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Numerical Methods for the Solution of Systems of Uncertain Differential Equations with Application in Numerical Modelling of Oil Recovery from Underground Reservoirs

by

M.P. Dainton

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Abstract

Direct numerical methods for the evaluation of uncertainties in the solutions of various partial differential equations, that contain uncertain parameters, are developed. The equations of particular interest are those which model the flow of a fluid in a porous medium whose properties are not known precisely, such as the modelling of oil in an underground reservoir. Some analytic work on the use of field theoretic methods to study flow in a heterogeneous medium is reviewed, and then extended to a simple time-dependent case. The first numerical model is a zero-dimensional transient mass-balance, where uncertainty is modelled as a single parameter, with a corresponding single-variate probability distribution function. This is treated, predominantly analytically, as a one-dimensional sensitivity analysis problem, with various plots of the development of the distribution function and mean value, of the analytic solution, being obtained and analysed. The next model is that of a simple single-phase steady-state flow model. The method involves a discretisation of the analytic equation, after which, perturbations about a mean of the uncertain parameter are considered. The distribution function, when treated in this way, is modelled as multivariate and dealt with accordingly. The final, and most significant, model is that of a two-dimensional, single-phase, dynamic one. Again, a perturbation expansion about a mean of the parameters is done, resulting in coupled equations for second order approximations to the mean at each point, and field covariance of the solution. These are then solved numerically. This method involves only one (albeit complicated) solution of the equations, and contrasts with the more usual Monte-Carlo approach, where many such solutions are required.

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Contents

Chapter 1

Overview

The objective of this research project is to investigate techniques for analysing uncertain systems of differential equations, with application to the problem of oil recovery from heterogeneous porous rock reservoirs.

Throughout this thesis, we concentrate on consideration of the effects of uncertainties in the permeability of the rock properties, with a specific concentration on how they affect the output properties of the oil field, such as pressure, when standard numerical discretisation techniques are applied to the differential equations modelling the flow through the porous media.

In the first chapter we introduce the basic model that is under consideration. This is obtained by a combination of Darcy's law [18] for flow of a fluid in a porous medium with the continuity equations. We consider three main model equations, with the effects of uncertainty, which are derived in this chapter. These are very much simplified versions of the types of equations that would be used in a practical context, but they are investigated in this form so as to illustrate clearly and pinpoint the main problems associated with treatment of the uncertainties in the permeabilities. In each case, single-phase flow is considered. The need for a probabilistic approach is also discussed.

We consider which methods might be used to solve the flow equations for the deterministic problem, which is the corresponding problem where uncertainties are disregarded. The emphasis throughout this research is on how these deterministic schemes may be used, in an adapted form, to assess the complete problem containing the uncertainties.

The second chapter is an introduction to the statistical ideas that are needed to develop the theory. Various probabilistic concepts are necessary. Most important are the ideas of distribution and expected value, out of which we can develop the crucial idea of moments. Specific forms of distributions are introduced, with the importance of lognormal distribution in a geostatistic context emphasised; and the generalisations to multivariate distribution functions, with corresponding correlations and moments, are explored. We also consider the effect of integral and differential operators on the uncertain variables, such as permeability.

The following chapter is a review of some of the background literature in the area of this research. We consider Monte-Carlo methods, and how a potential development of direct methods to obtain statistical moments might be far more flexible. There is some discussion of previous research in direct methods for stochastic p.d.e.s, including those of Adomian, and Schwydler and Matheron, whose perturbation series are studied further in later chapters. We also review some more theoretical work proving the existence of solutions of some equations under consideration. We then review some of the techniques employed in groundwater research for equations of the type we are interested in, and then briefly introduce the ideas behind the evaluation of effective permeabilities.

In chapter 5, an extension of the analytic work done by Dr. P. King, [33], on the use of field theoretic methods in the study of flow in a heterogeneous porous medium, is described. This work uses a perturbation formulation on the Fourier

transform of the Green's function equation for single-phase, steady-state flow in a heterogeneous medium. By use of field theoretic methods, Dr. King is able to sum up the entire perturbation series and compare with earlier work, which involved truncated summation processes and assumptions in single dimensions [7]. An effective overall permeability is thus obtained, together with the pressure variance, in terms of the correlation function and the geometric mean of the permeability.

We make an attempt to extend this work to a time-dependent version of the King model, which allows changes in effective permeability with respect to time to be observed. The Fourier transformed time-dependent equation version is shown to satisfy an identical integral equation to that in Dr. King's paper. The perturbation summation can then be performed in an identical way, so that the parallels in the results can be used to find expressions for the time-Fourier transformed effective permeability. Unfortunately, terms which then occur prove to be impossible to invert analytically. However, it is felt that future investigations into this approach might prove fruitful.

An extension of some preliminary work done for this project, which was performed in order to satisfy requirements for an MSc dissertation, [17], is described next. This work involves the investigation of a zero-dimensional mass-balance equation as a simplified model equation, with a single uncertain variable. Although very simplistic, this model provides considerable insight into the general problem of differential equations with uncertain variables. By using a closed form of the solution, the difference between the expected value of this solution and the deterministic value of the solution can be expressed in terms of moment generating functions of distributions. By manipulation of some basic probability theory, and the closed form solution, expressions for the probability distribution

function of the analytic solution, as a function of time, can be obtained. As an extension to this work, we are able to find a method of summing up the series terms that were not evaluated previously. The development in time of the mean of the solution, and in particular its behaviour with respect to the deterministic solution, can then be assessed more precisely.

In the following chapter, we consider the steady-state two-dimensional model equation. We primarily consider the discretised equation, using a perturbation method. When a perturbation formulation is applied to equations that have already been discretised, we are left with matrix equations involving mean value matrices and perturbed matrices that contain uncertain components. It is therefore necessary to develop some elementary theory for coping with matrices with uncertain components, such as distributivity and associativity of mean value operators. To begin with, we investigate the properties of a general admissible realisation, by developing general perturbation series. Expressions for the bounds on errors caused by truncating these series for general realisations are also obtained. When we take mean values over all possible realisations, we obtain expressions for mean values of the numerical solutions, which, when truncated at second order, involve just mean values and autocorrelation terms of the permeability, which are the generally available geostatistic quantities. Taking these mean values of the error terms also allows expressions to be obtained for the error introduced by truncating the mean value perturbation series. Other manipulations of the original perturbation series for general realisations allow us to develop second order approximation terms for the covariances of the numerical solution in terms of the same input geostatic quantities. Making use of the structure of these equations, we then are able to develop numerical procedures to evaluate the particular terms of interest. This development is followed by a presentation and discussion of the

results.

In the final chapter, the time-dependent model is considered. This model is again based on obtaining approximations to the mean value of the numerical solution using a perturbation series. We first develop a system of hierarchical equations for both standard-symmetric and lognormal distributions, where the effects of truncating these series at some arbitrary order of accuracy is investigated. The problems of taking mean values straight away are illustrated, leading to awkward terms that cannot be dealt with. Terms for an approximation to the variance can also be obtained, with the same problems preventing any attempt at solution. It has been found, however, that if the hierarchical terms are discretised, in some general sense, expressions allowing us to evaluate the required terms can be obtained. The result of these manipulations is to provide us with systems of coupled numerical differential equations, with all relevant quantities being solvable. A specific discretisation is then investigated in detail, with typical equations that occur from this process being presented. Some results of the application of this method are then presented and discussed, followed by comments on suggested extensions for the work.

Chapter 2

Introduction to the Model Equations

Difficulty in the mathematical and numerical modelling of physical systems may often arise when a precise knowledge of data is not available. Specifically, data that is crucial for describing the system, may only be known within certain limits of accuracy, or it may only be possible to specify certain statistical properties of the data. This may be due to inaccuracy in measuring equipment, or inaccessibility, or a high level of heterogeneity in materials whose parameters are involved in the model equations,

It is the effects of these latter sorts of uncertainty on the solutions of analytic and numerical systems which form the basis of this research project.

2.1 Darcy's Law

The starting point for any mathematical model of flow in a porous medium is Darcy's law [18]. This is an empirical law that states that the fluid flow velocity is proportional to the pressure gradient across the medium, with a gravitational

potential term included, and is given by,

$$\mathbf{u} = -\frac{K}{n\eta}\left(\nabla p - \varrho g \nabla d\right), \tag{2.1.1}$$

where $\mathbf{u}$ is the fluid flow velocity, K is the permeability tensor, η the fluid viscosity, p the pressure, n the volumetric porosity, ϱ the fluid density, and d the physical depth, [8]. Viscosity is just a property of the fluid, which ought to be known. Permeability is a property of the oil reservoir, and it is uncertainties in this that are of specific interest.

For multiphase flow considerations, the modified Darcy's law is used for each phase π, with an effectively reduced permeability, caused by the presence of other phases,

$$\mathbf{u}_\pi = -\frac{K\kappa_\pi}{\eta_\pi}\left(\nabla p_\pi - \varrho_\pi g \nabla d\right), \tag{2.1.2}$$

where $0 \leq \kappa_\pi \leq 1$ is the reduction factor associated with each phase, π.

For the purposes of this research, in order to highlight the effect of, and problems due to, uncertainty, we shall restrict our considerations to single-phase flow with gravitational effects neglected, so that we are modelling the flow in a horizontal plane, thus

$$\mathbf{u} = -\frac{K}{\eta}\nabla p. \tag{2.1.3}$$

In many practical cases of reservoir modelling, the oil-bearing rock can be assumed to be considerably larger in its horizontal dimensions than its vertical dimension. It it therefore quite a reasonable assumption to ignore the gravitational effects. Because of the linearity of equation (2.1.1), though, inclusion of the gravitational term may be incorporated into much of the analysis in a fairly straightforward way. For example, this may be done by substitution of a total potential form in equation (2.1.1).

It is hoped that a detailed investigation of this form of Darcy's law will provide

insight into how to treat uncertainties, which, in the future, will allow us to deal with more complicated equations, such as multi-phase flow with gravitational terms included.

2.2 Conservation Equations

We now combine the Darcy equation with the following form of the three-dimensional basic mass-balance equation, [8]

$$\frac{\partial n\,\varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{q}) = 0,$$

where $\mathbf{q}$ is the storage, defined as

$$\mathbf{q} = n\mathbf{u}.$$

Assuming constant porosity and dividing throughout by n, whilst combining these two equations gives

$$\frac{\partial \varrho}{\partial t} - \nabla \left(\varrho \frac{K}{\eta} \nabla p \right) = 0.$$

Multiplying throughout by $\frac{dp}{d\varrho}$, and assuming a constant compressibility,

$$\frac{dp}{d\varrho}\varrho = \frac{1}{\gamma},$$

gives the basic model equation for pressure that concerns us,

$$\gamma \frac{\partial p}{\partial t} - \frac{1}{\eta} \nabla \left(K \nabla p \right) = 0. \tag{2.2.1}$$

It is the effects of uncertainties in the permeability on the solution of this equation that is of primary interest in this research project.

We also consider the steady-state version of equation (2.2.1),

$$\nabla (K \nabla p) = 0, \tag{2.2.2}$$

with appropriate boundary conditions.

Consider a spatial discretisation of equation (2.2.1), simplified so that effects of compressibility and viscosity are neglected. This would result in the system of o.d.e.s

$$\frac{d\mathbf{p}}{dt} - \nabla_h \left(K \nabla_h \mathbf{p}\right) = \mathbf{f} : \qquad \mathbf{p}(0) = \mathbf{p_0}. \tag{2.2.3}$$

A standard method to approach this system of equations might be to diagonalise it, and study the resultant equations, which would have the simple form

$$\frac{dp}{dt} + \kappa p = f(t), \qquad p(0) = p_0. \tag{2.2.4}$$

At the time it was decided to study the behaviour of of this equation, where κ is an uncertain parameter, in the hope that it might give some useful insight into the general behaviour of the system of equations, 2.2.3, and ultimately indicate how to proceed with the study of numerical approaches for the analysis of the model equation (2.2.1).

Equation (2.2.4) will later be shown to be a good starting point for an introduction to this research.

2.3 Uncertainty

In all practical cases, the rocks that make up the oil reservoir are a considerable distance under the ground. This means that the corresponding properties, such as permeabilities and porosity are clearly not going to be accessible to any direct measurement.

The only information about the particular makeup of the rock is obtained by experiments performed on samples of the rocks that are brought up from recovery wells. It is not considered an efficient method of oil production to drill more boreholes than is necessary. The ideal number would be just two, one for

injection and one for extraction, with no extra wells drilled just to provide more information about the rock properties.

We are therefore left with the problem of how to cope with the uncertainties in the properties and structure of the rocks that lie between the primary wells.

One approach would be to try to interpolate for the values of permeabilities between the wells. There is, however, no straightforward way to do this, as rock strata are known to have varied and complex structures which would invalidate any simple linear interpolation between boreholes.

This leads to the necessity of developing a statistical approach to the problem.

If we consider one possible three-dimensional representation of a permeability function, or realisation, we may assign a probability value to that realisation. This allows us to think of the problem as consisting of sets of functions assigned with probability distributions. This is described quantitatively in the following chapter.

Given probability distributions for permeability, we would like to find procedures to characterise the probability distributions of the solutions to the model equation. This is certainly not a simple problem, as it effectively involves solving partial differential equations for each possible, or admissible, realisation. A continuous probability distribution functional model for permeability would suggest an infinite number of admissible realisations, which indicates the considerable complexity of the problem. This can be partially overcome by Monte-Carlo simulations, [41], with the need to generate large numbers of realisations, solving the system of equations for every realisation. The problem with this approach is the considerable number of times that the numerical procedure has to be applied in order to obtain accurate results that reflect the probability distributions of the solutions.

The approach we take in this research is to find more direct methods of finding quantities that characterise the p.d.f. of the solution. We are particularly interested in finding mean and variance, eventually to allow some prediction of yields and spread for ranges of yields that particular oil-fields may be expected to provide. Where possible, we develop separate equations that need only be solved once, in order to provide the important parameters. The effect of applying numerical discretisations to the model equations is investigated in detail.

We make the approximation of an isotropic permeability for K. This means that the tensor K can be approximated by a scalar, k, and allows for an easier description of numerical and analytic techniques for solving the p.d.e.s. More importantly in this context, it facilitates a straightforward description of the statistical properties of the permeability. We may then re-write the model equations (2.2.1) and (2.2.2),

$$\gamma \frac{\partial \varrho}{\partial t} - \frac{1}{\eta} \nabla \left(k \nabla p \right) = 0, \tag{2.3.1}$$

and

$$\nabla \left(k \nabla p \right) = 0. \tag{2.3.2}$$

2.4 Deterministic Equations

We introduce here an important distinction of terms used throughout this research. This is the concept of the deterministic problem.

Classically, a problem containing uncertainty might be approached by solving the deterministic version of the problem. This can be defined as the system of equations solved with the uncertain parameter replaced by its mean value. For our three model equations, these would be,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \left(\frac{\langle k \rangle}{\eta} \nabla p_0 \right) = f_0, \tag{2.4.1}$$

$$\nabla \left(\langle k \rangle \nabla p_0 \right) = f_0, \tag{2.4.2}$$

and,

$$\frac{dp_0}{dt} - \langle \kappa \rangle p_0 = f_0. \tag{2.4.3}$$

Procedures for solving these types of equations are well known in standard analysis.

The classical solutions to these equations might be accepted as reasonable approximations to the mean values of the solutions of the model equations, (2.2.1) to (2.2.4). The accuracy of this statement is one of the things we explore in detail in this thesis.

In many cases, we consider the difference between the mean value of the solutions to the stochastic problem and the deterministic solutions, both analytically and numerically. The behaviour of the difference of these two quantities governs much of the research about the p.d.f. of the solution.

In the next chapter we introduce the probabilistic concepts and notations that are necessary to understand the following chapters. Definitions of statistical properties of single-variate, and then multivariate, probability distributions are introduced, in order to analyse equations (2.2.1), (2.2.2), and (2.2.4), with particular emphasis on moments and their derivation.

$$\nabla \left(\langle k \rangle \nabla p_0 \right) = f_0, \tag{2.4.2}$$

and,

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Chapter 3

Statistical Concepts

Various statistical concepts and results are needed for the development of this research, and those used in this thesis are described in this chapter. The first six sections deal with univariate distributions, or those where there is a single, uncertain parameter. Most of the general results and properties obtained here are used in Chapter 6. The remaining chapters deal with results concerning multivariate distributions.

3.1 Distribution and Range

The starting point for any statistical consideration is that of a random variable, defined by two quantities, [32],

1. the set of possible values that X can take, also known as the set, or the phase-space. It can be discrete or continuous.
2. the probability distribution over the range.

