, by Lemma C.3.2, we have $B_m f_\tau \in A_\tau$ for $-n \leq \tau \leq n$. Hence,

$$\sum_{\tau=-n}^{n} f_\tau = f = B_m f = \sum_{\tau=-n}^{n} B_m f_\tau,$$

implying that $B_m f_\tau = f_\tau$. ☐

C.4 Proof of Theorem C.1.1: polynomials with non-negative coefficients

Proof of Theorem C.1.1. By Lemma C.3.3, it is enough to prove the theorem for $f \in H_{n,\tau}$, where $-n \leq \tau \leq n$. Since, by Lemma C.3.1 harmonic polynomials are fixed, without loss of generality, we assume that $B_m f = f$ where $f(z) = \sum_{j=1}^{n} a_j z^{j+\tau}\bar{z}^j \in H_{n,\tau}$ is a polynomial with $a_j \geq 0$ for all $j \geq 1$, $a_n > 0$, and $\tau \geq 0$. Hence, f is not harmonic. Similar to (C.3.2), one obtains that

$$K_m(z,z)B_m f(z) = m\sum_{l=0}^{\infty}\sum_{j=1}^{n} a_j \frac{d_{l+\tau}d_l}{d_{j+l+\tau}} z^{l+\tau}\bar{z}^l,$$

and

$$K_m(z,z)f(z) = m\sum_{l=0}^{\infty}\sum_{j=1}^{n} a_j d_l z^{j+\tau+l}\bar{z}^{j+l} = m\sum_{l=j}^{\infty}\sum_{j=1}^{n} a_j d_{l-j} z^{l+\tau}\bar{z}^l.$$

Note that by definition and Lemma C.3.2, $K_m(\cdot,\cdot)f$ and $K_m(\cdot,\cdot)B_m f$ both belong to A_τ, and hence their difference as well.

$$\begin{aligned} 0 = K_m(z,z)B_m f(z) - K_m(z,z)f(z) =& m\sum_{l=0}^{j-1}\sum_{j=1}^{n} a_j \frac{d_{l+\tau}d_l}{d_{j+\tau+l}} z^{l+\tau}\bar{z}^l \\ &+ m\sum_{l=j}^{\infty}\sum_{j=1}^{n} a_j \left(\frac{d_{l+\tau}d_l}{d_{j+\tau+l}} - d_{l-j}\right) z^{l+\tau}\bar{z}^l \end{aligned} \tag{C.4.1}$$

for all z. We note that $d_{j+\tau+l}$ is nonzero for any $j \geq 1$ because the gamma function has poles only on negative integers. We will use the following well known formula

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Taking $x = \frac{2}{m}$, and

$$d_l = \frac{1}{\Gamma\left(\frac{2(l+1)}{m}\right)} = \frac{1}{\Gamma((l+1)x)},$$

for every nonnegative j we obtain

$$\frac{d_{l+\tau}d_l}{d_{j+\tau+l}} - d_{l-j} = \frac{d_{l+\tau}d_l - d_{l-j}d_{j+\tau+l}}{d_{j+\tau+l}} = \Gamma((j+\tau+l+1)x)\Gamma((2l+\tau+2)x)\frac{B_{l,j,\tau}}{A_{l,j,\tau}}$$

where

$$\begin{aligned} A_{l,j,\tau} =& \Gamma((l+\tau+1)x)\Gamma((l+1)x)\Gamma((l-j+1)x)\Gamma((j+\tau+l+1)x), \\ B_{l,j,\tau} =& \beta((j+l+\tau+1)x,(l-j+1)x) - \beta((l+\tau+1)x,(l+1)x). \end{aligned}$$

In this case

$$k = (j+l+\tau+1)x + (l-j+1)x = (l+\tau+1)x + (l+1)x = (2l+\tau+2)x.$$

Recall that $\beta(y,k-y)$ is a convex function of y and takes its minimum at $y = k/2$. Hence $\beta(\alpha_1,k-\alpha_1) > \beta(\alpha_2,k-\alpha_2)$ if $\alpha_1 > \alpha_2 \geq k/2$. We choose $\alpha_1 = (j+l+\tau+1)x$ and $\alpha_2 = (l+\tau+1)x$. Then

$$(j+l+\tau+1)x > (l+\tau+1)x > (l+\frac{\tau}{2}+1)x = \frac{k}{2}.$$

Moreover, $k-\alpha_1=(l-j+1)x$ and $k-\alpha_2=(l+1)x$. Then for any $j\geq 1$, we have

$$\beta((j+\tau+l+1)x,(l-j+1)x)-\beta((l+\tau+1)x,(l+1)x)>0.$$

Hence, $a_n>0$ implies that

$$K_m(z,z)B_mf(z)-K_m(z,z)f(z)\not\equiv 0$$

which contradicts (C.4.1). Therefore, f is not a fixed point of the Berezin transform. ☐

C.5 Proof of Theorem C.1.2: the case $m=2$

When $m=2$, one can write the Berezin transform as

$$B_2f(z)=2\int_{\mathbb{C}}f(z+\xi)e^{-|\xi|^2}dA(\xi)$$

where $dA=rdrd\theta/2\pi$. We note that the since we normalize dA by 2π, our formula above has a multiple 2 instead of $1/\pi$ as in [79, section 3.2].

Lemma C.5.1. *Let $f(z)=z^{j+\tau}\bar{z}^j$ for $j\in\mathbb{N}$ and $\tau\in\mathbb{N}_0$.*

$$B_2f(z)=z^{j+\tau}\bar{z}^j+\frac{(j+\tau)j}{d_1}z^{j-1+\tau}\bar{z}^{j-1}+\textit{lower order terms.}$$

Furthermore, $B_2:H_{n,\tau}\to H_{n,\tau}$ is a bijection.

Proof. By the binomial expansion formula, we compute the Berezin transform for $f(z)=z^{j+\tau}\bar{z}^j$ as follows.

$$\begin{aligned}
B_2f(z)&=2\int_{\mathbb{C}}f(z+\xi)e^{-|\xi|^2}dA(\xi)\\
&=2\sum_{s=0}^{j}\sum_{t=0}^{j+\tau}\binom{j+\tau}{t}\binom{j}{s}z^t\bar{z}^s\int_{\mathbb{C}}\xi^{j+\tau-t}\bar{\xi}^{j-s}e^{-|\xi|^2}dA(\xi)\\
(t=s+\tau)\quad&=\frac{1}{\pi}\sum_{s=0}^{j}\binom{j+\tau}{s+\tau}\binom{j}{s}z^{s+\tau}\bar{z}^s\int_0^{2\pi}\int_0^{\infty}r^{2(j-s)+1}e^{-r^2}drd\theta\\
&=\sum_{s=0}^{j}\binom{j+\tau}{s+\tau}\binom{j}{s}\Gamma(j-s+1)z^{s+\tau}\bar{z}^s\\
&=z^{j+\tau}\bar{z}^j+\frac{(j+\tau)j}{d_1}z^{j-1+\tau}\bar{z}^{j-1}+\textit{lower order terms,}
\end{aligned}$$

where $d_1^{-1}=\Gamma(2)$, as defined before. Furthermore, the formula above shows that $B_2f\in H_{n,\tau}$ whenever $f\in H_{n,\tau}$.

