# Linear and Nonlinear Non-Divergence Elliptic Systems of Partial Differential Equations

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A thesis submitted for the degree of

*Doctor of Philosophy*

June 2018

# Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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June 2018

# Acknowledgements

My first thanks go to my supervisor, Associate Professor Dr. Nikos Katzourakis for his valuable support, motivation, encouragement, insightful comments, enthusiasm and continuous guidance during the development of this thesis. I am grateful for all his efforts and for helping me to constantly challenge and improve myself, and would like to thank him for giving the permission to use his own figures, as well as for all the comments and suggestions which improved the content of this thesis. He was always open to discussing and clarifying different concepts and made me feel part of the department and of his family, many thanks to him and to his family for being my family in UK, I cannot thank them enough for that.

I would also like to express my heartfelt gratefulness to my family. Without the support and encouragement of them, I would never have had the ability to finish this thesis, and for their belief in me I will be forever grateful and thankful. To my wonderful parents, who have been there for me all the time. Thank you for all you have done for me, I am so sorry for not been able to attend your funeral Dad, but I hope I have been able to make you proud.

I would like to express my thanks to the many people, who so generously contributed to the work presented in this thesis. On behalf of the joint authors of the paper presented in chapter four, we are indebted to G. Aronsson for his encouragement and support towards us. We would also like to thank Roger Moser and Giles Shaw for their comments and suggestions which improved the content of the paper mentioned above. We would also like to thank Jan Kristensen, Tristan Pryer and Enea Parini for several scientific discussions relevant to the results therein.

Similar, profound gratitude to Associate Professor Dr. Tristan Pryer and Dr. Titus Hilberdink who were my monitoring committee during my research study period along which they have evaluated my progress and have provided me with valuable feedback.

Many thanks to my sponsor the Iraqi government for the funding I have received. Without this funding this research would not have been possible. Many thanks also to the Iraqi Ministry of Higher Education and Scientific Research (MOHESR), Iraqi Cultural Attachè in London and to the Department of Mathematics, College of Science, University of Basra in Iraq, for the support which allowed me to undertake this research.

I would also like to thank the organisers and the group members of the South-West

Network in Generalised Solutions for Nonlinear PDEs for giving me the opportunity to present my work in some of the meetings and get feedback.

Last, but not the least, I would like to thank my fabulous friends in the office for keeping me smiling, the fantastic staff of the Department of Mathematics especially Peta Ann who was my fairy godmother during the first two years of my PhD study, Ruth Harris, Brigitte Calderon, Kristine Aldridge and other people at the university of Reading for their support and friendship, I am also indebted to them for their help, and everyone who has believed in me and supported me.

# Dedication

This thesis is dedicated to:

The soul of my father,

The souls of brave men and women who defend the world against the terrorism,

My role model and supervisor Nikos Katzourakis,

My Family and everybody who believed in me and supported me.

# Abstract

This thesis is a collection of published and submitted papers. Each paper presents a chapter of the thesis and in each paper we make progress in the field of nondivergence systems of nonlinear PDEs. The new progress includes proving the existence and uniqueness of strong solutions to first order elliptic systems in Chapter 2, proving the existence of absolute minimisers to a vectorial $1D$ minimisation problem in Chapter 3, proving geometric aspects of $p$-Harmonic maps in Chapter 4, proving new properties of classical solutions to the vectorial infinity Laplacian in Chapter 5.

In Chapter 2 of this thesis we present the joint paper with Katzourakis in which we extend the results of [43]. In the very recent paper [43], Katzourakis proved that for any $f \in L^2(\mathbb{R}^n, \mathbb{R}^N)$, the fully nonlinear first order system $F(\cdot, \mathrm{D}u) = f$ is well posed in the so-called J.L. Lions space and moreover the unique strong solution $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the problem satisfies a quantitative estimate. A central ingredient in the proof was the introduction of an appropriate notion of ellipticity for $F$ inspired by Campanato's classical work in the 2nd order case. Herein we extend the results of [43] by introducing a new strictly weaker ellipticity condition and by proving well posedness in the same "energy" space. In Chapter 3 of this thesis we present the joint paper with Katzourakis in which we prove the existence of vectorial Absolute Minimisers in the sense of Aronsson for the supremal functional $E_\infty(u, \Omega') = \|\mathscr{L}(\cdot, u, \mathrm{D}u)\|_{L^\infty(\Omega')}$, $\Omega' \Subset \Omega$, applied to $W^{1,\infty}$ maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ with given boundary values. The assumptions on $\mathscr{L}$ are minimal, improving earlier existence results previously established by Barron-Jensen-Wang and by Katzourakis. In Chapter 4 of this thesis we present the joint paper with Katzourakis and Ayanbayev in which we consider the PDE system of vanishing normal projection of the Laplacian for $C^2$ maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$:

$$[\![\mathrm{D}u]\!]^\perp \Delta u = 0 \quad \text{in } \Omega.$$

This system has discontinuous coefficients and geometrically expresses the fact that the Laplacian is a vector field tangential to the image of the mapping. It arises as a constituent component of the $p$-Laplace system for all $p \in [2, \infty]$. For $p = \infty$, the $\infty$-Laplace system is the archetypal equation describing extrema of supremal functionals in vectorial Calculus of Variations in $L^\infty$. Herein we show that the image of a solution $u$ is piecewise affine if the rank of $\mathrm{D}u$ is equal to one. As a consequence we obtain corresponding flatness results for $p$-Harmonic maps, $p \in [2, \infty]$. In Chapter 5 of this thesis we present a single authored paper in which we discuss an extension

of a recent paper [41]. In [41], among other interesting results, Katzourakis analysed the phenomenon of separation of the solutions $u:\mathbb{R}^2 \supseteq \Omega \longrightarrow \mathbb{R}^N$, to the $\infty$-Laplace system

$$\Delta_\infty u \,:=\, \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes I\Big) : \mathrm{D}^2 u \,=\, 0,$$

to phases with qualitatively different behaviour in the case of $n = 2 \leq N$. The solutions of the $\infty$-Laplace system are called the $\infty$-Harmonic mappings. Chapter 5 of this thesis present an extension of Katzourakis' result mentioned above to higher dimensions by studying the phase separation of $n$-dimensional $\infty$-Harmonic mappings in the case $N \geq n \geq 2$.

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# Chapter 1

# Background and motivations

## 1.1 Introduction

There is no doubt that PDEs in general, either linear or nonlinear, do not possess classical solutions, considering that not all derivatives which appear in the equation may actually exist. The modern approach to this problem consists of looking for appropriately defined *generalised solutions* for which the hope is that at least existence can be proved given certain boundary and/or initial conditions. Once existence is settled, subsequent considerations typically include uniqueness, qualitative properties, regularity and of course numerics.

This approach to PDE theory has been enormously successful, but unfortunately so far only equations and systems with fairly special structure have been considered. A standing idea consist of using integration by parts and duality of functional spaces in order to interpret rigorously derivatives which do not exist, by "passing them to test functions". This approach of Sobolev spaces and Schwartz's Distributions which dates back to the 1930s is basically restricted to equations which have divergence structure, like the Euler-Lagrange equation of Calculus of Variations or linear systems with *smooth* coefficients. Let us demonstrate that a solution $u \in C^2(\bar{\Omega})$ of the boundary-value problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

for Poisson's equation can be characterised as the minimiser of $E[u] = \min_{\omega \in \mathcal{A}} E[\omega]$, where $E[\omega]$ is the energy functional which we define as follows:

$$E[\omega] := \int_\Omega \frac{1}{2}|\mathrm{D}\omega|^2 - \omega f \, dx,$$

$\omega$ belonging to the admissible set:

$$\mathcal{A} := \{\omega \in C^2(\bar{\Omega}) \mid \omega = g \text{ on } \partial\Omega\}$$

A more recent theory discovered in the 1980s is that of viscosity solutions and builds on the idea that the maximum principle allows to "pass derivatives to test functions" without duality. The theory of viscosity solutions applies to fully nonlinear first and second order partial differential equations. For such equations, and in particular for second order ones, solutions are generally non-smooth and standard approaches in order to define a "weak solution" do not apply:classical, strong almost everywhere, weak, measure-valued and distributional solutions either do not exist or may not even be defined. The main reason for the latter failure is that, the standard idea of using integration by parts in order to pass derivatives to smooth test functions by duality, is not available for non-divergence structure PDE. This idea applies mostly to scalar solutions of single equations which support the maximum principle (elliptic or parabolic up to second order), but has been hugely successful because it includes *fully nonlinear* equations. For more information about viscosity solutions we refer to the reference [42]. A relevant notion of solution which bridges the gap between classical and generalised is that of strong solutions, where it is usually assumed that all derivatives appearing exist a.e. but in a weak sense.

This thesis is a collection of papers as we will explain in more details in section 1.3 of this chapter by giving a brief outline of the thesis structure. Some of these papers are joint papers with other researchers at the University of Reading. In these papers we developed theories in the nonlinear PDEs field of study mentioned above.

## 1.2 Literature review

Due to the vastness of the field, it is not easy to include a comprehensive literature review. A significant amount of the literature is reviewed in the introductions of the papers that are included in the chapters of this thesis. However, we will try to preview briefly the general outlines of the most important previous studies in this field, that inspired the new progress in this thesis. We will list these previous studies in an order corresponding to the order of the papers that inspired by them as they appear in the chapters of this thesis.

### 1.2.1 Near operators theory

In 1989, S. Campanato [22] has introduced the notion of "near operators" for studying the existence of solutions of elliptic differential equations and systems. In 1994, he has introduced in his work [25] a strong ellipticity condition which is a condition of nearness between operators. He also has presented a theory of nearness of mappings say $\mathcal{F}, \mathcal{A}$ defined on a set $\Omega \subseteq \mathbf{X}$ taking values in a Banach space $\mathbf{X}$. He has proved that $\mathcal{F}$ is injective, surjective or bijective if and only if $\mathcal{F}$ is near $\mathcal{A}$ with these properties. The "Campanato" ellipticity condition states that if $\mathcal{F}, \mathcal{A} : \mathfrak{X} \longrightarrow X$ are two mappings from the set $\mathfrak{X} \neq \emptyset$ into the Banach space $(X, \|\cdot\|)$. If there is a constant $K \in (0,1)$ such that

$$\Big\|\mathcal{F}[u]-\mathcal{F}[v]-\big(\mathcal{A}[u]-\mathcal{A}[v]\big)\Big\| \,\leq\, K\big\|\mathcal{A}[u]-\mathcal{A}[v]\big\|$$

for all $u, v \in \mathfrak{X}$ and if $\mathcal{A} : \mathfrak{X} \longrightarrow X$ is a bijection, it follows that $\mathcal{F} : \mathfrak{X} \longrightarrow X$ is a bijection as well.

In 1998, A. Tarsia [61] has studied a generalisation of the near operators theorem. And in 2000, he has made a developments of the Campanato's theory of near operators [62], therein he showed that the theory of near operators is also applicable in more general situations than those considered up to the time of his contribution. And also Another contribution of A. Tarsia was [63] in 2008, wherein he has gave a short survey of the Campanato's near operators theory and of its applications to fully nonlinear elliptic equations.

In 2015, N. Katzourakis [37] has introduced a new much weaker ellipticity notion for $\mathcal{F}$ than the Campanato-Tarsia condition and for the first time he has considered the case of global solutions on $\Omega = \mathbb{R}^n$. He applied the"K-Condition of ellipticity" to study the existence and uniqueness of strong solutions to fully nonlinear first order elliptic systems. We recall the "K-Condition of ellipticity" for the convenience of the reader in the paper presented in Chapter 2.

In 2016, N. Katzourakis [40] has applied the "K-Condition of ellipticity" to study the existence and uniqueness of global strong solutions to fully nonlinear second order elliptic systems. For such problems, he has considered the case of global solutions on $\Omega = \mathbb{R}^n$ for the first time. Also, therein he showed the connection between ellipticity and his K-Condition of ellipticity.

### 1.2.2 Calculus of Variations in $L^\infty$

In the early 1960s, the Calculus of Variations in $L^\infty$ has been initiated by G. Aronsson [6–10], namely the study of supremal functionals and of their associated equations describing critical points. It has a long history since then, but the theory was essentially restricted to the scalar case. By introducing the appropriate minimality notion for

$$E_\infty(u,\Omega') \,:=\, \underset{x\in\Omega'}{\mathrm{ess\,sup}}\, \mathscr{L}\left(x,u(x),\mathrm{D}u(x)\right), \qquad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N),\ \Omega' \Subset \Omega, \tag{1.2.1}$$

that of absolute minimisers which states that the map $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ is absolutely minimising for (1.2.1) when for all $\Omega \Subset \mathbb{R}^n$ and all $\phi \in W^{1,\infty}_0(\Omega',\mathbb{R}^N)$ we have

$$E_\infty(u,\Omega') \,\leq\, E_\infty(u+\phi,\Omega'), \tag{1.2.2}$$

Aronsson studied solutions $u \in C^2(\mathbb{R}^n)$ of what we now call "Aronsson's PDE" , in the case $N = 1$ and the Lagrangian $\mathscr{L}$ is $C^1$:

$$A_\infty u \; := \; \mathrm{D}\big(\mathscr{L}(\cdot, u, \mathrm{D}u)\big)\mathscr{L}_P(\cdot, u, \mathrm{D}u) \; = \; 0. \tag{1.2.3}$$

which is the equivalent of the Euler-Lagrange equation for supremal functionals. In Aronsson's PDE above, the subscript denotes the gradient of $\mathscr{L}(x, \eta, P)$ with respect to $P$ and, as it is customary, the equation is written for smooth solutions. Such maps miss information along a hyperplane when compared to tight maps. Katzourakis recovered the lost term which causes non-uniqueness and derived the complete Aronsson's system which has discontinuous coefficients. Indeed in the early 2010s, N. Katzourakis has started to initiate the systematic study of the vector-valued case in a series of papers [37–42, 44, 46–49], and in [37] N. Katzourakis has recovered, for the first time, the lost term which causes non-uniqueness, and has derived the complete Aronsson system which has discontinuous coefficients. One of the important outcomes of this systematic study of the vector-valued case is what we call the $\infty$-Laplacian. The ODE system associated to (1.2.1) for smooth maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ turns out to be

$$\mathcal{F}_\infty\big(\cdot, u, \mathrm{D}u, \mathrm{D}^2 u\big) \; = \; 0, \quad \text{on } \Omega, \tag{1.2.4}$$

where

$$\begin{aligned}
\mathcal{F}_\infty(x, \eta, P, X) \; := \Big[ & \mathscr{L}_P(x, \eta, P) \otimes \mathscr{L}_P(x, \eta, P) \\
& + \mathscr{L}(x, \eta, P)[\mathscr{L}_P(x, \eta, P)]^\perp \mathscr{L}_{PP}(x, \eta, P)\Big] X \\
& + \Big(\mathscr{L}_\eta(x, \eta, P) \cdot P \, + \, \mathscr{L}_x(x, \eta, P)\Big)\mathscr{L}_P(x, \eta, P) \\
& + \mathscr{L}(x, \eta, P)\big[\mathscr{L}_P(x, \eta, P)\big]^\perp \Big(\mathscr{L}_{P\eta}(x, \eta, P)P \\
& + \mathscr{L}_{Px}(x, \eta, P) \, - \, \mathscr{L}_\eta(x, \eta, P)\Big).
\end{aligned} \tag{1.2.5}$$

Quite unexpectedly, in the case $N \geq 2$ the Lagrangian needs to be $C^2$ for the equation to make sense, whilst the coefficients of the full system are *discontinuous*; for more details we refer to the papers cited above. In (1.2.5) the notation of subscripts symbolises derivatives with respect to the respective variables and $\big[\mathscr{L}_P(x, \eta, P)\big]^\perp$ is the orthogonal projection to the hyperplane normal to $\mathscr{L}_P(x, \eta, P) \in \mathbb{R}^N$:

$$\big[\mathscr{L}_P(x, \eta, P)\big]^\perp \; := \mathrm{I} - \operatorname{sgn}\big(\mathscr{L}_P(x, \eta, P)\big) \otimes \operatorname{sgn}\big(\mathscr{L}_P(x, \eta, P)\big). \tag{1.2.6}$$

which plays the role of the Euler-Lagrange equation and arises in connection with variational problems for supremal functional.

#### 1.2.2.1 Vectorial Absolute Minimisers

In the early 1960s, G.Aronsson has introduced the appropriate minimality notion in $L^\infty$ to the scalar case which is the "Absolute minimality" notion explained in (1.2.2). He considered to be the first to note the locality problems associated to supremal functional. He has proved the equivalence between the so-called Absolute Minimisers and solutions of the analogue of the Euler-Lagrange equation which is associated to supremal functional under $C^2$ smoothness hypotheses.

In 2001, Barron-Jensen-Wang [15, 16] have made a notable contribution. They have studied existence of Absolute Minimisers in the "rank-1" cases. However, their study was under a certain assumptions. More precisely, in [15] they studied the lower semicontinuity properties and existence of minimiser of the functional

$$F(u) \;=\; \operatorname*{ess\,sup}_{x\in\Omega} f\left(x, u(x), \mathrm{D}u(x)\right)$$

among other assumptions they assumed that for any $(x,\eta) \in \mathbb{R}^n \times \mathbb{R}^N$ the function $f(x,\eta,\cdot)$ is weak Morrey quasiconvex, which means for all $P \in \mathbb{R}^{Nn}$, and $\phi \in W^{1,\infty}(\Omega,\mathbb{R}^N)$ the measurable function $f:\mathbb{R}^{Nn} \longrightarrow \mathbb{R}$ satisfy

$$f(P) \;\leq\; \operatorname*{ess\,sup}_{x\in\Omega} f\left(P + \mathrm{D}\Phi(x)\right) \quad \text{on } W_0^{1,\infty}(\Omega,\mathbb{R}^N).$$

Also in [16] they proved that when $N = 1$ so that $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, or when $n = 1$ and $u : \mathbb{R} \longrightarrow \mathbb{R}^N$, there exists an absolute minimiser for $F$ under appropriat growth and coercivity assumptions on $f$. More precisely, the first assumption is that for each $(x,\eta) \in \mathbb{R}^n \times \mathbb{R}^N$, the function $P \longmapsto f(x,\eta,P)$ is quasiconvex. The second assumption is that there exist non-negative constants $C_1, C_2, C_3$, and $0 < q \leq r < +\infty$ and a positive locally bounded function $h : \mathbb{R}^n \times \mathbb{R} \longrightarrow [0,+\infty)$ such that for all $(x,\eta,P) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nn}$

$$C_1|P|^q \,-\, C_2 \;\leq\; f(x,\eta,P) \;\leq\; h(x,\eta)|P|^r \,+\, C_3.$$

They needed further assumption for the case $N > n = 1$, they assumed that the above hypotheses holds for $C_2 = C_3 = 0$, which implies

$$f(x,\eta,0) = 0, \quad \text{for all } \; (x,\eta) \in \mathbb{R} \times \mathbb{R}^N.$$

In 2012, N. Katzourakis [37] has established that Aronsson's notion of Absolute Minimals adapted to the vector case indeed leads to solutions of the tangential system

$$\begin{aligned}(\mathrm{A}_{\top,\infty}u)_\alpha \;:=&\; \big(\mathscr{L}_{P_{\alpha i}}(\cdot,u,\mathrm{D}u)\mathscr{L}_{P_{\beta j}}(\cdot,u,\mathrm{D}u)\big)\mathrm{D}^2_{ij}u \\ &+ \mathscr{L}_{P_{\alpha i}}(\cdot,u,\mathrm{D}u)\big(\mathscr{L}_{\eta\beta}(\cdot,u,\mathrm{D}u)\mathrm{D}_i u_\beta + \mathscr{L}_{x_i}(\cdot,u,\mathrm{D}u)\big) = 0,\end{aligned}$$

but the question of how to describe variationally the full $\infty$-Laplacian system remained open. He also showed that the tangential system is not sufficient for Absolute

Minimality.

In 2017, N. Katzourakis [47] has studied the problem of Absolutely minimising generalised solutions to the equations of one-dimensional vectorial calculus of variations in $L^\infty$, under certain different structural assumptions from that of Barron-Jensen-Wang. He assumed: strong convexity, smoothness and structural assumptions. By the structural assumptions we mean that he assumed the Lagrangian can be written in the following form

$$\mathscr{L}(x,\eta,P) := \mathscr{H}\Big(x,\eta,\frac{1}{2}\big|P-\mathscr{V}(x,\eta)\big|^2\Big).$$

For more details we refer to the introduction of the paper presented in Chapter 3.

#### 1.2.2.2 Structure of $\infty$-Harmonic maps

By the $\infty$-Harmonic maps we mean the solutions of the $\infty$-Laplacian.

Given a map $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$. The $\infty$-Laplace equation is the PDE

$$\Delta_\infty u := \mathrm{D}u\otimes\mathrm{D}u : \mathrm{D}^2u = 0 \quad \text{in } \Omega,$$

this equation was first derived by G. Aronsson [6–10] as the governing equation for the so-called absolute minimizer $u$ of the $L^\infty$ variational problem of minimizing

$$I[v] := \operatorname*{ess\,sup}_{\Omega} |\mathrm{D}v|,$$

among Lipschitz continuous functions $v$ taking prescribed boundary values on $\partial\Omega$.

For a map $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^N$, the $\infty$-Laplacian is the system

$$\Delta_\infty u := \Big(\mathrm{D}u\otimes\mathrm{D}u + |\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp\otimes\mathrm{I}\Big) : \mathrm{D}^2u = 0 \quad \text{in } \Omega.$$

The $\infty$-Laplacian plays the role of the Euler-Lagrange equation and arises in connexion with variational problems for the supremal functional

$$E_\infty(u,\Omega) := \|\mathrm{D}u\|_{L^\infty(\Omega)}, \quad u\in W^{1,\infty}(\Omega,\mathbb{R}^N).$$

In 2013, N. Katzourakis [38] constructed new explicit smooth solutions for the case when the dimensions of the domain and the target of the solution are $n=N=2$, namely smooth 2D $\infty$-Harmonic maps whose interfaces have triple junctions and non-smooth corners and are given by the explicit formula

$$u(x,y) := \int_y^x e^{iK(t)}dt. \tag{1.2.7}$$

Indeed, for $K\in C^1(\mathbb{R},\mathbb{R})$ with $\|K\|_{L^\infty(\mathbb{R})}<\frac{\pi}{2}$, (1.2.7) defines $C^2$ $\infty$-Harmonic map whose phases are as shown in Figures 1(a), 1(b) below,when $K$ qualitatively behaves as shown in the Figures 2(a), 2(b) respectively.