The probability distribution in the case of a continuous one-dimensional range is given by a probability density function (henceforth known as p.d.f.) which is

a non-negative normalised function $p(x) \geq 0$, such that

$$\int_R p(x)dx = 1, \tag{3.1.1}$$

where R is the range of possible values.

The probability that X takes a value in the incremental range x to $x + dx$, is then given by

$$p(x)dx,$$

and so the probability that X has some value in the range (x_1, x_2) is given by

$$\int_{x_1}^{x_2} p(x)dx.$$

3.2 Expected Value and Moments

We now introduce the concept of a mean, or expected value of a function of X, defined on some interval region R, $f(X)$, given by

$$\langle f(X) \rangle = \int_R f(x)p(x)dx, \tag{3.2.1}$$

The expected value of the random variable itself is then given by

$$\langle X \rangle = \int_R xp(x)dx, \tag{3.2.2}$$

which can be generalised so that the idea of moments can be defined as the expected value of X raised to some arbitrary m^{th} power,

$$\mu_m = \langle X^m \rangle = \int_R x^m p(x)dx. \tag{3.2.3}$$

So, μ_1 corresponds to the mean value, and the standard deviation, σ, and variance, σ^2, are both defined, [25], by

$$\begin{aligned} \sigma^2 &= \langle (X - \langle X \rangle)^2 \rangle \\ &= \mu_2 - \mu_1^2 \end{aligned} \tag{3.2.4}$$

This idea of a moment deviation from the mean can be generalised to the idea of shifted moments, which are defined by

$$\nu_m = \int_R (x - \langle x \rangle)^m p(x) dx, \tag{3.2.5}$$

so that

$$\nu_0 = 1,$$

$$\nu_1 = 0,$$

and

$$\nu_2 = \sigma^2,$$

for all distributions.

An important property of a symmetric distribution function is that m^{th} order corrected moments, where m is odd, are all equal to zero. This is because, for a symmetric distribution function, $p(x + \langle x \rangle)$ is an even function, and the region R can be assumed to be symmetric about $\langle x \rangle$ (most generally, from $-\infty$ to $+\infty$.) Therefore, we may write the m^{th} corrected moment as

$$\nu_m = \int_R (x - \langle x \rangle)^m p(x) dx,$$

and, by changing variable to $x' = x - \langle x \rangle$,

$$\nu_m = \int_R (x')^m p(x' + \langle x \rangle) dx'. \tag{3.2.6}$$

For m odd, the integrand is clearly odd, also, thus giving the zero result.

3.3 Moment Generating Function

This is defined, [32], as the expected value of the function e^{ikX},

$$\mathcal{G}(k) = \langle e^{ikX} \rangle = \int_R e^{ikx} p(x) dx, \tag{3.3.1}$$

which is also the Fourier transform of the p.d.f.

It generates the individual moments of the distribution, in the sense of the Taylor expansion

$$\begin{aligned} \langle e^{ikX} \rangle &= \langle 1 + ikX + \frac{(ikX)^2}{2!} + \frac{(ikX)^3}{3!} + \cdots \rangle \\ &= 1 + ik\langle X \rangle + \frac{(ik)^2}{2!}\langle X^2 \rangle + \frac{(ik)^3}{3!}\langle X^3 \rangle + \cdots \\ &= \sum_{m=0}^{\infty} \frac{(ik)^m}{m!}\mu_m. \end{aligned} \tag{3.3.2}$$

3.3.1 Cumulants

The moment generating function also serves to generate the cumulants, ξ_m of the distribution, defined in the sense that

$$log\ \mathcal{G}(k) = \sum_{m=0}^{\infty} \frac{(ik)^m}{m!}\xi_m. \tag{3.3.3}$$

By expanding the log of expression (3.3.1), these cumulants can be shown to be various combinations of the moments, for example,

$$\xi_1 = \mu_1, \tag{3.3.4}$$

$$\xi_2 = \mu_2 - \mu_1^2 = \sigma^2, \tag{3.3.5}$$

and

$$\xi_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3. \tag{3.3.6}$$

3.4 Examples of Univariate Distributions

3.4.1 Uniform Distribution

The most simple type of univariate distribution is the uniform distribution. The p.d.f. has the form,

$$\rho(x) = \begin{cases} \frac{1}{2A}, & \langle x \rangle - A \leq x \leq \langle x \rangle + A \\ 0, & \text{otherwise} \end{cases} \tag{3.4.1}$$

and can be interpreted as the parameter lying somewhere within a defined range, of size $2A$, with equal probability of it taking any value within that range, and no probability of it lying outside. Although a very simple-minded type of distribution function, it serves to illustrate a number of interesting properties, particularly where no more specific information about the shape of the distribution function is available.

In the MSc dissertation, [17], the following properties were found. An explicit formula exists for a general moment,

$$\mu_n = \frac{1}{2(n+1)A} \left\{ (\langle x \rangle + A)^{n+1} - (\langle x \rangle - A)^{n+1} \right\}, \tag{3.4.2}$$

and for the corresponding shifted moments,

$$\nu_n = \begin{cases} \frac{A^n}{n+1} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \tag{3.4.3}$$

It then follows that

$$\sigma^2 = \frac{A^2}{3}, \tag{3.4.4}$$

and so the standard deviation is $\frac{A}{\sqrt{3}}$, which is proportional to the size of the range.

The moment generating function of a uniform distribution takes the form of a sinc multiplied by an exponential function,

$$\mathcal{G}(k) = \frac{e^{ik\langle x \rangle} sin(kA)}{kA}. \tag{3.4.5}$$

3.4.2 Gaussian Distribution

The most commonly used distribution function is the Gaussian distribution, which has the form

$$\rho(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} exp\left[-\frac{(x-\langle x\rangle)^2}{2\sigma^2}\right], \tag{3.4.6}$$

and x takes possible values in the infinite range, $[-\infty, +\infty]$.

From [17], a recurrence relationship can be obtained for the moments,

$$\langle x^n\rangle = (n-1)\sigma^2\langle x^{n-2}\rangle + \langle x\rangle\langle x^{n-1}\rangle, \tag{3.4.7}$$

or,

$$\mu_n = (n-1)\sigma^2\mu_{n-2} + \mu_1\mu_{n-1}.$$

A formula for the corrected moments, which is much more useful in this context, can be derived fairly easily. Noting that a Gaussian is a symmetric distribution, so that all odd moments are zero, we can derive the even moments as

$$\nu_{2m} = \int_{-\infty}^{\infty} (x-\langle x\rangle)^{2m} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} exp\left[-\frac{(x-\langle x\rangle)^2}{2\sigma^2}\right] dx,$$

and, changing variable to $y = x - \langle x\rangle$, we have,

$$\begin{aligned} \nu_{2m} &= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} y^{2m} e^{-\frac{y^2}{2\sigma^2}} dy \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} y^{(2m-1)} \times y e^{-\frac{y^2}{2\sigma^2}} dy. \end{aligned} \tag{3.4.8}$$

Integrating by parts gives

$$\begin{aligned} &\nu_{2m} \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \left[-y^{(2m-1)}\sigma^2 e^{-\frac{y^2}{2\sigma^2}}\right]_{-\infty}^{\infty} + \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (2m-1) y^{(2m-2)} \sigma^2 e^{-\frac{y^2}{2\sigma^2}} dy \\ &= \sigma^2(n-1)\nu_{2m-2}, \end{aligned} \tag{3.4.9}$$

since the $e^{-\frac{y^2}{2\sigma^2}}$ term always dominates the $y^{(2m-2)}$ term at $\pm\infty$.

So, for the Gaussian,

$$\left.\begin{array}{rcl} \nu_{2m} & = & \frac{(2m-1)!\sigma^{2m}}{2^{(m-1)}(m-1)!} \\ \nu_{2m+1} & = & 0 \end{array}\right\} \forall m \in \mathbb{N}. \tag{3.4.10}$$

An important property of a Gaussian distribution is that all cumulants above second order are zero, ([17] and [32]). This is because the moment generating function can be written

$$\mathcal{G}(k) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{ikx} exp\left[\frac{-(x-\langle x\rangle)^2}{2\sigma^2}\right] dx. \tag{3.4.11}$$

By multiplying the integrand throughout by

$$\exp\left(\frac{2i\langle x\rangle\sigma^2\kappa - \sigma^4\kappa^2}{2\sigma^2}\right) \cdot \exp -\left(\frac{2i\langle x\rangle\sigma^2\kappa - \sigma^4\kappa^2}{2\sigma^2}\right)$$

we can complete the square in the exponential, as in [17], to give

$$\mathcal{G}(k) = exp\left[\frac{2\langle x\rangle ik - \sigma^2k^2}{2}\right],$$

so that

$$log(\mathcal{G}(k)) = ik\langle x\rangle + \frac{(ik)^2}{2}\sigma^2.$$

Equating this with the expansion term defining cumulants, (3.3.3), shows that for a general Gaussian distribution

$$\begin{array}{rcl} \xi_1 & = & \langle x\rangle \\ \xi_2 & = & \sigma^2 \\ \xi_m & = & 0 \quad \forall \quad m > 2. \end{array}$$

Cumulants of order greater than two, in distributions other than Gaussian, can be thought of as a measure of how dissimilar they are from Gaussian.

3.4.3 Lognormal Distribution

A commonly-used distribution function in geostatistics [10] is the log-normal distribution function. To describe a univariate log-normal function, we must define a new variable, y, where $y = \ln k$. If k is defined to have a log-normal distribution, then y is standard normally-distributed.

We define k_g to be the geometric mean,

$$k_g = e^{\langle y \rangle}. \tag{3.4.12}$$

Then there exist some well-known equations relating the statistics of y and k, [36],

$$\langle x \rangle = exp\left\{ \frac{\sigma_y^2}{2} + \langle y \rangle \right\}, \tag{3.4.13}$$

and

$$\sigma_x^2 = exp\left\{ 2(\sigma_y^2 + \langle y \rangle) \right\} - exp\left\{ \sigma_y^2 + 2\langle y \rangle \right\}. \tag{3.4.14}$$

3.5 Multivariate Distributions

Here we introduce the concept of a multivariate distribution, necessary for the development of any partial differential equation containing uncertain parameters, in this context.

A multivariate distribution function can be thought of as a function of many variables with corresponding ranges of possibles values for each variable,

$$F = f(x_1, x_2, x_3, ...x_N), \tag{3.5.1}$$

with ranges R_i for each x_i.

We can have a mean value for each variable,

$$\langle x_i \rangle = \int_R x_i f(x_1, x_2, ..., x_i, ..., x_N) d\mathbf{x}, \tag{3.5.2}$$

and its moments,

$$\langle x_i^n \rangle = \int_R x_i^n f(x_1, x_2, ..., x_i, ..., x_N) d\mathbf{x}. \tag{3.5.3}$$

The integration is performed over the entire space of variable, $\{x_i\}$ for all i. The correlation moment between two variables is also defined as

$$\begin{aligned} Co(x_i, x_j) &= \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle \\ &= \int_R (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) f(x_1, x_2, ..., x_i, ...x_j, .., x_N) d\mathbf{x} \end{aligned} \tag{3.5.4}$$

3.5.1 Permeability

In the previous chapter, we discussed the need for a statistical approach, specifically when considering the permeability.

The ideas for a multivariate distribution can be generalised to that of an uncertain function, which enables us to introduce formally the statistical description of the permeability function, when considered as a function of spatial position, $\mathbf{r}$. The statistics of the permeability are characterised, [21], by its mean value function,

$$k_0(\mathbf{r}) = \langle k(\mathbf{r}) \rangle, \tag{3.5.5}$$

its variance,

$$\sigma_k^2(\mathbf{r}) = \langle (k(\mathbf{r}) - k_0(\mathbf{r}))^2 \rangle, \tag{3.5.6}$$

and the permeability autocorrelation function, (P.A.F.), defined as a function of two spatial positions, $\mathbf{r_1}$ and $\mathbf{r_2}$

$$\rho(\mathbf{r_1}, \mathbf{r_2}) = \frac{\langle (k(\mathbf{r_1}) - k_0(\mathbf{r_1}))(k(\mathbf{r_2}) - k_0(\mathbf{r_2})) \rangle}{\sigma_k(\mathbf{r_1}) \sigma_k(\mathbf{r_2})}. \tag{3.5.7}$$

This is essentially a measure of the statistical relationship of neighbouring permeabilities.

Mathematically, the P.A.F. ought to satisfy certain requirements, which are

1. approach of $\rho(\mathbf{r_1}, \mathbf{r_2})$ to one, as $\mathbf{r_1}$ approaches $\mathbf{r_2}$

   -we would expect the permeability at two points to be perfectly correlated as those two points coincide,

2. approach of $\rho(\mathbf{r_1}, \mathbf{r_2})$ to zero, when $\mathbf{r_1}$ and $\mathbf{r_2}$ are separated by some characteristic distance

   -this characteristic distance is the correlation length, and is associated with a particular model.

The P.A.F. can usually be assumed to be a function of separation only,

$$\rho(\mathbf{r_1}, \mathbf{r_2}) = \rho(\mathbf{r_1} - \mathbf{r_2}), \tag{3.5.8}$$

and a further simplification that can be made is that the P.A.F. is (statistically) homogeneous,

$$\rho(\mathbf{r_1} - \mathbf{r_2}) = \rho(\mid \mathbf{r_1} - \mathbf{r_2} \mid). \tag{3.5.9}$$

The correlation length can be formally defined as

$$\lambda = \int_0^\infty \rho(x) dx, \tag{3.5.10}$$

and represents an intuitive measure of maximum distance over which the statistical properties of the permeability are strongly correlated.

We have now introduced most of the statistical concepts that are needed in the remainder of this thesis. Some additional theory concerning multivariate distribution functions is described in chapter 7, where it is used in its context. In the next chapter, we review methods for treating the problem of fluid flow in an uncertain medium that were developed elsewhere.

Chapter 4

Literature Survey

In this chapter, we review some of the literature that concerns the numerical solution of differential equations that contain uncertain parameters. In particular, we restrict our study to equations of a similar structure to those of our model equations (2.2.1), (2.2.2), and (2.2.4).

4.1 Monte-Carlo Methods

Most numerical work that has been done up to this point in the practical area of oil reservoir modelling, taking into account uncertainty in the equation parameters, has involved a Monte-Carlo approach. The first attempt to employ this technique was developed by Warren and Price, in their paper titled Flow in Heterogeneous Porous Media, [41]. Subsequent work has been done, specifically in the field of water flow in underground aquifers, in one-dimension by R. Allan Freeze, [23], and in two dimensions by Smith and Freeze, [24].

Monte-Carlo techniques involve multiple generations of large sets of realisations of rock properties. The model equations in numerical, or discretised, form can then be applied to each separate realisation in turn to give each particular

solution for whatever properties are of interest, such as pressure or flux, that are associated with that specific realisation. The statistical properties of the solution may then be directly obtained from the ensemble of solutions for the realisations. Those properties of usual interest are mean and variance/covariance, and also the effect of the statistical properties of the uncertain parameters on the statistical properties of the numerical solutions.

This is the general approach taken in [1], where 50 realisations of a two-dimensional cross-section are generated. The results given are found to be over optimistic estimations of the mean performance of the reservoir. In [22] different probabilities are assigned to each realisation, and estimations of the probability distributions are made. It is shown that finding the most probable realisation is not a simple problem.

The advantage of Monte-Carlo methods is that the same numerical scheme may be applied to each realisation, enabling the numerical errors to be assessed and identified. Once the numerical results for each realisation are obtained and collated, it is then a comparatively straightforward process to obtain the statistical properties of interest. For a reasonably large number of realisations, it is feasible that any statistical property, such as higher order moments, of the solution are available. This, in theory at least, allows a detailed picture of the distribution function to be built up.

Any set of realisations generated for the purposes of Monte-Carlo simulation must, of course, have the correct statistical properties. That is, their statistics are required to have a broad similarity with the known, or assumed, properties of the rocks under investigation. In most cases for the oil problem, we need to generate the realisations from data given in the form of a correlation function. One paper that succinctly describes this process is that by P. R. King, and P. J. Smith,

entitled Generation of Correlated Properties in Heterogeneous Porous Media, [34]. The general mathematical methods required to quantify and generate sets of correlated random fields, are described, particularly using the nearest neighbour model. The techniques investigated are especially appropriate when it is required that the set of fields generated are done so over a discrete numerical grid.

The problems with using Monte-Carlo methods are that the accuracy and convergence with the statistical solution parameters are partly controlled by the number of realisations that need to be generated. The number required might possibly be impracticably large, especially in terms of speed and storage for computational purposes.

For the purposes of this research, we have restricted our consideration to an investigation of more direct methods to find the statistical parameters of the solutions. The emphasis has been on trying to find methods that allow a single application of some numerical technique to produce directly an approximation to moments of the distribution function of the solution.