Let us choose $B=\{z^\tau,z^{1+\tau}\bar{z},\dots,z^{n+\tau}\bar{z}^n\}$ as a basis for $H_{n,\tau}$. Then the matrix representation $[B_2]$ for B_2 with respect to basis B can be computed as follows.

$$[B_2]=\begin{bmatrix}1&(1+\tau)/d_1&\cdots&\cdots&\cdots\\0&1&(2+\tau)2/d_1&\cdots&\cdots\\0&0&1&(3+\tau)3/d_1&\cdots\\\vdots&\vdots&\vdots&\vdots&\\0&\cdots&\cdots&0&1\end{bmatrix}_{(n+1)\times(n+1)}. \tag{C.5.1}$$

Therefore, B_2 is a bijection on $H_{n,\tau}$ because the matrix $[B_2]$ is nonsingular. ☐

Proof of Theorem C.1.2. Let f be a polynomial of z and $\bar{z}$ of order n. Then, by Lemma C.3.3, it is enough to prove the theorem for $f \in H_{n,\tau}$, where $-n \leq \tau \leq n$. Without loss of generality, we fix $\tau \geq 0$.

Using the notation in the proof of Lemma C.5.1, we observe that the matrix $[B_2]$ in (C.5.1) is an upper triangular matrix with 1 on the diagonal and $(j+\tau)j/d_1$ on the entries above the diagonal. Hence, $[B_2]-I$ is an upper triangular matrix with 0 on the diagonal and $(j+\tau)j/d_1$ on the entries above the diagonal. Then the first column and the last row of $[B_2]-I$ are composed of zero entries. That is

$$[B_2]-I = \left[\begin{array}{c|c} 0 & \\ \vdots & M \\ 0 & \\ \hline 0 & 0 \cdots 0 \end{array}\right],$$

where

$$M = \begin{bmatrix} (1+\tau)/d_1 & \cdots & \cdots & \cdots \\ 0 & (2+\tau)2/d_1 & \cdots & \cdots \\ 0 & 0 & (3+\tau)3/d_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & (n+\tau)n/d_1 \end{bmatrix}_{n \times n}$$

is the $n \times n$ submatrix of $[B_2]-I$. We note that M is upper triangular with positive entries on the diagonal. Hence M is of rank n as $\det(M) > 0$. That is, $rank([B_2]-I) \geq n$. We also know that z^τ is in the kernel of $[B_2]-I$. The rank-nullity theorem implies that

$$rank([B_2]-I) + dim(ker([B_2]-I)) = n+1.$$

Then, $rank([B_2]-I) = n$ and $dim(ker([B_2]-I)) = 1$. Therefore, if $f \in H_{n,\tau}$ such that $B_2 f = f$, then $f \in ker([B_2]-I)$. Namely, f is holomorphic. ☐

C.6 Proof of Theorem C.1.3: the case $m > 0$

Proof of Theorem C.1.3. By Lemma C.3.3, it is enough to prove the theorem for functions in $H_{n,\tau}$, where $0 \leq \tau \leq n$. Let us define

$$B_{m,n} : H_{n,\tau} \to C^\omega(\mathbb{C}, e^{-|z|^m} dA)$$

to be the Berezin transform of F^2_m restricted to $H_{n,\tau}$ and $T_{m,n} : H_{n,\tau} \to C^\omega(\mathbb{C}, e^{-|z|^m} dA)$ as

$$T_{m,n} f(z) = K_m(z,z) B_{m,n} f(z) - K_m(z,z) f(z)$$

for $f \in H_{n,\tau}$ and $z \in \mathbb{C}$. Then

$$ker(B_{m,n} - I) = ker(T_{m,n}) \supseteq span\{z^\tau\} = ker(T_{2,n}),$$

and

$$rank(B_{m,n} - I) = rank(T_{m,n}) \leq n = rank(T_{2,n}).$$

Then rank-nullity theorem implies that

$$rank(T_{m,n}) + dim(ker(T_{m,n})) = dim(H_{n,\tau}) = n+1$$

for all $m > 0$.

Let $f(z) = z^{j+\tau}\bar{z}^j \in H_{n,\tau}$. Then by (C.3.2) we have

$$K_m(z,z)B_m f(z) = m\sum_{l=0}^{\infty}\frac{d_{l+\tau}d_l}{d_{j+\tau+l}}z^{l+\tau}\bar{z}^l.$$

Moreover, $K_m(z,z)f(z) = m\sum_{l=0}^{\infty} d_l z^l \bar{z}^l z^{j+\tau}\bar{z}^j$, and hence

$$\begin{aligned}
T_{m,n}(z^{j+\tau}\bar{z}^j) &= m\sum_{l=0}^{\infty}\frac{d_{l+\tau}d_l}{d_{j+\tau+l}}z^{l+\tau}\bar{z}^l - m\sum_{l=0}^{\infty} d_l z^{j+\tau+l}\bar{z}^{j+l}\\
&= m\sum_{l=0}^{\infty}\frac{d_{l+\tau}d_l}{d_{j+\tau+l}}z^{l+\tau}\bar{z}^l - m\sum_{l=j}^{\infty} d_{l-j} z^{l+\tau}\bar{z}^{l}\\
&= m\sum_{l=0}^{j-1}\frac{d_{l+\tau}d_l}{d_{j+\tau+l}}z^{l+\tau}\bar{z}^l + m\sum_{l=j}^{\infty}\left(\frac{d_{l+\tau}d_l}{d_{j+\tau+l}} - d_{l-j}\right)z^{l+\tau}\bar{z}^l.
\end{aligned}$$

Hence,

$$T_{m,n}(z^{j+\tau}\bar{z}^j) = \sum_{k=0}^{\infty} a_{j,k}(m) z^{k+\tau}\bar{z}^k,$$

where each $a_{j,k}$ is holomorphic on $U = \{z \in \mathbb{C} : Re(z) > 0\}$, by properties of the gamma function. Then we will study the rank and nullity of the matrix $[a_{j,k}(m)]$ as a function of the complexified variable m. We note that the matrix $[a_{j,k}(m)]$ is of size $\infty \times (n+1)$.

Since, by Theorem C.1.2, $rank(T_{2,n}) = n$, there exists a submatrix S_n of $[a_{j,k}]$ of size $n \times n$ with entries holomorphic on U such that $\det(S_n(2)) \neq 0$. Let $S(m) = \det(S_n(m))$ and $Z_{n,\tau}$ denote the zero set of S. Then S is holomorphic on U and $Z_{n,\tau}$ is a discrete set with no accumulation point in U. Hence $rank(T_{m,n}) \geq n$ for $z \in U \setminus Z_{n,\tau}$ as $\det(S_n(m)) \neq 0$ for $z \in U \setminus Z_{n,\tau}$. However, $rank(T_{m,n}) \leq n$ as $dim(ker(T_{m,n})) \geq 1$ for all $0 < m < \infty$. Then, $rank(T_{m,n}) = n$ and $dim(ker(T_{m,n})) = 1$ for $m \in (0,\infty) \setminus Z_{n,\tau}$. Namely, $span\{z^\tau\} = ker(T_{m,n})$ for $m \in (0,\infty) \setminus Z_{n,\tau}$.