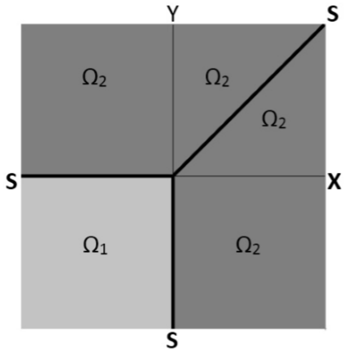


Figure 1(a).

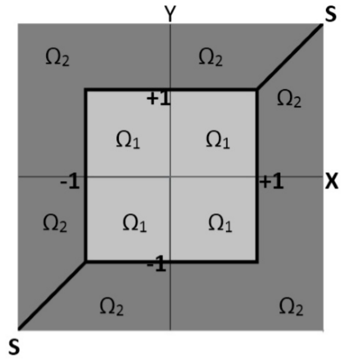


Figure 1(b).

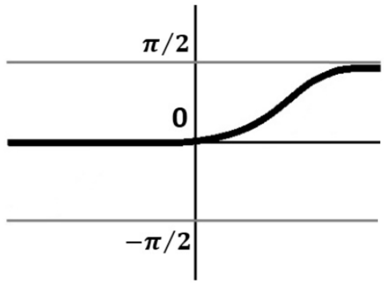


Figure 2(a).

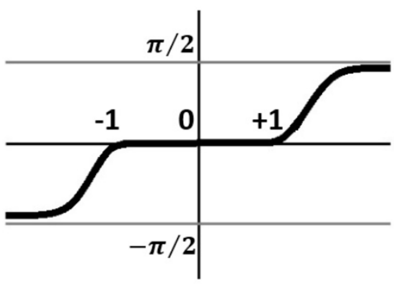


Figure 2(b).

Also, on the phase $\Omega_1$ the $\infty$-Harmonic map (1.2.7) is given by a scalar $\infty$-Harmonic function times a constant vector, and on the phase $\Omega_2$ it is a solution of the vectorial Eikonal equation. The high complexity of these solutions provides further understanding of the $\infty$-Laplacian and limits what might be true in future regularity considerations of the interfaces.

In 2014, N. Katzourakis [40] among other interesting things studied the variational structure of $\infty$-Harmonic maps. He introduced $L^\infty$ variational principle, and has established maximum and minimum principles for the gradient of $\infty$-Harmonic maps of full rank.

In 2014, N. Katzourakis [41] besides other interesting things he studied the structure of 2D $\infty$-Harmonic mappings. He has established a rigidity theorem for rank-one maps, and analysed a phenomenon of separation of the solutions to phases with qualitatively different behaviour.

In 2016, N. Katzourakis and T. Pryer [53] introduced numerical approximations of $\infty$-Harmonic mappings when the dimension of the domain of the solutions is n = 2 and the dimension of the target is N = 2, 3. This contribution demonstrate interesting and unexpected phenomena occurring in $L^\infty$ and provide insights on the structure of general solutions and the natural separation to phases they present.

For more details we refer to the introductions of the papers presented in Chapters

4 and 5.

## 1.3 Organisation of thesis

The main aim of this thesis is to advance and develop some new and recent ideas about the field of non-divergence systems of nonlinear PDEs. We have achieved this goal by submitting, publishing and having preprint papers in different aspects of the field of non-divergence systems of nonlinear PDEs. This thesis is a collection of these papers, and each paper presents a chapter starting from Chapter 2 as it will be explained in the outline of the thesis structure below.

Chapter 1 is dedicated for the background and motivations. We start the chapter with short introduction about the field of the study. Then, we give a brief literature review. And then the organisation of thesis.

In Chapter 2 we present the joint paper with Katzourakis [3]. The estimated percentage contribution is 50%. This paper has been published online in May 2016 in the journal Advances in Nonlinear Analysis (ANONA). In this paper, we work on the problem of proving the existence and uniqueness of global strong solutions $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to fully nonlinear first order elliptic system.

$$F(\cdot, \mathrm{D}u) \,=\, f, \quad \text{a.e. on } \mathbb{R}^n,$$

where $n \geq 3, N \geq 2$ and $F : \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ is a Carathéodory map. After a short introduction, we will firstly recall the theorem of existence - uniqueness - representation introduced in [43]. Then, we recall the strict ellipticity condition of the Katzourakis“ K-Condition of ellipticity” taken from [43] in an alternative form which is more convenient for our analysis. And also, we recall the theorem of Campanato. After that we introduce our new hypothesis of ellipticity which we refer to as the “AK-Condition” which states that if we have an elliptic reference linear map $\mathrm{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$, then we say that a Carathéodory map $F : \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ is elliptic with respect to A when there exists a positive function $\alpha$ with $\alpha, 1/\alpha \in L^\infty(\mathbb{R}^n)$ and $\beta, \gamma > 0$ with $\beta + \gamma < 1$ such that

$$\Big|\alpha(x)\Big[F(x, X+Y) - F(x,Y)\Big] \,-\, \mathrm{A} : X\Big| \,\leq\, \beta\,\nu(\mathrm{A})|X| \,+\, \gamma\,|\mathrm{A} : X|,$$

for all $X, Y \in \mathbb{R}^{Nn}$ and a.e. $x \in \mathbb{R}^n$. Here $\nu(\mathrm{A})$ is the ellipticity constant of A. Then for fixed constant $\alpha \in (0, 1/2]$ we give an example shows that there exist even linear constant“coefficients” $F(x,X) := \frac{1}{\alpha}\mathrm{A} : X$, which are elliptic in the sense of our AK-Condition but are not elliptic in the sense of K-Condition of ellipticity with respect to a specific elliptic reference linear map A which is the Cauchy-Riemann tensor

$$\mathrm{A} \,=\, \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right].$$

Then, for fixed $c, b > 0$ such that $c + b < 1$ and $\sqrt{2}c + b > 1$ and a unit vector $\eta \in \mathbb{R}^N$we give a more elaborate example the Lipschitz function $F \in C^0\big(\mathbb{R}^{2\times 2}\big)$, given by:

$$F(x, X) := \mathrm{A} : X \,+\, \eta \cdot \big(\, b\,\big|X\big| + c\,\big|\mathrm{A} : X\big|\big),$$

where A is again the Cauchy-Riemann tensor. This example shows that even if we ignore the rescaling function and normalise it, AK-Condition is still more general than K-Condition of ellipticity. Then, we introduce and prove a lemma in which we show that our ellipticity assumption can be seen as a notion of pseudo-monotonicity coupled by a global Lipschitz continuity property. Finally, we introduce and prove the main result of this paper which is the theorem of“ Existence-Uniqueness” states that for $n \geq 3$, $N \geq 2$ and a Carathéodory map $F : \mathbb{R}^n \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^N$ satisfying the “AK-Condition” with respect to an elliptic reference tensor A.

(1) For any two maps $v, u \in W^{1;2^*,2}(\mathbb{R}^n, \mathbb{R}^N)$, we have the estimate

$$\|v - u\|_{W^{1;2^*,2}(\mathbb{R}^n)} \,\leq\, C\big\|F(\cdot, Dv) - F(\cdot, Du)\big\|_{L^2(\mathbb{R}^n)}$$

for some $C > 0$ depending only on $F$. Hence, the PDE system $F(\cdot, \mathrm{D}u) = f$ has at most one solution.

(2) Suppose further that $F(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$. Then for any $f \in L^2(\mathbb{R}^n, \mathbb{R}^N)$, the system

$$F(\cdot, \mathrm{D}u) \,=\, f, \quad \text{a.e. on } \mathbb{R}^n,$$

has a unique solution $u$ in the space $W^{1;2^*,2}(\mathbb{R}^n, \mathbb{R}^N)$ which also satisfies the estimate

$$\|u\|_{W^{1;2^*,2}(\mathbb{R}^n)} \,\leq\, C\|f\|_{L^2(\mathbb{R}^n)}$$

for some $C > 0$ depending only on $F$.

In Chapter 3 we present the joint paper with Katzourakis [4]. The estimated percentage contribution is 50%. This paper has been published in December 2016 in Proceedings of the American Mathematical Society (AMS). In this paper we prove the existence of vectorial Absolute Minimisers with given boundary values to the supremal functional

$$E_\infty(u, \Omega') \,:=\, \underset{x\in\Omega'}{\mathrm{ess\,sup}}\, \mathscr{L}\left(x, u(x), \mathrm{D}u(x)\right), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N),\ \Omega' \Subset \Omega,$$

applied to maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$, $N \in \mathbb{N}$. First we give a brief introduction. Then, we introduce the main result of the paper the theorem of“ Existence of vectorial Absolute Minimisers”, which states that if $\Omega \subseteq \mathbb{R}$ is bounded open interval and

$$\mathscr{L} \,:\, \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow [0, \infty),$$

is a given continuous function with $N \in \mathbb{N}$. We assume that:

1. For each $(x, \eta) \in \overline{\Omega} \times \mathbb{R}^N$, the function $P \longmapsto \mathscr{L}(x, \eta, P)$ is level-convex, that

is for each $t \geq 0$ the sublevel set

$$\left\{P \in \mathbb{R}^N : \mathscr{L}(x, \eta, P) \leq t\right\}$$

is a convex set in $\mathbb{R}^N$.

2. there exist non-negative constants $C_1, C_2, C_3$, and $0 < q \leq r < +\infty$ and a positive locally bounded function $h : \mathbb{R} \times \mathbb{R}^N \longrightarrow [0, +\infty)$ such that for all $(x, \eta, P) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N$

$$C_1|P|^q - C_2 \leq \mathscr{L}(x, \eta, P) \leq h(x, \eta)|P|^r + C_3.$$

Then, for any affine map $b : \mathbb{R} \longrightarrow \mathbb{R}^N$, there exist a vectorial Absolute Minimiser $u^\infty \in W_b^{1,\infty}(\Omega, \mathbb{R}^N)$ of the supremal functional mentioned above.

After that, for the convenience of the reader we recall the Jensen's inequality for level-convex functions. And then we recall a lemma of [16] in which they proved the existence of a vectorial minimise. Finally we give the proof of the main result of the paper.

In Chapter 4 we present the joint preprint paper with Katzourakis and Ayanbayev [2]. The estimated percentage contribution is 30%. In this paper we study the rigidity and flatness of the image of certain classes of $\infty$-Harmonic and $p$-Harmonic maps. We start by giving a brief introduction. And we continue by recalling the $L^\infty$ variational principle introduced in [40]. As a generalisation of this theorem we then give our first main result which is the theorem of rigidity and flatness of rank-one maps with tangential Laplacian, which states that if $\Omega \subseteq \mathbb{R}^n$ is an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a solution to the nonlinear system $[\![\mathrm{D}u]\!]^\perp \Delta u = 0$ in $\Omega$, satisfying that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections).

Then, we give an example shows in general rank-one solutions for the system under consideration can not have affine image but only piecewise affine. After the example we give the theorem of the rigidity of $p$-Harmonic maps which is a consequence of the first main theorem, this consequence states that if $\Omega \subseteq \mathbb{R}^n$ is an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a $p$-Harmonic map in $\Omega$ for some $p \in [2, \infty)$. Suppose that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, the same result as in theorem above is true.

In addition, there exists a partition of $\Omega$ to at most countably many Borel sets, where each set of the partition is a non-empty open set with a (perhaps empty) boundary portion, such that, on each of these, $u$ can be represented as

$$u = \nu \circ f.$$

Here, $f$ is a scalar $C^2$ $p$-Harmonic function (for the respective $p \in [2, \infty)$), defined on an open neighbourhood of the Borel set, whilst $\nu : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a Lipschitz curve which is twice differentiable and with unit speed on the image of $f$.

At the end of the chapter we list the proofs of the main result and its consequence.

In Chapter 5 we present the single authored preprint paper [1]. in which we study the phase separation of $n-$dimensional $\infty$-Harmonic mappings. We start the chapter by giving a brief introduction. Then, we recall the theorem of the structure of 2D $\infty$ -Harmonic maps from [41]. Next to that we introduce the main result of this paper which generalise the results of [41] to higher dimensions, we refer to it by " Phase separation of $n$-dimensional $\infty$-Harmonic mappings", which states that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, and let $u : \Omega \longrightarrow \mathbb{R}^N$, $N \geq n \geq 2$, be an $\infty$-Harmonic map in $C^2\big(\Omega, \mathbb{R}^N\big)$, that is a solution to the $\infty$-Laplace system

$$\Delta_\infty u \, := \, \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes I\Big) : \mathrm{D}^2 u \, = \, 0, \quad \text{on } \Omega.$$

Then, there exist disjoint open sets $\big(\Omega_r\big)_{r=1}^n \subseteq \Omega$, and a closed nowhere dense set $S$ such that $\Omega \, = \, S \bigcup \Big( \bigcup\limits_{i=1}^n \Omega_i \Big)$ such that:

(i) On $\Omega_n$ we have $\mathrm{rk}(\mathrm{D}u) \, \equiv \, n$ and the map $u : \Omega_n \longrightarrow \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$|\mathrm{D}u|^2 = C^2 > 0.$$

The constant $C$ may vary on different connected components of $\Omega_n$.

(ii) On $\Omega_r$ we have $\mathrm{rk}(\mathrm{D}u) \, \equiv \, r$, where $r$ is integer in $\{2, 3, 4, ..., (n-1)\}$, and $|\mathrm{D}u(\gamma(t))|$ is constant along trajectories of the parametric gradient flow of $u(\gamma(t, x, \xi))$

$$\left\{ \begin{array}{l} \dot\gamma(t, x, \xi) = \xi^\top \mathrm{D}u\big(\gamma(t, x, \xi)\big), \quad t \in (-\varepsilon, 0) \bigcup (0, \varepsilon), \\ \gamma(0, x, \xi) = x, \end{array} \right.$$

where $\xi \in \mathbb{S}^{N-1}$, and $\xi \notin N\big(\mathrm{D}u\big(\gamma(t, x, \xi)\big)^\top\big)$.

(iii) On $\Omega_1$ we have $\mathrm{rk}(\mathrm{D}u) \, \leq \, 1$ and the map $u : \Omega_1 \longrightarrow \mathbb{R}^N$ is given by an essentially scalar $\infty$-Harmonic function $f : \Omega_1 \longrightarrow \mathbb{R}$:

$$u \, = \, a + \xi f, \;\; \Delta_\infty f \, = \, 0, \;\; a \in \mathbb{R}^N, \;\; \xi \in \mathbb{S}^{N-1}.$$

The vectors $a, \xi$ may vary on different connected components of $\Omega_1$.

(iv) On S, when $S \supseteq \partial\Omega_p \cap \partial\Omega_q \, = \, \emptyset$ for all $p$ and $q$ such that $2 \leq p < q \leq n-1$, then we have that $|\mathrm{D}u|^2$ is constant and also $\mathrm{rk}(\mathrm{D}u) \equiv 1$. Moreover on

$$\partial\Omega_1 \cap \partial\Omega_n \subseteq S,$$

(when both 1D and $n$D phases coexist), we have that $u : S \longrightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$u \;=\; a + \xi f, \;\; |\mathrm{D}f|^2 = C^2 > 0, \;\; a \in \mathbb{R}^N, \;\; \xi \in \mathbb{S}^{N-1}.$$

On the other hand, if there exist some $r$ and $q$ such that $2 \leq r < q \leq n-1$, then on $S \supseteq \partial\Omega_r \cap \partial\Omega_q \neq \emptyset$ (when both $r$D and $q$D phases coexist), we have that $\mathrm{rk}(\mathrm{D}u) \equiv r$ and we have same result as in (ii) above.

In the preliminaries section, for the convenience of the reader we recall the theorem of rigidity of rank-one maps, proved in [41], which will be used in the proof of the main result and we also recall the proposition of Gradient flows for tangentially $\infty$-Harmonic maps which introduced in [37] and its improved modification lemma in [40]. We end up the chapter by giving the proof of the main result of the paper.

In Chapter 6 we discuss the conclusions and the future work.

# Chapter 2

# On the Well-Posedness of Global Fully Nonlinear First Order Elliptic Systems

## 2.1 Introduction

In this chapter we present the joint paper with Katzourakis [3]. The estimated percentage contribution is 50%. This paper has been published online in May 2016 in the journal Advances in Nonlinear Analysis (ANONA). In this paper we consider the problem of existence and uniqueness of global strong solutions $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the fully nonlinear first order PDE system

$$F(\cdot, \mathrm{D}u) \,=\, f, \quad \text{a.e. on } \mathbb{R}^n, \tag{2.1.1}$$

where $n \geq 3, N \geq 2$ and $F : \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ is a Carathéodory map. The latter means that $F(\cdot, X)$ is a measurable map for all $X \in \mathbb{R}^{Nn}$ and $F(x, \cdot)$ is a continuous map for almost every $x \in \mathbb{R}^n$. The gradient $\mathrm{D}u : \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}$ of our solution $u = (u_1, ..., u_N)^\top$ is viewed as an $N \times n$ matrix-valued map $\mathrm{D}u = (\mathrm{D}_i u_\alpha)_{i=1...n}^{\alpha=1...N}$ and the right hand side $f$ is assumed to be in $L^2(\mathbb{R}^n, \mathbb{R}^N)$.

The method we use in this paper to study (2.1.1) follows that of the recent paper [43] of the second author. Therein the author introduced and employed a new perturbation method in order to solve (2.1.1) which is based on the solvability of the respective linearised system and a structural ellipticity hypothesis on $F$, inspired by the classical work of Campanato in the fully nonlinear second order case $\mathcal{F}(\cdot, \mathrm{D}^2 u) = f$ (see [20–27] and [61–63]). Loosely speaking, the ellipticity notion of [43] requires that $F$ is "not too far away" from a linear constant coefficient first order differential operator. In the linear case of constant coefficients, $F$ assumes the form

$$F(x,X) \;=\; \sum_{\alpha,\beta=1}^{N}\sum_{j=1}^{n} \mathrm{A}_{\alpha\beta j} X_{\beta j}\, e^{\alpha},$$

for some linear map $\mathrm{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^{N}$. We will follow almost the same conventions as in [43], for instance we will denote the standard bases of $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{N\times n}$ by $\{e^i\}$, $\{e^\alpha\}$ and $\{e^\alpha \otimes e^i\}$ respectively. In the linear case, (2.1.1) can be written as

$$\sum_{\beta=1}^{N}\sum_{j=1}^{n} \mathrm{A}_{\alpha\beta j}\mathrm{D}_j u_\beta \;=\; f_\alpha, \qquad \alpha = 1, ..., N,$$

and compactly in vector notation as

$$\mathrm{A} : \mathrm{D}u \;=\; f. \tag{2.1.2}$$

The appropriate well-known notion of ellipticity in the linear case is that the *nullspace of the linear map* A *contains no rank-one lines.* This requirement can be quantified as

$$|\mathrm{A} : \xi \otimes a| \;>\; 0, \quad \text{when} \;\; \xi \neq 0, \;\; a \neq 0 \tag{2.1.3}$$

which says that all rank-one directions $\xi \otimes a \in \mathbb{R}^{Nn}$ are transversal to the nullspace. A prototypical example of such operator $\mathrm{A} : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^2$ is given by

$$\mathrm{A} \;=\; \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \tag{2.1.4}$$

and corresponds to the Cauchy-Riemann PDEs. In [43] the system (2.1.1) was proved to be well-posed by solving (2.1.2) via Fourier transform methods and by utilising the following ellipticity notion: (2.1.1) is an elliptic system (or $F$ is elliptic) when there exists a linear map

$$\mathrm{A} \;:\; \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^{N}$$

which is elliptic in the sense of (2.1.3) and

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \sup_{X,Y\in\mathbb{R}^{Nn},X\neq Y} \frac{\big|\big[F(x,Y) - F(x,X)\big] - \mathrm{A} : (Y-X)\big|}{|Y-X|} \;<\; \nu(A), \tag{2.1.5}$$

where

$$\nu(\mathrm{A}) \;:=\; \min_{|\eta|=|a|=1} \big|\mathrm{A} : \eta \otimes a\big| \tag{2.1.6}$$

is the "ellipticity constant" of A. This notion was called "K-Condition" in [43]. The functional space in which well posedness was obtained is the so-called J.L. Lions space

$$W^{1;2^*,2}(\mathbb{R}^n,\mathbb{R}^N) \;:=\; \Big\{u \in L^{2^*}(\mathbb{R}^n,\mathbb{R}^N) \;:\; \mathrm{D}u \in L^2(\mathbb{R}^n,\mathbb{R}^{Nn})\Big\}. \tag{2.1.7}$$

Here $2^*$ is the conjugate Sobolev exponent

$$2^* = \frac{2n}{n-2},$$

where $n > 2$ (note that "$L^{2^*}$" means "$L^p$ for $p = 2^*$", not duality) and the natural norm of the space is

$$\|u\|_{W^{1;2^*,2}(\mathbb{R}^n)} := \|u\|_{L^{2^*}(\mathbb{R}^n)} + \|\mathrm{D}u\|_{L^2(\mathbb{R}^n)}.$$

In [43] only global strong a.e. solutions on the whole space were considered and for dimensions $n \geq 3$ and $N \geq 2$, in order to avoid the compatibility difficulties which arise in the case of the Dirichlet problem for first order systems on bounded domains and because the case $n = 2$ has been studied quite extensively.