4.2 Direct Methods

Much of the background work in the field of generalised linear stochastic operators is presented by G. Adomian in his paper Linear Stochastic Operators [2]. This paper concerns itself with equations of form,

$$Ly = x(t), \tag{4.2.1}$$

where

$$L = \sum_{\nu=0}^{n} a_\nu(t) \frac{d^\nu}{dt^\nu}. \tag{4.2.2}$$

In the most general case, there is uncertainty contained in the coefficients $a_\nu(t)$ which are stochastic quantities. The term x can be thought of as the input

function, which may, or may not be stochastic, and the general problem can be thought of as identifying the uncertainty caused in y due to the inverted form of the L operator acting on $x(t)$.

This gives a clarification of the nature of the type of problems that we are dealing with in this research. A typical stochastic problem [2] would involve the $x(t)$ term being a stochastic process, such as a white noise effect, and the coefficients $a_\nu(t)$ being deterministic. We would then wish to investigate the effect of the deterministic inverse operator L^{-1} on the stochastic process to see how it effects the uncertainties in the behaviour of $y(t)$. Our particular problem is different in that it is the coefficients $a_\nu(t)$ that cause the uncertainty in $y(t)$ and the input function may or may not be a stochastic term.

For the type of simplified flow problems we are dealing with, the coefficients, which stem from permeability and porosity, are independent of time.

Adomian is able to deal extensively with equations of this sort in an analytic sense, by consideration of the Green's function and integral form for equation (4.2.1). Some of the results he obtains are used in further research, particularly when an analytic form of the equation is under consideration, such as in the work of P. King, [33], which is reviewed in some detail in chapter 5.

Our background research on direct methods for stochastic solutions of the oil reservoir problem, due to random permeabilities, began with Dupuy's review paper entitled Some New Mathematical Approaches for Heterogeneous Porous Medium Studies [21]. In this paper, some techniques and results from two sets of research, done by Schwydler and Matheron, are collated and summarised.

The contributions made by Schwydler are published in papers [43] to [47]. He first formalises the idea of the permeability autocorrelation function, to describe

a stochastic permeability function,

$$Cov(\mathbf{r}) = \frac{\langle (k(\mathbf{x}) - k_0(\mathbf{x})) - (k(\mathbf{y}) - k_0(\mathbf{y})) \rangle}{\sigma_k^2}, \tag{4.2.3}$$

where $\mathbf{r} = |\mathbf{x} - \mathbf{y}|$.

For flow defined $\mathbf{u}$, and given by $\mathbf{u} = -k\nabla p$, he defines the probable flow u^*,

$$u^* = \langle u \rangle,$$

and flow deviation

$$\sigma_u = \left(\langle (u - u^*)^2 \rangle \right)^{\frac{1}{2}}.$$

Most importantly, the idea of relative standard deviation of permeability and flow,σ_k' and σ_u', respectively, are defined as

$$\sigma_k' = \frac{\sigma_k}{\langle k \rangle},$$

$$\sigma_u' = \frac{\sigma_u}{\langle u \rangle}.$$

It is argued that the variance reduction factor,

$$VRF = \frac{(\sigma_u')^2}{(\sigma_k')^2},$$

is a measure of the effective heterogeneity of the porous medium model with regard to the flow considered.

Schwydler considers the permeability as a perturbation about its homogeneous mean,

$$k(\mathbf{x}) = k_0 + \eta k_1(\mathbf{x}), \tag{4.2.4}$$

so that the pressure and flow may be expressed as a perturbation series about their unperturbed, or deterministic, solutions ,

$$p(\mathbf{x}) = p_0(\mathbf{x}) + \sum_{j=1}^{\infty} p_j(\mathbf{x}), \tag{4.2.5}$$

$$u(\mathbf{x}) = u_0(\mathbf{x}) + \sum_{j=1}^{\infty} u_j(\mathbf{x}). \tag{4.2.6}$$

By substitution into the flow equations, expressions for the analytic flow variance can be obtained, in terms of the analytic unperturbed pressure solutions, and the autocorrelation function. By dividing by the unperturbed flow, the relative flow variance can be obtained and compared with the relative permeability variance to give the variance reduction factor (V.R.F.).

The V.R.F.s for linear and radial flows are then formulated to give an idea of local flow heterogeneities for these types of processes. Schwydler is also able to apply these ideas, with some success, to a multi-well problem.

The work done by Matheron, with similar methods to Schwydler, is presented in papers [37], [38], and [39]. He was able to make a significant contribution to the evaluation of effective permeabilities.

By expressing the pressures and flows in terms of Green's functions for the flow equations, Matheron is able to make comparisons with an effective flow. These can be expressed in terms of series involving relative variance of permeability with geometric factors for different dimensions of flow under consideration. The sorts of results obtained are comparable to those by Warren and Price [41].

Many results obtained by the two authors Schwydler and Matheron, are confirmed in later studies. Some of the mathematical formulations have proven themselves to be invaluable for subsequent research, particularly some of the analytic perturbation techniques, which we further develop analytically, and also numerically, later in this thesis.

A more recent attempt at this problem is made by Dikow and Hornung, [19], who study some further analytic properties of the homogeneous version of the model equation, (2.2.2), and also develop series terms for the flux, but with slightly different assumptions about the form of the random permeability func-

tion. The precise form of equation they investigate is

$$\nabla(k\nabla p) = 0, \tag{4.2.7}$$

with mixed boundary conditions, $p = p_0$ on S, and $\frac{\partial p}{\partial n} = 0$ on T, where $S \cup T$ makes up the boundary of the region G under consideration. The assumption made on the values of the permeability k is that for all spatial positions it has the limits,

$$\frac{1}{\mu} \leq k \leq \mu,$$

where μ is some constant that defines the limits of all admissible realisations for permeability k. A unique weak solution can be shown to exist for all possible realisations under this constraint, [40].

The random flux, Q, is defined by

$$Q = \int k \frac{\partial p}{\partial \mathbf{n}} \cdot d\mathbf{S}.$$

The permeability is assumed to have the form

$$k(\mathbf{x}) = k_0(\mathbf{x}) exp(\xi(\mathbf{x})), \tag{4.2.8}$$

where $\xi(\mathbf{x})$ is a random function with mean value 0, and a given covariance

$$cov\,(\xi(\mathbf{x}), \xi(\mathbf{y})) = R(\mathbf{x} - \mathbf{y}), \tag{4.2.9}$$

such that

$$-\Lambda \leq \xi(\mathbf{x}) \leq \Lambda,$$

where

$$\Lambda = \ln \mu.$$

The random total flux can then be expressed as

$$Q(\xi) = Q_0 + Q_0'(\xi) + \frac{1}{2} Q_0'' + r, \tag{4.2.10}$$

where, Q_0', Q_0', and Q_0'' have explicit forms, Q_0' being a linear functional of ξ, Q_0'' being a quadratic functional of ξ, and an explicit bound existing for the remainder r. The theory developed in this paper enables these moment terms for the random total flux to be calculated explicitly. Also, an expression for the variance of Q_0' can be obtained.

The results achieved here are far more rigorous than those of Matheron and Schwydler, [21]. Knowledge of the existence of these quantities has proved invaluable in subsequent research, as we are now able to proceed to develop techniques for solving these quantities numerically.

4.3 Groundwater Modelling

There is far more literature connected with flow in uncertain media in the area of groundwater modelling than in the field of oil reservoir modelling. We mention here some of that work that has proven useful in our studies.

R. Allan Freeze [23] applies Monte-Carlo methods to one-dimensional groundwater flow problems in a non-uniform medium.

B. Sagar [42] considers flow though a random porous medium with a Galerkin finite element discretisation.

The equation under consideration is a time dependent version of the water flow equations,

$$\nabla(T\nabla h) - \frac{\partial h}{\partial t} = W, \tag{4.3.1}$$

where T is the transmissivity tensor, h the hydraulic head, S the coefficient of storage of the aquifer under investigation, and W is some forcing function.

By consideration of the weak form of these equations, a matrix equation may

be formed,

$$G\mathbf{B} + P\frac{d\mathbf{B}}{dt} = \mathbf{F}, \tag{4.3.2}$$

where $\mathbf{B}$ is the vector of solution parameters, and G is a matrix with coefficients dependent on the discretised uncertain transmissivity. Using a finite central difference discretisation in time, the equation is reduced to the form,

$$C\mathbf{u} = \mathbf{D} \tag{4.3.3}$$

where the problem becomes one of inverting the matrix C, which has uncertain elements whose statistical properties are known. We tackle a similar problem to this in Chapter 7, but Sagar in [42] employs rank one updates to change the inverse elementwise, so that the mean of C^{-1} may be found. The first two moments of the solution distribution function can then be approximated. A simple example is presented.

A few other authors have tried to approach the problem of finding inverse forms for the sort of uncertain matrix equations as in equation (4.3.3). Some of the more recent of these include Townley and Wilson, [49] and Hoeksema and Kitanidis, [31]. These techniques involve linearisation of the Taylor expansions for the inverse matrices and applying first order sensitivity analysis. This allows for numerical results for mean value and covariance matrices of head values to be obtained. Townley, [48], extended the approach to include second-order terms in the mean head values, which permits a better estimate of how accurate the technique may or may not be. A similar line of work has been employed to attempt to solve the steady-state oil problem presented in chapter 7, where the second order terms are again included in the probabilistic approach. In this case the emphasis is placed on solving larger sets of equations.

McKinney and Loucks [35] apply the idea of first order uncertainty analysis to

the Galerkin finite element discretised water flow model equations, using a sensitivity matrix. Approximations for mean and variance/covariance of the solutions are found, and favourable comparisons are made with Monte-Carlo simulations.

Graham and McLaughlin, [30], explore, in detail, transport equations for a conservative flow in an uncertain environment

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x_i}(v_i c) - \frac{\partial}{\partial x_i}\left[D_{ij}\frac{\partial c}{\partial x_i}\right] = 0, \tag{4.3.4}$$

where $c(\mathbf{x}, t)$ is concentration, a non-stationary random function. The pore velocity, $v_i(\mathbf{x})$, is a random function.

Expressing the pore velocity as a perturbation function about its mean value, writing the concentration as a perturbation also, and substituting these two terms into the transport equation allows coupled equations for subsequent moments to be developed. These analytical equations can then be solved and compared with Monte-Carlo simulations.

Much detailed and varied work has been done in the area of groundwater flows, with uncertainty taken into account, by G. Dagan in papers such as [13] to [15], and many others. Some of the principles are summarised in the book entitled Flow and Transport in Porous Formations [16].

The broad approach Dagan employs in this field is a predominantly analytic one, and can be best illustrated by a look at [13]. Here he explores the equations for solving for the random head field, h,

$$\mathbf{u} = -k\nabla h, \tag{4.3.5}$$

and the continuity equation,

$$\nabla \cdot \mathbf{u} = R(\mathbf{x}), \tag{4.3.6}$$

on a domain D, subject to appropriate boundary conditions.

Combining these gives

$$\nabla(k\nabla h) = -R(\mathbf{x})$$

or in logarithmic form,

$$\nabla^2 h + \nabla Y \cdot \nabla h = -e^{-Y} R(\mathbf{x}), \tag{4.3.7}$$

where Y is the log of the transmissivity, K, and R is rate of recharge.

These equations can be written in integral form using Greens' functions, for example, [11],

$$\begin{aligned} h(\mathbf{x}) - \int \left[\nabla Y(\mathbf{x}) \cdot \nabla h(\mathbf{x}')\right] G(\mathbf{x}, \mathbf{x}') d\mathbf{x}' \\ = \quad - \int h_b(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} d\mathbf{x}' + \int e^{-Y(\mathbf{x}')} R(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d\mathbf{x}' \end{aligned} \tag{4.3.8}$$

where $h_b(\mathbf{x})$ is the boundary value.

Dagan forms a perturbation expansion of this equation, to give arbitrarily high order terms in the expansion. As before, this method allows explicit formulae for expansion terms which can be averaged to give analytic terms for mean and variance. The contribution of this paper is to consider conditional, as opposed to the usual unconditional, probability distribution function. This is in order to represent points at which a complete knowledge of the field is precisely available, usually at a limited number of points, such as where wells have been drilled. By application of conditional p.d.f.s and moments to the equations for the perturbation series terms, conditional means and variances are developed.

Figures for effective conductivity in the conditional and unconditional cases are obtained and compared. The difference between the two quantities is of particular interest.

The calculations are done in one and two dimensions, with both linear and radial flows, and finally in three dimensions.

The introduction of conditional probabilities into this area of study, so that conditional effective permeabilities and variances may be compared with the unconditional versions, is very interesting in the context of this research. Application in oil reservoir modelling would clearly be of interest in cases where knowledge of rock properties from borehole samples from a small number of production wells needs to be taken into account. The reduction in variance is particularly important.

In subsequent papers, Dagan further investigates uncertainty problems. Many of his analytic results may prove interesting to investigate in a more practical or numerical context in the future. These analytic results are based on an implicit assumption of the existence, or at least availability, of analytic solutions to the deterministic case. In this thesis the assumption that the analytic solution is readily available is not necessarily made, which we feel allows a more physical approach to problems, as encountered in a practical sense.

In recent years there has also been some considerable work done in specifically spectral-based perturbation techniques. Some of these include those by Gelhar and Vomvoris, [28], Gelhar and Axness, [27], Vomvoris, [50], and many others.

Gelhar and Vomvoris, [28], employ a spectral based perturbation technique to find concentration variability in a three-dimensional aquifer. This results in a highly anisotropic covariance function for the concentration, with very large correlation lengths aligned to the mean flow direction. The authors were able to show a significant decrease in the unusually large longitudinal correlation length when the input $\ln K$ spectra had their high wave number values omitted. It is noted that some of the analytical results obtained for the covariance function here can be tentatively confirmed by much of the general numerical behaviour observed by Graham and McLaughlin, [30]. It is felt that more practical work in

this area could be pursued in the future.

In the paper of Bellin, Salandin and Rinaldo, [9], an effort is made to draw together some of the differing approaches employed in this area and to make comparisons of their effective usefulness. Some aspects of numerical analysis of dispersion of flow in a heterogeneous porous medium are discussed. Of particular interest is the accuracy of approximations in some of the previously developed first order theories, such as that of Dagan, [16], and the convergence of some of the computational results. For example, different rates of convergence were found for Monte-Carlo simulations for different spatial moments, and assessments of the required number of simulations for convergence to be obtained for second order moments is made and found to be large, even in the case of relatively mild heterogeneity. Unexpectedly favourable comparisons between linear theories for moderately heterogeneous formations and Monte-Carlo simulations when solving for variances are found, and some explanations for these observations are obtained.

4.4 Effective Permeability

A number of people have done research with the emphasis on establishing mathematical formulations of effective values of permeabilities for flow in porous media. Gutjhar et al, [6], found that the effective conductivity is the harmonic mean in one-dimensional flow, the geometric mean for two-dimensional flow, and $(1+\sigma_f^2)$ times the geometric mean for three-dimensional flow. Some of their results are replicated, only more rigorously and in more general cases, by P. King, in [33]. This paper is be reviewed in more detail in Chapter 5. Drummond and Horgan, [20], are able to replicate some of the values for effective permeability by two different methods. One is a perturbative method, as before, but they are also

able to apply a simulation method that is shown to give good agreement with other methods. It is envisaged that the simulation method might be appropriate in cases where the standard perturbation methods cannot be implemented.

4.5 Conclusions

There is a great deal of background literature in the field of porous flow in uncertain media stemming from groundwater flow models. Although most of these ideas have not been extensively applied to oil reservoir modelling, there is no reason why this could not be done.

However, virtually all this literature adopts analytic methods for the solution of the model equations. The assumption behind this is that analytic solutions for these equations are freely available, something that is not necessarily true in most practical applications. It is the aim of this project to produce numerical methods, adapted from simulation techniques for deterministic equations, to evaluate statistical data about the solutions. Present literature indicates, to some extent, how the jump from deterministic to stochastic solutions may be made, but there is hardly any that demonstrates precisely how this can be done when working with numerical simulation models for oil reservoirs.

There have recently been published several books that review the entire subject, especially from the point of view of groundwater modelling in an uncertain environment. Among the best of these is that by Dagan, [16], which has already been mentioned, and L. W. Gelhar's Stochastic Subsurface Hydrology, [26]. Gelhar's book contains chapters reviewing much of the current work already described in this chapter, especially that of the author, with extra chapters that emphasize the transport processes in strongly heterogeneous media. Rather

importantly, some limited indication of numerical extensions to all the analytic methods introduced here are reviewed also. This at least allows the reader to gain some limited insight into the important jump from numerical deterministic approaches to full probabilistic numerical techniques, which in the context of this research, can be regarded as the ultimate aim of this field of study.