Let $Z_n^h = \cup_{\tau=0}^{n} Z_{n,\tau}$. Hence Z_n^h is a discrete set with no cluster in $(0,\infty)$. Furthermore, we showed that only the holomorphic polynomial is fixed whenever $0 \leq \tau \leq n$ and $m \in (0,\infty) \setminus Z_n^h$. Similarly, we can show that there exists a discrete set Z_n^c such that when $-n \leq \tau < 0$ and $m \in (0,\infty) \setminus Z_n^c$ only the conjugate holomorphic polynomial is fixed. Let $Z_n = Z_n^h \cup Z_n^c$. Therefore, if $m \in (0,\infty) \setminus Z_n$ and $B_m f = f$ for a polynomial f of degree at most n, then f is harmonic. □

C.7 Proof of Theorem C.1.6: binomial fixed points

First, we focus on a single monomial.

Proposition C.7.1. *Let $m > 0$ and $f(z) = z^p\bar{z}^q$ be a fixed point for the Berezin transform B_m. Then either p or q must be zero. That is, f is either a holomorphic or a conjugate holomorphic monomial.*

Proof. By Lemma C.3.1, z^p and $\bar{z}^q$ are fixed under B_m. We assume that $B_m f = f$ and $p, q \geq 1$. Then (C.3.1) implies that

$$K_m(z,z)z^p\bar{z}^q = K_m(z,z)B_m f(z) = \int_{\mathbb{C}} w^p\bar{w}^q |K_m(w,z)|^2 e^{-|w|^m} dA(w).$$

Hence,

$$\begin{aligned} m\sum_{k=0}^{\infty} d_k z^p \bar{z}^q z^k \bar{z}^k &= m^2 \sum_{k,l=0}^{\infty} d_k d_l \int_{\mathbb{C}} w^p \bar{w}^q z^k \bar{w}^k \bar{z}^l w^l e^{-|w|^m} dA(w) \\ &= m^2 \sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \int_{\mathbb{C}} w^{p+l} \bar{w}^{q+k} e^{-|w|^m} dA(w). \end{aligned}$$

First, assume that $p \geq q$. The above integral is nonzero only if $k = p + l - q$. Therefore,

$$\begin{aligned} m\sum_{k=0}^{\infty} d_k z^{p+k} \bar{z}^{q+k} &= m^2 \sum_{l=0}^{\infty} d_{p+l-q} d_l z^{p+l-q} \bar{z}^l \int_0^{\infty} r^{2(p+l)+1} e^{-r^m} dr \\ &= m^2 \sum_{l=0}^{\infty} d_{p+l-q} d_l z^{p+l-q} \bar{z}^l \frac{1}{m} \Gamma\left(\frac{2(p+l)+2}{m}\right) \qquad \text{(C.7.1)} \\ &= m \sum_{l=0}^{\infty} d_{p+l-q} d_l z^{p+l-q} \bar{z}^l \frac{1}{d_{p+l}}. \end{aligned}$$

For the above equation to hold, every corresponding l-term on each side must agree. Note that the zeroth term on the right hand side is holomorphic, while there is no holomorphic term on the left hand side. Hence, they can not be equal.

When $p < q$, similarly we obtain

$$m\sum_{k=0}^{\infty} d_k z^{p+k} \bar{z}^{q+k} = m \sum_{k=0}^{\infty} d_k d_{q-p+k} z^k \bar{z}^{q-p+k} \frac{1}{d_{q+k}}.$$

The zeroth term on the right hand side is conjugate holomorphic, while there is no conjugate holomorphic term on the left hand side. Hence, they are not equal.

Therefore, we can conclude that any monomial which is a fixed point of the Berezin transformation should be either of the form z^p or $\bar{z}^q$. ☐

Proof of Theorem C.1.6. Since the case of monomials was considered in Proposition C.7.1, we can assume that $B_m f = f$ and a, b, c, d are positive integers such that $a \neq c$ or $b \neq d$. Then by (C.3.1) we get

$$\begin{aligned} K_m(z,z)(c_1 z^a \bar{z}^b + c_2 z^c \bar{z}^d) &= K_m(z,z) B_m f(z) \\ &= \int_{\mathbb{C}} (c_1 w^a \bar{w}^b + c_2 w^c \bar{w}^d) |K_m(w,z)|^2 e^{-|w|^m} dA(w). \end{aligned}$$

Therefore,

$$\begin{aligned} m\sum_{k=0}^{\infty} d_k z^k \bar{z}^k (c_1 z^a \bar{z}^b + c_2 z^c \bar{z}^d) &= m^2 \sum_{k,l=0}^{\infty} d_k d_l \int_{\mathbb{C}} (c_1 w^a \bar{w}^b + c_2 w^c \bar{w}^d) z^k \bar{w}^k \bar{z}^l w^l e^{-|w|^m} dA(w) \\ &= c_1 m^2 \sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \int_{\mathbb{C}} w^{a+l} \bar{w}^{b+k} e^{-|w|^m} dA(w) \qquad \text{(C.7.2)} \\ &\quad + c_2 m^2 \sum_{k,l=0}^{\infty} d_k d_l z^k \bar{z}^l \int_{\mathbb{C}} w^{c+l} \bar{w}^{d+k} e^{-|w|^m} dA(w). \end{aligned}$$

We consider two different cases $a \geq b \geq 1, c \geq d \geq 1$ and $a \geq b \geq 1, 1 \leq c < d$. The other cases turn into one of these simply by conjugation.

First let us consider the case $a \geq b \geq 1$ and $c \geq d \geq 1$. Note that the first integral on the right hand side of (C.7.2) is nonzero only if $k = a + l - b$, and the second integral in nonzero only if $k = c + l - d$. Then similarly to the monomial case as (C.7.1), we can write the right hand side of (C.7.2) as

$$c_1 m \sum_{l=0}^{\infty} d_{a-b+l} d_l z^{a-b+l} \bar{z}^l \frac{1}{d_{a+l}} + c_2 m \sum_{l=0}^{\infty} d_{c-d+l} d_l z^{c-d+l} \bar{z}^l \frac{1}{d_{c+l}}.$$

Hence if f is a fixed point, then

$$c_1 \sum_{l=0}^{\infty} d_l z^{a+l} \bar{z}^{b+l} + c_2 \sum_{l=0}^{\infty} d_l z^{c+l} \bar{z}^{d+l} = c_1 \sum_{l=0}^{\infty} \frac{d_{a-b+l} d_l}{d_{a+l}} z^{a-b+l} \bar{z}^l + c_2 \sum_{l=0}^{\infty} \frac{d_{c-d+l} d_l}{d_{c+l}} z^{c-d+l} \bar{z}^l. \tag{C.7.3}$$

Let $a - b = j$ and $c - d = k$. We can write (C.7.3) as

$$c_1 \sum_{l=0}^{\infty} d_l z^{a+l} \bar{z}^{b+l} + c_2 \sum_{l=0}^{\infty} d_l z^{c+l} \bar{z}^{d+l} = c_1 \sum_{l=0}^{\infty} \frac{d_{j+l} d_l}{d_{a+l}} z^{j+l} \bar{z}^l + c_2 \sum_{l=0}^{\infty} \frac{d_{k+l} d_l}{d_{c+l}} z^{k+l} \bar{z}^l.$$

Note that the zeroth terms in the sums on the right hand side above are holomorphic, while there is no holomorphic term on the left hand side. Hence if $j \neq k$ and f is a fixed point, the zeroth terms on the right hand side must be zero, implying that

$$c_1 \frac{d_j d_0}{d_a} = 0 \quad \text{and} \quad c_2 \frac{d_k d_0}{d_c} = 0.$$

Since $d_i \neq 0$ for any $i \geq 0$, we can conclude that $c_1 = c_2 = 0$.