In this paper we follow the method introduced in [43] and we prove well-posedness of (2.1.1) in the space (2.1.7) for the same dimensions $n \geq 3$ and $N \geq 2$. This is the content of our Theorem 2.4.1, whilst we also obtain an a priori quantitative estimate in the form of a "comparison principle" for the distance of two solutions in terms of the distance of the respective right hand sides of (2.1.1). *The main advance in this paper which distinguishes it from the results obtained in [43] is that we introduce a new notion of ellipticity for* (2.1.1) *which is strictly weaker than* (2.1.5)*, allowing for more general nonlinearities $F$ to be considered.* Our new hypothesis of ellipticity is inspired by an other recent work of the second author [45] on the second order case. We will refer to our condition as the "AK-Condition" (Definition 2.3.1). In Examples 2.3.2, 2.3.3 we demonstrate that the new condition is genuinely weaker and hence our results indeed generalise those of [43]. Further, motivated by [45] we also introduce a related notion which we call pseudo-monotonicity and examine their connection (Lemma 2.3.4). The idea of the proof of our main result Theorem 2.4.1 is based, as in [43], on the solvability of the linear system, our ellipticity assumption and on a fixed point argument in the form of Campanato's near operators, which we recall later for the convenience of the reader (Theorem 2.2.3).

We conclude this introduction with some comments which contextualise the standing of the topic and connect to previous contributions by other authors. Linear elliptic PDE systems of the first order are of paramount importance in several branches of Analysis like for instance in Complex and Harmonic Analysis. Therefore, they have been extensively studied in several contexts (see e.g. Buchanan-Gilbert [35], Begehr-Wen [17]), including regularity theory of PDE (see chapter 7 of Morrey's exposition [58] of the Agmon-Douglis-Nirenberg theory), Differential Inclusions and Compensated Compactness theory (Di Perna [32], Müller [57]), as well as Geometric Analysis and the theory of differential forms (Csató-Dacorogna-Kneuss [29]).

However, except for the paper [43] the fully nonlinear system (2.1.1) is much less studied and understood. By using the Baire category method of the Dacorogna-Marcellini [31] (which is the analytic counterpart of Gromov's geometric method of

Convex Integration), it can be shown that the Dirichlet problem

$$\left\{ \begin{aligned} F(\cdot, \mathrm{D}u) \,&=\, f, \quad && \text{in } \Omega, \\ u \,&=\, g, \quad && \text{on } \partial\Omega, \end{aligned} \right. \tag{2.1.8}$$

has *infinitely many* strong a.e. solutions in $W^{1,\infty}(\Omega, \mathbb{R}^N)$, for $\Omega \subseteq \mathbb{R}^n$, $g$ a Lipschitz map and under certain structural *coercivity* and compatibility assumptions. However, roughly speaking ellipticity and coercivity of $F$ are mutually exclusive. In particular, it is well known that the Dirichlet problem (2.1.8) is not well posed when $F$ is either linear or elliptic.

Further, it is well known that single equations, let alone systems of PDE, in general do not have *classical* solutions. In the scalar case $N = 1$, the theory of Viscosity Solutions of Crandall-Ishii-Lions (we refer to [42] for a pedagogical introduction of the topic) furnishes a very successful setting of *generalised* solutions in which Hamilton-Jacobi PDE enjoy strong existence-uniqueness theorems. However, there is no counterpart of this essentially scalar theory for (non-diagonal) systems. The general approach of this paper is inspired by the classical work of Campanato quoted earlier and in a nutshell consists of imposing an appropriate condition that allows to prove well-posedness in the setting of the intermediate theory of *strong a.e.* solutions. Notwithstanding, very recently the second author in [48] has proposed a new theory of generalised solutions in the context of which he has already obtained existence and uniqueness theorems for second order degenerate elliptic systems. We leave the study of the present problem in the context of "$\mathcal{D}$-solutions" introduced in [48] for future work.

## 2.2 Preliminaries

In this section we collect some results taken from our references which are needed for the main results of this paper. The first one below concerns the existence and uniqueness of solutions to the linear first order system with constant coefficient

$$\mathrm{A} : \mathrm{D}u \,=\, f, \quad \text{a.e. on } \mathbb{R}^n,$$

with $\mathrm{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ elliptic in the sense of (2.1.3), namely when the nullspace of A does not contain rank-one lines. By the compactness of the torus, it can be rewritten equivalently as

$$|\mathrm{A} : \xi \otimes a| \,\geq\, \nu\, |\xi||a|, \quad \xi \in \mathbb{R}^N,\, a \in \mathbb{R}^n, \tag{2.2.1}$$

for some constant $\nu > 0$, which can be chosen to be the *ellipticity constant of* A given by (2.1.6). One can easily see that (2.2.1) can be rephrased as

$$\min_{|a|=1} \big| \det(\mathrm{A}a) \big| \,>\, 0, \tag{2.2.2}$$

where $\mathrm{A}a$ is the $N \times N$ matrix given by

$$\mathrm{A}a := \sum_{\alpha,\beta=1}^{N} \sum_{j=1}^{n} (\mathrm{A}_{\alpha\beta j}\, a_j)\, e^\alpha \otimes e^\beta .$$

It is easy to exhibit examples of tensors A satisfying (2.2.1). A map $\mathrm{A} : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^2$ satisfying it is

$$\mathrm{A} = \left[\begin{array}{cc|cc} \kappa & 0 & 0 & \lambda \\ 0 & -\mu & \nu & 0 \end{array}\right],$$

where $\kappa, \lambda, \mu, \nu > 0$. A higher dimensional example of map $\mathrm{A} : \mathbb{R}^{4\times 3} \longrightarrow \mathbb{R}^4$ is

$$\mathrm{A} = \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right]$$

which corresponds to the electron equation of Dirac in the case where there is no external force. For more details we refer to [43].

### 2.2.1 Theorem [Existence-Uniqueness-Representation, cf.[43]]

Let $n \geq 3$, $N \geq 2$, $\mathrm{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ a linear map satisfying (2.2.1) and $f \in L^2(\mathbb{R}^n, \mathbb{R}^N)$. Then, the system

$$\mathrm{A} : \mathrm{D}u = f, \quad \text{a.e. on } \mathbb{R}^n,$$

has a unique solution $u$ in the space $W^{1;2^*,2}(\mathbb{R}^n, \mathbb{R}^N)$ (see (2.1.7)), which also satisfies the estimate

$$\|u\|_{W^{1;2^*,2}(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{2.2.3}$$

for some $C > 0$ depending only on A. Moreover, the solution can be represented explicitly as:

$$u = -\frac{1}{2\pi i} \lim_{m\to\infty} \left\{ \widehat{h_m} * \left[ \frac{\mathrm{cof}\,(\mathrm{Asgn})^\top}{\det(\mathrm{Asgn})} \check{f} \right]^\wedge \right\}. \tag{2.2.4}$$

In (2.2.4), $(h_m)_1^\infty$ is any sequence of even functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$0 \leq h_m(x) \leq \frac{1}{|x|} \quad \text{and} \quad h_m(x) \longrightarrow \frac{1}{|x|}, \ \text{for a.e. } x \in \mathbb{R}^n, \ \text{ as } m \to \infty.$$

The limit in (2.2.4) is meant in the weak $L^{2^*}$ sense as well as a.e. on $\mathbb{R}^n$, and $u$ is independent of the choice of sequence $(h_m)_1^\infty$.

In the above statement, "sgn" symbolise the sign function on $\mathbb{R}^n$, namely $\mathrm{sgn}(x) = x/|x|$ when $x \neq 0$ and $\mathrm{sgn}(0) = 0$. , "cof" and "det" symbolise the cofactor and the determinant on $\mathbb{R}^{N\times N}$ respectively. Although the formula (2.2.4) involves complex quantities, $u$ above is a *real* vectorial solution. Moreover, the symbol "$\widehat{\ }$" stands for Fourier transform (with the conventions of [34]) and "$^{\vee}$" stands for its inverse.

Next, we recall the strict ellipticity condition of the second author taken from [43] in an alternative form which is more convenient for our analysis. We will relax it in the next section. Let

$$\mathrm{A} \ : \ \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^{N}$$

be a fixed reference linear map satisfying (2.2.1).

### 2.2.2 Definition [K-Condition of ellipticity, cf. [43]]

Let $F : \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ be a Carathéodory map. We say that $F$ is elliptic with respect to A when there exists $0 < \beta < 1$ such that for all $X, Y \in \mathbb{R}^{Nn}$ and a.e. $x \in \mathbb{R}^n$, we have

$$\left| \Big[ F(x, X+Y) - F(x, X) \Big] - \mathrm{A} : Y \right| \ \leq \ \beta\, \nu(\mathrm{A})\, |Y|, \tag{2.2.5}$$

where $\nu(\mathrm{A})$ is given by (2.1.6).

Finally, we recall the next classical result of Campanato taken from [25] which is needed for the proof of our main result Theorem 2.4.1:

### 2.2.3 Theorem [Campanato]

Let $\mathcal{F}, \mathcal{A} : \mathfrak{X} \longrightarrow X$ be two mappings from the set $\mathfrak{X} \neq \emptyset$ into the Banach space $(X, \|\cdot\|)$. If there is a constant $K \in (0,1)$ such that

$$\Big\| \mathcal{F}[u] - \mathcal{F}[v] - \big( \mathcal{A}[u] - \mathcal{A}[v] \big) \Big\| \ \leq \ K \big\| \mathcal{A}[u] - \mathcal{A}[v] \big\| \tag{2.2.6}$$

for all $u, v \in \mathfrak{X}$ and if $\mathcal{A} : \mathfrak{X} \longrightarrow X$ is a bijection, it follows that $\mathcal{F} : \mathfrak{X} \longrightarrow X$ is a bijection as well.

## 2.3 The AK-Condition of Ellipticity for Fully Nonlinear First Order Systems

In this section we introduce and study a new ellipticity condition for the PDE system (2.1.1)which relaxes the K-Condition Definition 2.2.2 and still allows to prove existence and uniqueness of strong solutions to

$$F(\cdot, \mathrm{D}u) \ = \ f, \qquad \text{a.e. on } \mathbb{R}^n$$

in the functional space (2.1.7). Let

$$\mathrm{A} \ : \ \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^{N}$$

be an elliptic reference linear map satisfying (2.2.1).

### 2.3.1 Definition [The AK-Condition of ellipticity]

Let $n, N \geq 2$ and

$$F \ : \ \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$$

a Carathéodory map. We say that $F$ is elliptic with respect to A when there exists a positive function $\alpha$ with $\alpha, 1/\alpha \in L^\infty(\mathbb{R}^n)$ and $\beta, \gamma > 0$ with $\beta + \gamma < 1$ such that

$$\Big|\alpha(x)\Big[F(x, X+Y) - F(x,Y)\Big] \, - \, \mathrm{A} : X\Big| \, \leq \, \beta\,\nu(\mathrm{A})|X| \, + \, \gamma\,|\mathrm{A} : X|. \qquad (2.3.1)$$

for all $X, Y \in \mathbb{R}^{Nn}$ and a.e. $x \in \mathbb{R}^n$. Here $\nu(\mathrm{A})$ is the ellipticity constant of $A$ given by (2.1.6).

Nontrivial fully nonlinear examples of maps $F$ which are elliptic in the sense of the Definition 2.3.1 above are easy to find. Consider any fixed map $\mathrm{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ for which $\nu(\mathrm{A}) > 0$ and any Carathéodory map

$$L \ : \ \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$$

which is Lipschitz with respect to the second variable and

$$\big\|L(x,\cdot)\big\|_{C^{0,1}(\mathbb{R}^{Nn})} \, \leq \, \beta\,\nu(\mathrm{A}), \quad \text{for a.e. } x \in \mathbb{R}^n$$

for some $0 < \beta < 1$. Let also $\alpha$ be a positive essentially bounded function with $1/\alpha$ essentially bounded as well. Then, the map $F : \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ given by

$$F(x, X) \, := \, \frac{1}{\alpha(x)}\Big(\mathrm{A} : X \, + \, L(x, X)\Big)$$

satisfies Definition 2.3.1, since

$$\begin{aligned}
\Big|\alpha(x)\Big[F(x, X+Y) - F(x,Y)\Big] - \mathrm{A} : X\Big| \, &\leq \, \big|L(x, X+Y) - L(x,Y)\big| \\
&\leq \, \beta\,\nu(\mathrm{A})|X| \\
&\leq \, \beta\,\nu(\mathrm{A})|X| \, + \, \frac{1-\beta}{2}|\mathrm{A} : X|.
\end{aligned}$$

As a consequence, $F$ satisfies the AK-Condition for the same function $\alpha(\cdot)$ and for the constants $\beta$ and $\gamma = (1-\beta)/2$.

The following example shows that, given a reference tensor A, there exist even

**linear constant** *"coefficients"* $F$ which are elliptic with respect to A in the sense of our AK-Condition Definition 2.3.1 but which are **not** elliptic with respect to A in the sense of Definition 2.2.2 of [43].

### 2.3.2 Example

Fix a constant $\alpha \in (0, 1/2]$ and consider the linear map $F$ given by

$$F(x, X) := \frac{1}{\alpha}\mathrm{A} : X,$$

where $A$ is the Cauchy-Riemann tensor of (2.1.4). Then, $F$ is elliptic in the sense of Definition 2.3.1 with respect to A for $\alpha(\cdot) \equiv \alpha$ and any $\beta, \gamma > 0$ with $\beta + \gamma < 1$, but it is not elliptic with respect to A in the sense of Definition 2.2.2. Indeed for any $X, Y \in \mathbb{R}^{Nn}$ we have:

$$\begin{aligned}
\Big|\alpha\Big[F(\cdot, X+Y) - F(\cdot, Y)\Big] - \mathrm{A} : X\Big| &= \left|\alpha\left[\frac{1}{\alpha}\mathrm{A} : (X+Y) - \frac{1}{\alpha}\mathrm{A} : Y\right] - \mathrm{A} : X\right| \\
&= 0 \\
&\leq \beta\nu(\mathrm{A})|X| + \gamma\,|\mathrm{A} : X|.
\end{aligned}$$

On the other hand, by (2.1.4) and (2.1.6) we have that $\nu(\mathrm{A}) = 1$. Moreover, for

$$X_0 := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

we have $|X_0| = 2$ and $|\mathrm{A} : X_0| = 2$. Hence, for any $Y \in \mathbb{R}^{Nn}$ we have

$$\begin{aligned}
\Big|\Big[F(\cdot, X_0+Y) - F(\cdot, Y)\Big] - \mathrm{A} : X_0\Big| &= \left|\left[\frac{1}{\alpha}\mathrm{A} : (X_0+Y) - \frac{1}{\alpha}\mathrm{A} : Y\right] - \mathrm{A} : X_0\right| \\
&= \left|\frac{1}{\alpha}\mathrm{A} : X_0 \; - \mathrm{A} : X_0\right| \\
&= \big|\mathrm{A} : X_0\big| \left|\frac{1}{\alpha} \; - 1\right| \\
&= 2\left(\frac{1}{\alpha} \; - 1\right) \\
&\geq 2 \\
&= \nu(A)\,|X_0|,
\end{aligned}$$

where we have used that $(1/\alpha) - 1 \geq 1$. Our claim ensues.

The essential point in the above example that makes Definition 2.3.1 more general than Definition 2.2.2 was the introduction of the *rescaling function* $\alpha(\cdot)$. Now we give a more elaborate example which shows that *even if we ignore the rescaling function*

*$\alpha$ and normalise it to $\alpha(\cdot) \equiv 1$, Definition 2.3.1 is still more general than Definition 2.2.2 with respect to the same fixed reference tensor* A.

### 2.3.3 Example

Fix $c, b > 0$ such that $c + b < 1$ and $\sqrt{2}c + b > 1$ and a unit vector $\eta \in \mathbb{R}^N$. Consider the Lipschitz function $F \in C^0\big(\mathbb{R}^{2\times 2}\big)$, given by:

$$F(x, X) := \mathrm{A} : X \,+\, \eta \cdot \big(\, b\,\big|X\big| + c\,\big|\mathrm{A} : X\big|\big), \tag{2.3.2}$$

where A is again the Cauchy-Riemann tensor (2.1.4). Then, this $F$ satisfies

$$\Big|\Big[F(\cdot, X+Y) - F(\cdot, X)\Big] - \mathrm{A} : Y\Big| \,\leq\, \beta\,\nu(\mathrm{A})|Y| \,+\, \gamma\,\big|\mathrm{A} : Y\big|, \tag{2.3.3}$$

for some $\beta, \gamma > 0$ with $\beta + \gamma < 1$, but does not satisfy (2.3.3) with $\gamma = 0$ **for any** $0 < \beta < 1$ for the same A. Hence, $F$ satisfies Definition 2.3.1 (even if we fix $\alpha(\cdot) \equiv 1$) but it does not satisfy Definition 2.2.2. Indeed we have:

$$\begin{aligned}
&\Big|\mathrm{A} : Y - \Big[F(\cdot, X+Y) - F(\cdot, X)\Big]\Big| \\
&\qquad = \Big|\mathrm{A} : Y \,-\, \mathrm{A} : Y \,-\, b\eta\Big(|X+Y| - |X|\Big) \,-\, c\eta\Big(\big|\mathrm{A} : (X+Y)\big| - |\mathrm{A} : X|\Big)\Big| \\
&\qquad \leq b|\eta|\Big||X+Y| - |X|\Big| \,+\, c|\eta|\Big||\mathrm{A} : X + \mathrm{A} : Y| - |\mathrm{A} : X|\Big| \\
&\qquad \leq b|Y| \,+\, c|\mathrm{A} : Y|
\end{aligned}$$

and hence (2.3.3) holds for $\beta = b$ and $\gamma = c$. On the other hand, we choose

$$X_0 \,:=\, 0, \qquad Y_0 \,:=\, \left[\begin{array}{cc} 1 & \zeta \\ \zeta & 1 \end{array}\right], \qquad \zeta \,:=\, \frac{1-b}{\sqrt{2c^2 - (1-b)^2}}.$$

This choice of $\zeta$ is admissible because our assumption $\sqrt{2}c + b > 1$ implies $2c^2 - (1 - b)^2 > 0$. For these choices of $X$ and $Y$, we calculate:

$$\begin{aligned}
\Big|\mathrm{A} : Y_0 - \Big[F(\cdot, X_0+Y_0) - F(\cdot, X_0)\Big]\Big| \,&=\, \Big|\mathrm{A} : Y_0 \,-\, F(\cdot, Y_0)\Big| \\
&=\, \Big|\mathrm{A} : Y_0 \,-\, \mathrm{A} : Y_0 \,-\, \eta\Big(b\,|Y_0| \,+\, c|\mathrm{A} : Y_0|\Big)\Big| \\
&=\, |\eta|\Big|b|Y_0| \,+\, c|\mathrm{A} : Y_0|\Big| \\
&=\, b\,|Y_0| \,+\, c\,|\mathrm{A} : Y_0|.
\end{aligned}$$

We now show that

$$b\,|Y_0| \,+\, c|\mathrm{A} : Y_0| \,=\, |Y_0|$$

and this will allow us to conclude that (2.3.3) can not hold for any $\beta < 1$ if we impose $\gamma = 0$. Indeed, since $|Y_0|^2 = 2 + 2\zeta^2$ and $|\mathrm{A} : Y_0|^2 = 4\zeta^2$, we have

$$\begin{aligned}
\big(1-b\big)^2|Y_0|^2 \,-\, c^2|\mathrm{A} : Y_0|^2 \,&=\, \big(1-b\big)^2 2\big(1+\zeta^2\big) \,-\, c^2\, 4\zeta^2 \\
&=\, 2\big(1-b\big)^2 + 2\Big(\big(1-b\big)^2 - 2c^2\Big)\zeta^2 \\
&=\, 2\big(1-b\big)^2 + 2\Big(\big(1-b\big)^2 - 2c^2\Big)\frac{(1-b)^2}{2c^2-(1-b)^2} \\
&=\, 0.
\end{aligned}$$

We now show that our ellipticity assumption implies a condition of pseudo-monotonicity coupled by a global Lipschitz continuity property. The statement and the proof are modelled after a similar result appearing in [45] which however was in the second order case.

### 2.3.4 Lemma [AK-Condition of ellipticity as Pseudo-Monotonicity]

Definition 2.3.1 implies the following statements:

There exist $\lambda > \kappa > 0$, a linear map $\mathrm{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ satisfying (2.1.3) a positive function $\alpha$ such that $\alpha, 1/\alpha \in L^\infty(\mathbb{R}^n)$ with respect to which $F$ satisfies

$$(\mathrm{A} : Y)^\top \Big[F(x, X+Y) - F(x, X)\Big] \;\geq\; \frac{\lambda}{\alpha(x)}|\mathrm{A} : Y|^2 \,-\, \frac{\kappa}{\alpha(x)}\nu(\mathrm{A})^2|Y|^2, \qquad (2.3.4)$$

for all $X, Y \in \mathbb{R}^{Nn}$ and a.e. $x \in \mathbb{R}^n$. In addition, $F(x,\cdot)$ is Lipschitz continuous on $\mathbb{R}^{Nn}$, essentially uniformly in $x \in \mathbb{R}^n$; namely, there exists $M > 0$ such that

$$\big|F(x,X) \,-\, F(x,Y)\big| \;\leq\; M|X-Y| \qquad (2.3.5)$$

for a.e. $x \in \mathbb{R}^n$ and all $X, Y \in \mathbb{R}^{Nn}$.