In the next chapter, we review and then attempt to extend the work of P. King. This is one of the first steps taken in this research project. The general difference between the jump from deterministic to stochastic in an analytic case and a numerical case is illustrated by the way in which this chapter differs from subsequent chapters.

Chapter 5

Extension of P. King's Work

In this chapter, we review, and then attempt to extend, the work done by Dr. P. King [33], entitled "The use of field theoretic methods for the study of flow in a heterogeneous porous medium".

In this paper, an integral equation for the Green's function is developed. A perturbation series is obtained, and this is expressed using field theory by diagrammatic means. This allows the entire series to be averaged and summed up. The averaged Green's function is then equated with the deterministic form to obtain an effective permeability in a renormalised form.

As an extension to this work, we consider the extension to a time- dependent version of this model equation. By considering the Green's function of this equation, Fourier transformed in time, we are able to show that it satisfies the same equation as the steady-state Green's function. The same diagrammatic series technique could then be employed to perform the entire summation of the averaged series. It was intended that this would enable us to develop a time-dependent effective permeability, the main problem being how to invert the Fourier time-transformed effective permeability. We attempted to invert this with contour integration, but found eventually that this was not a suitable approach. It was

concluded that there may be ways of making use of this technique in the future, but they have yet to be explored.

5.1 P. King's work

Throughout this chapter only, the permeability is denoted by K, instead of k, in order to maintain consistency with the notation of King.

The model equation under consideration is the steady-state equation (2.2.2),

$$\nabla \left(K \nabla p \right) = 0.$$

The Green's function for this equation is defined by

$$\nabla_{\mathbf{r}} \left(K(\mathbf{r}) \nabla_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') \right) = \delta(\mathbf{r} - \mathbf{r}'). \tag{5.1.1}$$

The pressure is then given as

$$p(\mathbf{r}) = \mathbf{q} \cdot \int G(\mathbf{r}, \mathbf{r}') d\mathbf{S}'. \tag{5.1.2}$$

To solve equation (5.1.1), King considers a perturbation $y(\mathbf{r})$ about a homogeneous medium, K_0, which is assumed to have a corresponding Green's function, known throughout as the deterministic, or "bare" Green's function, $G_0(\mathbf{r} - \mathbf{r}')$. This has the familiar form, as in standard Green's theory, for example, $ln|\mathbf{r} - \mathbf{r}'|$ in two-dimensional problems, and satisfies

$$K_0 \nabla^2_{\mathbf{r}} G_0(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{5.1.3}$$

So then, the full Green's function satisfies,

$$K_0 \nabla^2_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') - K_0 \nabla_{\mathbf{r}} \left(y(\mathbf{r}) \nabla_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') \right). \tag{5.1.4}$$

By multiplying equation (5.1.4) throughout by $G_0(\mathbf{r} - \mathbf{r}'')$, and integrating over all $\mathbf{r}''$ space, this equation can be transformed into an integral equation of the

form

$$G(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r}-\mathbf{r}') - \int G_0(\mathbf{r}-\mathbf{r}'')K_0\nabla_{\mathbf{r}''}y(\mathbf{r}'')G(\mathbf{r}'',\mathbf{r}')d\mathbf{r}''. \tag{5.1.5}$$

It should be stressed that this action of multiplying by the Green's function and integrating is equivalent to the inverse operation of the operator $K_0\nabla^2_{\mathbf{r}}$, stemming from the form of equation (5.1.3).

Written in Fourier transformed form, this gives the form required,

$$\hat{G}(\mathbf{j},\mathbf{k}) = \hat{G}_0(\mathbf{j})\delta(\mathbf{j}+\mathbf{k}) + \hat{G}_0(\mathbf{j})\int d\mathbf{l}d\mathbf{m}K_0\left((\mathbf{l}+\mathbf{m}).\mathbf{m}\right)\delta(\mathbf{l}+\mathbf{m}-\mathbf{j})\hat{y}(\mathbf{l})\hat{G}(\mathbf{m},\mathbf{k}), \tag{5.1.6}$$

where the Fourier transforms of G and y are denoted by $\hat{G}$ and $\hat{y}$ respectively. This enables a series to be generated.

The deterministic Green's function for the homogeneous mean is known to have the following form in Fourier transform space,

$$\hat{G}_0(\mathbf{k}) = -\frac{1}{K_0k^2}. \tag{5.1.7}$$

It is argued that if the mean behaviour of the heterogeneous medium is that of a homogeneous one with an effective permeability, then the averaged Green's function would have the analogous form,

$$\langle\hat{G}(\mathbf{k})\rangle = -\frac{1}{K_{eff}k^2}, \tag{5.1.8}$$

where K_{eff} is the effective permeability of the entire heterogeneous medium.

Evaluating terms for moments of the lognormal distribution, allows the series generated by equation (5.1.6) to be summed diagrammatically, [33], giving derived solutions of the form

$$\langle\hat{G}(\mathbf{k})\rangle^{-1} = \hat{G}_0^{-1}(\mathbf{k}) - \sum\mathbf{k}, \tag{5.1.9}$$

where $\sum \mathbf{k}$ is called the self-energy term, [33]. The expression for the self-energy term obtained is

$$\sum \mathbf{k} = K_0^2 \int d\mathbf{j} (\mathbf{k} \cdot \mathbf{j})^2 \hat{\rho}(\mathbf{k} - \mathbf{j}) \hat{G}_0(\mathbf{j}). \tag{5.1.10}$$

Here, $\hat{\rho}(\mathbf{k} - \mathbf{j})$ is the Fourier-transformed spatial correlation function considered in this context to be a real function with a real argument.

So, by equating equations (5.1.8) and (5.1.9), King gives the form for the effective permeability of a heterogeneous medium, [33], as

$$K_{eff} = K_0 + \frac{\sum(\mathbf{k})}{k^2}. \tag{5.1.11}$$

King found this term, when evaluated, to agree perfectly with that obtained by Gutjahr at al [7]. That result, however, is obtained by a truncated perturbation series in limited dimensions. The significant aspect of King's results in [33] is that no limiting assumption about the dimensions is made, and that the second order truncated series is shown to be stable to higher orders.

Writing the expression for covariance,

$$\sigma_p^2(\mathbf{r_1}, \mathbf{r_2}) = \langle p(\mathbf{r_1}) p(\mathbf{r_2}) \rangle - \langle p(\mathbf{r_1}) \rangle \langle p(\mathbf{r_2}) \rangle,$$

in terms of Green's functions, from (5.1.2), enables series terms for the covariance to be obtained also.

A degree of agreement with previously-obtained results, such as those from Bakr et al [5], is again observed.

5.2 Extension of Concept

In this section we consider the effects of introducing some time-dependence into King's approach. Our main aim is to find a time-dependent effective permeability,

using parallel steps to those in [33]. We consider the following extended equation,

$$\frac{\partial p}{\partial t} + \nabla(K(\mathbf{r})\nabla p) = 0, \tag{5.2.1}$$

which is a simplified and adapted version of our model equation, (2.2.1).

Again, we are trying to find the Green's function for this equation. This is defined to satisfy

$$\frac{\partial G}{\partial t} + \nabla_{\mathbf{r}}(K(\mathbf{r})\nabla_{\mathbf{r}} G) = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{5.2.2}$$

where the problem is assumed to be static, and

$$G = G(\mathbf{r}, \mathbf{r}', t-t').$$

Consider $K(\mathbf{r})$ to be a perturbation about some mean value, of the form

$$K(\mathbf{r}) = K_0(1 + y(\mathbf{r})). \tag{5.2.3}$$

The bare Green's function, in this context, can be defined to satisfy,

$$\frac{\partial G_0}{\partial t} + K_0\nabla^2_{\mathbf{r}} G_0 = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{5.2.4}$$

where we assume $G_0 = G_0(\mathbf{r}-\mathbf{r}', t-t')$.

Taking the Fourier transform of (5.2.4) in space and time gives

$$i\omega\hat{\tilde{G}}_0 - K_0 k^2 \hat{\tilde{G}}_0 = 1. \tag{5.2.5}$$

We define the time Fourier transform of G to be $\tilde{G}$, and the Fourier transform in space and time of G as $\hat{\tilde{G}}$. Thus,

$$\hat{\tilde{G}}_0(\mathbf{k},\omega) = \frac{1}{i\omega - K_0 k^2}. \tag{5.2.6}$$

So, we are seeking a form for the "full" Green's function, that, when averaged over all possible realisations for y, is of the form,

$$\langle \hat{\tilde{G}}(\mathbf{k},\omega)\rangle = \frac{1}{i\omega - K_{eff} k^2}. \tag{5.2.7}$$

Returning to equation (5.2.2), with $K(\mathbf{r})$ substituted as $K_0(1+y(\mathbf{r}))$, we find

$$\frac{\partial G}{\partial t} + K_0\nabla^2_{\mathbf{r}}G + K_0\nabla_{\mathbf{r}}(y(\mathbf{r})\nabla_{\mathbf{r}}G) = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{5.2.8}$$

so

$$\frac{\partial G}{\partial t} + K_0\nabla^2_{\mathbf{r}}G = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t') - K_0\nabla_{\mathbf{r}}(y\nabla_{\mathbf{r}}G), \tag{5.2.9}$$

which can be re-expressed as

$$\left(\frac{\partial}{\partial t} + K_0\nabla^2_{\mathbf{r}}\right)G = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t') - K_0\nabla_{\mathbf{r}}(y\nabla_{\mathbf{r}}G). \tag{5.2.10}$$

Now, from (5.2.4), we see that the action of the inverse operator $(\frac{\partial}{\partial t} + K_0\nabla^2_{\mathbf{r}})^{-1}$ is the same as multiplication by the bare Green's function and integration with respect to space and time. It can therefore be deduced that another form of equation (5.2.9), obtained by multiplying by $(\frac{\partial}{\partial t} + K_0\nabla^2_{\mathbf{r}})^{-1}$ on both sides, is

$$\begin{aligned} &G(\mathbf{r},\mathbf{r}',t,t') \\ &\quad = \quad G_0(\mathbf{r}-\mathbf{r}',t-t') - K_0\int\int G_0(\mathbf{r}-\mathbf{r}'',t-t'')\nabla_{\mathbf{r}''}y(\mathbf{r}'')\nabla_{\mathbf{r}''}G(\mathbf{r}'',\mathbf{r}',t'',t')d^3\mathbf{r}''dt''. \end{aligned} \tag{5.2.11}$$

Making the assumption that the full problem is static, so that $K(\mathbf{r})$ does not vary in time, we have

$$G(\mathbf{r},\mathbf{r}',t,t') = G(\mathbf{r},\mathbf{r}',t-t'),$$

so

$$\begin{aligned} &G(\mathbf{r},\mathbf{r}',t-t') \\ &\quad = \quad G_0(\mathbf{r}-\mathbf{r}',t-t') - K_0\int\int G_0(\mathbf{r}-\mathbf{r}'',t-t'')\nabla_{\mathbf{r}''}y\nabla_{\mathbf{r}''}G(\mathbf{r}'',\mathbf{r}',t''-t')d^3\mathbf{r}''dt''. \end{aligned} \tag{5.2.12}$$

We now consider taking the time Fourier transform of the Green's function,

$$\int G_0(\mathbf{r}-\mathbf{r}',t-t')e^{-i\omega(t-t')}d(t-t') = \tilde{G_0}(\mathbf{r}-\mathbf{r}',\omega).$$

with great care taken when changing variable. Equation (5.2.12) may then be re-written,

$$\begin{aligned}\tilde{G}(\mathbf{r},\mathbf{r}',\omega) &= \tilde{G}_0(\mathbf{r}-\mathbf{r}',\omega) \\ &- K_0 \int \left\{ \int\int G_0(\mathbf{r}-\mathbf{r}', t-t'')\nabla_{\mathbf{r}''} y \nabla_{\mathbf{r}''} G(\mathbf{r}'',\mathbf{r},t''-t')d^3\mathbf{r}''dt'' \right\} e^{-i\omega\tau}d\tau,\end{aligned} \tag{5.2.13}$$

where $\tau = t - t'$. Now, consider the integral I, where

$$I = \int\int G_0(\mathbf{r}-\mathbf{r}', t-t'')\nabla_{\mathbf{r}''} y \nabla_{\mathbf{r}''} G(\mathbf{r}'',\mathbf{r},t''-t')d^3\mathbf{r}''dt''.$$

If we call

$$z = t'' - t',$$

then

$$dz = dt'',$$

and

$$\begin{aligned} t - t'' &= t - t' - t'' + t' \\ &= t - t' - z \\ &= \tau - z. \end{aligned}$$

Therefore

$$I(\tau) = \int\int G_0(\mathbf{r}-\mathbf{r}', \tau - z)\nabla_{\mathbf{r}''} y(\mathbf{r}'') \nabla_{\mathbf{r}''} G(\mathbf{r}'',\mathbf{r}',z)d^3\mathbf{r}''dz. \tag{5.2.14}$$

But note, this is in the form of a simple convolution, and when we take the Fourier transform of $I(\tau)$, we are just left with the product of the two Fourier transforms within the integral. Thus,

$$F.T.[I] = \int \tilde{G}_0(\mathbf{r}-\mathbf{r}'',\omega)\nabla_{\mathbf{r}''} y(\mathbf{r}'') \nabla_{\mathbf{r}''} \tilde{G}(\mathbf{r}'',\mathbf{r}',\omega)d^3\mathbf{r}''. \tag{5.2.15}$$

So, if we write out (5.2.13) in full, we obtain,

$$\begin{aligned}\tilde{G}(\mathbf{r},\mathbf{r}',\omega) &= \tilde{G}_0(\mathbf{r}-\mathbf{r}',\omega) \\ &- \int \tilde{G}_0(\mathbf{r}-\mathbf{r}'',\omega)K_0\nabla_{\mathbf{r}''}y(\mathbf{r}'')\nabla_{\mathbf{r}''}\tilde{G}(\mathbf{r}'',\mathbf{r}',\omega)d^3\mathbf{r}''\end{aligned} \tag{5.2.16}$$

Thus, the Fourier transform in time of the time dependent Green's function exactly satisfies the same equation as the Green's function for the steady-state equation (5.1.5), in [33]. This is considered a very interesting and significant result. Equation (5.2.16) is quite general, and so any techniques used in solving the steady-state problem in [33], may be re-applied to each frequency mode ω to give a solution for the averaged Green's function as in the steady-state case.

We apply the results from equation (5.1.9)

$$\langle\hat{G}(\mathbf{k})\rangle^{-1} = \hat{G_0}^{-1} - \sum \mathbf{k},$$

where,

$$\sum \mathbf{k} = K_0^2 \int (\mathbf{k}\cdot\mathbf{j})^2 \hat{\rho}(\mathbf{k}-\mathbf{j})\hat{G_0}(\mathbf{j})d^3\mathbf{j}.$$

Using exactly the same techniques and arguments, and treating the frequency as a parameter (or each frequency mode separately), as in [33], we obtain the form of this equation for the time-dependent case as

$$\langle\hat{\tilde{G}}(\mathbf{k},\omega)\rangle^{-1} = \hat{\tilde{G}}_0^{-1}(\mathbf{k},\omega) - \sum \mathbf{k}, \tag{5.2.17}$$

where

$$\begin{aligned}\sum \mathbf{k} &= K_0^2 \int d^3\mathbf{j}(\mathbf{k}\cdot\mathbf{j})^2\hat{\rho}(\mathbf{k}-\mathbf{j})\hat{\tilde{G}}_0(\mathbf{k},\omega) \\ &= K_0^2 \int d^3\mathbf{j}(\mathbf{k}\cdot\mathbf{j})^2 \frac{1}{i\omega - K_0 j^2}.\end{aligned} \tag{5.2.18}$$

5.2.1 One-dimensional Example

We consider here a one-dimensional example of this idea. It ought to be fairly easy to extend the results, for this case, to three dimensions. The equation governing the behaviour is

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(K(x)\frac{\partial p}{\partial x}\right) = 0, \tag{5.2.19}$$

$K(x)$ being the one-dimensional heterogeneous permeability, and K_0 its homogeneous mean value. The main problem is in calculating the self-energy term, equation (5.2.18). The added problem now is that it contains both a real and imaginary part. It is hoped that the imaginary part can be included with the $i\omega$ term, in the bare Green's function, in some way to give a fully averaged Green's function of the form,

$$\langle \hat{\tilde{G}}(k,\omega)\rangle = \frac{1}{i(\omega+\alpha) - K_{eff}k^2}. \tag{5.2.20}$$

Consider (5.2.18) for the one-dimensional case,

$$\begin{aligned}
\sum \mathbf{k} &= K_0^2 \int k^2 j^2 \hat{\rho}(k-j)\frac{1}{i\omega - K_0 j^2}dj \\
&= K_0 k^2 \int_{-\infty}^{\infty} \frac{K_0 j^2}{i\omega - K_0 j^2}\hat{\rho}(k-j)dj \\
&= -K_0 k^2 \int_{-\infty}^{\infty} \left\{ \left(\frac{i\omega - K_0 j^2}{i\omega - K_0 j^2}\right)\hat{\rho}(k-j) - \frac{i\omega}{i\omega - K_0 j^2}\hat{\rho}(k-j)\right\} dj \\
&= -K_0 k^2 \left\{ \int_{-\infty}^{\infty} \hat{\rho}(k-j)dj - \int_{-\infty}^{\infty} \frac{i\omega}{i\omega - K_0 j^2}\hat{\rho}(k-j)dj \right\}.
\end{aligned} \tag{5.2.21}$$

The first term in (5.2.21) is easy to cope with because, if I_1 is defined as

$$I_1 = \int_{-\infty}^{\infty} \hat{\rho}(k-j)dj,$$

then, by a change of variables to $l = k - j$,

$$\begin{aligned}
I_1 &= -\int_{\infty}^{-\infty} \hat{\rho}(l)dl \\
&= \int_{-\infty}^{\infty} \hat{\rho}(l)dl \\
&= \rho(0),
\end{aligned} \tag{5.2.22}$$

(that is, $I_1 = \rho(r = 0)$)by inverse Fourier transform, as in the appendix of [33].