From now on we assume that $j = k$. Then, we take $B_m f = f$ with $a - b = c - d = j \geq 0$. Hence (C.7.3) can be written as

$$c_1 \sum_{l=0}^{\infty} d_l z^{a+l} \bar{z}^{b+l} + c_2 \sum_{l=0}^{\infty} d_l z^{c+l} \bar{z}^{d+l} = \sum_{l=0}^{\infty} d_{j+l} d_l \left(\frac{c_1}{d_{a+l}} + \frac{c_2}{d_{c+l}} \right) z^{j+l} \bar{z}^l. \tag{C.7.4}$$

Again the zeroth term on the right hand side above is holomorphic, while there is no holomorphic term on the left hand side. So for $B_m f = f$ to hold, we must assume that the holomorphic term on the right hand side is zero. This is equivalent to

$$\frac{c_1}{d_a} + \frac{c_2}{d_c} = 0. \tag{C.7.5}$$

Writing $a = j + b$ and $c = j + d$, we can write (C.7.4) as

$$c_1 \sum_{l=0}^{\infty} d_l z^{j+b+l} \bar{z}^{b+l} + c_2 \sum_{l=0}^{\infty} d_l z^{j+d+l} \bar{z}^{d+l} = \sum_{l=0}^{\infty} d_{j+l} d_l \left(\frac{c_1}{d_{a+l}} + \frac{c_2}{d_{c+l}} \right) z^{j+l} \bar{z}^l. \tag{C.7.6}$$

Without loss of generality, we assume that $d > b$ and we define $r = d - b \geq 1$. Recall that when $r = 0$, f is a monomial as $a - b = c - d$ and that was considered in Proposition C.7.1. Hence (C.7.6) can be written as the following.

$$c_1 \sum_{l=0}^{\infty} d_l z^{j+b+l} \bar{z}^{b+l} + c_2 \sum_{l=0}^{\infty} d_l z^{j+b+r+l} \bar{z}^{b+r+l} = \sum_{l=0}^{\infty} d_{j+l} d_l \left(\frac{c_1}{d_{a+l}} + \frac{c_2}{d_{c+l}} \right) z^{j+l} \bar{z}^l. \tag{C.7.7}$$

To get a better idea about the series above, we can write (C.7.7) as

$$\begin{aligned} c_1 d_0 z^{j+b}\bar{z}^b &+ c_1 d_1 z^{j+b+1}\bar{z}^{b+1} + \cdots + c_1 d_{r-1} z^{j+b+r-1}\bar{z}^{b+r-1} \\ &+ c_1 \sum_{l=r}^{\infty} d_l z^{j+b+l}\bar{z}^{b+l} + c_2 \sum_{l=0}^{\infty} d_l z^{j+b+r+l}\bar{z}^{b+r+l} \\ = d_j d_0 \left(\frac{c_1}{d_a} + \frac{c_2}{d_c}\right) z^j &+ d_{j+1} d_1 \left(\frac{c_1}{d_{a+1}} + \frac{c_2}{d_{c+1}}\right) z^{j+1}\bar{z} + \cdots \\ &+ d_{j+b+r-1} d_{b+r-1} \left(\frac{c_1}{d_{a+b+r-1}} + \frac{c_2}{d_{c+b+r-1}}\right) z^{j+b+r-1}\bar{z}^{b+r-1} \\ &+ \sum_{l=b+r}^{\infty} d_{j+l} d_l \left(\frac{c_1}{d_{a+l}} + \frac{c_2}{d_{c+l}}\right) z^{j+l}\bar{z}^l. \end{aligned}$$

Then we can rewrite it as

$$\begin{aligned} c_1 d_0 z^{j+b}\bar{z}^b &+ c_1 d_1 z^{j+b+1}\bar{z}^{b+1} + \cdots + c_1 d_{r-1} z^{j+b+r-1}\bar{z}^{b+r-1} \\ &+ c_1 \sum_{l=0}^{\infty} d_{r+l} z^{j+b+r+l}\bar{z}^{b+r+l} + c_2 \sum_{l=0}^{\infty} d_l z^{j+b+r+l}\bar{z}^{b+r+l} \\ = d_j d_0 \left(\frac{c_1}{d_a} + \frac{c_2}{d_c}\right) z^j &+ d_{j+1} d_1 \left(\frac{c_1}{d_{a+1}} + \frac{c_2}{d_{c+1}}\right) z^{j+1}\bar{z} + \cdots \\ &+ d_{j+b+r-1} d_{b+r-1} \left(\frac{c_1}{d_{a+b+r-1}} + \frac{c_2}{d_{c+b+r-1}}\right) z^{j+b+r-1}\bar{z}^{b+r-1} \\ &+ \sum_{l=0}^{\infty} d_{j+b+r+l} d_{b+r+l} \left(\frac{c_1}{d_{a+b+r+l}} + \frac{c_2}{d_{c+b+r+l}}\right) z^{j+b+r+l}\bar{z}^{b+r+l}, \end{aligned}$$

where the sums on both the left and right hand sides are in terms of $z^{j+b+r+l}\bar{z}^{b+r+l}$. Comparing the coefficients of $z^{j+b+r-1}\bar{z}^{b+r-1}$ on both sides, we obtain

$$c_1 d_{r-1} = d_{j+b+r-1} d_{b+r-1} \left(\frac{c_1}{d_{a+b+r-1}} + \frac{c_2}{d_{c+b+r-1}}\right) \tag{C.7.8}$$

Comparing later terms, we must have

$$c_1 d_{r+l} + c_2 d_l = d_{j+b+r+l} d_{b+r+l} \left(\frac{c_1}{d_{a+b+r+l}} + \frac{c_2}{d_{c+b+r+l}}\right) \text{ for all } l \in \mathbb{N}_0.$$

Considering (C.7.5) and (C.7.8), we will show that $c_1 = c_2 = 0$. Equivalently, we would like to show that the following determinant

$$A_m = \det \begin{bmatrix} \frac{1}{d_a} & \frac{1}{d_c} \\ \frac{d_{j+b+r-1} d_{b+r-1}}{d_{a+b+r-1}} - d_{r-1} & \frac{d_{j+b+r-1} d_{b+r-1}}{d_{c+b+r-1}} \end{bmatrix}$$

is non-zero. Using $d_l = \frac{1}{\Gamma\left(\frac{2l+2}{m}\right)}$, we can write the determinant above as

$$\begin{aligned} A_m &= \frac{d_{j+b+r-1} d_{b+r-1}}{d_a d_{c+b+r-1}} - \frac{d_{j+b+r-1} d_{b+r-1}}{d_c d_{a+b+r-1}} + \frac{d_{r-1}}{d_c} \\ &= \frac{\Gamma\left(\frac{2a+2}{m}\right)\Gamma\left(\frac{2(c+b+r-1)+2}{m}\right)}{\Gamma\left(\frac{2(j+b+r-1)+2}{m}\right)\Gamma\left(\frac{2(b+r-1)+2}{m}\right)} - \frac{\Gamma\left(\frac{2c+2}{m}\right)\Gamma\left(\frac{2(a+b+r-1)+2}{m}\right)}{\Gamma\left(\frac{2(j+b+r-1)+2}{m}\right)\Gamma\left(\frac{2(b+r-1)+2}{m}\right)} + \frac{\Gamma\left(\frac{2c+2}{m}\right)}{\Gamma\left(\frac{2(r-1)+2}{m}\right)}. \end{aligned}$$