### 2.3.5 Proof of Lemma 2.3.4.

Suppose that Definition 2.3.1 holds for some constant $\beta, \gamma > 0$ with $\beta + \gamma < 1$, some positive function $\alpha$ with $\alpha, 1/\alpha \in L^\infty(\mathbb{R}^n)$ and some linear map $\mathrm{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$

satisfying (2.1.3). Fix $\varepsilon > 0$. Then, for a.e. $x \in \mathbb{R}^N$ and all $X, Y \in \mathbb{R}^{Nn}$ we have:

$$\begin{aligned}
|\mathrm{A} : Y|^2 \, + \, \alpha(x)^2 \Big| F(x, X+Y) - F(x,X) \Big|^2 & \\
- \, 2\alpha(x)\, (\mathrm{A} : Y)^\top \Big[ F(x, X+Y) - F(x,X) \Big] & \\
\leq \, \beta^2 \nu(\mathrm{A})^2 |Y|^2 \, + \, \gamma^2 |\mathrm{A} : Y|^2 \, + \, 2\beta\nu(\mathrm{A})|Y|\, \gamma |\mathrm{A} : Y| &
\end{aligned}$$

which implies

$$\begin{aligned}
|\mathrm{A} : Y|^2 - 2\alpha(x)\, (\mathrm{A} : Y)^\top \Big[ F(x, X+Y) - F(x,X) \Big] & \\
\leq \, \beta^2 \nu(\mathrm{A})^2 |Y|^2 + \gamma^2 |\mathrm{A} : Y|^2 \, + \, \frac{\beta^2 \nu(\mathrm{A})^2 |Y|^2}{\varepsilon} \, + \, \varepsilon \gamma^2 |\mathrm{A} : Y|^2. &
\end{aligned}$$

Hence,

$$\begin{aligned}
(\mathrm{A} : Y)^\top \Big[ F(x, X+Y) - F(x,X) \Big] & \\
\geq \, \frac{1}{\alpha(x)} \left( \frac{1 - \gamma^2 - \varepsilon\gamma^2}{2} \right) |\mathrm{A} : Y|^2 \, - \, \frac{1}{\alpha(x)} \left( \frac{\varepsilon\beta^2 + \beta^2}{2\varepsilon} \right) \nu(\mathrm{A})^2 |Y|^2. &
\end{aligned}$$

By choosing $\varepsilon := \beta/\gamma$, from the above inequality we obtain (2.3.4) for the values

$$\lambda \, := \, \frac{1 - \gamma(\gamma + \beta)}{2}, \qquad \kappa \, := \, \frac{\beta(\gamma + \beta)}{2}.$$

These are admissible because $\kappa > 0$ and $\lambda > \kappa$ since

$$\lambda \, - \, \kappa \, = \, \frac{1 - (\beta + \gamma)^2}{2} \, > \, 0.$$

In addition, again by (2.3.1) we have:

$$\alpha(x) \Big| F(x,X) - F(x,Y) \Big| \, \leq \, \beta\nu(\mathrm{A})|X - Y| \, + \, \gamma \left| \mathrm{A} : (X - Y) \right| \, + \, \left| \mathrm{A} : (X - Y) \right|,$$

and hence,

$$\begin{aligned}
\Big| F(x,X) - F(x,Y) \Big| \, &\leq \, \frac{1}{\alpha(x)} \Big( (1+\gamma) \big| \mathrm{A} : (X - Y) \big| \, + \, \beta\nu(\mathrm{A}) \big| X - Y \big| \Big) \\
&\leq \, \left\{ \left\| \frac{1}{\alpha(\cdot)} \right\|_{L^\infty(\mathbb{R}^n)} \Big( (1+\gamma)|\mathrm{A}| + \beta\nu(\mathrm{A}) \Big) \right\} |X - Y|
\end{aligned}$$

for a.e. $x \in \mathbb{R}^N$ and all $X, Y \in \mathbb{R}^{Nn}$, which immediately leads to (2.3.5) and the proposition ensues. $\square$

## 2.4 Well-Posedness of Global Fully Nonlinear First Order Elliptic Systems

In this section we state and prove the main result of this paper which is the following:

### 2.4.1 Theorem [Existence-Uniqueness]

Assume that $n \geq 3$, $N \geq 2$ and let $F : \mathbb{R}^n \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^N$ be a Carathéodory map, satisfying Definition 2.3.1 with respect to a reference tensor A which satisfies (2.2.1).

(1) For any two maps $v, u \in W^{1;2^*,2}(\mathbb{R}^n, \mathbb{R}^N)$ (see (2.1.7)), we have the estimate

$$\|v - u\|_{W^{1;2^*,2}(\mathbb{R}^n)} \leq C\big\|F(\cdot, Dv) - F(\cdot, Du)\big\|_{L^2(\mathbb{R}^n)} \tag{2.4.1}$$

for some $C > 0$ depending only on $F$. Hence, the PDE system $F(\cdot, \mathrm{D}u) = f$ has at most one solution.

(2) Suppose further that $F(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$. Then for any $f \in L^2(\mathbb{R}^n, \mathbb{R}^N)$, the system

$$F(\cdot, \mathrm{D}u) = f, \quad \text{a.e. on } \mathbb{R}^n,$$

has a unique solution $u$ in the space $W^{1;2^*,2}(\mathbb{R}^n, \mathbb{R}^N)$ which also satisfies the estimate

$$\|u\|_{W^{1;2^*,2}(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{2.4.2}$$

for some $C > 0$ depending only on $F$.

### 2.4.2 Proof of Theorem 2.4.1.

(1) Let $\alpha$ and A be as in Definition 2.3.1 and fix $u, v \in W^{1;2^*,2}(\mathbb{R}^n, \mathbb{R}^N)$. Since A satisfies (2.2.1), by Plancherel's theorem (see e.g. [34]) we have:

$$\begin{aligned}
\frac{1}{\nu(\mathrm{A})}\big\|\mathrm{A} : \big(\mathrm{D}v - \mathrm{D}u\big)\big\|_{L^2(\mathbb{R}^n)} &= \frac{1}{\nu(\mathrm{A})}\big\|\mathrm{A} : \big(\widehat{Dv} - \widehat{\mathrm{D}u}\big)\big\|_{L^2(\mathbb{R}^n)} \\
&= \frac{1}{\nu(\mathrm{A})}\big\|\mathrm{A} : \big(\widehat{v} - \widehat{u}\big) \otimes (2\pi i \mathrm{Id})\big\|_{L^2(\mathbb{R}^n)} \\
&\geq \big\|\big(\widehat{v} - \widehat{u}\big) \otimes (2\pi i \mathrm{Id})\big\|_{L^2(\mathbb{R}^n)} \qquad (2.4.3) \\
&= \big\|\widehat{\mathrm{D}v} - \widehat{\mathrm{D}u}\big\|_{L^2(\mathbb{R}^n)} \\
&= \big\|\mathrm{D}v - \mathrm{D}u\big\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where we symbolised the identity map by "Id", which means $\mathrm{Id}(x) := x$. Further, by Definition 2.3.1 also we have

$$\begin{aligned}\Big\|\alpha(\cdot)\big[F(\cdot,\mathrm{D}u)-F(\cdot,\mathrm{D}v)\big]\;-\;\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\Big\|_{L^2(\mathbb{R}^n)}&\\ \leq\;\beta\nu(\mathrm{A})\big\|\mathrm{D}u-\mathrm{D}v\big\|_{L^2(\mathbb{R}^n)}+\gamma\big\|\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\big\|_{L^2(\mathbb{R}^n)}&\end{aligned}$$

Using the estimate (2.4.3) above this gives:

$$\begin{aligned}\Big\|\alpha(\cdot)\big[F(\cdot,\mathrm{D}u)-F(\cdot,Dv)\big]\;-\;\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\Big\|_{L^2(\mathbb{R}^n)}&\\ \leq\;\beta\big\|\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\big\|_{L^2(\mathbb{R}^n)}+\gamma\big\|\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\big\|_{L^2(\mathbb{R}^n)}&\\ \leq\;\big(\beta+\gamma\big)\big\|\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\big\|_{L^2(\mathbb{R}^n)}&\end{aligned}\tag{2.4.4}$$

and hence

$$\begin{aligned}\big(\beta+\gamma\big)\big\|\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\big\|_{L^2(\mathbb{R}^n)}&\\ \geq\;\Big\|\mathrm{A}:\big(\mathrm{D}u-Dv\big)\;-\;\alpha(\cdot)\big[F(\cdot,\mathrm{D}u)-F(\cdot,\mathrm{D}v)\big]\Big\|_{L^2(\mathbb{R}^n)}&\\ \geq\;\big\|\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\big\|_{L^2(\mathbb{R}^n)}-\;\Big\|\alpha(\cdot)\big[F(\cdot,\mathrm{D}u)-F(\cdot,\mathrm{D}v)\big]\Big\|_{L^2(\mathbb{R}^n)}&\end{aligned}$$

which implies the following estimate:

$$\begin{aligned}\Big\|\alpha(\cdot)\big[F(\cdot,\mathrm{D}u)-F(\cdot,\mathrm{D}v)\big]\Big\|_{L^2(\mathbb{R}^n)}&\geq\big[1-(\beta+\gamma)\big]\big\|\mathrm{A}:\big(\mathrm{D}u-\mathrm{D}v\big)\big\|_{L^2(\mathbb{R}^n)}\\ &\geq\big[1-(\beta+\gamma)\big]\nu(\mathrm{A})\big\|\mathrm{D}u-\mathrm{D}v\big\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

Since $\beta+\gamma<1$, we have the estimate:

$$\frac{\|\alpha(\cdot)\|_{L^\infty(\mathbb{R}^n)}}{\big[1-(\beta+\gamma)\big]\nu(\mathrm{A})}\big\|F(\cdot,\mathrm{D}u)-F(\cdot,\mathrm{D}v)\big\|_{L^2(\mathbb{R}^n)}\;\geq\;\big\|\mathrm{D}u-\mathrm{D}v\big\|_{L^2(\mathbb{R}^n)}.\tag{2.4.5}$$

By (2.4.5), and the fact that $n\geq 3$, the Gagliardo-Nirenberg-Sobolev inequality gives the estimate

$$\|u-v\|_{W^{1;2^*,2}(\mathbb{R}^n)}\;\leq\;C\big\|F(\cdot,\mathrm{D}u)-F(\cdot,\mathrm{D}v)\big\|_{L^2(\mathbb{R}^n)}\tag{2.4.6}$$

where $C>0$ depends only on $F$.

(2) By our assumptions on $F$ and that $F(x,0)=0$, Lemma 2.3.4 implies that there exists an $M>0$ depending only on $F$, such that for any $u\in W^{1;2^*,2}(\mathbb{R}^n,\mathbb{R}^N)$, we

have the estimates

$$
\begin{aligned}
\big\|\alpha(\cdot)F(\cdot,\mathrm{D}u)\big\|_{L^2(\mathbb{R}^n)} \, &= \, \Big\|\alpha(\cdot)\big[F(\cdot,0+\mathrm{D}u)-F(\cdot,0)\big]\Big\|_{L^2(\mathbb{R}^n)} \\
&= \, M\|\alpha(\cdot)\|_{L^\infty(\mathbb{R}^n)}\|\mathrm{D}u\|_{L^2(\mathbb{R}^n)} \\
&\leq \, M\|\alpha(\cdot)\|_{L^\infty(\mathbb{R}^n)}\|u\|_{W^{1;2^*,2}(\mathbb{R}^n)}
\end{aligned}
\tag{2.4.7}
$$

and also

$$
\|\mathrm{A}:\mathrm{D}u\|_{L^2(\mathbb{R}^n)} \, \leq \, \|\mathrm{A}\| \, \|\mathrm{D}u\|_{L^2(\mathbb{R}^n)} \, \leq \, \|\mathrm{A}\|\|u\|_{W^{1;2^*,2}(\mathbb{R}^n)}. \tag{2.4.8}
$$

We conclude from (2.4.7) and (2.4.8) that the differential operators

$$
\left\{
\begin{array}{rcl}
\mathscr{A}[u] & := & \mathrm{A}:\mathrm{D}u, \\
\mathscr{F}[u] & := & \alpha(\cdot)F(\cdot,\mathrm{D}u),
\end{array}
\right.
$$

map the functional space $W^{1;2^*,2}(\mathbb{R}^n,\mathbb{R}^N)$ into the space $L^2(\mathbb{R}^n,\mathbb{R}^N)$. Note that Theorem 2.2.1 proved in [43] implies that the linear operator

$$
A \ : \ W^{1;2^*,2}(\mathbb{R}^n,\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^n,\mathbb{R}^N)
$$

is a bijection. Hence, in view of inequality (2.4.4) above and the fact that $\beta+\gamma<1$, Campanato's nearness Theorem 2.2.3 implies that $F$ is a bijection as well. As a result, for any $g\in L^2(\mathbb{R}^n,\mathbb{R}^N)$, the PDE system

$$
\alpha(\cdot)F(\cdot,\mathrm{D}u) \, = \, g, \quad \text{a.e. on } \mathbb{R}^n,
$$

has a unique solution $u\in W^{1;2^*,2}(\mathbb{R}^n,\mathbb{R}^N)$. Since $\alpha(\cdot),1/\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, by selecting $g=\alpha(\cdot)f$, we conclude that the problem

$$
F(\cdot,\mathrm{D}u) \, = \, f, \quad \text{a.e. on } \mathbb{R}^n,
$$

has a unique solution in $W^{1;2^*,2}(\mathbb{R}^n,\mathbb{R}^N)$. The proof of the theorem is now complete. $\square$

# Chapter 3

# Existence of $1D$ Vectorial Absolute Minimisers in $L^\infty$ under Minimal Assumptions

## 3.1 Introduction

In this chapter we present the joint paper with Katzourakis [4]. The estimated percentage contribution is 50%. This paper has been published in December 2016 in Proceedings of the American Mathematical Society (AMS). The main goal of this paper is to prove the existence of a Vectorial Absolute Minimiser to the supremal functional

$$E_\infty(u,\Omega') := \operatorname*{ess\,sup}_{x\in\Omega'} \mathscr{L}\left(x,u(x),\mathrm{D}u(x)\right), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N),\ \Omega' \Subset \Omega, \tag{3.1.1}$$

applied to maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$, $N \in \mathbb{N}$, where $\Omega$ is an open interval and $\mathscr{L} \in C(\Omega\times\mathbb{R}^N\times\mathbb{R}^N)$ is a non-negative continuous function which we call Lagrangian and whose arguments will be denoted by $(x,\eta,P)$. By Absolute Minimiser we mean a map $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ such that

$$E_\infty(u,\Omega') \le E_\infty(u+\phi,\Omega'), \tag{3.1.2}$$

for all $\Omega' \Subset \Omega$ and all $\phi \in W^{1,\infty}_0(\Omega',\mathbb{R}^N)$. This is the appropriate minimality notion for supremal functionals of the form (3.1.1); requiring at the outset minimality on all subdomains is necessary because of the lack of additivity in the domain argument. The study of (3.1.1) was pioneered by Aronsson in the 1960s [6–10] who considered the case $N = 1$. Since then, the (higher dimensional) scalar case of $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ has developed massively and there is a vast literature on the topic (see for instance the lecture notes [5, 42]). In the case the Lagrangian is $C^1$, of particular interest has been the study of the (single) equation associated to (3.1.1), which is the equivalent of the Euler-Lagrange equation for supremal functionals and is known as the "Aronsson equation":

$$A_\infty u \,:=\, \mathrm{D}\big(\mathscr{L}(\cdot,u,\mathrm{D}u)\big)\mathscr{L}_P(\cdot,u,\mathrm{D}u) \,=\, 0. \tag{3.1.3}$$

In (3.1.3) above, the subscript denotes the gradient of $\mathscr{L}(x,\eta,P)$ with respect to $P$ and, as it is customary, the equation is written for smooth solutions. Herein we are interested in the vectorial case $N \geq 2$ but in one spatial dimension. Unlike the scalar case, the literature for $N \geq 2$ is much more sparse and starts much more recently. Perhaps the first most important contributions were by Barron-Jensen-Wang [15, 16] who among other deep results proved the existence of Absolute Minimisers for (3.1.1) under certain assumptions on $\mathscr{L}$ which we recall later. However, their contributions were at the level of the functional and the appropriate (non-obvious) vectorial analogue of the Aronsson equation was not known at the time. The systematic study of the vectorial case of (3.1.1) (actually in the general case of maps $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^N$) together with its associated system of equations begun in the early 2010s by the second author in a series of papers, see [36–41, 44, 46–49] (and also the joint contributions with Croce, Pisante and Pryer [28, 53, 54]). The ODE system associated to (3.1.1) for smooth maps $u:\Omega\subseteq\mathbb{R}\longrightarrow\mathbb{R}^N$ turns out to be

$$\mathcal{F}_\infty\big(\cdot,u,\mathrm{D}u,\mathrm{D}^2u\big) \,=\, 0, \quad \text{on } \Omega, \tag{3.1.4}$$

where

$$\begin{aligned}
\mathcal{F}_\infty(x,\eta,P,X) \,:=\, & \Big[\mathscr{L}_P(x,\eta,P)\otimes\mathscr{L}_P(x,\eta,P) \\
& +\, \mathscr{L}(x,\eta,P)[\mathscr{L}_P(x,\eta,P)]^\perp\mathscr{L}_{PP}(x,\eta,P)\Big]X \\
& +\, \Big(\mathscr{L}_\eta(x,\eta,P)\cdot P \,+\, \mathscr{L}_x(x,\eta,P)\Big)\mathscr{L}_P(x,\eta,P) \\
& +\, \mathscr{L}(x,\eta,P)\big[\mathscr{L}_P(x,\eta,P)\big]^\perp\Big(\mathscr{L}_{P\eta}(x,\eta,P)P \\
& +\, \mathscr{L}_{Px}(x,\eta,P) \,-\, \mathscr{L}_\eta(x,\eta,P)\Big).
\end{aligned} \tag{3.1.5}$$

Quite unexpectedly, in the case $N \geq 2$ the Lagrangian needs to be $C^2$ for the equation to make sense, whilst the coefficients of the full system are *discontinuous*; for more details we refer to the papers cited above. In (3.1.5) the notation of subscripts symbolises derivatives with respect to the respective variables and $\big[\mathscr{L}_P(x,\eta,P)\big]^\perp$ is the orthogonal projection to the hyperplane normal to $\mathscr{L}_P(x,\eta,P)\in\mathbb{R}^N$:

$$\big[\mathscr{L}_P(x,\eta,P)\big]^\perp \,:=\, \mathrm{I} - \,\mathrm{sgn}\big(\mathscr{L}_P(x,\eta,P)\big)\otimes\mathrm{sgn}\big(\mathscr{L}_P(x,\eta,P)\big). \tag{3.1.6}$$

The system (3.1.4) reduces to the equation (3.1.3) when $N=1$. In the paper [47] the existence of an absolutely minimising *generalised solution* to (3.1.4) was proved, together with extra partial regularity and approximation properties. Since (3.1.4) is a quasilinear non-divergence degenerate system with discontinuous coefficients, a notion of appropriately defined "weak solution" is necessary because in general solutions are non-smooth. To this end, the general new approach of $\mathcal{D}$-solutions which has recently been proposed in [48] has proven to be the appropriate setting for vectorial Calculus of Variations in $L^\infty$ (see [46–48]), replacing to some extent viscosity solutions which essentially apply only in the scalar case.

Herein we are concerned with the existence of absolute minimisers to (3.1.1) without drawing any connections to the differential system (3.1.4). Instead, we are interested in obtaining existence under the weakest possible assumptions. Accordingly, we establish the following result.

### 3.1.1 Theorem [Existence of vectorial Absolute Minimisers]

Let $\Omega \subseteq \mathbb{R}$ be a bounded open interval and let also

$$\mathscr{L} \ : \ \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow [0,\infty),$$

be a given continuous function with $N \in \mathbb{N}$. We assume that:

1. For each $(x,\eta) \in \overline{\Omega} \times \mathbb{R}^N$, the function $P \longmapsto \mathscr{L}(x,\eta,P)$ is level-convex, that is for each $t \geq 0$ the sublevel set
$$\big\{P \in \mathbb{R}^N \ : \ \mathscr{L}(x,\eta,P) \leq t\big\}$$
is a convex set in $\mathbb{R}^N$.
2. there exist non-negative constants $C_1, C_2, C_3$, and $0 < q \leq r < +\infty$ and a positive locally bounded function $h : \mathbb{R} \times \mathbb{R}^N \longrightarrow [0,+\infty)$ such that for all $(x,\eta,P) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N$
$$C_1|P|^q \,-\, C_2 \,\leq\, \mathscr{L}(x,\eta,P) \,\leq\, h(x,\eta)|P|^r \,+\, C_3.$$

Then, for any affine map $b : \mathbb{R} \longrightarrow \mathbb{R}^N$, there exist a vectorial Absolute Minimiser $u^\infty \in W^{1,\infty}_b(\Omega,\mathbb{R}^N)$ of the supremal functional (3.1.1) (Definition (3.1.2)).

Theorem 3.1.1 generalises two respective results in the both the papers [16] and [47]. On the one hand, *in [16] Theorem 3.1.1 was established under the extra assumption $C_2 = C_3 = 0$ which forces $\mathscr{L}(x,\eta,0) = 0$, for all $(x,\eta) \in \mathbb{R} \times \mathbb{R}^N$.* Unfortunately this requirement is incompatible with important applications of (3.1.1) to problems of $L^\infty$-modelling of variational Data Assimilation (*4DVar*) arising in the Earth Sciences and especially in Meteorology (see [18, 19, 47]). An explicit model of $\mathscr{L}$ is given by

$$\mathscr{L}(x,\eta,P) \,:=\, \big|k(x) - K(\eta)\big|^2 + \big|P - \mathscr{V}(x,\eta)\big|^2, \tag{3.1.7}$$

and describes the "error" in the following sense: consider the problem of finding the solution $u$ to the following ODE coupled by a pointwise constraint:

$$\mathrm{D}u(t) = \mathscr{V}\big(t,u(t)\big) \quad \& \quad K(u(t)) = k(t), \quad t \in \Omega.$$

Here $\mathscr{V} : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a time-dependent vector field describing the law of motion of a body moving along the orbit described by $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ (e.g. Newtonian forces, Galerkin approximation of the Euler equations, etc), $k : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^M$ is some partial "measurements" in continuous time along the orbit and $K : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a submersion which corresponds to some component of the orbit that is observed. We interpret the problem as that $u$ should satisfy the law of motion and also be compatible with the measurements along the orbit. Then minimisation of (3.1.1) with $\mathscr{L}$ as given by (3.1.7) leads to a uniformly optimal approximate solution without "spikes" of large deviation of the prediction from the actual orbit.

On the other hand, in the paper [47] Theorem 3.1.1 was proved under assumptions allowing to model Data Assimilation but strong convexity, smoothness and structural assumptions were imposed, allowing to obtain stronger results accordingly. In particular, the Lagrangian was assumed to be radial in $P$, which means it can be written in the form

$$\mathscr{L}(x, \eta, P) := \mathscr{H}\Big(x, \eta, \frac{1}{2}\big|P - \mathscr{V}(x, \eta)\big|^2\Big).$$

In this paper we relax the hypotheses of both the aforementioned results.