Now consider the second term of (5.2.21), defining

$$I_2 = \int_{-\infty}^{\infty} \frac{i\omega}{i\omega - K_0 j^2} \hat{\rho}(k-j)dj.$$

We use calculus of residues to attempt to evaluate this. So,

$$\begin{aligned} I_2 &= \frac{i\omega}{K_0} \int_{-\infty}^{\infty} \frac{\hat{\rho}(k-j)}{(\frac{i\omega}{K_0} - j^2)} dj \\ &= \frac{i\omega}{K_0} \int_{-\infty}^{\infty} \frac{\hat{\rho}(k-j)}{\left(\sqrt{\frac{\omega}{2K_0}}(i+1) - j\right)\left(\sqrt{\frac{\omega}{2K_0}}(i+1) + j\right)} dj, \end{aligned}$$

or, by setting $a_0 = \sqrt{\frac{\omega}{2K_0}}(i+1)$,

$$I_2 = \frac{i\omega}{K_0} \int_{-\infty}^{\infty} \frac{\hat{\rho}(k-j)}{(a_0 - j)(a_0 + j)} dj. \tag{5.2.23}$$

To perform the integration, we need to integrate around the edge of the top half of the complex plane, along an infinite semi-circle, denoted as C. We assume the semi-circle has radius R, and that $R \to \infty$. Note here, that $\hat{\rho}$ is being treated as a complex function. This is due to the fact that it is now a function of a complex argument.

We have considered the two most straightforward cases for the form that $\hat{\rho}$ might take:

1. $\hat{\rho}(k-j)$ is wholly analytic within the region enclosed by C (this is not likely).
2. $\hat{\rho}(k-j)$ has a finite number of simple poles in the region enclosed by the contour C, denoted by a_1, a_2, a_3,...a_n.

If results for these two cases can be obtained, it is hoped that we can generalise the results to the more complicated cases which would be:

- finite number of multiple poles,

- infinite number of simple poles, and
- infinite number if multiple poles.

Case 1 implies that the integrand has one simple pole contained within the upper half of the complex plane, at a_0. The residue is then given by

$$\begin{aligned} & \lim_{j \to a_0} (j - a_0) \frac{\hat{\rho}(k-j)}{(a_0 - j)(a_0 + j)} \qquad (5.2.24) \\ &= -\frac{\hat{\rho}(k - a_0)}{2a_0} \\ &= -\frac{\hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}}\right)}{(i+1)\sqrt{\frac{2\omega}{K_0}}} \\ &= -\sqrt{\frac{K_0}{2\omega}} \frac{(1-i)}{2} \;\; \hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}}\right). \end{aligned}$$

(5.2.25)

And so,

$$\begin{aligned} \oint_C \frac{\hat{\rho}(k-j)}{(a_0 - j)(a_0 + j)} &= 2\pi i \times residue \\ &= \pi i \sqrt{\frac{K_0}{2\omega}} (i-1) \;\; \hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}}\right). \end{aligned}$$

(5.2.26)

If we assume Jordan's lemma [3] is satisfied, then the contribution of the infinite semi-circle to the integral is zero, and

$$\oint_C = \int_{-\infty}^{\infty}.$$

Then

$$\oint_C \frac{\hat{\rho}(k-j)}{(a_0 - j)(a_0 + j)} = \pi \sqrt{\frac{K_0}{2\omega}} (-1-i) \;\; \hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}}\right). \qquad (5.2.27)$$

Hence

$$I_2 = \frac{i\omega}{K_0} \pi \sqrt{\frac{K_0}{2\omega}} (-1-i) \;\; \hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}}\right),$$

and

$$\sum \mathbf{k} = -K_0 k^2 \left\{ \rho(0) + (i-1)\pi \sqrt{\frac{\omega}{2K_0}} \;\; \hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}} \right) \right\}. \qquad (5.2.28)$$

In case 2, $\hat{\rho}(k-j)$ has a finite number of simple poles at points a_1, a_2, a_3,...a_n, with corresponding residues b_1, b_2, b_3,...b_n. Then, by an expansion theorem and use of the formula

$$\oint \frac{\hat{\rho}(k-j)}{i\omega - K_0 j^2} dj = 2\pi i \times \sum residues, \qquad (5.2.29)$$

we find

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\hat{\rho}(k-j)}{i\omega - K_0 j^2} dj \\ &= \quad 2\pi i \left\{ -\sqrt{\frac{K_0}{2\omega}} \left(\frac{1-i}{2} \right) \hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}} \right) + \sum_{n=1}^{N} \frac{b_n}{\frac{i\omega}{K_0} - a_n^2} \right\}, \qquad (5.2.30) \\ & \qquad (5.2.31) \end{aligned}$$

where

$$b_n = \lim_{j \to a_n} (j - a_n) \hat{\rho}(k-j).$$

This result is known formally as Mittag Leffler's expansion theorem, [3] and [4]

Note the extra set of terms in expression (5.2.31) compared to (5.2.28) are the summation terms,

$$\begin{aligned} &\frac{i\omega}{K_0} 2\pi i \sum \frac{b_n}{\frac{i\omega}{K_0} - a_n^2} \\ &= \quad -\frac{2\pi\omega}{K_0} \sum \frac{\left(\frac{i\omega}{K_0} + a_n^2 \right) b_n}{-\frac{\omega^2}{K_0^2} - a_n^4} \\ &= \quad \frac{2\pi\omega}{K_0} \sum_n \frac{\left(a_n^2 + \frac{i\omega}{K_0} \right)}{\left(\frac{\omega^2}{K_0^2} + a_n^4 \right)} \;\; b_n \qquad (5.2.32) \end{aligned}$$

Although fairly complicated, this expression makes little difference to the structure of the self-energy term in equation (5.2.28).

A sensible model for the $\hat{\rho}(k-j)$ function might contain one simple pole in the upper half of the complex plane, in which case the self-energy takes on the following form,

$$\sum \mathbf{k} = -K_0 k^2 \left\{ \rho(0) + (i-1)\pi \sqrt{\frac{\omega}{2K_0}} \hat{\rho}\left(k - (i+1)\sqrt{\frac{\omega}{2K_0}}\right) + \frac{2\pi\omega}{K_0} \frac{\left(a^2 + \frac{i\omega}{K_0}\right)}{\left(\frac{\omega^2}{K_0^2} + a^4\right)} \quad b_n \right\}. \tag{5.2.33}$$

Further useful progress is not now possible without assuming a specific form for the autocorrelation function. We find that this presents a difficult problem in that the types of function assumed in [33], notably of the form e^{-z^2}, do not satisfy the Jordan's lemma, so that the contribution from the infinite semi-circle is patently non-zero.

We must either consider alternative forms for the autocorrelation function, or an alternative method of evaluating the self-energy term. This work was not pursued further, but it was certainly felt that this line of approach ought to be continued in the future, and may give useful results.

There is also the possibility of using a Laplace transform in time, as an alternative. This was explored, but no useful preliminary results were obtained; however, a fresh approach might prove useful.

In the following chapter, we review and extend work done on an ordinary differential equation, containing uncertainty. The main interest is, again, on how the uncertainty in the solution is effected and changed by the progression of the solution, both analytic, and numerical, in time. We then go on to consider in detail the behaviour of the discretised model equations for both the steady-state and time-dependent cases.

Chapter 6

Analysis of a Stochastic Ordinary Differential Equation

In this chapter, we summarise, and extend some earlier work that was completed in order to satisfy the requirements for an MSc dissertation, [17]. All the work summarised in section 6.1 represents a review of that from the MSc dissertation, [17], and it is not intended that this section forms part of the requirements for the research project.

The work in this chapter provides an introduction to the research project as a whole, and turns out to give some valuable insights into the general statistical behaviour of these sorts of equations. Some of these ideas are included in further consideration of the more complicated partial differential equations studied in subsequent chapters.

6.1 Mass Balance Model

We consider the simple form of the model equation (2.2.4)

$$\dot{y} + \kappa y = f(t) \qquad\qquad y(0) = 0. \tag{6.1.1}$$

Due to the comparative simplicity of this o.d.e., we can obtain directly a closed form of the solution,

$$y = \int_0^t e^{\kappa(\tau - t)} f(\tau) d\tau. \tag{6.1.2}$$

6.1.1 Mean Value of Solution

The mean value on either side of equation (6.1.2) can be taken, to allow the mean value of the analytic solution to be expressed as

$$\langle y(t) \rangle = \int_0^t \langle e^{\kappa(\tau - t)} \rangle f(\tau) d\tau. \tag{6.1.3}$$

We note that (6.1.3) can be written in terms of the moment generating function of the distribution,

$$\langle y(t) \rangle = \int_0^t \mathcal{G}(-i(\tau - t)) f(\tau) d\tau, \tag{6.1.4}$$

from the basic definition of the moment generating function, (3.3.1). This can then be compared to the quantity defined in the introductory chapter, chapter 2, as the intuitive or deterministic solution,

$$\tilde{y}(t) = \int_0^t e^{\langle \kappa \rangle (\tau - t)} f(\tau) d\tau. \tag{6.1.5}$$

The difference between the mean value of the solution, $\langle y(t) \rangle$, and the deterministic solution $\tilde{y}(t)$, is defined by

$$\begin{aligned} e_y(t) &= \int_0^t \left\{ \langle e^{\kappa(\tau - t)} \rangle - e^{\langle \kappa \rangle (\tau - t)} \right\} f(\tau) d\tau \\ &= \int_0^t \left\{ \sum_{\mu=2}^{\infty} (\langle \kappa^\mu \rangle - \langle \kappa \rangle^\mu) \frac{(\tau - t)^\mu}{\mu!} \right\} f(\tau) d\tau. \end{aligned} \tag{6.1.6}$$

To proceed, we need to evaluate the moment terms in equation (6.1.6), for which, in general, we do not have explicit forms. However, if we limit considerations to specific distribution types, such as Gaussian, we can at least obtain

recurrence relationships for successive moments. For example, for a Gaussian it is found, [17],

$$\langle x^n \rangle = (n-1)\sigma^2 \langle x^{n-2} \rangle + \langle x \rangle \langle x^{n-1} \rangle. \tag{6.1.7}$$

This allows the first few terms in equation (6.1.6) to be summed up, giving

$$e_y(t) = \int_0^t \left\{ \sigma^2 \frac{(\tau - t)^2}{2} + \frac{3\sigma^2 \langle \kappa \rangle}{6}(\tau - t)^3 + \frac{(3\sigma^4 + 6\sigma^2 \langle \kappa \rangle^2)}{24}(\tau - t)^4 + \cdots \right\} f(\tau) d\tau. \tag{6.1.8}$$

This series can be considered quantitatively to see that $e_y(t)$ cannot be guaranteed to be equal to zero at any time, particularly as $t \to \infty$.

At this point, it is not possible to sum up these infinite series completely. However, later work provides better insight into their general behaviour.

For the Gaussian case, we are also able to make use of the property of the cumulants, ξ_n, in that,

$$\begin{aligned} log(\mathcal{G}(x)) &= ix\xi_1 - \frac{x^2}{2}\xi_2 \\ &= ix\langle \kappa \rangle - \frac{x^2}{2}\sigma^2. \end{aligned}$$

Therefore,

$$\mathcal{G}(x) = exp\left(ix\langle \kappa \rangle - \frac{x^2}{2}\sigma^2 \right). \tag{6.1.9}$$

So if, for simplicity, we take $f(t) = 1$, for all time, so that

$$\langle y(t) \rangle = \int_0^t exp\left\{ \langle \kappa \rangle (\tau - t) + \frac{\sigma^2 (\tau - t)^2}{2} \right\} d\tau, \tag{6.1.10}$$

and we compare this with the deterministic solution,

$$\tilde{y}(t) = \frac{1}{\langle \kappa \rangle}(1 - e^{-\langle \kappa \rangle t}),$$

then we could argue that $\tilde{y}(t)$ was a valid approximation to $\langle y(t) \rangle$, but only in the region $[0, t_1]$, where

$$t_1 \ll \frac{2\langle \kappa \rangle}{\sigma^2}. \tag{6.1.11}$$

6.1.2 Development of Probability Distribution Function

We also explored in [17] the analytic development of the p.d.f. of the solution of model equation (2.2.4) in time.

First, we assumed that an incremental probability that κ takes a value within the range κ and $\kappa + d\kappa$ corresponds to the probability that y lies between y and $y + dy$, at time t, so that

$$\rho_\kappa(\kappa)d\kappa = \rho_y(y)dy. \tag{6.1.12}$$

Then, we can obtain an equation for the p.d.f. of the solution, of the type

$$\begin{aligned} \rho_y(y) &= \rho_\kappa(\kappa)\frac{d\kappa}{dy} \\ &= \frac{\rho_\kappa(\kappa)}{\frac{dy}{d\kappa}}. \end{aligned} \tag{6.1.13}$$

The term $\frac{dy}{d\kappa}$ can be extracted from the closed form of the solution, (6.1.2), so that

$$\rho_y(y) = \frac{\rho_\kappa(\kappa)}{\int_0^t (\tau - t)e^{\kappa(\tau - t)}f(\tau)d\tau}. \tag{6.1.14}$$

In [17] quadrature rules are used to obtain experimental results for plots of p.d.f. of the (analytic) solution with respect to time.

6.1.3 Experimental Results

We present here some examples of the plots obtained in [17]. In each case, figure (a) shows plots of the p.d.f. of the solution at different time intervals. The figures (b) show the behaviour of the mean value of the solution, and the deterministic solution plotted against time.

Figures 6.1.1(a) and 6.1.1(b) present the results when κ is assumed to have a Gaussian distribution, with mean value 0.0 and variance $\sigma^2 = 4.0$, and forcing function $f(t)$ equal to 1.0 for all time. The p.d.f. plots are plotted at time intervals of 0.125 seconds each.

In figures 6.1.2(a) and 6.1.2(b) we see a similar case to 6.1, but with much smaller variance, $\sigma^2 = 0.25$.

In figures 6.1.3(a) and 6.1.3(b) we see a similar case again, but this time the mean of κ is 1.0, with a Gaussian p.d.f., of variance, $\sigma^2 = 0.25$.

In figures 6.1.4(a) and 6.1.4(b) the forcing function is of the form $F(t) = t^2$, with $\langle\kappa\rangle = 1.0$, with a Gaussian p.d.f., variance $\sigma^2 = 0.25$.

The last figures, 6.1.5(a) and 6.1.5(b) κ has a uniform distribution form, with mean value $\langle\kappa\rangle = 1.0$, and variance $\sigma^2 = 0.25$. The forcing function is a constant one, $F(t) = 1.0$.

Figure 6.1.1(a) Solution p.d.f. plotted at 0.125 sec. intervals, $\langle\kappa\rangle = 0.0$, $\sigma_\kappa^2 = 4.0$ Gaussian distribution and $f(t) = 1.0$.

Figure 6.1.1(b) Plot of $\langle y\rangle$ and $\tilde{y}$ against time.

Figure 6.1.2(a) Solution p.d.f. plotted at 0.125 sec. intervals, $\langle\kappa\rangle = 0.0$, $\sigma_\kappa^2 = 0.25$ Gaussian distribution and $f(t) = 1.0$.

Figure 6.1.2(b) Plot of $\langle y\rangle$ and $\tilde{y}$ against time.

Figure 6.1.3(a) Solution p.d.f. plotted at 0.125 sec. intervals, $\langle\kappa\rangle = 1.0$, $\sigma_\kappa^2 = 0.25$ Gaussian distribution and $f(t) = 1.0$.