Letting $x = \frac{2}{m}$, we get

$$A_m = \frac{\Gamma((a+1)x)\Gamma((c+b+r)x)}{\Gamma((j+b+r)x)\Gamma((b+r)x)} - \frac{\Gamma((c+1)x)\Gamma((a+b+r)x)}{\Gamma((j+b+r)x)\Gamma((b+r)x)} + \frac{\Gamma((c+1)x)}{\Gamma(rx)}.$$

One can show that the contribution of the first two terms is already positive. Indeed,

$$\frac{\Gamma((a+1)x)\Gamma((c+b+r)x)}{\Gamma((j+b+r)x)\Gamma((b+r)x)} - \frac{\Gamma((c+1)x)\Gamma((a+b+r)x)}{\Gamma((j+b+r)x)\Gamma((b+r)x)} > 0$$

if and only if

$$\Gamma((a+1)x)\Gamma((c+b+r)x) - \Gamma((c+1)x)\Gamma((a+b+r)x) > 0$$

which is equivalent to

$$\frac{\Gamma((a+1)x)\Gamma((c+b+r)x)}{\Gamma((a+b+c+r+1)x)} > \frac{\Gamma((c+1)x)\Gamma((a+b+r)x)}{\Gamma((a+b+c+r+1)x)}.$$

However, using $(a+1)x + (c+b+r)x = (c+1)x + (a+b+r)x = k$ we have

$$\begin{aligned} \beta((a+1)x, k-(a+1)x) =& \frac{\Gamma((a+1)x)\Gamma((c+b+r)x)}{\Gamma((a+b+c+r+1)x)} \\ \beta((c+1)x, k-(c+1)x) =& \frac{\Gamma((c+1)x)\Gamma((a+b+r)x)}{\Gamma((a+b+c+r+1)x)}. \end{aligned}$$

We recall that $a < c$. So we would like to show that

$$\beta((a+1)x, k-(a+1)x) > \beta((c+1)x, k-(c+1)x). \tag{C.7.9}$$

Since $\beta(y, k-y)$ is convex with minimum at $k/2 = (a+b+c+r+1)x/2$, it is enough to show that

$$(a+1)x < (c+1)x \leq \frac{k}{2}.$$

The first inequality above is clear as $a < c$. The second inequality is equivalent to

$$c+1 \leq a+b+r = a+d$$

as $r = d-b$. However, $a-b = c-d$. Then the inequality above is equivalent to

$$a-b+1 \leq a$$

which correct as $b \geq 1$. Hence, the inequality (C.7.9) is satisfied and A_m is non-singular for all $m > 0$. Therefore, $c_1 = c_2 = 0$.

We finish the proof by considering the second case $a \geq b \geq 1$ and $1 \leq c < d$. The first integral on the right hand side of (C.7.2) is nonzero when $k = a-b+l$ and the second integral is nonzero when $l = d-c+k$. Therefore, (C.7.2) can be written as

$$\begin{aligned} &c_1 \sum_{k=0}^{\infty} d_k z^{a+k} \bar{z}^{b+k} + c_2 \sum_{k=0}^{\infty} d_k z^{c+k} \bar{z}^{d+k} \\ &= c_1 \sum_{l=0}^{\infty} \frac{d_{a-b+l} d_l}{d_{a+l}} z^{a-b+l} \bar{z}^l + c_2 \sum_{k=0}^{\infty} \frac{d_k d_{d-c+k}}{d_{d+k}} z^k \bar{z}^{d-c+k}. \end{aligned} \tag{C.7.10}$$

The zeroth terms in the sums on the right hand side are harmonic, while there is no harmonic term on the left hand side. Hence (C.7.10) cannot hold unless $c_1 = c_2 = 0$. □

Bibliography

[1] Patrick Ahern. “On the range of the Berezin transform”. In: *J. Funct. Anal.* 215.1 (2004), pp. 206–216. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2003.08.007. URL: https://doi.org/10.1016/j.jfa.2003.08.007.

[2] Patrick Ahern, Manuel Flores, and Walter Rudin. “An invariant volume-mean-value property”. In: *J. Funct. Anal.* 111.2 (1993), pp. 380–397. ISSN: 0022-1236,1096-0783. DOI: 10.1006/jfan.1993.1018. URL: https://doi.org/10.1006/jfan.1993.1018.

[3] L. Ambrosio, M. Miranda Jr., and D. Pallara. “Special functions of bounded variation in doubling metric measure spaces”. In: *Calculus of variations: topics from the mathematical heritage of E. De Giorgi*. Vol. 14. Quad. Mat. Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 1–45. ISBN: 88-7999-414-X.

[4] J. Arazy and M. Engliš. “Iterates and the boundary behavior of the Berezin transform”. In: *Ann. Inst. Fourier (Grenoble)* 51.4 (2001), pp. 1101–1133. ISSN: 0373-0956,1777-5310. DOI: 10.5802/aif.1847. URL: https://doi.org/10.5802/aif.1847.

[5] N. Aronszajn. “Theory of reproducing kernels”. In: *Trans. Amer. Math. Soc.* 68 (1950), pp. 337–404. ISSN: 0002-9947,1088-6850. DOI: 10.2307/1990404. URL: https://doi.org/10.2307/1990404.

[6] Hicham Arroussi, Ghazaleh Asghari, and Jani Virtanen. “Toeplitz operators on large vector-valued Fock spaces”. In: *arXiv preprint arXiv:2504.15239* (2025).

[7] Hicham Arroussi and Cezhong Tong. “Weighted composition operators between large Fock spaces in several complex variables”. In: *J. Funct. Anal.* 277.10 (2019), pp. 3436–3466. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2019.04.008. URL: https://doi.org/10.1016/j.jfa.2019.04.008.

[8] Hicham Arroussi et al. “Toeplitz operators between large Fock spaces”. In: *Banach J. Math. Anal.* 16.2 (2022), Paper No. 32, 32. ISSN: 2662-2033,1735-8787. DOI: 10.1007/s43037-022-00187-5. URL: https://doi.org/10.1007/s43037-022-00187-5.

[9] Ghazaleh Asghari, Zeljko Cuckovic, and Sonmez Sahutoglu. “Fixed points of the Berezin transform on Fock-type spaces”. In: *arXiv preprint arXiv:2508.13115* (2025).

[10] Ghazaleh Asghari, Zhangjian Hu, and Jani A. Virtanen. “Schatten class Hankel operators on doubling Fock spaces and the Berger-Coburn phenomenon”. In: *J. Math. Anal. Appl.* 540.2 (2024), Paper No. 128596, 32. ISSN: 0022-247X,1096-0813. DOI: 10.1016/j.jmaa.2024.128596. URL: https://doi.org/10.1016/j.jmaa.2024.128596.

[11] Rohan Attele. “Bounded analytic functions and the little Bloch space”. In: *Internat. J. Math. Math. Sci.* 13.1 (1990), pp. 193–198. ISSN: 0161-1712,1687-0425. DOI: 10.1155/S016117129000028X. URL: https://doi.org/10.1155/S016117129000028X.

[12] Sheldon Axler. "The Bergman space, the Bloch space, and commutators of multiplication operators". In: *Duke Math. J.* 53.2 (1986), pp. 315–332. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-86-05320-2. URL: https://doi.org/10.1215/S0012-7094-86-05320-2.