### 3.1.2 Theorem [Jensen's inequality for level-convex functions]

Let $\mathscr{L} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be lower semi-continuous and level-convex and let $\mu$ be a probability measure on $\mathbb{R}^n$ supported on $\Omega$. Let $\psi \in L^1_\mu(\Omega)$ be a given function. Then

$$\mathscr{L}\left(\int_\Omega \psi d\mu\right) \leq (\mu)\text{ess}\sup_{x\in\Omega} \mathscr{L}\big(\psi(x)\big).$$

The $(\mu)$-essential supremum means we exclude sets of $\mu$ measure zero.

## 3.2 The Proof of the existence of vectorial Absolute Minimisers

In this section we prove our main theorem 3.1.1, in which we claimed that under the assumptions (1)-(2) there exists a vectorial absolute minimiser $u^\infty \in W^{1,\infty}_b(\Omega, \mathbb{R}^N)$ of the supremal functional (3.1.1). For the first part of the proof which is the existence of a vectorial minimiser for the supremal functional (3.1.1) it suffices to recall a lemma of [16] in which they proved the following result which we recall right bellow for the convenience of the reader.

### 3.2.1 Lemma [Existence of a vectorial minimiser] (cf. [16])

In the setting of theorem 3.1.1 and under the same hypotheses, for a fixed affine map $b : \mathbb{R} \longrightarrow \mathbb{R}^N$, set

$$\begin{aligned} C_m &:= \inf\Big\{E_m(u,\Omega)\ :\ u \in W^{1,qm}_b(\Omega,\mathbb{R}^N)\Big\},\\ C_\infty &:= \inf\Big\{E_\infty(u,\Omega)\ :\ u \in W^{1,\infty}_b(\Omega,\mathbb{R}^N)\Big\}. \end{aligned}$$

where $E_\infty$ is as in (3.1.1) and

$$E_m(u,\Omega) := \int_\Omega \mathscr{L}\big(x,u(x),\mathrm{D}u(x)\big)^m\,dx. \tag{3.2.1}$$

Then, there exist $u^\infty \in W^{1,\infty}_b(\Omega,\mathbb{R}^N)$ which is a (mere) minimiser of (3.1.1) over $W^{1,\infty}_b(\Omega,\mathbb{R}^N)$ and a sequence of approximate minimisers $\{u^m\}_{m=1}^\infty$ of (3.2.1) in the spaces $W^{1,qm}_b(\Omega,\mathbb{R}^N)$ such that, for any $s \geq 1$,

$$u^m \longrightarrow u^\infty, \text{ weakly as } m \to \infty \text{ in } W^{1,s}(\Omega,\mathbb{R}^N)$$

along a subsequence. Moreover,

$$E_\infty(u^\infty,\Omega)\ =\ C_\infty\ =\ \lim_{m\to\infty}(C_m)^{\frac{1}{m}}. \tag{3.2.2}$$

By approximate minimiser we mean that $u^m$ satisfies

$$\big|E_m(u^m,\Omega) - C_m\big|\ <\ 2^{-m^2}, \tag{3.2.3}$$

Finally, for any $A \subseteq \Omega$ measurable of positive measure the following lower semicontinuity inequality holds

$$E_\infty(u^\infty,A)\ \leq\ \liminf_{m\to\infty} E_m(u^m,A)^{\frac{1}{m}}. \tag{3.2.4}$$

The idea of the proof of (3.2.3) is based on the use of Young measures in order to bypass the lack of convexity for the approximating $L^m$ minimisation problems (recall that $\mathscr{L}(x,\eta,\cdot)$ is only assumed to be level-convex); without weak lower-semicontinuity of $E_m$, the relevant infima of the approximating functionals may not be realised. For details we refer to [16] (this method of [16] has most recently been applied to higher order $L^\infty$ problems, see [54]). We also note that (3.2.4) has been established in p. 264 of [16] in slightly different guises, whilst the scaling of the functionals $E_m$ is also slightly different therein. However, it is completely trivial for the reader to check that their proofs clearly establish our Lemma 3.2.1.

### 3.2.2 Proof of Theorem 3.1.1.

Our goal now is to prove that the candidate $u^\infty$ of Lemma 3.2.1 above is actually an Absolute Minimiser of (3.1.1), which means we need to prove $u^\infty$ satisfies (3.1.2).

The method we utilise follows similar lines to those of [47], although technically has been slightly simplified. The main difference is that due to the weaker assumptions than those of [47], we invoke the general Jensen's inequality for level-convex functions Theorem 3.1.2. In [47] the Lagrangian was assumed to be radial in the third argument, a condition necessary and sufficient for the symmetry of the coefficient matrix multiplying the second derivatives in (3.1.4); this special structure of $\mathscr{L}$ led to some technical complications. Also, herein we have reduced the number of auxiliary parameters in the energy comparison map (defined below) by invoking a diagonal argument.

Let us fix $\Omega' \Subset \Omega$. Since $\Omega'$ is a countable disjoint union of open intervals, then there is no loss of generality in assuming that $\Omega'$ itself is an open interval, and by simple rescaling argument, it suffices to assume that $\Omega' = (0,1) \Subset \mathbb{R}$. Let $\phi \in W^{1,\infty}_0((0,1),\mathbb{R}^N)$ be an arbitrary variation and set $\psi^\infty := u^\infty + \phi$. In order to conclude, it suffices to establish

$$E_\infty\big(u^\infty,(0,1)\big) \,\leq\, E_\infty\big(\psi^\infty,(0,1)\big).$$

Obviously, $u^\infty(0) = \psi^\infty(0)$ and $u^\infty(1) = \psi^\infty(1)$. We define the energy comparison function $\psi^{m,\delta}$, for any fixed $0<\delta<1/3$ as

$$\psi^{m,\delta}(x) \,:=\, \begin{cases} \left(\dfrac{\delta - x}{\delta}\right) u^m(0) + \left(\dfrac{x}{\delta}\right)\psi^\infty(\delta), & x \in (0,\delta], \\ \psi^\infty(x), & x\in(\delta,1-\delta), \\ \left(\dfrac{1-x}{\delta}\right)\psi^\infty(1-\delta) + \left(\dfrac{x-(1-\delta)}{\delta}\right) u^m(1), & x\in[1-\delta,1), \end{cases}$$

where $m \in \mathbb{N}\cup\{\infty\}$. Then, $\psi^{m,\delta} - u^m \in W^{1,\infty}_0\big((0,1),\mathbb{R}^N\big)$ and

$$\mathrm{D}\psi^{m,\delta}(x) \,=\, \begin{cases} \dfrac{\psi^\infty(\delta) - u^m(0)}{\delta}, & \text{on } (0,\delta), \\ \mathrm{D}\psi^\infty, & \text{on } (\delta,1-\delta), \\ \dfrac{\psi^\infty(1-\delta) - u^m(1)}{-\delta}, & \text{on } (1-\delta,1). \end{cases}$$

Now, note that

$$\psi^{m,\delta} \longrightarrow \psi^{\infty,\delta} \text{ in } W^{1,\infty}\big((0,1),\mathbb{R}^N\big), \text{ as } m\to\infty, \tag{3.2.5}$$

because $\psi^{m,\delta} \longrightarrow \psi^{\infty,\delta}$ in $L^\infty\big((0,1),\mathbb{R}^N\big)$ and for a.e. $x \in (0,1)$ we have

$$\begin{aligned}
\Big|\mathrm{D}\psi^{m,\delta}(x) - \mathrm{D}\psi^{\infty,\delta}(x)\Big| \;&=\; \chi_{(0,\delta)}\frac{|u^\infty(0) - u^m(0)|}{\delta} \;+\; \chi_{(1-\delta,1)}\frac{|u^\infty(1) - u^m(1)|}{\delta} \\
&\le\; \left(\frac{1}{\delta} + \frac{1}{\delta}\right)\|u^m - u^\infty\|_{L^\infty(\Omega)} \\
&=\; o(1),
\end{aligned}$$

as $m \to \infty$ along a subsequence. Now, recall that $\psi^{m,\delta} = u^m$ at the endpoints $\{0,1\}$. Let us also remind to the reader that after the rescaling simplification, $(0,1)$ is a subinterval of $\Omega \subseteq \mathbb{R}$ whilst (3.2.3) holds only for the whole of $\Omega$. Since $u^m$ is an approximate minimiser of (3.2.1) over $W^{1,m}_b(\Omega,\mathbb{R}^N)$ for each $m \in \mathbb{N}$, by utilising the approximate minimality of $u^m$ (given by (3.2.3)), the additivity of $E_m$ with respect to its second argument, we obtain the estimate

$$E_m\big(u^m,(0,1)\big) \;\le\; E_m\big(\psi^{m,\delta},(0,1)\big) \;+\; 2^{-m^2}.$$

Hence, by Hölder inequality

$$\begin{aligned}
E_m\big(u^m,(0,1)\big)^{\frac{1}{m}} \;&\le\; E_m\big(\psi^{m,\delta},(0,1)\big)^{\frac{1}{m}} \;+\; 2^{-m} \\
&\le\; E_\infty\big(\psi^{m,\delta},(0,1)\big) \;+\; 2^{-m}.
\end{aligned} \tag{3.2.6}$$

On the other hand, we have

$$\begin{aligned}
E_\infty\big(\psi^{m,\delta},(0,1)\big) \;= \max\Big\{&E_\infty\big(\psi^{m,\delta},(0,\delta)\big), \\
&E_\infty\big(\psi^{m,\delta},(\delta,1-\delta)\big), \\
&E_\infty\big(\psi^{m,\delta},(1-\delta,1)\big)\Big\}
\end{aligned}$$

and since $\psi^{m,\delta} = \psi^\infty$ on $(\delta, 1-\delta)$, we have

$$\begin{aligned}
E_\infty\big(\psi^{m,\delta},(0,1)\big) \;\le \max\Bigg\{&E_\infty\big(\psi^{m,\delta},(0,\delta)\big),\; E_\infty\big(\psi^{\infty},(0,1)\big), \\
&E_\infty\big(\psi^{m,\delta},(1-\delta,1)\big)\Bigg\}.
\end{aligned} \tag{3.2.7}$$

Combining (3.2.5)-(3.2.7) and (3.2.4), we get

$$\begin{aligned}
E_\infty\big(u^\infty,(0,1)\big) \;&\le\; \liminf_{m\to\infty}\Big(\max\Big\{E_\infty\big(\psi^{m,\delta},(0,\delta)\big),\; E_\infty\big(\psi^{\infty},(0,1)\big), \\
&\qquad\qquad E_\infty\big(\psi^{m,\delta},(1-\delta,1)\big)\Big\}\Big) \\
&\le\; \max\Big\{E_\infty\big(\psi^{\infty},(0,1)\big),\; E_\infty\big(\psi^{\infty,\delta},(0,\delta)\big), \\
&\qquad E_\infty\big(\psi^{\infty,\delta},(1-\delta,1)\big)\Big\}.
\end{aligned} \tag{3.2.8}$$

Let us now denote the difference quotient of a function $v : \mathbb{R} \longrightarrow \mathbb{R}^N$ as $\mathrm{D}^{1,t}v(x) := \frac{1}{t}[v(x+t) - v(x)]$. Then, we may write

$$\begin{aligned} \mathrm{D}\psi^{\infty,\delta}(x) &= \mathrm{D}^{1,\delta}\psi^{\infty}(0), \qquad x \in (0,\delta), \\ \mathrm{D}\psi^{\infty,\delta}(x) &= \mathrm{D}^{1,-\delta}\psi^{\infty}(1), \quad x \in (1-\delta,1), \end{aligned}$$

Note now that

$$\left\{ \begin{aligned} E_\infty\big(\psi^{\infty,\delta},(0,\delta)\big) &= \max_{0\le x\le\delta} \mathscr{L}\Big(x,\psi^{\infty,\delta}(x),\mathrm{D}^{1,\delta}\psi^{\infty}(0)\Big), \\ E_\infty\big(\psi^{\infty,\delta},(1-\delta,1)\big) &= \max_{1-\delta\le x\le 1} \mathscr{L}\Big(x,\psi^{\infty,\delta}(x),\mathrm{D}^{1,-\delta}\psi^{\infty}(1)\Big). \end{aligned} \right. \tag{3.2.9}$$

In view of (3.2.8)-(3.2.9), it is suffices to prove that there exist an infinitesimal sequence $(\delta_i)_{i=1}^{\infty}$ such that

$$E_\infty\big(\psi^{\infty},(0,1)\big) \ge \max\left\{ \begin{aligned} &\limsup_{i\to\infty} \max_{[0,\delta_i]} \mathscr{L}\Big(\cdot,\psi^{\infty,\delta_i},\mathrm{D}^{1,\delta_i}\psi^{\infty}(0)\Big), \\ &\limsup_{i\to\infty} \max_{[1-\delta_i,1]} \mathscr{L}\Big(\cdot,\psi^{\infty,\delta_i},\mathrm{D}^{1,-\delta_i}\psi^{\infty}(1)\Big) \end{aligned} \right\}. \tag{3.2.10}$$

The rest of the proof is devoted to establishing (3.2.10). Let us begin by recording for later use that

$$\left\{ \begin{aligned} \max_{0\le x\le\delta}\Big|\psi^{\infty,\delta}(x) - \psi^{\infty}(0)\Big| &\longrightarrow 0, \quad \text{as } \delta\to 0, \\ \max_{1-\delta\le x\le 1}\Big|\psi^{\infty,\delta}(x) - \psi^{\infty}(1)\Big| &\longrightarrow 0, \quad \text{as } \delta\to 0. \end{aligned} \right. \tag{3.2.11}$$

Fix a generic $u \in W^{1,\infty}(\Omega,\mathbb{R}^N)$, $x \in [0,1]$ and $0 < \varepsilon < 1/3$ and define

$$A_\varepsilon(x) := [x-\varepsilon, x+\varepsilon] \cap [0,1].$$

We claim that there exist an increasing modulus of continuity $\omega \in C(0,\infty)$ with $\omega(0^+) = 0$ such that

$$E_\infty\big(u, A_\varepsilon(x)\big) \ge \operatorname*{ess\,sup}_{y\in A_\varepsilon(x)} \mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big) - \omega(\varepsilon). \tag{3.2.12}$$

Indeed for a.e. $y \in A_\varepsilon(x)$ we have $|x-y| \le \varepsilon$ and by the continuity of $\mathscr{L}$ and the essential boundedness of the derivative $Du$, there exist $\omega$ such that

$$\Big|\mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big) - \mathscr{L}\Big(y,u(y),\mathrm{D}u(y)\Big)\Big| \le \omega(\varepsilon)$$

for a.e. $y \in A_\varepsilon(x)$, leading directly to (3.2.12). Now, we show that

$$\sup_{A_\varepsilon(x)}\left\{\limsup_{t\to 0} \mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big)\right\} \le \operatorname*{ess\,sup}_{A_\varepsilon(x)} \mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big). \tag{3.2.13}$$

Indeed, for any Lipschitz function $u$, we have

$$\mathrm{D}^{1,t}u(y) \;=\; \frac{u(y+t)-u(y)}{t} \;=\; \int_0^1 Du(y+\lambda t)\,d\lambda, \tag{3.2.14}$$

when $y, y+t \in A_\varepsilon(x)$, $t \neq 0$. Further, for any $x \in \Omega$ the function $\mathscr{L}(x,u(x),\cdot)$ is level-convex and the Lebesgue measure on $[0,1]$ is a probability measure, thus Jensen's inequality for level-convex functions (see e.g. [15, 16]) yields

$$\begin{aligned}\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big) \;&=\; \mathscr{L}\left(x,u(x),\int_0^1 Du(y+\lambda t)\,d\lambda\right)\\ &\leq \operatorname*{ess\,sup}_{0\leq\lambda\leq 1} \mathscr{L}\Big(x,u(x),\mathrm{D}u(y+\lambda t)\Big),\end{aligned}$$

when $y \in A_\varepsilon(x)$, $0<x<1$. Consequently,

$$\begin{aligned}&\sup_{A_\varepsilon(x)}\left\{\limsup_{t\to0}\,\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big)\right\}\\ &\qquad\leq \sup_{A_\varepsilon(x)}\left\{\limsup_{t\to0}\left[\operatorname*{ess\,sup}_{0\leq\lambda\leq1}\mathscr{L}\Big(x,u(x),\mathrm{D}u(y+\lambda t)\Big)\right]\right\}\\ &\qquad\leq \sup_{A_\varepsilon(x)}\left\{\lim_{s\to0}\left[\operatorname*{ess\,sup}_{y-s\leq z\leq y+s}\mathscr{L}\Big(x,u(x),\mathrm{D}u(z)\Big)\right]\right\}\\ &\qquad= \operatorname*{ess\,sup}_{A_\varepsilon(x)}\mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big),\end{aligned}$$

as desired. Above we have used the following known property of the $L^\infty$ norm (see e.g. [5])

$$\|f\|_{L^\infty(\Omega)} \;=\; \sup_{x\in\Omega}\left(\lim_{\varepsilon\to0}\left\{\operatorname*{ess\,sup}_{(x-\varepsilon,x+\varepsilon)}|f|\right\}\right).$$

Note now that by (3.2.12) we have

$$E_\infty\big(u,(0,1)\big) \;\geq\; E_\infty\big(u,A_\varepsilon(x)\big) \;\geq\; \operatorname*{ess\,sup}_{A_\varepsilon(x)}\mathscr{L}\left(x,u(x),\mathrm{D}u(y)\right) \;-\;\omega(\varepsilon)$$

which combined with (3.2.13) leads to

$$\begin{aligned}E_\infty\big(u,(0,1)\big) \;&\geq\; \sup_{A_\varepsilon(x)}\left(\limsup_{t\to0}\;\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big)\right) \;-\;\omega(\varepsilon)\\ &\geq\; \limsup_{t\to0}\;\left(\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(x)\Big)\right)\;-\;\omega(\varepsilon).\end{aligned}$$

By passing to the limit as $\varepsilon\to0$ we get

$$E_\infty\big(u,(0,1)\big) \;\geq\; \limsup_{t\to0}\;\left(\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(x)\Big)\right), \tag{3.2.15}$$

for any fixed $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ and $x \in [0,1]$. Now, since

$$\big|\mathrm{D}^{1,t}u(x)\big| \,\leq\, \|\mathrm{D}u\|_{L^\infty(\Omega)}, \qquad x \in (0,1),\ t \neq 0,$$

for any *finite* set of points $x \in (0,1)$, there is a common infinitesimal sequence $(t_i(x))_{i=1}^{\infty}$ such that

$$\text{the limit vectors } \lim_{i\to\infty} \mathrm{D}^{1,t_i(x)}u(x) \ \text{ exists in } \mathbb{R}^N. \tag{3.2.16}$$

Utilising the continuity of $\mathscr{L}$ together with (3.2.15)-(3.2.16) we obtain

$$\begin{aligned} E_\infty\big(u,(0,1)\big) \,&\geq\, \limsup_{i\to\infty}\ \mathscr{L}\Big(x, u(x), \mathrm{D}^{1,t_i(x)}u(x)\Big) \\ &=\, \mathscr{L}\left(x, u(x), \lim_{i\to\infty} \mathrm{D}^{1,t_i(x)}u(x)\right). \end{aligned} \tag{3.2.17}$$

Now we apply (3.2.17) to $u = \psi^\infty$ and $x \in \{0,1\}$ to deduce the existence of a sequence $(\delta_i)_{i=1}^{\infty}$ along which

$$\text{the limit vectors } \lim_{i\to\infty} \mathrm{D}^{1,\delta_i}\psi^\infty(0),\ \lim_{i\to\infty} \mathrm{D}^{1,-\delta_i}\psi^\infty(1) \quad \text{exist in } \mathbb{R}^N \tag{3.2.18}$$

and also

$$\begin{aligned} E_\infty\big(\psi^\infty,(0,1)\big) \,\geq\, \max\Bigg\{ &\mathscr{L}\left(0, \psi^\infty(0), \lim_{i\to\infty} \mathrm{D}^{1,\delta_i}\psi^\infty(0)\right), \\ &\mathscr{L}\left(1, \psi^\infty(1), \lim_{i\to\infty} \mathrm{D}^{1,-\delta_i}\psi^\infty(1)\right)\Bigg\}. \end{aligned} \tag{3.2.19}$$

By recalling (3.2.9), (3.2.11) and (3.2.18), for $\delta = \delta_i$ we obtain

$$\begin{aligned} \lim_{i\to\infty} E_\infty\big(\psi^{\infty,\delta_i},(0,\delta_i)\big) \,&=\, \lim_{i\to\infty} \max_{[0,\delta_i]} \mathscr{L}\Big(\cdot, \psi^{\infty,\delta_i}, \mathrm{D}^{1,\delta_i}\psi^\infty(0)\Big) \\ &=\, \mathscr{L}\left(0, \psi^\infty(0), \lim_{i\to\infty} \mathrm{D}^{1,\delta_i}\psi^\infty(0)\right), \end{aligned} \tag{3.2.20}$$

and also

$$\begin{aligned} \lim_{i\to\infty} E_\infty\big(\psi^{\infty,\delta_i},(1-\delta_i,1)\big) \,&=\, \lim_{i\to\infty} \max_{[1-\delta_i,1]} \mathscr{L}\Big(\cdot, \psi^{\infty,\delta_i}, \mathrm{D}^{1,-\delta_i}\psi^\infty(1)\Big) \\ &=\, \mathscr{L}\left(1, \psi^\infty(1), \lim_{i\to\infty} \mathrm{D}^{1,-\delta_i}\psi^\infty(1)\right). \end{aligned} \tag{3.2.21}$$

By putting together (3.2.19)-(3.2.21), (3.2.10) ensues and we conclude the proof. $\square$

# Chapter 4

# Rigidity and flatness of the image of certain classes of $\infty$-Harmonic and $p$-Harmonic maps

## 4.1 Introduction

In this chapter we present the joint preprint paper with Katzourakis and Ayanbayev [2]. The estimated percentage contribution is 30%. Suppose that $n, N$ are integers and $\Omega$ an open subset of $\mathbb{R}^n$. In this paper we study geometric aspects of the image $u(\Omega) \subseteq \mathbb{R}^N$ of certain classes of $C^2$ vectorial solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to the following nonlinear degenerate elliptic PDE system:

$$[\![\mathrm{D}u]\!]^{\perp} \Delta u = 0 \quad \text{in } \Omega. \tag{4.1.1}$$