Figure 6.1.3(b) Plot of $\langle y\rangle$ and $\tilde{y}$ against time.

Figure 6.1.4(a) Solution p.d.f. plotted at 0.125 sec. intervals, $\langle\kappa\rangle = 1.0$, $\sigma_\kappa^2 = 0.25$ Gaussian distribution and $f(t) = t^2$.

Figure 6.1.4(b) Plot of $\langle y\rangle$ and $\tilde{y}$ against time.

Figure 6.1.5(a) Solution p.d.f. plotted at 0.125 sec. intervals, $\langle\kappa\rangle = 1.0$, $\sigma_\kappa^2 = 0.25$ uniform distribution and $f(t) = 1.0$.

Figure 6.1.5(b) Plot of $\langle y\rangle$ and $\tilde{y}$ against time.

In figure 6.1.1 we see data where the mean value of κ is zero and there is a comparatively large dispersion, $\sigma^2 = 4.0$. As expected, the mean value of the solution diverges away from the deterministic solution, $\tilde{y}$, almost immediately, with $\tilde{y}$ being analytically linear. The initial movement of the p.d.f. is in a positive direction, accompanied by its dispersion. After a while, it comes to a halt, with a peak value of 0.5, which is rather surprising, since it implies a high probability of the solution tending towards this value as the time increases, even though the mean and variance increase unboundedly.

In figure 6.1.2, the mean value of κ is again zero, with a comparatively smaller variance than in 6.1.1. Accordingly, $\langle y(t) \rangle$ diverges away from $\tilde{y}(t)$ much less drastically at first, owing to the higher value of t_1, in equation (6.1.11), when plotted on the same scale. This behaviour also accounts for the fact that the p.d.f. moves further before coming to a stop, with peak at about 2.3.

Figure 6.1.3 shows behaviour when $\langle \kappa \rangle$ has a non-zero value, with the same variance as 6.1.2. The deterministic solution has the analytic solution

$$\tilde{y}(t) = 1 - e^{-t},$$

and tends asymptotically to 1.0. For time less than 1.0, the deterministic solution does approximate $\langle y(t) \rangle$, in accord with expression (6.1.11), but there is rather extreme divergence after this. The p.d.f., plotted at time intervals of 0.25, tends to a fixed profile, as in the previous two figures, reflecting the behaviour of $\tilde{y}(t)$, There is less dispersion, though, as this is to some extent, connected with the distance travelled by the p.d.f.

In figure 6.1.4 $\tilde{y}(t)$ tends asymptotically to $t^2 - 2t + 2$. This demonstrates a case where both the deterministic solution, and the mean value grow unboundedly with respect to time. They do so, however, at considerably different rates.

In the last figures, 6.1.5, we see what is arguably the most interesting be-

haviour. There is convergence of both the deterministic solution and the mean value, but to different values.

For some of the results, notably those in figures 6.1.1(a) and 6.1.1(b), and figures 6.1.3(a) and 6.1.3(b), we can see an apparent stabilisation of the solution p.d.f. whilst the mean increases unboundedly. This is due to the effects of the tail of the distribution always making a contribution to the mean value as time increases. It could be argued in this sort of case that the deterministic solution $\tilde{y}$ gives a better reflection of the dynamic behaviour of the distribution function than the actual mean value $\langle y \rangle$.

6.1.4 Numerical Approach

The effect of applying a numerical scheme to the model equation is also investigated in [17].

It is shown that when applying a numerical scheme to both the mean value of the solution and the p.d.f. of the solution, the convergence is of same order as the convergence achieved when applying the same scheme to a deterministic problem [17].

6.2 Further Analysis of Model O.D.E.

The work in this section is an extension to the previous MSc work, [17], and was done as part of the main research project. We use a form of one-dimensional sensitivity analysis, which enables the series terms of the type in (6.1.6) to be investigated further. For a general value of κ, consider the solution y for a fixed time as an expansion about $\langle \kappa \rangle$,

$$y(\kappa) = y(\langle \kappa \rangle) \quad + \quad (\kappa - \langle \kappa \rangle)\frac{\partial y(\langle \kappa \rangle)}{\partial \kappa} + \frac{(\kappa - \langle \kappa \rangle)^2}{2!}\frac{\partial^2 y(\langle \kappa \rangle)}{\partial \kappa^2}$$

$$+ \quad \cdots + \frac{(\kappa - \langle\kappa\rangle)^n}{n!}\frac{\partial^n y(\langle\kappa\rangle)}{\partial\kappa^n} + \cdots \tag{6.2.1}$$

Assuming uniform convergence of the series in equation (6.2.1), we can take the mean over all possible values of κ, on either side of this equation, to obtain

$$\begin{aligned}\langle y(\kappa)\rangle = y(\langle\kappa\rangle) \quad &+ \quad \frac{\langle(\kappa - \langle\kappa\rangle)^2\rangle}{2!}\frac{\partial^2 y(\langle\kappa\rangle)}{\partial\kappa^2} \\ &+ \quad \cdots + \frac{\langle(\kappa - \langle\kappa\rangle)^n\rangle}{n!}\frac{\partial^n y(\langle\kappa\rangle)}{\partial\kappa^n} + \cdots \end{aligned} \tag{6.2.2}$$

The term of particular interest is the difference between the mean value of the solution and the deterministic solution,

$$\langle y(\kappa)\rangle - y(\langle\kappa\rangle) = \sum_{n=2}^{\infty}\frac{\nu_n}{n!}\frac{\partial^n y(\langle\kappa\rangle)}{\partial\kappa^n}, \tag{6.2.3}$$

where ν_n is the n^{th} corrected moment, $\langle(\kappa - \langle\kappa\rangle)^n\rangle$. The time dependence is contained within the derivatives of y, and must be considered carefully. This is an example of one dimensional uncertain stability analysis, as in [35].

Equation (6.2.3) is true for any general κ distribution, defined by mean $\langle\kappa\rangle$, and corrected moments $\{\nu_n\}$. It is now illustrative to consider the effect of substituting specific distributions. If the distribution is symmetric about the mean, the shifted moments for n odd are equal to zero from chapter 3, equation (3.4.13).

So, for a symmetric distribution function, equation(6.2.3) becomes

$$\langle y(\kappa)\rangle - y(\langle\kappa\rangle) = \sum_{m=1}^{\infty}\frac{\nu_{2m}}{2m!}\frac{\partial^{2m} y(\langle\kappa\rangle)}{\partial\kappa^{2m}}. \tag{6.2.4}$$

For simple distribution functions, fairly straightforward, explicit formulae for the $2m^{th}$ shifted moments can be obtained. For a Gaussian distribution, with variance σ^2, from [17],

$$\nu_{2m} = \frac{(2m-1)!\sigma^{2m}}{2^{m-1}(m-1)!} \qquad \forall m \in \mathbb{N}. \tag{6.2.5}$$

For a Uniform distribution, defined as

$$\rho(\kappa) = \begin{cases} \frac{1}{2A}, & \langle\kappa\rangle - A \leq \kappa \leq \langle\kappa\rangle + A \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\nu_{2m} = \frac{A^{2m}}{2m+1} \qquad \forall m \in \mathbb{N}. \tag{6.2.6}$$

6.2.1 Example: A Uniform Distribution

Considering the behaviour for a uniform distribution first, we find that equation (6.2.4) becomes

$$\begin{aligned} \langle y(\kappa)\rangle - y(\langle\kappa\rangle) &= \sum_{m=1}^{\infty} \frac{A^{2m}}{2m!(2m+1)} \frac{\partial^{2m} y(\langle\kappa\rangle)}{\partial\kappa^{2m}} \\ &= \sum_{m=1}^{\infty} \frac{A^{2m}}{(2m+1)!} \frac{\partial^{2m} y(\langle\kappa\rangle)}{\partial\kappa^{2m}}. \end{aligned} \tag{6.2.7}$$

The important question is does this series converge? The answer can easily be obtained by looking at successive terms in the series and employing the ratio test. If we find that the differential term $\frac{\partial^{2m} y}{\partial\kappa^{2m}}$ equals zero for one particular, and all successively higher terms, we are left with a finite series which is, by definition, convergent. It is important to state when employing the ratio test that it is only the limit of successive terms that we must consider, and cases where individual terms are zero result in a truncated series, where convergence can be assessed by consideration of the limit of the ratio.

If the series is written as $\sum_{m=1}^{\infty} a_m$, then, the ratio of absolute value of successive terms can be written

$$\begin{aligned} \frac{|a_{m+1}|}{|a_m|} &= \frac{A^{2m+2}}{(2m+3)!} \left| \frac{\partial^{(2m+2)} y(\langle\kappa\rangle)}{\partial\kappa^{(2m+2)}} \right| \times \frac{(2m+1)!}{A^{2m}} \frac{1}{\left| \frac{\partial^{2m} y(\langle\kappa\rangle)}{\partial\kappa^{2m}} \right|} \\ &= \frac{A^2}{(2m+3)(2m+2)} \frac{\left| \frac{\partial^{(2m+2)} y(\langle\kappa\rangle)}{\partial\kappa^{(2m+2)}} \right|}{\left| \frac{\partial^{2m} y(\langle\kappa\rangle)}{\partial\kappa^{2m}} \right|}. \end{aligned}$$

If we substitute b_n for $\left| \frac{\partial^n y(\langle\kappa\rangle)}{\partial\kappa^n} \right|$, then

$$\frac{|a_{m+1}|}{|a_m|} = \frac{A^2}{(2m+3)(2m+2)} \frac{b_{2m+2}}{b_{2m}}, \tag{6.2.8}$$

where A^2 is finite, by definition of the original distribution. Now let us make the assumption that $\frac{b_{2m+2}}{b_{2m}}$ is bounded for all values of m. Therefore, $A^2\frac{b_{2m+2}}{b_{2m}}$ is a finite number. So there must exist an integer, M, such that

$$(2M+3)(2M+2) > A^2\frac{b_{2m+2}}{b_{2m}},$$

which implies

$$\frac{A^2}{(2M+3)(2M+2)}\frac{b_{2m+2}}{b_{2m}} < 1, \tag{6.2.9}$$

and what's more,

$$\frac{A^2}{(2n+3)(2n+2)}\frac{b_{2n+2}}{b_{2n}} < 1, \qquad \forall n \geq M. \tag{6.2.10}$$

So the series $\sum_{m=1}^{\infty} a_m$ is convergent under these conditions, due to the ratio test.

For a finite time value, the assumption that $\frac{\partial^{2m+2}y(\langle\kappa\rangle)}{\partial\kappa^{2m+2}}/\frac{\partial^{2m}y(\langle\kappa\rangle)}{\partial\kappa^{2m}}$ is bounded is a fairly reasonable one, and so it is not expected that $\langle y(\kappa)\rangle - y(\langle\kappa\rangle)$ diverges to infinity for a finite time, under standard conditions. It may, however, converge to a very large number. The interesting behaviour, though, is that for $t \to \infty$.

Looking at a specific example, consider the model equation (6.1.1), and its closed form solution (6.1.2),

$$y = \int_0^t e^{\kappa(\tau-t)}f(\tau)d\tau.$$

This implies that

$$\frac{\partial y}{\partial\kappa} = \int_0^t (\tau-t)e^{\kappa(\tau-t)}f(\tau)d\tau, \tag{6.2.11}$$

and so

$$\frac{\partial^n y(\langle\kappa\rangle)}{\partial\kappa^n} = \int_0^t (\tau-t)^n e^{\langle\kappa\rangle(\tau-t)}f(\tau)d\tau. \tag{6.2.12}$$

An explicit formula for this can be calculated, if we make a specific assumption about the form of $f(t)$. Let us consider $f(t) = 1$, and then define I_n to be

$$I_n = \int_0^t (\tau-t)^n e^{\langle\kappa\rangle(\tau-t)}d\tau.$$

Integrating by parts gives

$$I_n = -\frac{(-t)^n}{\langle\kappa\rangle}e^{-\langle\kappa\rangle t} - \frac{n}{\langle\kappa\rangle}I_{n-1}. \tag{6.2.13}$$

If $n = 0$,

$$\begin{aligned} I_0 &= \int_0^t e^{\langle\kappa\rangle(\tau-t)}d\tau \\ &= \frac{1}{\langle\kappa\rangle}\left(1 - e^{-\langle\kappa\rangle t}\right), \end{aligned}$$

and if $n \geq 1$,

$$\begin{aligned} I_1 &= -\frac{1}{\langle\kappa\rangle^2}\left(1 - e^{-\langle\kappa\rangle t} - \langle\kappa\rangle t e^{-\langle\kappa\rangle t}\right), \\ &\vdots \\ &\vdots \end{aligned}$$

In general, it can be proved by simple induction that

$$I_n = \frac{(-1)^n n!}{\langle\kappa\rangle^{n+1}}\left(1 - e^{-\langle\kappa\rangle t}\sum_{m=0}^{n}\frac{(\langle\kappa\rangle t)^m}{m!}\right). \tag{6.2.14}$$

Looking at the general case for a uniform distribution, we find

$$\langle y(\kappa)\rangle - y(\langle\kappa\rangle) = \sum_{m=1}^{\infty}\frac{A^{2m}}{(2m+1)\langle\kappa\rangle^{2m+1}}\left(1 - e^{-\langle\kappa\rangle t}\sum_{n=0}^{2m}\frac{(\langle\kappa\rangle t)^n}{n!}\right) \tag{6.2.15}$$

As $t \to \infty$, if $\langle\kappa\rangle$ is strictly negative, then the $e^{-\langle\kappa\rangle t}$ always dominates the $\sum_{n=0}^{2m}\frac{(\langle\kappa\rangle t)^n}{n!}$ terms, and always blows up. So,

$$\langle\kappa\rangle < 0 \Rightarrow \langle y(\kappa)\rangle - y(\langle\kappa\rangle) \to \infty,$$

as $t \to \infty$.

If $\langle\kappa\rangle$ is greater than, or equal to 0, then as $t \to \infty$ we find that $e^{-\langle\kappa\rangle t}$ eventually dominates the $\frac{(\langle\kappa\rangle t)^n}{n!}$ terms, and therefore,

$$\begin{aligned} \langle y(\kappa)\rangle - y(\langle\kappa\rangle) &\to \sum_{m=1}^{\infty}\frac{A^{2m}}{(2m+1)\langle\kappa\rangle^{2m+1}} \\ &= \sum_{m=1}^{\infty}\frac{1}{\langle\kappa\rangle}\left(\frac{A}{\langle\kappa\rangle}\right)^{2m}\frac{1}{2m+1} \\ &= \frac{1}{\langle\kappa\rangle}\sum_{m=1}^{\infty}\left(\frac{A}{\langle\kappa\rangle}\right)^{2m}\frac{1}{2m+1}. \end{aligned} \tag{6.2.16}$$

A necessary and sufficient condition for this series to converge is just

$$\frac{A}{\langle\kappa\rangle} < 1.$$

Therefore, the condition for convergence of the series is $A < \langle\kappa\rangle$, and there is divergence if $A \geq \langle\kappa\rangle$. In the case where $A = \langle\kappa\rangle$, the series becomes $\frac{1}{\langle\kappa\rangle}\sum_{m=1}^{\infty}\frac{1}{2m+1}$, which is divergent, albeit very slowly.

So, to summarise the case for a uniform distribution, $\langle y(\kappa)\rangle - y(\langle\kappa\rangle)$ diverges if either $\langle\kappa\rangle$ is negative, or $A \geq \langle\kappa\rangle$. This corresponds to a constraint that all possible values for κ about $\langle\kappa\rangle$ must lie in the positive half-plane. This corresponds with the intuitive result that any possible negative value for κ contributes to an eventual blow up of the mean value.

A more rigorous proof of this is as follows, obtained by considering series (6.2.15) again,

$$\langle y(\kappa)\rangle - y(\langle\kappa\rangle) = \sum_{m=1}^{\infty}\frac{A^{2m}}{(2m+1)\langle\kappa\rangle^{2m+1}}\left(1 - e^{-\langle\kappa\rangle t}\sum_{n=0}^{2m}\frac{(\langle\kappa\rangle t)^n}{n!}\right).$$

If the terms in the series are

$$a_m = \frac{A^{2m}}{(2m+1)\langle\kappa\rangle^{2m+1}}\left(1 - e^{-\langle\kappa\rangle t}\sum_{n=0}^{2m}\frac{(\langle\kappa\rangle t)^n}{n!}\right),$$

then, as $2m \to \infty$, so that $\sum_{n=0}^{2m}\frac{(\langle\kappa\rangle t)^n}{n!} \to e^{\langle\kappa\rangle t}$,

$$\begin{aligned} a_m &\to \frac{A^{2m}}{(2m+1)\langle\kappa\rangle^{2m+1}}\left(1 - e^{-\langle\kappa\rangle t}e^{\langle\kappa\rangle t}\right) \\ &= \frac{A^{2m}}{(2m+1)\langle\kappa\rangle^{2m+1}} \times (0)\,. \end{aligned} \tag{6.2.17}$$

So, $a_m \to 0$.