[13] Sheldon Axler and Željko Čučković. "Commuting Toeplitz operators with harmonic symbols". In: *Integral Equations Operator Theory* 14.1 (1991), pp. 1–12. ISSN: 0378-620X,1420-8989. DOI: 10.1007/BF01194925. URL: https://doi.org/10.1007/BF01194925.

[14] Wolfram Bauer. "Hilbert-Schmidt Hankel operators on the Segal-Bargmann space". In: *Proc. Amer. Math. Soc.* 132.10 (2004), pp. 2989–2996. ISSN: 0002-9939,1088-6826. DOI: 10.1090/S0002-9939-04-07264-8. URL: https://doi.org/10.1090/S0002-9939-04-07264-8.

[15] Wolfram Bauer, Boo Rim Choe, and Hyungwoon Koo. "Commuting Toeplitz operators with pluriharmonic symbols on the Fock space". In: *J. Funct. Anal.* 268.10 (2015), pp. 3017–3060. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2015.03.003. URL: https://doi.org/10.1016/j.jfa.2015.03.003.

[16] C. A. Berger and L. A. Coburn. "Heat flow and Berezin-Toeplitz estimates". In: *Amer. J. Math.* 116.3 (1994), pp. 563–590. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2374991. URL: https://doi.org/10.2307/2374991.

[17] C. A. Berger and L. A. Coburn. "Toeplitz operators on the Segal-Bargmann space". In: *Trans. Amer. Math. Soc.* 301.2 (1987), pp. 813–829. ISSN: 0002-9947,1088-6850. DOI: 10.2307/2000671. URL: https://doi.org/10.2307/2000671.

[18] Hélène Bommier-Hato. "Lipschitz estimates for the Berezin transform". In: *J. Funct. Spaces Appl.* 8.2 (2010), pp. 103–128. ISSN: 0972-6802. DOI: 10.1155/2010/461315. URL: https://doi.org/10.1155/2010/461315.

[19] Hélène Bommier-Hato and Olivia Constantin. "Big Hankel operators on vector-valued Fock spaces in $\mathbb{C}^d$". In: *Integral Equations Operator Theory* 90.1 (2018), Paper No. 2, 25. ISSN: 0378-620X,1420-8989. DOI: 10.1007/s00020-018-2433-y. URL: https://doi.org/10.1007/s00020-018-2433-y.

[20] Hélène Bommier-Hato and El Hassan Youssfi. "Hankel operators on weighted Fock spaces". In: *Integral Equations and Operator Theory* 59.1 (2007), pp. 1–17.

[21] Hélène Bommier-Hato, El Hassan Youssfi, and Kehe Zhu. "Sarason's Toeplitz product problem for a class of Fock spaces". In: *Bull. Sci. Math.* 141.5 (2017), pp. 408–442. ISSN: 0007-4497,1952-4773. DOI: 10.1016/j.bulsci.2017.03.002. URL: https://doi.org/10.1016/j.bulsci.2017.03.002.

[22] Irène Casseli. "Fixed points of the Berezin transform of multidimensional polyanalytic Fock spaces". In: *J. Math. Anal. Appl.* 481.2 (2020), pp. 123479, 15. ISSN: 0022-247X,1096-0813. DOI: 10.1016/j.jmaa.2019.123479. URL: https://doi.org/10.1016/j.jmaa.2019.123479.

[23] Jeff Cheeger, Mikhail Gromov, and Michael Taylor. "Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds". In: *J. Differential Geometry* 17.1 (1982), pp. 15–53. ISSN: 0022-040X,1945-743X. URL: `http://projecteuclid.org/euclid.jdg/1214436699`.

[24] Michael Christ. "On the $\overline{\partial}$ equation in weighted L^2 norms in $\mathbf{C}^1$". In: *J. Geom. Anal.* 1.3 (1991), pp. 193–230. ISSN: 1050-6926,1559-002X. DOI: `10.1007/BF02921303`. URL: `https://doi.org/10.1007/BF02921303`.

[25] L. A. Coburn. "A Lipschitz estimate for Berezin's operator calculus". In: *Proc. Amer. Math. Soc.* 133.1 (2005), pp. 127–131. ISSN: 0002-9939,1088-6826. DOI: `10.1090/S0002-9939-04-07476-3`. URL: `https://doi.org/10.1090/S0002-9939-04-07476-3`.

[26] John B. Conway. *A course in functional analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.

[27] Carl C. Cowen and Christopher Felder. "Convexity of the Berezin range". In: *Linear Algebra Appl.* 647 (2022), pp. 47–63. ISSN: 0024-3795,1873-1856. DOI: `10.1016/j.laa.2022.04.003`. URL: `https://doi.org/10.1016/j.laa.2022.04.003`.

[28] Željko Čučković and Bo Li. "Berezin transform, Mellin transform and Toeplitz operators". In: *Complex Anal. Oper. Theory* 6.1 (2012), pp. 189–218. ISSN: 1661-8254,1661-8262. DOI: `10.1007/s11785-010-0051-z`. URL: `https://doi.org/10.1007/s11785-010-0051-z`.

[29] Željko Čučković and Sönmez Şahutoğlu. "Berezin regularity of domains in $\mathbb{C}^n$ and the essential norms of Toeplitz operators". In: *Trans. Amer. Math. Soc.* 374.4 (2021), pp. 2521–2540. ISSN: 0002-9947,1088-6850. DOI: `10.1090/tran/8201`. URL: `https://doi.org/10.1090/tran/8201`.

[30] Gian Maria Dall'Ara. "Pointwise estimates of weighted Bergman kernels in several complex variables". In: *Adv. Math.* 285 (2015), pp. 1706–1740. ISSN: 0001-8708,1090-2082. DOI: `10.1016/j.aim.2015.06.024`. URL: `https://doi.org/10.1016/j.aim.2015.06.024`.

[31] Vishwa Dewage and Mishko Mitkovski. "A quantum harmonic analysis approach to the Berger-Coburn theorem". In: *New York J. Math.* 31 (2025), pp. 887–901. ISSN: 1076-9803.

[32] J. Diestel and J. J. Uhl Jr. *Vector measures*. Vol. No. 15. Mathematical Surveys. With a foreword by B. J. Pettis. American Mathematical Society, Providence, RI, 1977, pp. xiii+322.

[33] Jianxiang Dong, Chunxu Xu, and Yufeng Lu. "Positive operator-valued Toeplitz operators on vector-valued generalized Fock spaces". In: *Oper. Matrices* 17.4 (2023), pp. 925–938. ISSN: 1846-3886,1848-9974. DOI: `10.7153/oam-2023-17-61`. URL: `https://doi.org/10.7153/oam-2023-17-61`.

[34] Oliver Dragičević. "Weighted estimates for powers and the Ahlfors-Beurling operator". In: *Proc. Amer. Math. Soc.* 139.6 (2011), pp. 2113–2120. ISSN: 0002-9939,1088-6826. DOI: `10.1090/S0002-9939-2010-10645-7`. URL: `https://doi.org/10.1090/S0002-9939-2010-10645-7`.

[35] Harry Dym and Santanu Sarkar. “Multiplication operators with deficiency indices (p, p) and sampling formulas in reproducing kernel Hilbert spaces of entire vector valued functions”. In: *J. Funct. Anal.* 273.12 (2017), pp. 3671–3718. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2017.09.007. URL: https://doi.org/10.1016/j.jfa.2017.09.007.