Here, for the map $u$ with components $(u_1, ..., u_N)^{\top}$ the notation $\mathrm{D}u$ symbolises the gradient matrix

$$\mathrm{D}u(x) = \big(\mathrm{D}_i u_\alpha(x)\big)_{i=1...n}^{\alpha=1...N} \in \mathbb{R}^{N\times n} \ , \quad \mathrm{D}_i \equiv \partial/\partial x_i,$$

$\Delta u$ stands for the Laplacian

$$\Delta u(x) = \sum_{i=1}^{n} \mathrm{D}^2_{ii} u(x) \in \mathbb{R}^N$$

and for any $X \in \mathbb{R}^{N\times n}$, $[\![X]\!]^{\perp}$ denotes the orthogonal projection on the orthogonal complement of the range of linear map $X : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![X]\!]^{\perp} := \mathrm{Proj}_{\mathrm{R}(X)^{\perp}}. \tag{4.1.2}$$

Our general notation will be either self-explanatory, or otherwise standard as e.g. in [30, 33]. Note that, since the rank is a discontinuous function, the map $[\![ \cdot ]\!]^\perp$ is discontinuous on $\mathbb{R}^{N\times n}$; therefore, the PDE system (4.1.1) has *discontinuous coefficients*. The geometric meaning of (4.1.1) is that *the Laplacian vector field* $\Delta u$ *is tangential to the image* $u(\Omega)$ and hence (4.1.1) is equivalent to the next statement: there exists a vector field

$$\mathrm{A} \,:\, \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^n$$

such that

$$\Delta u = \mathrm{D}u\,\mathrm{A} \quad \text{in } \Omega.$$

Our interest in (4.1.1) stems from the fact that it is a constituent component of the $p$-Laplace PDE system for all $p \in [2,\infty]$. Further, contrary perhaps to appearances, (4.1.1) is in itself a *variational PDE system* but in a non-obvious way. Deferring temporarily the specifics of how exactly (4.1.1) arises and what is the variational principle associated with it, let us recall that, for $p \in [2,\infty)$, the celebrated $p$-Laplacian is the divergence system

$$\Delta_p u := \mathrm{Div}\big(|\mathrm{D}u|^{p-2}\mathrm{D}u\big) = 0 \quad \text{in } \Omega \tag{4.1.3}$$

and comprises the Euler-Lagrange equation which describes extrema of the model $p$-Dirichlet integral functional

$$E_p(u) := \int_\Omega |\mathrm{D}u|^p, \quad u \in W^{1,p}(\Omega,\mathbb{R}^N), \tag{4.1.4}$$

in conventional vectorial Calculus of Variations. Above and subsequently, for any $X \in \mathbb{R}^{N\times n}$, the notation $|X|$ symbolises its Euclidean (Frobenius) norm:

$$|X| = \left(\sum_{\alpha=1}^{N}\sum_{i=1}^{n} (\mathbf{X}_{\alpha i})^2\right)^{1/2}.$$

The pair (4.1.3)-(4.1.4) is of paramount importance in applications and has been studied exhaustively. The extremal case of $p \to \infty$ in (4.1.3)-(4.1.4) is much more modern and intriguing. It turns out that one then obtains the following nondivergence PDE system

$$\Delta_\infty u := \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes \mathrm{I}\Big) : \mathrm{D}^2 u = 0 \quad \text{in } \Omega, \tag{4.1.5}$$

which is known as the $\infty$-Laplacian. In index from, (4.1.5) reads

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n} \Big(\mathrm{D}_i u_\alpha\, \mathrm{D}_j u_\beta + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp_{\alpha\beta}\, \delta_{ij}\Big)\mathrm{D}^2_{ij} u_\beta \;=\; 0, \quad \alpha = 1,...,N.$$

The system (4.1.5) plays the role of the Euler-Lagrange equation and arises in connexion with variational problems for the supremal functional

$$E_\infty(u,\mathcal{O}) := \|\mathrm{D}u\|_{L^\infty(\mathcal{O})}, \quad u \in W^{1,\infty}(\Omega,\mathbb{R}^N),\ \mathcal{O} \Subset \Omega. \tag{4.1.6}$$

The scalar case of $N=1$ in (4.1.5)-(4.1.6) was pioneered by G. Aronsson in the 1960s [6–10] who initiated the field of Calculus of Variations in $L^\infty$, namely the study of supremal functionals and of their associated equations describing critical points. Since then, the field has developed tremendously and there is an extensive relevant literature (for a pedagogical introduction see e.g. [42] and [5]). The vectorial case $N \geq 2$ began much more recently and first arose in work of the third author in the early 2010s [37]. The area is developing very rapidly due to both the mathematical significance as well as the importance for applications, particularly in Data Assimilation (see [38–41, 44, 46–49] and the joint contributions with the first and second author, Croce, Manfredi, Moser, Parini, Pisante and Pryer [4, 13, 28, 50–55]), as well as the paper of Papamikos and Pryer [60].

In this paper we focus on the $C^2$ case and establish the geometric rigidity and flatness of the images of solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to the nonlinear system (4.1.1), under the assumption that $\mathrm{D}u$ has rank at most 1. As a consequence, we obtain corresponding flatness results for the images of solutions to (4.1.3) and (4.1.5). Both aforementioned classes of solutions furnish very important particular examples which provide substantial intuition for the behaviour or general extremal maps in Calculus of Variations in $L^\infty$, see [5, 11, 12, 38, 41, 42, 53, 54]. Obtaining further information for the still largely mysterious behaviour of $\infty$-Harmonic maps is perhaps the greatest driving force to isolate and study the particular nonlinear system (4.1.1).

Let us note that the rank-one case includes the scalar and the one-dimensional case (i.e. when $\min\{n,N\}=1$), although in the case of $N=1$ (in which the single $\infty$-Laplacian reduces to $\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2u = 0$) (4.1.1) has no bearing since it vanishes identically at any non-critical point.

*The effect of* (4.1.1) *to the flatness of the image can be seen through the $L^\infty$ variational principle introduced in [40]*, where it was shown that solutions to (4.1.1) of constant rank can be characterised as those having minimal area with respect to (4.1.6)-(4.1.4). More precisely, therein the following result was proved:

### 4.1.1 Theorem [cf. [40, Theorem 2.7, Lemma 2.2]]

Given $N \geq n \geq 1$, let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a $C^2$ immersion defined on the open set $\Omega$ (more generally $u$ can be a map with constant rank of its gradient on $\Omega$). Then, the following statements are equivalent:

1. The map $u$ solves the PDE system (4.1.1) on $\Omega$.
2. For all $p \in [2,\infty]$, for all compactly supported domains $\mathcal{O} \Subset \Omega$ and all $C^1$ vector fields $\nu : \mathcal{O} \longrightarrow \mathbb{R}^N$ which are normal to the image $u(\mathcal{O}) \subseteq \mathbb{R}^N$ (without

requiring to vanish on $\partial\mathcal{O}$), namely those for which $\nu = [\![\mathrm{D}u]\!]^{\perp}\nu$ in $\mathcal{O}$, we have

$$\|\mathrm{D}u\|_{L^p(\mathcal{O})} \leq \|\mathrm{D}u + \mathrm{D}\nu\|_{L^p(\mathcal{O})}.$$

3. The same statement as in item (2) holds, but only for *some* $p \in [2, \infty]$.

If in addition $p < \infty$ in (2)-(3), then we may further restrict the class of normal vector fields to those satisfying $\nu|_{\partial\mathcal{O}} = 0$.

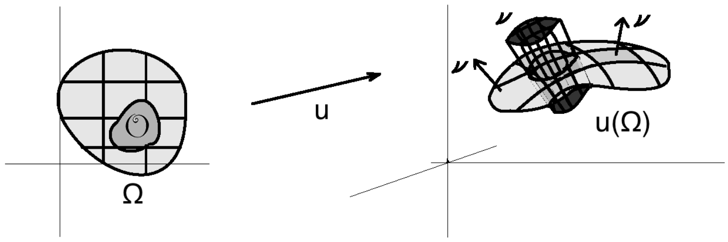


Figure 1. Illustration of the variational principle characterising (4.1.1).

In the paper [40], it was also shown that in the conformal class, (4.1.1) *expresses the vanishing of the mean curvature vector of* $u(\Omega)$.

The effect of (4.1.1) to the flatness of the image can be easily seen in the case of $n = 1 \leq N$ as follows: since

$$[\![u']\!]^{\perp}u'' = 0 \quad \text{in } \Omega \subseteq \mathbb{R}$$

and in one dimension we have

$$[\![u']\!]^{\perp} = \begin{cases} \mathrm{I} - \dfrac{u' \otimes u'}{|u'|^2}, & \text{on } \{u' \neq 0\}, \\ \mathrm{I}, & \text{on } \{u' = 0\}, \end{cases}$$

we therefore infer that $u'' = fu'$ on the open set $\{u' \neq 0\} \subseteq \mathbb{R}$ for some function $f$, readily yielding after an integration that $u(\Omega)$ is necessarily contained in a piecewise polygonal line of $\mathbb{R}^N$. As a generalisation of this fact, our first main result herein is the following:

### 4.1.2 Theorem [Rigidity and flatness of rank-one maps with tangential Laplacian]

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a solution to the nonlinear system (4.1.1) in $\Omega$, satisfying that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections).

Let us note that the rank-one assumption for $\mathrm{D}u$ is equivalent to the existence of two vector fields $\xi : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ and $a : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^n$ such that $\mathrm{D}u = \xi \otimes a$ in $\Omega$.

Example 4.1.3 below shows that Theorem 4.1.2 is optimal and in general rank-one solutions to the system (4.1.1) can not have affine image but only piecewise affine.

### 4.1.3 Example

Consider the $C^2$ rank-one map $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$u(x, y) = \left\{ \begin{array}{ll} (-x^4, x^4), & x \leq 0,\ y \in \mathbb{R}, \\ (+x^4, x^4), & x > 0,\ y \in \mathbb{R}. \end{array} \right.$$

Then, $u = \nu \circ f$ with $\nu : \mathbb{R} \longrightarrow \mathbb{R}^2$ given by $\nu(t) = (t, |t|)$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x, y) = \mathrm{sgn}(x)x^4$. It follows that $u$ solves (4.1.1) on $\mathbb{R}^2$: indeed, $\Delta u$ is a non-vanishing vector field on $\{x \neq 0\}$, being tangential to the image thereon since it is parallel to the derivative $\nu'(t) = (1, \pm 1)$ for $t \neq 0$. On the other hand, on $\{x = 0\}$ we have that $\Delta u = 0$. However, the image of $u$ is piecewise affine but not affine (see Figure 2 below).

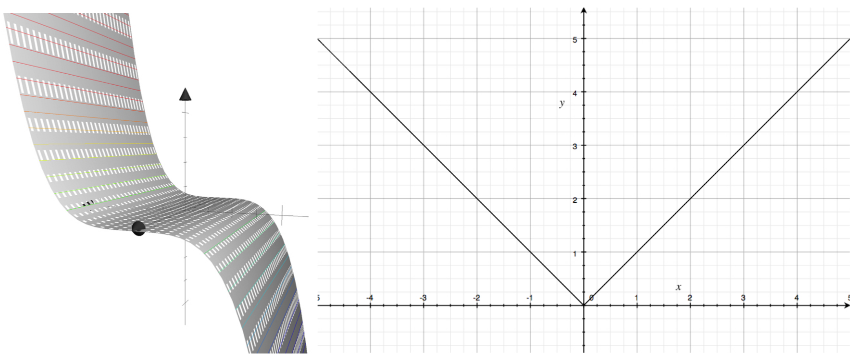

*Figure 2. The function $f$ and the image of the curve $\nu$ comprising the function $u$.*

As a consequence of Theorem 4.1.2, we obtain the next result regarding the rigidity of $p$-Harmonic maps for $p \in [2, \infty)$ which complements one of the results in the paper [41] where the case $p = \infty$ was considered.

### 4.1.4 Corollary [Rigidity of $p$-Harmonic maps, cf. [41]]

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a $p$-Harmonic map in $\Omega$ for some $p \in [2, \infty)$, that is $u$ solves (4.1.3). Suppose that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, the same result as in Theorem 4.1.2 is true.

In addition, there exists a partition of $\Omega$ to at most countably many Borel sets, where each set of the partition is a non-empty open set with a (perhaps empty) boundary portion, such that, on each of these, $u$ can be represented as

$$u = \nu \circ f.$$

Here, $f$ is a scalar $C^2$ $p$-Harmonic function (for the respective $p \in [2, \infty)$), defined on an open neighbourhood of the Borel set, whilst $\nu : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a Lipschitz curve which is twice differentiable and with unit speed on the image of $f$.

In this paper we try to keep the exposition as simple as possible and therefore we refrain from discussing generalised solutions to (4.1.1) and (4.1.5) (or (4.1.3)). We confine ourselves to merely mentioning that in the scalar case, $\infty$-Harmonic functions are understood in the viscosity sense of Crandall-Ishii-Lions (see e.g. [5, 42]), whilst in the vectorial case a new candidate theory for systems has been proposed in [48] which has already borne significant fruit in [13, 28, 46–48, 51, 53–55].

We now expound on how exactly the nonlinear system (4.1.1) arises from (4.1.3) and (4.1.5). By expanding the derivatives in (4.1.3) and normalising, we arrive at

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u \,+\, \frac{|\mathrm{D}u|^2}{p-2} \Delta u = 0. \tag{4.1.7}$$

For any $X \in \mathbb{R}^{N \times n}$, let $[\![X]\!]^{\parallel}$ denote the orthogonal projection on the range of the linear map $X : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![X]\!]^{\parallel} \,:=\, \mathrm{Proj}_{\mathrm{R}(X)}. \tag{4.1.8}$$

Since the identity of $\mathbb{R}^N$ splits as $\mathrm{I} = [\![\mathrm{D}u]\!]^{\parallel} + [\![\mathrm{D}u]\!]^{\perp}$, by expanding $\Delta u$ with respect to these projections,

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u \,+\, \frac{|\mathrm{D}u|^2}{p-2} [\![\mathrm{D}u]\!]^{\parallel} \Delta u = -\frac{|\mathrm{D}u|^2}{p-2} [\![\mathrm{D}u]\!]^{\perp} \Delta u.$$

The mutual perpendicularity of the vector fields of the left and right hand side leads via a renormalisation argument (see e.g. [37, 40, 41]) to the equivalence of the $p$-Laplacian with the pair of systems

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u \,+\, \frac{|\mathrm{D}u|^2}{p-2} [\![\mathrm{D}u]\!]^{\parallel} \Delta u = 0 \;, \quad |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^{\perp} \Delta u = 0. \tag{4.1.9}$$

The $\infty$-Laplacian corresponds to the limiting case of (4.1.9) as $p \to \infty$, which takes the form

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u = 0 \;, \quad |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^{\perp} \Delta u = 0. \tag{4.1.10}$$

Hence, the $\infty$-Laplacian (4.1.5) actually consists of the two independent systems in (4.1.10) above. The second system in (4.1.9)-(4.1.10) is, at least on $\{\mathrm{D}u \neq 0\}$, equivalent to our PDE system (4.1.1). Note that in the scalar case of $N = 1$ as well as in the case of submersion solutions (for $N \leq n$), the second system trivialises.

We conclude the introduction with a geometric interpretation of the nonlinear system (4.1.1), which can be expressed in a more geometric language as follows:[1] Suppose that $u(\Omega)$ is a $C^2$ manifold and let $\mathbf{A}(u)$ denote its second fundamental form. Then

$$[\![\mathrm{D}u]\!]^{\perp}\Delta u \,=\, -\operatorname{tr}\mathbf{A}(u)(\mathrm{D}u,\mathrm{D}u).$$

The tangential part $[\![\mathrm{D}u]\!]^{\|}\Delta u$ of the Laplacian is commonly called the *tension field* in the theory of *Harmonic maps* and is symbolised by $\tau(u)$ (see e.g. [59]). Hence, we have the orthogonal decomposition

$$\Delta u \,=\, \tau(u) \,-\, \operatorname{tr}\mathbf{A}(u)(\mathrm{D}u,\mathrm{D}u).$$

Therefore, in the case of higher regularity of the image of $u$, we obtain that the nonlinear system

$$\Delta u \,=\, \tau(u) \quad \text{in } \Omega, \tag{4.1.11}$$

is a further geometric reformulation of our PDE system (4.1.1).

## 4.2 Proofs

In this section we prove the main results of this paper. The main analytical tool needed in the proof of Theorem 4.1.2 is the next rigidity theorem for maps whose gradient has rank at most one. It was established in [41] and we recall it below for the convenience of the reader and only in the case needed in this paper.

### 4.2.1 Theorem [Rigidity of Rank-One maps, cf. [41]]

Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set and $u$ is in $C^2(\Omega,\mathbb{R}^N)$. Then, the following are equivalent:

(i) The map $u$ satisfies that $\mathrm{rk}(Du) \leq 1$ on $\Omega$. Equivalently, there exist vector fields $\xi : \Omega \longrightarrow \mathbb{R}^N$ and $a : \Omega \longrightarrow \mathbb{R}^n$ with $a \in C^1(\Omega,\mathbb{R}^n)$ and $\xi \in C^1(\Omega \setminus \{a = 0\},\mathbb{R}^N)$ such that

$$\mathrm{D}u = \xi \otimes a, \quad \text{on } \Omega.$$

(ii) There exists Borel subsets $\{B_i\}_{i\in\mathbb{N}}$ of $\Omega$ such that

$$\Omega = \bigcup_{i=1}^{\infty} B_i$$

and each $B_i$ equals a non-empty connected open set with a (possibly empty) boundary portion, functions $\{f_i\}_{i\in\mathbb{N}} \in C^2(\Omega)$ and curves $\{\nu_i\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ such that, on

[1] This fact has been brought to our attention by Roger Moser.

each $B_i$ the map $u$ has the form

$$u = \nu_i \circ f_i, \quad \text{on } B_i. \tag{4.2.1}$$

Moreover, $|\nu_i'| \equiv 1$ on the interval $f_i(B_i)$, $\nu_i' \equiv 0$ on $\mathbb{R} \setminus f_i(B_i)$ and $\nu_i''$ exists everywhere on $f_i(B_i)$, interpreted as 1-sided derivative on $\partial f_i(B_i)$ (if $f_i(B_i)$ is not open). Also,

$$\left\{ \begin{array}{rcll} \mathrm{D}u & = & (\nu_i' \circ f_i) \otimes \mathrm{D}f_i \ , & \text{on } B_i, \\ \mathrm{D}^2 u & = & (\nu_i'' \circ f_i) \otimes \mathrm{D}f_i \otimes \mathrm{D}f_i \ + \ (\nu_i' \circ f_i) \otimes \mathrm{D}^2 f_i \ , & \text{on } B_i. \end{array} \right. \tag{4.2.2}$$

In addition, the local functions $(f_i)_1^\infty$ extend to a global function $f \in C^2(\Omega)$ with the same properties as above if $\Omega$ is contractible (namely, homotopically equivalent to a point).

We may now prove our first main result.

### 4.2.2 Proof of Theorem 4.1.2.

Suppose that $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is a solution to the nonlinear system (4.1.1) in $C^2(\Omega, \mathbb{R}^N)$ which in addition satisfies that $\mathrm{rk}(\mathrm{D}u) \leq 1$ in $\Omega$. Since $\{\mathrm{D}u = 0\}$ is closed, necessarily its complement in $\Omega$ which is $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ is open.

By invoking Theorem 4.2.1, we have that there exists a partition of the open subset $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ to countably many Borel sets $(B_i)_1^\infty$ with respective functions $(f_i)_1^\infty$ and curves $(\nu_i)_1^\infty$ as in the statement such that (4.2.1)-(4.2.2) hold true and in addition

$$\mathrm{D}f_i \neq 0 \quad \text{on } B_i, \ i \in \mathbb{N}.$$

Consequently, on each $B_i$ we have

$$\begin{aligned} [\![\mathrm{D}u]\!]^\perp \ &= \ [\![(\nu_i' \circ f_i) \otimes \mathrm{D}f_i]\!]^\perp \ = \ \mathrm{I} \ - \ \frac{(\nu_i' \circ f_i) \otimes (\nu_i' \circ f_i)}{|\nu_i' \circ f_i|^2}, \\ \Delta u \ &= \ (\nu_i'' \circ f_i)|\mathrm{D}f_i|^2 \ + \ (\nu_i' \circ f_i)\Delta f_i. \end{aligned}$$

Hence, (4.1.1) becomes

$$\left[\mathrm{I} \ - \ \frac{(\nu_i' \circ f_i) \otimes (\nu_i' \circ f_i)}{|\nu_i' \circ f_i|^2}\right] \Big( (\nu_i'' \circ f_i)|\mathrm{D}f_i|^2 \ + \ (\nu_i' \circ f_i)\Delta f_i \Big) = 0,$$

on $B_i$. Since $|\nu_i|^2 \equiv 1$ on $f_i(B_i)$, we have that $\nu_i''$ is orthogonal to $\nu_i'$ thereon and therefore the above equation reduces to

$$(\nu_i'' \circ f_i)|\mathrm{D}f_i|^2 = 0 \quad \text{on } B_i, \ i \in \mathbb{N}.$$

Therefore, $\nu_i$ is affine on the interval $f_i(B_i) \subseteq \mathbb{R}$ and as a result $u(B_i) = \nu_i(f_i(B_i))$ is contained in an affine line of $\mathbb{R}^N$, for each $i \in \mathbb{N}$. On the other hand, since

$$u(\Omega) \ = \ u\big(\{\mathrm{D}u = 0\}\big) \bigcup_{i\in\mathbb{N}} u(B_i)$$

and $u$ is constant on each connected component of the interior of $\{\mathrm{D}u = 0\}$, the conclusion ensues by the regularity of $u$ because $u\big(\{\mathrm{D}u = 0\}\big)$ is also contained in the previous union of affine lines. The result ensues. $\square$

Now we establish Corollary 4.1.4 by following similar lines to those of the respective result in [41].