However, this does not necessarily imply convergence of the series. The important factor is again to consider the ratio of successive terms as $m \to \infty$. This is given by (6.2.8),

$$\frac{|a_{m+1}|}{|a_m|} = \frac{A^2}{(2m+3)(2m+2)}\frac{\left|\frac{\partial^{(2m+2)}y(\langle\kappa\rangle)}{\partial\kappa^{(2m+2)}}\right|}{\left|\frac{\partial^{2m}y(\langle\kappa\rangle)}{\partial\kappa^{2m}}\right|}$$

$$= \frac{A^2(2m+1)}{\langle\kappa\rangle^2(2m+3)} \frac{\left(1 - e^{-\langle\kappa\rangle t}\sum_{n=0}^{2m+2}\frac{(\langle\kappa\rangle t)^n}{n!}\right)}{\left(1 - e^{-\langle\kappa\rangle t}\sum_{n=0}^{2m}\frac{(\langle\kappa\rangle t)^n}{n!}\right)}. \tag{6.2.18}$$

Now, we put

$$\sum_{n=0}^{2m}\frac{(\langle\kappa\rangle t)^n}{n!} = e^{\langle\kappa\rangle t} - \sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!},$$

so that,

$$\begin{aligned}
\frac{|a_{m+1}|}{|a_m|} &= \frac{A^2(2m+1)}{\langle\kappa\rangle^2(2m+3)} \frac{(1 - e^{-\langle\kappa\rangle t}(e^{\langle\kappa\rangle t} - \sum_{2m+3}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}))}{(1 - e^{-\langle\kappa\rangle t}(e^{\langle\kappa\rangle t} - \sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}))} \\
&= \frac{A^2(2m+1)}{\langle\kappa\rangle^2(2m+3)} \frac{e^{-\langle\kappa\rangle t}\sum_{2m+3}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}}{e^{-\langle\kappa\rangle t}\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}} \\
&= \frac{A^2(2m+1)}{\langle\kappa\rangle^2(2m+3)} \frac{\sum_{2m+3}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}}{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}} \\
&= \frac{A^2(2m+1)}{\langle\kappa\rangle^2(2m+3)} \frac{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!} - \frac{(\langle\kappa\rangle t)^{2m+1}}{(2m+1)!} - \frac{(\langle\kappa\rangle t)^{2m+2}}{(2m+2)!}}{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}} \\
&= \frac{A^2(2m+1)}{\langle\kappa\rangle^2(2m+3)} \left(\frac{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}}{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}} - \frac{\frac{(\langle\kappa\rangle t)^{2m+1}}{(2m+1)!}}{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}} - \frac{\frac{(\langle\kappa\rangle t)^{2m+2}}{(2m+2)!}}{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}}\right) \\
&= \frac{A^2(2m+1)}{\langle\kappa\rangle^2(2m+3)} \left(1 - \frac{\frac{(\langle\kappa\rangle t)^{2m+1}}{(2m+1)!}}{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}} - \frac{\frac{(\langle\kappa\rangle t)^{2m+2}}{(2m+2)!}}{\sum_{2m+1}^{\infty}\frac{(\langle\kappa\rangle t)^n}{n!}}\right).
\end{aligned} \tag{6.2.19}$$

As $m \to \infty$, the last two terms in the brackets of equation (6.2.19) go to zero, and thus

$$\frac{|a_{m+1}|}{|a_m|} \to \frac{A^2}{\langle\kappa\rangle^2},$$

for all times under consideration.

So, convergence for any value of t has been proved, as long as $\frac{A^2}{\langle\kappa\rangle^2} < 1$ and, $\langle\kappa\rangle > 0$. This gives a slightly more rigorous proof than before.

One specific example, given experimentally in the dissertation, [17], is figure 6.5(b), where $\langle\kappa\rangle = 1.0$, and $\sigma^2 = 0.25$, and hence, $A = 0.5 \times \sqrt{3} = \frac{\sqrt{3}}{2}$.

Consider the asymptotic behaviour of the series as $t \to \infty$,

$$\langle y(\kappa)\rangle - y(\langle\kappa\rangle) = \sum_{m=1}^{\infty}\frac{A^{2m}}{(2m+1)!}\frac{\partial^{2m}y(\langle\kappa\rangle)}{\partial\kappa^{2m}}$$

$$= \sum_{m=1}^{\infty} \frac{(\frac{\sqrt{3}}{2})^{2m}}{(2m+1)!} \frac{(-1)^{2m} 2m!}{\langle\kappa\rangle^{2m+1}} \left(1 - e^{-\langle\kappa\rangle t} \sum_{n=0}^{2m} \frac{(\langle\kappa\rangle t)^n}{n!}\right). \tag{6.2.20}$$

Putting $\langle\kappa\rangle = 1.0$ implies

$$\begin{aligned} \langle y(\kappa)\rangle - y(\langle\kappa\rangle) &= \sum_{m=1}^{\infty} \frac{3^m}{2^{2m}(2m+1)} \left(1 - e^{-\langle\kappa\rangle t} \sum_{n=0}^{2m} \frac{(\langle\kappa\rangle t)^n}{n!}\right) \\ &= \sum_{m=1}^{\infty} \frac{3^m}{2^{2m}(2m+1)} \left(1 - e^{-t} \sum_{n=0}^{2m} \frac{(t)^n}{n!}\right). \end{aligned} \tag{6.2.21}$$

As $t \to \infty$, e^{-t} eventually dominates all the terms in the series $\sum_{n=0}^{2m} \frac{(t)^n}{n!}$.

So asymptotically,

$$\begin{aligned} \langle y(\kappa)\rangle - y(\langle\kappa\rangle) &\to \sum_{m=1}^{\infty} \frac{3^m}{2^{2m}(2m+1)} \\ &= \sum_{m=1}^{\infty} \frac{(0.75)^m}{2m+1}. \end{aligned} \tag{6.2.22}$$

This series clearly does converge, in comparison, for example, with a geometric series. By computation, it was found that this series converges to 0.521, to three significant figures, which compares extremely well with the measured difference on figure 6.1.5(b), of 0.51, for a large value of t.

6.2.2 Gaussian Distribution

For a Gaussian distribution function, the shifted moments have the following form [17]

$$\left.\begin{aligned} \nu_{2m} &= \tfrac{(2m-1)!\sigma^{2m}}{2^{m-1}(m-1)!} \\ \nu_{2m-1} &= 0 \end{aligned}\right\} \forall m \in \mathbb{N}.$$

Applying these terms to the previous work,

$$\begin{aligned} \langle y(\kappa)\rangle - y(\langle\kappa\rangle) &= \sum_{m=1}^{\infty} \frac{\nu_{2m}}{2m!} \frac{\partial^{2m} y(\langle\kappa\rangle)}{\partial\kappa^{2m}} \\ &= \sum_{m=1}^{\infty} \frac{(2m-1)!\sigma^{2m}}{2^{m-1}2m!(m-1)!} \left\{ \frac{2m!}{\langle\kappa\rangle^{2m+1}} (1 - e^{-\langle\kappa\rangle t} \sum_{n=0}^{2m} \frac{\langle\kappa\rangle t)^n}{n!}) \right\} \end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{(2m-1)!}{2^{m-1}(m-1)!} \left(\frac{\sigma}{\langle\kappa\rangle}\right)^{2m} \frac{1}{\langle\kappa\rangle} \left\{1 - e^{-\langle\kappa\rangle t} \sum_{n=0}^{2m} \frac{(\langle\kappa\rangle t)^n}{n!}\right\}. \tag{6.2.23}$$

As before, we can consider the ratio of successive absolute terms,

$$\begin{aligned}
\frac{|a_{m+1}|}{|a_m|} &= \frac{\frac{(2m+1)!}{2^m m!} \frac{\sigma^{2m+2}}{\langle\kappa\rangle^{2m+3}} \left\{1 - e^{-\langle\kappa\rangle t} \sum_{n=0}^{2m+2} \frac{(\langle\kappa\rangle t)^n}{n!}\right\}}{\frac{(2m-1)!}{2^{m-1}(m-1)!} \frac{\sigma^{2m}}{\langle\kappa\rangle^{2m+1}} \left\{1 - e^{-\langle\kappa\rangle t} \sum_{n=0}^{2m+2} \frac{(\langle\kappa\rangle t)^n}{n!}\right\}} \\
&= \frac{(2m+1)2m\sigma^2}{2^2 m\langle\kappa\rangle^2} \frac{\sum_{2m+3}^{\infty} \frac{(\langle\kappa\rangle t)^n}{n!}}{\sum_{2m+1}^{\infty} \frac{(\langle\kappa\rangle t)^n}{n!}} \\
&= \frac{2m+1}{2} \frac{\sigma^2}{\langle\kappa\rangle^2} \frac{\sum_{2m+3}^{\infty} \frac{(\langle\kappa\rangle t)^n}{n!}}{\sum_{2m+1}^{\infty} \frac{(\langle\kappa\rangle t)^n}{n!}}.
\end{aligned} \tag{6.2.24}$$

It has already been shown that as $m \to \infty$,

$$\frac{\sum_{2m+3}^{\infty} \frac{(\langle\kappa\rangle t)^n}{n!}}{\sum_{2m+1}^{\infty} \frac{(\langle\kappa\rangle t)^n}{n!}} \to 1,$$

so,

$$\begin{aligned}
\frac{|a_{m+1}|}{|a_m|} &\to \frac{2m+1}{2} \frac{\sigma^2}{\langle\kappa\rangle^2} \\
&\to \infty.
\end{aligned}$$

Hence, there is divergence of this series for all possible choices of $\frac{\sigma^2}{\langle\kappa\rangle^2}$. This confirms precisely what was observed in the dissertation, [17], for Gaussian distributions of κ - that $\langle y(\kappa)\rangle - y(\langle\kappa\rangle)$ always diverges to infinity, irrespective of the choice of variance and mean.

6.3 Summary

The first part of the work in this chapter on the stochastic model o.d.e., set out in detail in the dissertation, [17], was of predominantly academic interest. We were able to see how the probability distribution functions changed with respect to time, and how the mean value of the solution developed. In the second part,

by consideration of the high-order corrected moments, we were able to analyse the behaviour of the mean value of the solution with respect to the deterministic solution, $\tilde{y}(t)$. These relative behaviours allow us to conclude that there are some situations where the deterministic solution may be an acceptable approximation to mean of the solution, but in the majority of cases this is not an assumption that can be made without further investigation. This means that further research must involve a careful consideration of this behaviour.

In further chapters we consider perturbation techniques on the more complicated model p.d.e.s, (2.2.1), and (2.2.2). The perturbations, as in this chapter, are also done about various means of permeability, but due to the difference between these equations and the more straightforward model o.d.e., (2.2.4), under consideration in this chapter, the expansions performed turn out to be multi-dimensional ones. We begin with the steady-state case, (2.2.2).

Chapter 7

Steady-State Model

In this chapter, we investigate in detail the behaviour of the model steady-state partial differential equation, equation (2.2.2), in two dimensions, on a rectangular region, with various types of boundary conditions imposed. We consider uncertainty, specifically in the value of the permeability, which is considered to be a random spatial function.

7.1 Properties of a General Realisation

It is illustrative to consider one general realisation as an example; and by investigation of the properties associated with this case, a great deal of information can be obtained about the statistical properties of the complete problem, taken over all realisations. The work is done here in two dimensions, but a generalisation to three dimensions is fairly straightforward.

We consider a discretisation, with a simple five-point difference scheme, on a uniform stencil of mesh size h, of equation(2.2.2) A numerical approximation to the equation is

$$p_{i,j+1}k_{i,j+\frac{1}{2}} + p_{i+1,j}k_{i+\frac{1}{2},j} - (k_{i+\frac{1}{2},j} + k_{i-\frac{1}{2},j} + k_{i,j+\frac{1}{2}} + k_{i,j-\frac{1}{2}})p_{i,j}$$

$$+ p_{i-1,j}k_{i-\frac{1}{2},j} + p_{i,j-1}k_{i,j-\frac{1}{2}} = 0 \qquad (7.1.1)$$

We note that the discretised values for the permeability field are those evaluated at the half way points, between the nodes.

In the complete statistical problem, the permeability $k(x, y)$ is a random spatial function with a known mean value, which may or may not be assumed to be homogeneous (i.e. spatially constant), and a spatial autocorrelation function which relates the statistical properties of the function at different points. When the permeability is discretised, as in the case above, the function is represented as a set of statistical variables, each having its own mean value, which are all equal, if the assumption of a homogeneous mean value function is made. Each statistical variable is also correlated to the others, reflecting a discretisation of the original spatial, autocorrelation function. This means that points close together have a high correlation in their discretised permeability values, and points at large distances apart have correlations approaching zero - i.e. the correlation value is related to the separation of the two half grid-points. In the case of an isotropic autocorrelation function the correlation is a function of distance only; and this is the situation investigated here.

We consider a perturbation representation of the permeability function. This can be written

$$k(x, y) = k_0(x, y) + k_1(x, y), \qquad (7.1.2)$$

where the perturbation $k_1(x, y)$ is taken about the mean value function, $k_0(x, y)$. In the discretised form this can be directly translated to

$$k_{ij} = k^0_{ij} + k^1_{ij}. \qquad (7.1.3)$$

If the simplification that the mean value of the permeability is homogeneous is made, then equations (7.1.2) and (7.1.3) become respectively

$$k(x,y) = k_0(1 + d(x,y)), \tag{7.1.4}$$

and

$$k_{ij} = k_0(1 + d_{i,j}), \tag{7.1.5}$$

where $d(x,y)$ is a random function, with mean value zero, that is assumed small.

We now re-write the permeability autocorrelation function, as defined in equation (3.5.7) and assuming the variance is homogeneous, in terms of these perturbations,

$$\begin{aligned}
\rho(\mathbf{r},\mathbf{r}') &= \frac{\langle (k(\mathbf{r}) - \langle k(\mathbf{r})\rangle)(k(\mathbf{r}') - \langle k(\mathbf{r}')\rangle)\rangle}{\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}} \\
&= \frac{\langle (k_0(1+d(\mathbf{r})) - k_0)(k_0(1+d(\mathbf{r}')) - k_0)\rangle}{\sigma^2} \\
&= \frac{k_0^2}{\sigma^2}\langle d(\mathbf{r})d(\mathbf{r}')\rangle,
\end{aligned} \tag{7.1.6}$$

and for the discretised version, in 2-D

$$\rho_{i,j,\ i',j'} = \frac{k_0^2}{\sigma^2}\langle d_{i,j}\ d_{i',j'}\rangle. \tag{7.1.7}$$

The homogeneous mean assumption is made throughout, for simplicity. However, it is shown later that various conclusions about the generalised case (i.e. with spatially varying mean) can be obtained from consideration of the simplified case. Substituting equation (7.1.5) into (7.1.1), we get

$$\begin{aligned}
&-p_{i,j+1} - p_{i+1,j} + 4p_{i,j} - p_{i-1,j} - p_{i,j-1} - d_{i+\frac{1}{2},j}p_{i+1,j} - d_{i,j+\frac{1}{2}}p_{i,j+1} \\
&+(d_{i+\frac{1}{2},j}+d_{i-\frac{1}{2},j}+d_{i,j+\frac{1}{2}}+d_{i,j-\frac{1}{2}})p_{i,j} - d_{i-\frac{1}{2},j}p_{i-1,j} - d_{i,j-\frac{1}{2}}p_{i,j-1} = 0 \quad .
\end{aligned} \tag{7.1.8}$$

This leads to a system of equations, which, when combined with the relevant boundary conditions for the problem, can be written as the matrix equation

$$A\mathbf{p} + D\mathbf{p} = \mathbf{b}, \tag{7.1.9}$$

where $\mathbf{b}$ contains the boundary conditions, and is quite sparse. Due to the perturbation formulation, $\mathbf{b}$ can always be split into two parts:

$$\mathbf{b} = \mathbf{b_0} + \mathbf{b_d}, \tag{7.1.10}$$

where $\mathbf{b_0}$ corresponds to the right hand side vector of the deterministic problem, and $\mathbf{b_d}$ contains various linear combinations of the uncertain statistical variables, $\{d_{ij}\}$.

For the purposes of the following examples, all on a square region, the Dirichlet case has boundary conditions a_S, a_E, a_N, a_W on respective south, east, north, and west boundaries; and the mixed case has a_N, a_S on the north and south boundaries, with normal gradients v_E, and v_W on the east and west ones.