[36] Miroslav Engliš. “Functions invariant under the Berezin transform”. In: *J. Funct. Anal.* 121.1 (1994), pp. 233–254. ISSN: 0022-1236,1096-0783. DOI: 10.1006/jfan.1994.1048. URL: https://doi.org/10.1006/jfan.1994.1048.

[37] Miroslav Engliš. “Singular Berezin transforms”. In: *Complex Anal. Oper. Theory* 1.4 (2007), pp. 533–548. ISSN: 1661-8254,1661-8262. DOI: 10.1007/s11785-007-0023-0. URL: https://doi.org/10.1007/s11785-007-0023-0.

[38] Miroslav Engliš. “Toeplitz operators on Bergman-type spaces”. PhD thesis. 1991.

[39] Quanlei Fang and Jingbo Xia. “Hankel operators on weighted Bergman spaces and norm ideals”. In: *Complex Anal. Oper. Theory* 12.3 (2018), pp. 629–668. ISSN: 1661-8254,1661-8262. DOI: 10.1007/s11785-017-0710-4. URL: https://doi.org/10.1007/s11785-017-0710-4.

[40] Loukas Grafakos. *Modern Fourier analysis*. Second. Vol. 250. Graduate Texts in Mathematics. Springer, New York, 2009, pp. xvi+504. ISBN: 978-0-387-09433-5. DOI: 10.1007/978-0-387-09434-2. URL: https://doi.org/10.1007/978-0-387-09434-2.

[41] Raffael Hagger and Jani A. Virtanen. “Compact Hankel operators with bounded symbols”. In: *J. Operator Theory* 86.2 (2021), pp. 317–329. ISSN: 0379-4024,1841-7744. DOI: 10.7900/jot. URL: https://doi.org/10.7900/jot.

[42] Einar Hille and Ralph S. Phillips. *Functional analysis and semi-groups*. Vol. Vol. 31. American Mathematical Society Colloquium Publications. rev. ed. American Mathematical Society, Providence, RI, 1957.

[43] Zhangjian Hu and Xiaofen Lv. “Positive Toeplitz operators between different doubling Fock spaces”. In: *Taiwanese J. Math.* 21.2 (2017), pp. 467–487. ISSN: 1027-5487,2224-6851. DOI: 10.11650/tjm/7031. URL: https://doi.org/10.11650/tjm/7031.

[44] Zhangjian Hu and Xiaofen Lv. “Toeplitz operators from one Fock space to another”. In: *Integral Equations Operator Theory* 70.4 (2011), pp. 541–559. ISSN: 0378-620X,1420-8989. DOI: 10.1007/s00020-011-1887-y. URL: https://doi.org/10.1007/s00020-011-1887-y.

[45] Zhangjian Hu and Xiaofen Lv. “Toeplitz operators on Fock spaces $F^p(\varphi)$”. In: *Integral Equations Operator Theory* 80.1 (2014), pp. 33–59. ISSN: 0378-620X,1420-8989. DOI: 10.1007/s00020-014-2168-3. URL: https://doi.org/10.1007/s00020-014-2168-3.

[46] Zhangjian Hu, Xiaofen Lv, and Brett D. Wick. “Localization and compactness of operators on Fock spaces”. In: *J. Math. Anal. Appl.* 461.2 (2018), pp. 1711–1732. ISSN: 0022-247X,1096-0813. DOI: 10.1016/j.jmaa.2017.12.046. URL: https://doi.org/10.1016/j.jmaa.2017.12.046.

[47] Zhangjian Hu and Jani A. Virtanen. “Corrigendum to “Schatten class Hankel operators on the Segal-Bargmann space and the Berger-Coburn phenomenon””. In: *Trans. Amer. Math. Soc.* 376.8 (2023), pp. 6011–6014. ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/8857. URL: https://doi.org/10.1090/tran/8857.

[48] Zhangjian Hu and Jani A. Virtanen. “Fredholm Toeplitz operators on doubling Fock spaces”. In: *J. Geom. Anal.* 32.4 (2022), Paper No. 106, 29. ISSN: 1050-6926,1559-002X. DOI: 10.1007/s12220-021-00761-7. URL: https://doi.org/10.1007/s12220-021-00761-7.

[49] Zhangjian Hu and Jani A. Virtanen. “IDA and Hankel operators on Fock spaces”. In: *Anal. PDE* 16.9 (2023), pp. 2041–2077. ISSN: 2157-5045,1948-206X. DOI: 10.2140/apde.2023.16.2041. URL: https://doi.org/10.2140/apde.2023.16.2041.

[50] Zhangjian Hu and Jani A. Virtanen. “On the Berger-Coburn Phenomenon”. In: *Operator and matrix theory, function spaces, and applications*. Vol. 295. Oper. Theory Adv. Appl. Birkhäuser/Springer, Cham, [2024] ©2024, pp. 223–234. ISBN: 978-3-031-50612-3; 978-3-031-50613-0. DOI: 10.1007/978-3-031-50613-0_9. URL: https://doi.org/10.1007/978-3-031-50613-0_9.

[51] Zhangjian Hu and Jani A. Virtanen. “Schatten class Hankel operators on the Segal-Bargmann space and the Berger-Coburn phenomenon”. In: *Trans. Amer. Math. Soc.* 375.5 (2022), pp. 3733–3753. ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/8638. URL: https://doi.org/10.1090/tran/8638.

[52] Zhangjian Hu and Ermin Wang. “Hankel operators between Fock spaces”. In: *Integral Equations Operator Theory* 90.3 (2018), Paper No. 37, 20. ISSN: 0378-620X,1420-8989. DOI: 10.1007/s00020-018-2459-1. URL: https://doi.org/10.1007/s00020-018-2459-1.

[53] Josh Isralowitz and Kehe Zhu. “Toeplitz operators on the Fock space”. In: *Integral Equations Operator Theory* 66.4 (2010), pp. 593–611. ISSN: 0378-620X,1420-8989. DOI: 10.1007/s00020-010-1768-9. URL: https://doi.org/10.1007/s00020-010-1768-9.

[54] Miroljub Jevtić. “Fixed points of an integral operator”. In: *J. Anal. Math.* 91 (2003), pp. 123–141. ISSN: 0021-7670,1565-8538. DOI: 10.1007/BF02788784. URL: https://doi.org/10.1007/BF02788784.

[55] Steven G. Krantz. *Function theory of several complex variables*. Reprint of the 1992 edition. AMS Chelsea Publishing, Providence, RI, 2001, pp. xvi+564. ISBN: 0-8218-2724-3. DOI: 10.1090/chel/340. URL: https://doi.org/10.1090/chel/340.

[56] Yohann Le Floch. *A brief introduction to Berezin-Toeplitz operators on compact Kähler manifolds*. CRM Short Courses. Springer, Cham, 2018, pp. viii+140. ISBN: 978-3-319-94681-8; 978-3-319-94682-5. DOI: 10.1007/978-3-319-94682-5. URL: https://doi.org/10.1007/978-3-319-94682-5.

[57] Jaesung Lee. “Some properties of the Berezin transform in the bidisc”. In: *Commun. Korean Math. Soc.* 32.3 (2017), pp. 779–787. ISSN: 1225-1763,2234-3024. DOI: 10.4134/CKMS.c160158. URL: https://doi.org/10.4134/CKMS.c160158.