### 4.2.3 Proof of Corollary 4.1.4.

Suppose $u$ is as in the statement of the corollary. By Theorem 4.2.1, there exists, a partition of $\Omega$ to Borel sets $\{B_i\}_{i\in\mathbb{N}}$, functions $f_i \in C^2(\Omega)$ and Lipschitz curves $\{\nu_i\}_{i\in\mathbb{N}} : \mathbb{R} \longrightarrow \mathbb{R}^N$ with $|\nu_i'| \equiv 1$ on $f_i(B_i)$, $|\nu_i'| \equiv 0$ on $\mathbb{R} \setminus f_i(B_i)$ and twice differentiable on $f_i(B_i)$, such that $u|_{B_i} = \nu_i \circ f_i$ and (4.2.2) holds as well. Since on each $B_i$ we have

$$|\mathrm{D}u| = \big|(\nu_i' \circ f_i) \otimes \mathrm{D}f_i\big| = |\mathrm{D}f_i|,$$

by (4.1.7) and the above, we obtain

$$\begin{aligned}\big((\nu_i' \circ f_i) \otimes \mathrm{D}f_i\big) \otimes \big((\nu_i' \circ f_i) \otimes \mathrm{D}f\big) : \Big[(\nu_i'' \circ f_i) \otimes \mathrm{D}f_i \otimes \mathrm{D}f_i \, + \, (\nu_i' \circ f_i) \otimes \mathrm{D}^2 f_i\Big] \\ + \, \frac{|\mathrm{D}f_i|^2}{p-2}\Big\{(\nu_i' \circ f_i)\, \Delta f_i \, + \, (\nu_i'' \circ f)|\mathrm{D}f_i|^2\Big\} \ = \ 0,\end{aligned}$$

on $B_i$. Since $\nu_i''$ is orthogonal to $\nu_i'$ and also $\nu_i'$ has unit length, the above reduces to

$$(\nu_i' \circ f_i)\Big[\mathrm{D}f_i \otimes Df_i : \mathrm{D}^2 f_i \, + \, \frac{|\mathrm{D}f_i|^2}{p-2}\, \Delta f_i\Big] \, + \, \frac{1}{p-2}(\nu_i'' \circ f_i)|\mathrm{D}f_i|^4 \ = \ 0,$$

on $B_i$. Again by orthogonality, the above is equivalent to the pair of independent systems

$$(\nu_i' \circ f_i)\Big[\mathrm{D}f_i \otimes Df_i : \mathrm{D}^2 f_i \, + \, \frac{|\mathrm{D}f_i|^2}{p-2}\, \Delta f_i\Big] \ = \ 0 \ , \quad (\nu_i'' \circ f_i)|\mathrm{D}f_i|^4 \ = \ 0,$$

on $B_i$. Since $|\nu_i'| \equiv 1$ of $f_i(B_i)$, it follows that $\Delta_p f_i = 0$ on $B_i$ and since $(B_i)_1^\infty$ is a partition of $\Omega$ of the form described in the statement, the result ensues by invoking Theorem 4.1.2. $\square$

# Chapter 5

# Phase separation of n-dimensional $\infty$-Harmonic mappings

## 5.1 Introduction

In this chapter we present a submitted single authored paper [1]. In this paper we study the phase separation of $n$-dimensional $\infty$-Harmonic mappings $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ by which we mean the classical solutions to the $\infty$-Laplace system

$$\Delta_\infty u \;:=\; \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes I\Big) : \mathrm{D}^2 u \;=\; 0, \quad \text{on } \Omega, \tag{5.1.1}$$

where $n, N$ are integers such that $N \geq n \geq 2$ and $\Omega$ an open subset of $\mathbb{R}^n$. Here, for the map $u$ with components $(u_1, ..., u_N)^\top$ the notation $\mathrm{D}u$ symbolises the gradient matrix

$$\mathrm{D}u(x) = \big(\mathrm{D}_i u_\alpha(x)\big)_{i=1...n}^{\alpha=1...N} \in \mathbb{R}^{N\times n}\ , \quad \mathrm{D}_i \equiv \partial/\partial x_i, \tag{5.1.2}$$

and for any $X \in \mathbb{R}^{N\times n}$, $[\![X]\!]^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of linear map $X : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![X]\!]^\perp := \mathrm{Proj}_{\mathrm{R}(X)^\perp}. \tag{5.1.3}$$

In index form, the system (5.1.1) reads

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n} \Big(\mathrm{D}_i u_\alpha\, \mathrm{D}_j u_\beta + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp_{\alpha\beta}\, \delta_{ij}\Big) \mathrm{D}^2_{ij} u_\beta \;=\; 0, \quad \alpha = 1, ..., N.$$

Our general notation will be either self-explanatory, or otherwise standard as e.g. in [30, 33, 56]. Throughout this paper we reserve $n, N \in \mathbb{N}$ for the dimensions of Euclidean spaces and $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$.

Speaking about the system (5.1.1), we would like to mention that the system

(5.1.1) is called the"$\infty$-Laplacian" and it arises as a sort of Euler-Lagrange PDE of vectorial variational problems in $L^\infty$ for the supremal functional

$$E_\infty(u,\mathcal{O}) := \|H(\mathrm{D}u)\|_{L^\infty(\mathcal{O})}, \quad u \in W^{1,\infty}_{loc}(\Omega,\mathbb{R}^N),\ \mathcal{O} \Subset \Omega, \tag{5.1.4}$$

when the Hamiltonian (the non-negative function $H \in C^2\big(\mathbb{R}^{N\times n}\big)$) is chosen to be $H(\mathrm{D}u) = \frac{1}{2}\big|\mathrm{D}u\big|^2$, where $\big|.\big|$ is the Euclidean norm on the space $\mathbb{R}^{N\times n}$. the $\infty$-Laplacian is a special case of the system

$$\Delta_\infty u \;:=\; \Big(H_P \otimes H_P + H[\![H_P]\!]^\perp H_{PP}\Big)(\mathrm{D}u) : \mathrm{D}^2 u \;=\; 0, \tag{5.1.5}$$

which was first formally derived by Katzourakis [37] as the limit of the Euler-Lagrange equations of the integral functionals $E_m(u,\Omega) \;:=\; \int_\Omega (H(\mathrm{D}u))^p$ as $p \longrightarrow \infty$.

The structure of weak solutions are complicated to understand even though the theory of weak solutions has witnessed a significant development so far, particularly the new theory of "$\mathcal{D}$-solutions" introduced by Katzourakis [48], which applies to nonlinear PDE systems of any order and allows for merely measurable maps to be rigorously interpreted and studied as solutions of PDE systems fully nonlinear and with discontinuous coefficients. In this paper, we restrict our attention to classical solutions which might be helpful to imagine and understand the behavior and the structure of the weak solutions.

For the $\infty$-Laplace system (5.1.1) the orthogonal projection on the orthogonal complement of the range, $[\![\mathrm{D}u]\!]^\perp$, coincides with the projection on the geometric normal space of the image of the solution.

It is worth noting that $\infty$-Harmonic maps are affine when $n = 1$ since in this case the system (5.1.1) simplifies to

$$\Delta_\infty u \;=\; \Big(u' \otimes u'\Big)u'' + |u'|^2\Big(I - \frac{u' \otimes u'}{|u'|^2}\Big)u'' \;=\; |u'|^2 u'', \tag{5.1.6}$$

and hence no interesting phenomena arise when $n = 1$.

For the case $N = 1$, the system (5.1.1) reduced to the single $\infty$-Laplace PDE

$$\Delta_\infty u \;:=\; \Big(\mathrm{D}u \otimes \mathrm{D}u\Big) : \mathrm{D}^2 u \;=\; 0, \tag{5.1.7}$$

since the normal coefficient $|\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp$ vanishes identically. This also happen when $u$ is submersion. The single $\infty$-Laplacian PDE (5.1.7), and the related scalar $L^\infty$-variational problems, started being studied in the '60s by Aronsson in [8, 9]. Aronsson studied solutions $u \in C^2(\mathbb{R}^n)$ of what we now call "Aronsson's PDE" , in the case $N = 1$ and the Lagrangian $\mathscr{L}$ is $C^1$:

$$A_\infty u \;:=\; \mathrm{D}\big(\mathscr{L}(\cdot,u,\mathrm{D}u)\big)\mathscr{L}_P(\cdot,u,\mathrm{D}u) \;=\; 0. \tag{5.1.8}$$

which is the equivalent of the Euler-Lagrange equation for supremal functionals $E_\infty(u,\Omega) = \underset{x\in\Omega\subseteq\mathbb{R}^n}{\operatorname{ess\,sup}} \mathscr{L}(x,u(x),\mathrm{D}u(x))$. In Aronsson's PDE above, the subscript denotes the gradient of $\mathscr{L}(x,\eta,P)$ with respect to $P$ and, as it is customary, the equation is written for smooth solutions.

Today it is being studied in the context of Viscosity Solutions (see for example Crandall [5], Barron-Evans-Jensen [14] and Katzourakis [42]). In particular, for $N=1$ and $H(p) := \frac{1}{2}|P|^2$, there is a triple equivalence among viscosity solutions $u \in C^{0,1}(\mathbb{R}^n)$ of the $\infty$-Laplacian (5.1.7), absolute minimizers of $E_\infty(u,\Omega) = \frac{1}{2}\|\mathrm{D}u\|^2_{L^\infty(\Omega)}$ and the so-called optimal Lipschitz extensions, namely functions $u \in C^{0,1}(\mathbb{R}^n)$ satisfying $\mathrm{Lip}(u,\Omega) = \mathrm{Lip}(u,\partial\Omega)$ for all $\Omega \Subset \mathbb{R}^n$, where Lip is the Lipschitz functional

$$\mathrm{Lip}(u,K) = \sup_{x,y\in K,\, x\neq y} \frac{|u(x)-u(y)|}{\mathrm{dist}(x,y)}, \quad K \subseteq \mathbb{R}^n.$$

The vectorial case $N \geq 2$ first arose in the early 2010s in the work of Katzourakis [37]. Due to both the mathematical significance as well as the importance for applications particularly in Data Assimilation, the area is developing very rapidly (see [4, 13, 28, 38–41, 44, 46–55]).

In a joint work with Katzourakis and Ayanbayev [2], among other results, we have proved that the image $u(\Omega)$ of a solution $u \in C^2(\Omega,\mathbb{R}^N)$ to the nonlinear system (5.1.1) satisfying that the rank of its gradient matrix is at most one, $\mathrm{rk}(\mathrm{D}u) \leq 1$ in $\Omega$, is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections). Hence the component $[\![\mathrm{D}u]\!]^\perp \Delta u$ of $\Delta_\infty$ forces flatness of the image of solutions.

Interestingly, even when the operator $\Delta_\infty$ is applied to $C^\infty$ maps, which may even be solutions, (5.1.1) may have discontinuous coefficients. This further difficulty of the vectorial case is not present in the scalar case. As an example consider

$$u(x,y) := e^{ix} - e^{iy}, \qquad u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2. \tag{5.1.9}$$

Katzourakis has showed in [37] that even though (5.1.9) is a smooth solution of the $\infty$-Laplacian near the origin, still the coefficient $|\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp$ of (5.1.1) is discontinuous. This is because when the dimension of the image changes, the projection $[\![\mathrm{D}u]\!]^\perp$ "jumps". More precisely, for (5.1.9) the domain splits to three components according to the $\mathrm{rk}(\mathrm{D}u)$, the "2D phase $\Omega_2$", whereon $u$ is essentially 2D, the"1D phase $\Omega_1$", whereon $u$ is essentially $1D$ (which is empty for (5.1.9))and the "interface $S$" where the coefficients of $\Delta_\infty$ become discontinuous.

In [38] Katzourakis constructed additional examples, which are more intricate than (5.1.9), namely smooth 2D $\infty$-Harmonic maps whose interfaces have triple junctions and corners and are given by the explicit formula

$$u(x,y) := \int_y^x e^{iK(t)}dt. \tag{5.1.10}$$

Indeed, for $K \in C^1(\mathbb{R},\mathbb{R})$ with $\|K\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}$, (5.1.10) defines $C^2$ $\infty$-Harmonic map whose phases are as shown in Figures 1(a), 1(b) below, when $K$ qualitatively behaves as shown in the Figures 2(a), 2(b) respectively. Also, on the phase $\Omega_1$ the $\infty$-Harmonic map (5.1.10) is given by a scalar $\infty$-Harmonic function times a constant vector, and on the phase $\Omega_2$ it is a solution of the vectorial Eikonal equation.

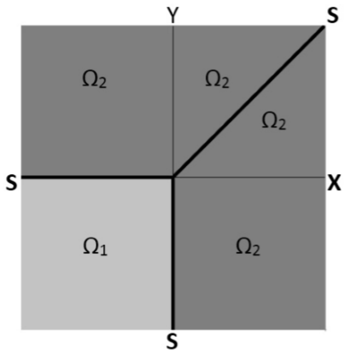


Figure 1(a).

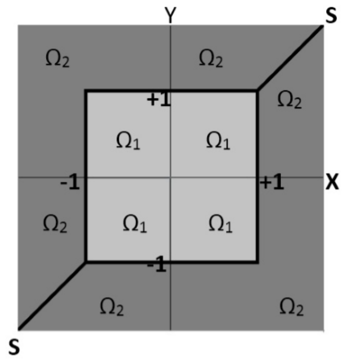


Figure 1(b).

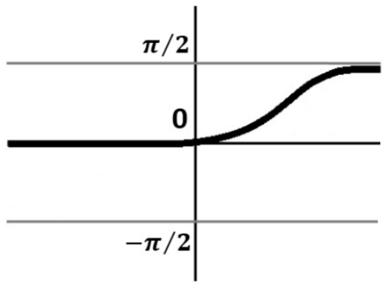


Figure 2(a).

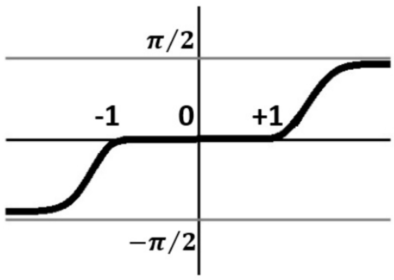


Figure 2(b).

One of the interesting results in [41] was that this phase separation is a general phenomena for smooth 2D $\infty$-Harmonic maps. Therein the author proves that on each phase the dimension of the tangent space is constant and these phases are separated by interfaces whereon $[\![\mathrm{D}u]\!]^\perp$ becomes discontinuous. Accordingly the author established the next result:

### 5.1.1 Theorem [Structure of 2D $\infty$-Harmonic maps, cf. [41]]

Let $u : \mathbb{R}^2 \supseteq \Omega \longrightarrow \mathbb{R}^N$ be an $\infty$-Harmonic map in $C^2\big(\Omega, \mathbb{R}^N\big)$, that is a solution to (5.1.1). Let also $N \geq 2$. Then, there exist disjoint open sets $\Omega_1$, $\Omega_2 \subseteq \Omega$, and a closed nowhere dense set $S$ such that $\Omega = \Omega_1 \bigcup S \bigcup \Omega_2$ and:

(i) On $\Omega_2$ we have $\mathrm{rk}(\mathrm{D}u) = 2$ , and the map $u : \Omega_2 \longrightarrow \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$|\mathrm{D}u|^2 = C^2 > 0. \tag{5.1.11}$$

The constant $C$ may vary on different connected components of $\Omega_2$.

(ii) On $\Omega_1$ we have $\mathrm{rk}(\mathrm{D}u) = 1$ and the map $u : \Omega_1 \longrightarrow \mathbb{R}^N$ is given by an essentially scalar $\infty$-Harmonic function $f : \Omega_1 \longrightarrow \mathbb{R}$:

$$u = a + \xi f, \ \ \Delta_\infty f = 0, \ \ a \in \mathbb{R}^N, \ \ \xi \in \mathbb{S}^{N-1}. \tag{5.1.12}$$

The vectors $a, \xi$ may vary on different connected components of $\Omega_1$.

(iii) On S, $|\mathrm{D}u|^2$ is constant and also $\mathrm{rk}(\mathrm{D}u) = 1$. Moreover if $S = \partial\Omega_1 \cap \partial\Omega_2$ (that is if both the 1D and 2D phases coexist) then $u : S \longrightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$u = a + \xi f, \ \ |\mathrm{D}f|^2 = C^2 > 0, \ \ a \in \mathbb{R}^N, \ \ \xi \in \mathbb{S}^{N-1}. \tag{5.1.13}$$

The main result of this paper is to generalise these results to higher dimension $N \geq n \geq 2$. The principle result in this paper in the following extension of theorem 5.1.1:

### 5.1.2 Theorem[ Phase separation of $n$-dimensional $\infty$-Harmonic mappings]

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $u : \Omega \longrightarrow \mathbb{R}^N$, $N \geq n \geq 2$, be an $\infty$-Harmonic map in $C^2\big(\Omega, \mathbb{R}^N\big)$, that is a solution to the $\infty$-Laplace system (5.1.1). Then, there exist disjoint open sets $\big(\Omega_r\big)_{r=1}^{n} \subseteq \Omega$, and a closed nowhere dense set $S$ such that $\Omega = S \bigcup \Big( \bigcup\limits_{i=1}^{n} \Omega_i \Big)$ such that:

(i) On $\Omega_n$ we have $\mathrm{rk}(\mathrm{D}u) \equiv n$ and the map $u : \Omega_n \longrightarrow \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$|\mathrm{D}u|^2 = C^2 > 0. \tag{5.1.14}$$

The constant $C$ may vary on different connected components of $\Omega_n$.

(ii) On $\Omega_r$ we have $\mathrm{rk}(\mathrm{D}u) \equiv r$, where $r$ is integer in $\{2, 3, 4, ..., (n-1)\}$, and $|\mathrm{D}u(\gamma(t))|$ is constant along trajectories of the parametric gradient flow of $u(\gamma(t,$

$x, \xi))$

$$\begin{cases} \dot{\gamma}(t,x,\xi) = \xi^\top \mathrm{D}u\big(\gamma(t,x,\xi)\big), & t \in (-\varepsilon, 0) \bigcup (0, \varepsilon), \\ \gamma(0,x,\xi) = x, \end{cases} \tag{5.1.15}$$

where $\xi \in \mathbb{S}^{N-1}$, and $\xi \notin N\big(\mathrm{D}u\big(\gamma(t,x,\xi)\big)^\top\big)$.

(iii) On $\Omega_1$ we have $\mathrm{rk}(\mathrm{D}u) \le 1$ and the map $u : \Omega_1 \longrightarrow \mathbb{R}^N$ is given by an essentially scalar $\infty$-Harmonic function $f : \Omega_1 \longrightarrow \mathbb{R}$:

$$u = a + \xi f, \ \ \Delta_\infty f = 0, \ \ a \in \mathbb{R}^N, \ \ \xi \in \mathbb{S}^{N-1}. \tag{5.1.16}$$

The vectors $a, \xi$ may vary on different connected components of $\Omega_1$.

(iv) On S, when $S \supseteq \partial\Omega_p \cap \partial\Omega_q = \emptyset$ for all $p$ and $q$ such that $2 \le p < q \le n-1$, then we have that $|\mathrm{D}u|^2$ is constant and also $\mathrm{rk}(\mathrm{D}u) \equiv 1$. Moreover on

$$\partial\Omega_1 \cap \partial\Omega_n \subseteq S,$$

(when both 1D and $n$D phases coexist), we have that $u : S \longrightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$u = a + \xi f, \ \ |\mathrm{D}f|^2 = C^2 > 0, \ \ a \in \mathbb{R}^N, \ \ \xi \in \mathbb{S}^{N-1}. \tag{5.1.17}$$

On the other hand, if there exist some $r$ and $q$ such that $2 \le r < q \le n-1$, then on $S \supseteq \partial\Omega_r \cap \partial\Omega_q \neq \emptyset$ (when both $r$D and $q$D phases coexist), we have that $\mathrm{rk}(\mathrm{D}u) \equiv r$ and we have same result as in (ii) above.

## 5.2 Preliminaries

For the convenience of the reader, in this section we recall without proof a theorem of rigidity of rank-one maps, proved in [41], which will be used in the proof of the main result of this paper in section 5.3. We also recall the proposition of Gradient flows for tangentially $\infty$-Harmonic maps which introduced in [37] and its improved modification lemma in [40].

### 5.2.1 Theorem [Rigidity of Rank-One maps, cf. [41]]

Suppose $\Omega \subseteq \mathbb{R}^n$ is open and contractible and $u : \Omega \longrightarrow \mathbb{R}^N$ is in $C^2\big(\Omega, \mathbb{R}^N\big)$. Then the following are equivalent:

(i) $u$ is a Rank-One map, that is $rk(\mathrm{D}u) \leq 1$ on $\Omega$ or equivalently there exist maps $\xi : \Omega \longrightarrow \mathbb{R}^N$ and $w : \Omega \longrightarrow \mathbb{R}^n$ with $w \in C^1(\Omega, \mathbb{R}^n)$ and $\xi \in C^1\big(\Omega \setminus \{w = 0\}, \mathbb{R}^N\big)$ such that $\mathrm{D}u = \xi \otimes w$.