[58] Jaesung Lee. “Weighted Berezin transform in the polydisc”. In: *J. Math. Anal. Appl.* 338.2 (2008), pp. 1489–1493. ISSN: 0022-247X,1096-0813. DOI: 10.1016/j.jmaa.2007.06.048. URL: https://doi.org/10.1016/j.jmaa.2007.06.048.

[59] Sam Looi. "A counterexample to the Berger–Coburn conjecture". In: *arXiv preprint arXiv:2601.20859* (2026).

[60] Daniel H. Luecking. "Characterizations of certain classes of Hankel operators on the Bergman spaces of the unit disk". In: *J. Funct. Anal.* 110.2 (1992), pp. 247–271. ISSN: 0022-1236,1096-0783. DOI: 10.1016/0022-1236(92)90034-G. URL: https://doi.org/10.1016/0022-1236(92)90034-G.

[61] Daniel H. Luecking. "Trace ideal criteria for Toeplitz operators". In: *J. Funct. Anal.* 73.2 (1987), pp. 345–368. ISSN: 0022-1236. DOI: 10.1016/0022-1236(87)90072-3. URL: https://doi.org/10.1016/0022-1236(87)90072-3.

[62] Xiaofen Lv. "Bergman projections on weighted Fock spaces in several complex variables". In: *J. Inequal. Appl.* (2017), Paper No. 286, 10. ISSN: 1029-242X. DOI: 10.1186/s13660-017-1560-3. URL: https://doi.org/10.1186/s13660-017-1560-3.

[63] Xiaofen Lv and Ermin Wang. "Hankel operators on doubling Fock spaces". In: *J. Math. Anal. Appl.* 531.1 (2024), Paper No. 127780, 17. ISSN: 0022-247X,1096-0813. DOI: 10.1016/j.jmaa.2023.127780. URL: https://doi.org/10.1016/j.jmaa.2023.127780.

[64] N. Marco, X. Massaneda, and J. Ortega-Cerdà. "Interpolating and sampling sequences for entire functions". In: *Geom. Funct. Anal.* 13.4 (2003), pp. 862–914. ISSN: 1016-443X,1420-8970. DOI: 10.1007/s00039-003-0434-7. URL: https://doi.org/10.1007/s00039-003-0434-7.

[65] Jordi Marzo and Joaquim Ortega-Cerdà. "Pointwise estimates for the Bergman kernel of the weighted Fock space". In: *J. Geom. Anal.* 19.4 (2009), pp. 890–910. ISSN: 1050-6926,1559-002X. DOI: 10.1007/s12220-009-9083-x. URL: https://doi.org/10.1007/s12220-009-9083-x.

[66] Charles A. Micchelli and Massimiliano Pontil. "Learning the kernel function via regularization". In: *J. Mach. Learn. Res.* 6 (2005), pp. 1099–1125. ISSN: 1532-4435,1533-7928. URL: https://jmlr.org/papers/v6/micchelli05a.html.

[67] Hà Quang Minh, Loris Bazzani, and Vittorio Murino. "A unifying framework in vector-valued reproducing kernel Hilbert spaces for manifold regularization and co-regularized multi-view learning". In: *J. Mach. Learn. Res.* 17 (2016), Paper No. 25, 72. ISSN: 1532-4435,1533-7928. URL: https://jmlr.csail.mit.edu/papers/v17/14-036.html.

[68] Roc Oliver and Daniel Pascuas. "Toeplitz operators on doubling Fock spaces". In: *J. Math. Anal. Appl.* 435.2 (2016), pp. 1426–1457. ISSN: 0022-247X,1096-0813. DOI: 10.1016/j.jmaa.2015.11.023. URL: https://doi.org/10.1016/j.jmaa.2015.11.023.

[69] N. V. Rao. "The range of the Berezin transform". In: *J. Math. Sci. (N.Y.)* 228.6 (2018), pp. 684–694. ISSN: 1072-3374,1573-8795. DOI: 10.1007/s10958-017-3656-1. URL: https://doi.org/10.1007/s10958-017-3656-1.

[70] Walter Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424. ISBN: 0-07-054236-8.

[71] Walter Rudin. *Real and complex analysis*. Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416. ISBN: 0-07-054234-1.

[72] Georg Schneider. “Hankel operators with antiholomorphic symbols on the Fock space”. In: *Proc. Amer. Math. Soc.* 132.8 (2004), pp. 2399–2409. ISSN: 0002-9939,1088-6826. DOI: 10.1090/S0002-9939-04-07362-9. URL: https://doi.org/10.1090/S0002-9939-04-07362-9.

[73] Alexander P. Schuster and Dror Varolin. “Toeplitz operators and Carleson measures on generalized Bargmann-Fock spaces”. In: *Integral Equations Operator Theory* 72.3 (2012), pp. 363–392. ISSN: 0378-620X,1420-8989. DOI: 10.1007/s00020-011-1939-3. URL: https://doi.org/10.1007/s00020-011-1939-3.

[74] Kristian Seip and El Hassan Youssfi. “Hankel operators on Fock spaces and related Bergman kernel estimates”. In: *J. Geom. Anal.* 23.1 (2013), pp. 170–201. ISSN: 1050-6926,1559-002X. DOI: 10.1007/s12220-011-9241-9. URL: https://doi.org/10.1007/s12220-011-9241-9.

[75] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Vol. 43. Princeton Mathematical Series. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993, pp. xiv+695. ISBN: 0-691-03216-5.

[76] Jingbo Xia. “The Berger-Coburn phenomenon for Hankel operators on the Fock space”. In: *Complex Anal. Oper. Theory* 17.3 (2023), Paper No. 35, 59. ISSN: 1661-8254,1661-8262. DOI: 10.1007/s11785-023-01338-8. URL: https://doi.org/10.1007/s11785-023-01338-8.

[77] Jingbo Xia and Dechao Zheng. “Standard deviation and Schatten class Hankel operators on the Segal-Bargmann space”. In: *Indiana Univ. Math. J.* 53.5 (2004), pp. 1381–1399. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.2004.53.2434. URL: https://doi.org/10.1512/iumj.2004.53.2434.

[78] Chunxu Xu, Jianxiang Dong, and Tao Yu. “Toeplitz and Hankel operators on vector-valued Fock-type spaces”. In: *Complex Anal. Oper. Theory* 18.6 (2024), Paper No. 133, 33. ISSN: 1661-8254,1661-8262. DOI: 10.1007/s11785-024-01575-5. URL: https://doi.org/10.1007/s11785-024-01575-5.

[79] Kehe Zhu. *Analysis on Fock spaces*. Vol. 263. Graduate Texts in Mathematics. Springer, New York, 2012, pp. x+344. ISBN: 978-1-4419-8800-3. DOI: 10.1007/978-1-4419-8801-0. URL: https://doi.org/10.1007/978-1-4419-8801-0.

[80] Kehe Zhu. *Operator theory in function spaces*. Second. Vol. 138. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007, pp. xvi+348. ISBN: 978-0-8218-3965-2. DOI: 10.1090/surv/138. URL: https://doi.org/10.1090/surv/138.

[81] Zhengyuan Zhuo and Zengjian Lou. “Sampling measure on doubling Fock spaces”. In: *J. Geom. Anal.* 33.10 (2023), Paper No. 313, 18. ISSN: 1050-6926,1559-002X. DOI: 10.1007/s12220-023-01380-0. URL: https://doi.org/10.1007/s12220-023-01380-0.