(ii) There exist $f \in C^2(\Omega, \mathbb{R})$, a partition $\big\{B_i\big\}_{i\in\mathbb{N}}$ of $\Omega$ into Borel sets where each $B_i$ equals a connected open set with a boundary portion and Lipschitz curves $\big\{\mathcal{V}^i\big\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{\mathrm{loc}}(\Omega)^N$ such that on each $B_i$ u equals the composition of $\mathcal{V}^i$ with $f$:

$$u \,=\, \mathcal{V}^i \circ f \quad , \qquad \text{on}\, B_i \subseteq \Omega. \tag{5.2.1}$$

Moreover, $|\dot{\mathcal{V}}^i| \equiv 1$ on $f(B_i)$, $\dot{\mathcal{V}}^i \equiv 0$ on $\mathbb{R} \setminus f(B_i)$ and there exist $\ddot{\mathcal{V}}^i$ on $f(B_i)$, interpreted as 1-sided on $\partial f(B_i)$, if any. Also,

$$\mathrm{D}u \,=\, (\mathcal{V}^i \circ f) \otimes \mathrm{D}f \quad , \qquad \text{on}\, B_i \subseteq \Omega, \tag{5.2.2}$$

and the image $u(\Omega)$ is an 1-rectifiable subset of $\mathbb{R}^N$:

$$u(\Omega) \,=\, \bigcup_{i=1}^{\infty} \mathcal{V}^i(f(B_i)) \subseteq \mathbb{R}^N. \tag{5.2.3}$$

### 5.2.2 Proposition [Gradient flows for tangentially $\infty$-Harmonic maps, cf. [37]]

Let $u \in C^2\big(\mathbb{R}^n, \mathbb{R}^N\big)$. Then, $\mathrm{D}u\,\mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big)^{\top} \,=\, 0$ on $\Omega \Subset \mathbb{R}^n$ if and only if the flow map $\gamma : \mathbb{R} \times \Xi \longrightarrow \Omega$ with $\Xi := \{(x, \xi) \mid \xi^{\top}\mathrm{D}u(x) \neq 0\} \subseteq \Omega \times \mathbb{S}^{N-1}$ of

$$\left\{ \begin{array}{l} \dot{\gamma}(t, x, \xi) = \xi^{\top}\mathrm{D}u\big(\gamma(t, x, \xi)\big), \\ \gamma(0, x, \xi) = x, \end{array} \right. \tag{5.2.4}$$

satisfies along trajectories

$$\left\{ \begin{array}{l} \big|\mathrm{D}u\big(\gamma(t, x, \xi)\big)\big| = \big|\mathrm{D}u\big(\gamma(x)\big)\big|, \quad t \in \mathbb{R} \\ t \longmapsto \xi^{\top}u\big(\gamma(t, x, \xi)\big) \;\; \text{is increasing.} \end{array} \right. \tag{5.2.5}$$

The following lemma is improved modification of proposition 5.2.2

### 5.2.3 Lemma [cf. [40]]

Let $u:\mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be in $u \in C^2\big(\Omega, \mathbb{R}^N\big)$. Consider the gradient flow

$$\begin{cases} \dot{\gamma}(t,x,\xi) = \Big(\frac{|\mathrm{D}u|^2}{|\xi^\top \mathrm{D}u|^2}\xi^\top \mathrm{D}u\Big)\big(\gamma(t,x,\xi)\big), \quad t \neq 0 \\ \gamma(0,x,\xi) = x, \end{cases} \tag{5.2.6}$$

for $x \in \Omega$, $\xi \in \mathbb{S}^{N-1} \setminus [\![\mathrm{D}u]\!]^\perp$. Then, we have the differential identities

$$\frac{d}{dt}\big(\frac{1}{2}|\mathrm{D}u\big(\gamma(t,x,\xi)\big)|^2\big) = \Big(\frac{|\mathrm{D}u|^2}{|\xi^\top \mathrm{D}u|^2}\xi^\top \mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u\Big)\big(\gamma(t,x,\xi)\big), \tag{5.2.7}$$

$$\frac{d}{dt}\big(\xi^\top \mathrm{D}u\big(\gamma(t,x,\xi)\big)\big) = |\mathrm{D}u\big(\gamma(t,x,\xi)\big)|^2, \tag{5.2.8}$$

which imply $\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u = 0$ on $\Omega$ if and only if $|\mathrm{D}u\big(\gamma(t,x,\xi)\big)|$ is constant along trajectories $\gamma$ and $t \longmapsto \xi^\top u\big(\gamma(t,x,\xi)\big)$ is affine.

## 5.3 Proof of the main result

In this section we present the proof of the main result of this paper, theorem 5.1.2

### 5.3.1 Proof of Theorem 5.1.2

Let $u \in C^2\big(\Omega, \mathbb{R}^N\big)$ be a solution to the $\infty$-Laplace system (5.1.1). Note that the PDE system can be decoupled to the following systems

$$\mathrm{D}u\,\mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big)^\top \;=\; 0, \tag{5.3.1}$$

$$|\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp \Delta u \;=\; 0. \tag{5.3.2}$$

Set $\Omega_1 := \ \mathrm{int}\{\mathrm{rk}(\mathrm{D}u) \leq 1\}$, $\Omega_r := \ \mathrm{int}\{\mathrm{rk}(\mathrm{D}u) \equiv r\}$ and $\Omega_n := \{\mathrm{rk}(\mathrm{D}u) \equiv n\}$. Then:

On $\Omega_n$ we have $\mathrm{rk}(\mathrm{D}u) = \dim(\Omega_n \subseteq \mathbb{R}^n) = n$. Since $N \geq n$ and hence the map $u:\Omega_n \longrightarrow \mathbb{R}^N$ is an immersion (because its derivative has constant rank equal to the dimension of the domain, the arguments in the case of $\mathrm{rk}(\mathrm{D}u) \equiv n$ follows the same lines as in [41, theorem 1.1] but we provide them for the sake of completeness). This means that $\mathrm{D}u$ is injective. Thus, $\mathrm{D}u(x)$ possesses a left inverse $(\mathrm{D}u(x))^{-1}$ for all $x \in \Omega_n$. Therefore, the system (5.3.1) implies

$$(\mathrm{D}u)^{-1}\mathrm{D}u\,\mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big)^\top \;=\; 0, \tag{5.3.3}$$

and hence $\mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big) \,=\, 0$ on $\Omega_n$, or equivalently

$$|\mathrm{D}u|^2 \,=\, C^2, \tag{5.3.4}$$

on each connected component of $\Omega_n$. Moreover, (5.3.4) holds on the common boundary of $\Omega_n$ with any other component of the partition.

On $\Omega_r$ we have $\mathrm{rk}(\mathrm{D}u) \,\equiv\, r$, where $r$ is an integer in $\{2, 3, 4, ..., (n-1)\}$. Consider the gradient flow (5.2.6). Giving that (5.3.1) holds, then by the proposition of Gradient flows for tangentially $\infty$-Harmonic maps [37] and its improved modification lemma [40] which we recalled in the preliminaries, we must have that $|\mathrm{D}u\big(\gamma(t,x,\xi)\big)|$ is constant along trajectories $\gamma$ and $t \longmapsto \xi^\top u\big(\gamma(t,x,\xi)\big)$ is affine. Moreover, if there exist some $r$ and $q$ such that $2 \leq r < q \leq n-1$, and $\partial\Omega_r \cap \partial\Omega_q \neq \emptyset$. Then a similar thing happen on $\partial\Omega_r \cap \partial\Omega_q \subseteq S$ (when both $r$D and $q$D phases coexist), because in this case we also have that $\mathrm{rk}(\mathrm{D}u) \,\equiv\, r$ and we have same result as above.

The proof of the remaining claims of the theorem is very similar to [41, theorem 1.1], which we give below for the sake of completeness:

On $\Omega_1 := \,\mathrm{int}\{rk(\mathrm{D}u) \,\leq\, 1\}$ we have $rk(\mathrm{D}u) \,\leq\, 1$. Hence there exist vector fields $\xi : \mathbb{R}^n \supseteq \Omega_1 \longrightarrow \mathbb{R}^N$ and $w : \mathbb{R}^n \supseteq \Omega_1 \longrightarrow \mathbb{R}^n$ such that $\mathrm{D}u \,=\, \xi \otimes w$. Suppose first that $\Omega_1$ is contractible. Then, by the Rigidity Theorem 5.2.1, there exist a function $f \in C^2(\Omega_1, \mathbb{R})$, a partition of $\Omega_1$ to Borel sets $\big\{B_i\big\}_{i\in\mathbb{N}}$ and Lipschitz curves $\big\{\mathcal{V}^i\big\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{\mathrm{loc}}(\Omega)^N$ with $|\dot{\mathcal{V}}^i| \,\equiv\, 1$ on $f(B_i)$, $|\dot{\mathcal{V}}^i| \,\equiv\, 0$ on $\mathbb{R}\backslash f(B_i)$ twice differentiable on $f(B_i)$, such that $u \,=\, \mathcal{V}^i \circ f$ on each $B_i \subseteq \Omega$ and hence $\mathrm{D}u \,=\, (\mathcal{V}^i \circ f) \otimes \mathrm{D}f$. By (5.3.1) , we obtain

$$\Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) \otimes \Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) : \\ : \Big[\big(\ddot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f \otimes \mathrm{D}f + \big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}^2 f\Big] \,=\, 0, \tag{5.3.5}$$

on $B_i \subseteq \Omega_1$. Since $|\dot{\mathcal{V}}^i| \equiv 1$ on $f(B_i)$, we have that $\ddot{\mathcal{V}}^i$ is normal to $\dot{\mathcal{V}}^i$ and hence

$$\Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) \otimes \Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) : \Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}^2 f\Big) \,=\, 0, \tag{5.3.6}$$

on $B_i \subseteq \Omega_1$. Hence, by using again that $|\dot{\mathcal{V}}^i|^2 \equiv 1$ on $f(B_i)$ we get

$$\Big(\mathrm{D}f \otimes \mathrm{D}f : \mathrm{D}^2 f\Big)\big(\dot{\mathcal{V}}^i \circ f\big) \,=\, 0, \tag{5.3.7}$$

on $B_i \subseteq \Omega_1$. Thus, $\Delta_\infty f \,=\, 0$ on $B_i$. By (5.3.2) and again since $|\dot{\mathcal{V}}^i|^2 \equiv 1$ on $f(B_i)$, we have $[\![\mathrm{D}u]\!]^\perp \,=\, [\![\dot{\mathcal{V}}^i \circ f]\!]^\perp$ and hence

$$|\mathrm{D}f|^2 [\![\dot{\mathcal{V}}^i \circ f]\!]^\perp \mathrm{Div}\Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) \,=\, 0, \tag{5.3.8}$$

on $B_i \subseteq \Omega_1$. Hence,

$$|\mathrm{D}f|^2 [\![\dot{\mathcal{V}}^i \circ f]\!]^\perp \Big( \big(\ddot{\mathcal{V}}^i \circ f\big) |\mathrm{D}f|^2 + \big(\dot{\mathcal{V}}^i \circ f\big) \Delta f \Big) \;=\; 0, \tag{5.3.9}$$

on $B_i$, which by using once again $|\dot{\mathcal{V}}^i|^2 \equiv 1$ gives

$$|\mathrm{D}f|^4 \big(\ddot{\mathcal{V}}^i \circ f\big) \;=\; 0, \tag{5.3.10}$$

on $B_i$. Since $\Delta_\infty f \;=\; 0$ on $B_i$ and $\Omega_1 \;=\; \cup_1^\infty B_i$, $f$ is $\infty$-Harmonic on $\Omega_1$. Thus, by Aronsson's theorem in [9] , either $|\mathrm{D}f| > 0$ or $|\mathrm{D}f| \equiv 0$ on $\Omega_1$.

If the first alternative holds, then by (5.3.10) we have $\ddot{\mathcal{V}}^i \equiv 0$ on $f(B_i)$ for all $i$ and hence, $\mathcal{V}^i$ is affine on $f(B_i)$, that is $\mathcal{V}^i \;=\; t\xi^i + a^i$ for some $|\xi^i| \;=\; 1, a^i \in \mathbb{R}^N$. Thus, since $u \;=\; \mathcal{V}^i \circ f$ and $u \in C^2\big(\Omega_1, \mathbb{R}^N\big)$, all $\xi^i$ and all $a^i$ coincide and consequently $u \;=\; \xi f + a$ for $\xi \in \mathbb{S}^{N-1}, a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1, \mathbb{R})$.

If the second alternative holds, then $f$ is constant on $\Omega_1$ and hence, by the representation $u \;=\; \mathcal{V}^i \circ f$, $u$ is piecewise constant on each $B_i$. Since $u \in C^2\big(\Omega_1, \mathbb{R}^N\big)$ and $\Omega_1 \;=\; \cup_i^\infty B_i$, necessarily $u$ is constant on $\Omega_1$. But then $|\mathrm{D}u|_{\Omega_2}| \;=\; |\mathrm{D}f|_{\mathscr{S}}| \;=\; 0$ and necessarily $\Omega_2 \;=\; \phi$. Hence, $|\mathrm{D}u| \;\equiv\; 0$ on $\Omega$, that is $u$ is affine on each of the connected components of $\Omega$.

If $\Omega_1$is not contractible, cover it with balls $\{\mathbb{B}_m\}_{m\in\mathbb{N}}$ and apply the previous argument. Hence, on each $\mathbb{B}_m$, we have $u \;=\; \xi^m f^m + a^m, \xi^m \in \mathbb{S}^{N-1}, a^m \in \mathbb{R}^N$ and $f^m \in C^2(\mathbb{B}_m, \mathbb{R})$ with $\Delta_\infty f^m \;=\; 0$ on $\mathbb{B}_m$ and hence either $|\mathrm{D}f^m| > 0$ or $|\mathrm{D}f^m| \equiv 0$. Since $C^2\big(\Omega_1, \mathbb{R}^N\big)$, on the other overlaps of the balls the different expressions of $u$ must coincide and hence, we obtain $u \;=\; \xi f + a$ for $\xi \in \mathbb{S}^{N-1}, a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1, \mathbb{R})$ where $\xi$ and $a$ may vary on different connected components of $\Omega_1$. The theorem follows.

□

# Chapter 6

# Conclusions and future work

## 6.1 Conclusions

We would like to conclude this thesis by mention that the work included in the papers presented in the chapters of this thesis is an original work. This work consists of new progress in the field of non-divergence systems of nonlinear PDEs. The new results are varied to include: introduce new conditions, relaxe and advance existed conditions. Some of them improve previous theorems to make them valid in higher dimensions/vectorial cases. The thesis is a collection of four papers, the first two of them are joint work with my supervisor Dr. N. Katzourakis. The third paper is a joint paper with my supervisor Dr. N. Katzourakis and my colleague B. Ayanbayev. While the fourth paper is a single authored work.

The main result of the first paper, which we presented in Chapter 2 of this thesis, is that we introduce a new notion of ellipticity for the fully nonlinear first order elliptic system

$$F(\cdot, \mathrm{D}u) \,=\, f, \quad \text{a.e. on } \mathbb{R}^n.$$

This new notion is strictly weaker than a previous one introduced in [43]. Our new ellipticity notion allowing for more general nonlinearities $F$ to be considered. We refer to our new hypothesis of ellipticity as the "AK-Condition", which states that if we have an elliptic reference linear map $\mathrm{A} \,:\, \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$, then we say that a Carathéodory map $F \,:\, \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ is elliptic with respect to A when there exists a positive function $\alpha$ with $\alpha, 1/\alpha \in L^\infty(\mathbb{R}^n)$ and $\beta, \gamma > 0$ with $\beta + \gamma < 1$ such that

$$\left|\alpha(x)\Big[F(x, X+Y) - F(x,Y)\Big] \,-\, \mathrm{A} : X\right| \,\leq\, \beta\,\nu(\mathrm{A})|X| \,+\, \gamma\,|\mathrm{A} : X|,$$

for all $X, Y \in \mathbb{R}^{Nn}$ and a.e. $x \in \mathbb{R}^n$. Here $\nu(\mathrm{A})$ is the ellipticity constant of A.

The main outcome of the second paper, which we presented in Chapter 3 of this thesis, is that we prove the existence of vectorial Absolute Minimisers with given boundary values to the supremal functional

$$E_\infty(u,\Omega') := \operatorname*{ess\,sup}_{x\in\Omega'} \mathscr{L}\left(x, u(x), \mathrm{D}u(x)\right), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N),\ \Omega' \Subset \Omega,$$

applied to maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$, $N \in \mathbb{N}$.

We studying the vectorial case $N \geq 2$ but in one spatial dimension. The existence of an absolutely minimising *generalised solution* was proved in [47] , together with extra partial regularity and approximation properties. What makes our results distinguishable from the previous results in [47] is that we are obtaining existence under the weakest possible assumptions. The main result of the paper is the theorem of" Existence of vectorial Absolute Minimisers", which states that if $\Omega \subseteq \mathbb{R}$ is bounded open interval and

$$\mathscr{L} \;:\; \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow [0,\infty),$$

is a given continuous function with $N \in \mathbb{N}$. We assume that:

1. For each $(x,\eta) \in \overline{\Omega} \times \mathbb{R}^N$, the function $P \longmapsto \mathscr{L}(x,\eta,P)$ is level-convex, that is for each $t \geq 0$ the sub-level set

$$\left\{P \in \mathbb{R}^N \,:\, \mathscr{L}(x,\eta,P) \leq t\right\}$$

   is a convex set in $\mathbb{R}^N$.

2. there exist non-negative constants $C_1, C_2, C_3$, and $0 < q \leq r < +\infty$ and a positive locally bounded function $h : \mathbb{R} \times \mathbb{R}^N \longrightarrow [0,+\infty)$ such that for all $(x,\eta,P) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N$

$$C_1|P|^q - C_2 \leq \mathscr{L}(x,\eta,P) \leq h(x,\eta)|P|^r + C_3.$$

Then, for any affine map $b : \mathbb{R} \longrightarrow \mathbb{R}^N$, there exist a vectorial Absolute Minimiser $u^\infty \in W^{1,\infty}_b(\Omega, \mathbb{R}^N)$ of the supremal functional mentioned above.

In the third paper which we presented in Chapter 4 of this thesis, we introduce and prove new theorems that study the PDE system of vanishing normal projection of the Laplacian for $C^2$ maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$:

$$[\![\mathrm{D}u]\!]^\perp \Delta u = 0 \quad \text{in } \Omega.$$

We are showing that the image of a solution $u$ is piecewise affine if the rank of $\mathrm{D}u$ is equal to one. The main result is theorem of rigidity and flatness of rank-one maps with tangential Laplacian, which states that if $\Omega \subseteq \mathbb{R}^n$ is an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a solution to the nonlinear system $[\![\mathrm{D}u]\!]^\perp \Delta u = 0$ in $\Omega$, satisfying that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections).

As a consequence we obtain corresponding flatness results for $p$-Harmonic maps, $p \in [2, \infty]$.

In the fourth paper which we presented in Chapter 5 of this thesis, we introduce and prove a modified version of the theorem of the structure of 2D $\infty$-Harmonic maps introduced in [41]. It was one of the interesting results in [41] shows that the phase separation is a general phenomena for smooth 2D $\infty$-Harmonic maps. We advanced this theorem by introducing a new version of it studying the phase separation of $n$-dimensional $\infty$-Harmonic mappings. The main result of this paper which generalise the results of [41] to higher dimensions, is the theorem that we refer to it by " Phase separation of $n$-dimensional $\infty$-Harmonic mappings", which states that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, and let $u : \Omega \longrightarrow \mathbb{R}^N$, $N \geq n \geq 2$, be an $\infty$-Harmonic map in $C^2\big(\Omega, \mathbb{R}^N\big)$, that is a solution to the $\infty$-Laplace system

$$\Delta_\infty u \;:=\; \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes I\Big) : \mathrm{D}^2 u \;=\; 0, \quad \text{on } \Omega.$$

Then, there exist disjoint open sets $\big(\Omega_r\big)_{r=1}^{n} \subseteq \Omega$, and a closed nowhere dense set $S$ such that $\Omega \;=\; S \bigcup \Big(\bigcup\limits_{i=1}^{n} \Omega_i\Big)$ such that:

(i) On $\Omega_n$ we have $\mathrm{rk}(\mathrm{D}u) \;\equiv\; n$ and the map $u : \Omega_n \longrightarrow \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$|\mathrm{D}u|^2 = C^2 > 0.$$

The constant $C$ may vary on different connected components of $\Omega_n$.

(ii) On $\Omega_r$ we have $\mathrm{rk}(\mathrm{D}u) \;\equiv\; r$, where $r$ is integer in $\{2, 3, 4, ..., (n-1)\}$, and $|\mathrm{D}u(\gamma(t))|$ is constant along trajectories of the parametric gradient flow of $u(\gamma(t, x, \xi))$

$$\begin{cases} \dot{\gamma}(t, x, \xi) = \xi^\top \mathrm{D}u\big(\gamma(t, x, \xi)\big), & t \in (-\varepsilon, 0) \bigcup (0, \varepsilon), \\ \gamma(0, x, \xi) = x, \end{cases}$$

where $\xi \in \mathbb{S}^{N-1}$, and $\xi \notin N\Big(\mathrm{D}u\big(\gamma(t, x, \xi)\big)^\top\Big)$.

(iii) On $\Omega_1$ we have $\mathrm{rk}(\mathrm{D}u) \;\leq\; 1$ and the map $u : \Omega_1 \longrightarrow \mathbb{R}^N$ is given by an essentially scalar $\infty$-Harmonic function $f : \Omega_1 \longrightarrow \mathbb{R}$:

$$u \;=\; a + \xi f, \;\; \Delta_\infty f \;=\; 0, \;\; a \in \mathbb{R}^N, \;\; \xi \in \mathbb{S}^{N-1}.$$

The vectors $a, \xi$ may vary on different connected components of $\Omega_1$.

(iv) On S, when $S \supseteq \partial\Omega_p \cap \partial\Omega_q \;=\; \emptyset$ for all $p$ and $q$ such that $2 \leq p < q \leq n-1$, then we have that $|\mathrm{D}u|^2$ is constant and also $\mathrm{rk}(\mathrm{D}u) \;\equiv\; 1$. Moreover on

$$\partial\Omega_1 \cap \partial\Omega_n \subseteq S,$$

(when both 1D and $n$D phases coexist), we have that $u : S \longrightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$u = a + \xi f, \ \ |\mathrm{D}f|^2 = C^2 > 0, \ \ a \in \mathbb{R}^N, \ \ \xi \in \mathbb{S}^{N-1}.$$

On the other hand, if there exist some $r$ and $q$ such that $2 \leq r < q \leq n-1$, then on $S \supseteq \partial\Omega_r \cap \partial\Omega_q \neq \emptyset$ (when both $r$D and $q$D phases coexist), we have that $\mathrm{rk}(\mathrm{D}u) \equiv r$ and we have same result as in (ii) above.

## 6.2 Future work

We believe that the work in this field is interesting and there are still many open problems one can work on, for example:

1. Since the theory of near operators allows us to obtain a generalisation of some important results, one can work on the same problem of Chapter 2 considering the new theory of *generalised solutions* (see [48]).
2. One can study the existence of vectorial Absolute Minimisers in higher dimensions.
3. One can modify the result of Chapter 5, and prove that the images of the solutions are curvature along some trajectories, which we couldn't prove due to the lack of time.

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