In a general case, the vector $\mathbf{b}$ can be written:

$$\mathbf{b} = \begin{pmatrix} \mathbf{b^1} \\ \mathbf{b^2} \\ \vdots \\ \mathbf{b^i} \\ \vdots \\ \mathbf{b^n} \end{pmatrix} = \begin{pmatrix} \mathbf{b_o^1} \\ \mathbf{b_o^2} \\ \vdots \\ \mathbf{b_o^i} \\ \vdots \\ \mathbf{b_o^n} \end{pmatrix} + \begin{pmatrix} \mathbf{b_d^1} \\ \mathbf{b_d^2} \\ \vdots \\ \mathbf{b_d^i} \\ \vdots \\ \mathbf{b_d^n} \end{pmatrix}. \tag{7.1.11}$$

For simple Dirichlet conditions, the component vectors are

$$\mathbf{b^1} = \begin{pmatrix} a_S + a_W \\ a_S \\ a_S \\ \vdots \\ a_S \\ a_S + a_E \end{pmatrix} + \begin{pmatrix} d_{\frac{1}{2},1} a_S + d_{1,\frac{1}{2}} a_W \\ d_{\frac{1}{2},2} a_S \\ d_{\frac{1}{2},3} a_S \\ \vdots \\ d_{\frac{1}{2},n-1} a_S \\ d_{\frac{1}{2},n} a_S + d_{1,n+\frac{1}{2}} a_E \end{pmatrix}, \tag{7.1.12}$$

$$\mathbf{b^n} = \begin{pmatrix} a_W + a_N \\ a_N \\ a_N \\ \vdots \\ a_N \\ a_N + a_E \end{pmatrix} + \begin{pmatrix} d_{n,\frac{1}{2}} a_W + d_{n+\frac{1}{2},1} a_N \\ d_{n+\frac{1}{2},2} a_N \\ d_{n+\frac{1}{2},3} a_N \\ \vdots \\ d_{n+\frac{1}{2},n-1} a_N \\ d_{n+\frac{1}{2},n} a_N + d_{n,n+\frac{1}{2}} a_E \end{pmatrix}, \tag{7.1.13}$$

and

$$\mathbf{b^i} = \begin{pmatrix} a_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ a_E \end{pmatrix} + \begin{pmatrix} d_{2,\frac{1}{2}} a_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ d_{2,n+\frac{1}{2}} a_E \end{pmatrix}, \quad for \;\; i = 2, 3, ...(n-1). \tag{7.1.14}$$

For simple mixed boundary conditions, these are

$$\mathbf{b^1} = \begin{pmatrix} a_S + 2hv_W \\ a_S \\ a_S \\ \vdots \\ a_S \\ a_S - 2hv_E \end{pmatrix} + \begin{pmatrix} d_{\frac{1}{2},1} a_S + 2d_{1,1} hv_W \\ d_{\frac{1}{2},2} a_S \\ d_{\frac{1}{2},3} a_S \\ \vdots \\ d_{\frac{1}{2},n+1} a_S \\ d_{\frac{1}{2},n+2} a_S - 2d_{1,n+2} hv_E \end{pmatrix}, \tag{7.1.15}$$

$$\mathbf{b^n} = \begin{pmatrix} a_N + 2hv_W \\ a_N \\ a_N \\ \vdots \\ a_N \\ a_N - 2d_{n,n+2} hv_E \end{pmatrix} + \begin{pmatrix} d_{n+\frac{1}{2},1} a_N + 2d_{n,1} hv_W \\ d_{n+\frac{1}{2},2} a_N \\ d_{n+\frac{1}{2},3} a_N \\ \vdots \\ d_{n+\frac{1}{2},n+1} a_N \\ d_{n+\frac{1}{2},n+2} a_N - 2d_{n,n+2} hv_E \end{pmatrix}, \tag{7.1.16}$$

and

$$\mathbf{b^i} = \begin{pmatrix} 2hv_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2hv_E \end{pmatrix} + \begin{pmatrix} 2d_{i,1}hv_W \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2d_{i,n+2}hv_E \end{pmatrix}, \quad for \;\; i = 2, 3, ...(n-1). \tag{7.1.17}$$

The matrix A is the usual block tridiagonal matrix for the simple 5-point difference scheme, and can be assumed to be irreducibly diagonally dominant, with the form

$$A = \begin{bmatrix} A_1 & -I & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ -I & A_2 & -I & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & -I & A_3 & -I & 0 & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & 0 & -I & A_i & -I & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & 0 & -I & A_{n-2} & -I & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & -I & A_{n-1} & -I \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & -I & A_n \end{bmatrix}. \tag{7.1.18}$$

For Dirichlet boundary conditions, the component diagonal matrices are n by n, symmetric, and tridiagonal with the form

$$A_i = \begin{bmatrix} 4 & -1 & 0 & \dots & \dots \\ -1 & 4 & -1 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -1 & 4 & -1 \\ \dots & \dots & 0 & -1 & 4 \end{bmatrix} \quad for \;\; i = 1, 2, ...n, \tag{7.1.19}$$

and for mixed boundary conditions, they are $(n+2)$ by $(n+2)$, and tridiagonal:

$$A_i = \begin{bmatrix} 4 & -2 & 0 & \dots & \dots \\ -1 & 4 & -1 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -1 & 4 & -1 \\ \dots & \dots & 0 & -2 & 4 \end{bmatrix}, \quad for \quad i = 1, 2, \dots n. \tag{7.1.20}$$

The matrix D has identical structure to A:

$$D = \begin{bmatrix} D_{1,1} & D_{1,2} & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ D_{2,1} & D_{2,2} & D_{2,3} & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & D_{3,2} & D_{3,3} & D_{3,4} & 0 & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & 0 & D_{i,i-1} & D_{i,i} & D_{i,i+1} & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & 0 & D_{n-2,n-3} & D_{n-2,n-2} & D_{n-2,n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & D_{n-1,n-2} & D_{n-1,n-1} & D_{n-1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & D_{n,n-1} & D_{n,n} \end{bmatrix}, \tag{7.1.21}$$

and the component matrices for the Dirichlet problem are n by n:

$$D_{i,i} = \begin{bmatrix} \Delta_{i,1} & -d_{i,1\frac{1}{2}} & 0 & \dots & \dots \\ -d_{i,1\frac{1}{2}} & \Delta_{i,2} & -d_{i,2\frac{1}{2}} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -d_{i,n-\frac{3}{2}} & \Delta_{i,n-1} & -d_{i,n-\frac{1}{2}} \\ \dots & \dots & 0 & -d_{i,n-\frac{1}{2}} & \Delta_{i,n} \end{bmatrix}, \quad for \quad i = 1, 2, \dots n, \tag{7.1.22}$$

where

$$\Delta_{i,j} = d_{i-\frac{1}{2},j} + d_{i,j+\frac{1}{2}} + d_{i+\frac{1}{2},j} + d_{i,j-\frac{1}{2}} \quad ,$$

and

$$D_{i,i-1} = D_{i-1,i} = \begin{bmatrix} -d_{i-\frac{1}{2},1} & 0 & \dots & 0 & \dots & \vdots \\ 0 & -d_{i-\frac{1}{2},2} & \dots & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -d_{i-\frac{1}{2},j} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & \dots & -d_{i-\frac{1}{2},n} \end{bmatrix}, \quad for \quad i = 2,3,...n. \tag{7.1.23}$$

For mixed boundary conditions, they are $(n+2)$ by $(n+2)$, and tridiagonal:

$$D_{i,i} = \begin{bmatrix} \Delta_{i,1} & -2d_{i,1\frac{1}{2}} & 0 & \dots & \dots \\ -d_{i,1\frac{1}{2}} & \Delta_{i,2} & -d_{i,2\frac{1}{2}} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -d_{i,n+\frac{1}{2}} & \Delta_{i,n+1} & -d_{i,n+\frac{3}{2}} \\ \dots & \dots & 0 & -2d_{i,n+\frac{3}{2}} & \Delta_{i,n+2} \end{bmatrix}, \quad for \quad i = 1,2,...n, \tag{7.1.24}$$

where

$$\begin{aligned} \Delta_{i,1} &= d_{i-\frac{1}{2},1} + 2d_{i,1\frac{1}{2}} + d_{i+\frac{1}{2},1}, \\ \Delta_{i,j} &= d_{i-\frac{1}{2},j} + d_{i,j+\frac{1}{2}} + d_{i+\frac{1}{2},j} + d_{i,j-\frac{1}{2}}, \qquad for \quad j = 2,3,...(n+1), \\ \Delta_{i,n+2} &= d_{i-\frac{1}{2},n+2} + 2d_{i,n+\frac{3}{2}} + d_{i+\frac{1}{2},1}, \end{aligned}$$

and

$$D_{i,i-1} = D_{i-1,i} = \begin{bmatrix} -d_{i-\frac{1}{2},1} & 0 & \dots & 0 & \dots & \vdots \\ 0 & -d_{i-\frac{1}{2},2} & \dots & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -d_{i-\frac{1}{2},j} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & \dots & -d_{i-\frac{1}{2},n+2} \end{bmatrix}, \tag{7.1.25}$$

for $i = 2,3,...(n+2)$.

We consider one specific realisation for the set of perturbations where the matrix D is denoted by D_n, the vector $\mathbf{b}$ by $\mathbf{b_n}$, and $\mathbf{b_d}$ by $\mathbf{b_{dn}}$. The corresponding expression to equation (7.1.9) is then

$$A\mathbf{p_n} + D_n\mathbf{p_n} = \mathbf{b_0} + \mathbf{b_{dn}}. \tag{7.1.26}$$

This can be re-arranged to give

$$\mathbf{p_n} = A^{-1}(\mathbf{b_0} + \mathbf{b_{dn}}) - A^{-1}D_n\mathbf{p_n}. \tag{7.1.27}$$

This can then be formed into a perturbation series, and, due to fact that D_n and $\mathbf{b_{dn}}$ are both linear in the $\{d_{ij}\}$ terms, the second order approximation to the perturbation series is

$$\mathbf{p_{sn}} = A^{-1}(\mathbf{b_0} + \mathbf{b_{dn}}) - A^{-1}D_nA^{-1}(\mathbf{b_0} + \mathbf{b_{dn}}) + A^{-1}D_nA^{-1}D_nA^{-1}\mathbf{b_0}. \tag{7.1.28}$$

The equivalent full perturbation series is

$$\mathbf{p_n} = A^{-1}\sum_{i=0}^{\infty}(-D_nA^{-1})^i\mathbf{b_n}. \tag{7.1.29}$$

This series is convergent elementwise if, and only if,

$$Sr(D_nA^{-1}) < 1, \tag{7.1.30}$$

where $Sr(M)$ denotes the spectral radius of the matrix M [18].

Alternatively, convergence with respect to a general norm, is satisfied for realisation n, if

$$\|(D_nA^{-1})^i\| \longrightarrow 0 \qquad\qquad as\ i \rightarrow \infty \quad ,$$

So a necessary and sufficient condition for convergence in realisation n is

$$\|D_nA^{-1}\| < 1.$$

Since

$$\|D_n A^{-1}\| \leq \|D_n\| \cdot \|A^{-1}\|,$$

this condition is satisfied if,

$$\|D_n\| \cdot \|A^{-1}\| < 1.$$

So

$$\|D_n\| < \frac{1}{\|A^{-1}\|},$$

implies

$$\|D_n A^{-1}\| < 1.$$

So, a necessary condition for convergence of the series is

$$\|D_n\| < \|A^{-1}\|^{-1}. \tag{7.1.31}$$

This implies a restriction on the size of all possible admissible realisations for the perturbations - they must be at least finite, and bounded. This could seem a rather limiting factor in this technique, particularly when we may be dealing with distributions that are not finite, such as Gaussian or lognormal type. As was demonstrated in Chapter 6, sometimes when an infinite distribution is under consideration, contributions from the far end of the tail may tend to lead to strange unphysical results. In any practical approach to obtaining a sensible measure of the statistical properties of the solution, such as implementation of a Monte-Carlo method, it is not possible to include all the admissible realisations. Due to the limiting process of taking a finite number, there is inevitably a loss of some amount of the tail of the distribution. The technique we are using here has an implied truncation of the tail, where the cut-off tail contains all the realisations that do not satisfy the condition in equation (7.1.31).

Now consider the effect of truncating the series (7.1.29) to second order,

$$\mathbf{p_{s_n}} = A^{-1} \sum_{j=0}^{2} (-D_n A^{-1})^j \mathbf{b_n}. \tag{7.1.32}$$

This introduces an error between this expression and (7.1.29) given by,

$$\begin{aligned} \mathbf{e_n} &= A^{-1} \sum_{j=3}^{\infty} (-D_n A^{-1})^j \mathbf{b_n} \\ &= A^{-1} (-D_n A^{-1})^3 \sum_{j=0}^{\infty} (-D_n A^{-1})^j \mathbf{b_n}. \end{aligned} \tag{7.1.33}$$

If it can be shown that there exists a bound on $D_n A^{-1}$ in realisation n such that

$$\| D_n A^{-1} \| \leq \nu_n \quad , \tag{7.1.34}$$

where $0 \leq \nu_n < 1$, then a bound on the error expression in (7.1.33) can be obtained quite straightforwardly,

$$\| \mathbf{e_n} \| < \frac{\| A^{-1} \| \quad \nu_n^3 \quad \| \mathbf{b_n} \|}{1 - \nu_n}. \tag{7.1.35}$$

Let us assume there is a bound on the maximum relative value that each perturbation can take, given by δ, where $\delta < 1$. This gives an absolute bound on the entries of a general D_n-type matrix,

$$|D_n| \leq \delta |A|, \qquad \forall \ \ n. \tag{7.1.36}$$

Here we are using the convention that if for two matrices M and N, of the same order, we have

$$M_{ij} \leq N_{ij}, \qquad \forall \ \ i, j,$$

then we write

$$M \leq N,$$

and we define the absolute value matrix, $\|M\|$, of any given matrix, M to be such that,

$$|M|_{ij} = |M_{ij}| \qquad \forall \ \ i, j.$$

Now consider $\|D_n A^{-1}\|_p$ for general $p \in \mathbb{N}$, and for all admissible realisations, n, defined in the usual way,

$$\|D_n A^{-1}\|_p = \max_{\mathbf{x}} \frac{\|D_n A^{-1}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \tag{7.1.37}$$

where the maximum is performed over all vectors, $\mathbf{x}$. Because of the properties for a general p-norm [29], we know that

$$\|D_n A^{-1}\|_p \leq \|D_n\|_p \|A^{-1}\|_p$$

Now, consider $\|D_n\|$, for all n. We have

$$\begin{aligned}
\|D_n\|_p^p &= \max_{\mathbf{x}} \frac{\|D_n\mathbf{x}\|_p^p}{\|\mathbf{x}\|_p^p} \\
&= \max_{\mathbf{x}} \frac{\sum_i |\sum_j (D_n)_{ij} x_j|^p}{\sum_i |x_i|^p} \\
&\leq \max_{\mathbf{x}} \frac{\sum_i (\sum_j |(D_n)_{ij} x_j|)^p}{\sum_i |x_i|^p} \\
&\leq \max_{\mathbf{x}} \frac{\sum_i |\sum_j |(D_n)_{ij}||x_j||^p}{\sum_i |x_i|^p} \\
&= \max_{\mathbf{x}} \frac{\||D_n||\mathbf{x}|\|_p^p}{\|\mathbf{x}\|_p^p} \\
&\leq \max_{\mathbf{x}} \frac{\||D_n|\|_p^p \||\mathbf{x}|\|_p^p}{\|\mathbf{x}\|_p^p} \\
&= \||D_n|\|_p^p,
\end{aligned}$$

since $\|\mathbf{x}\|_p = \||\mathbf{x}|\|_p$. Therefore,

$$\begin{aligned}
\|D_n A^{-1}\|_p &\leq \|D_n\|_p \|A^{-1}\|_p \\
&\leq \||D_n|\|_p \|A^{-1}\|_p
\end{aligned}$$

Combining this result with (7.1.36), we have

$$\|D_n A^{-1}\|_p \leq \delta \||A|\|_p \|A^{-1}\|_p, \qquad \forall \ \ n. \tag{7.1.38}$$

So, from (7.1.31), we find that

$$\delta < \frac{1}{\||A|\|_p \|A^{-1}\|_p} \quad ,$$

is a sufficient condition for convergence of the approximate series for all the relevant realisations.

In equations (7.1.11) to (7.1.17), it is seen that the vector $\mathbf{b_n}$ can be written as $\mathbf{b_0} + \mathbf{b_{d_n}}$, where the vector $\mathbf{b_{d_n}}$ contains linear combinations of the $\{d_{ij}\}$ perturbations as its components. By again taking the maximal value for all these perturbations, each component of $\mathbf{b_n}$ can be shown to satisfy

$$-2(\mathbf{b_0})_i \quad \leq \quad (\mathbf{b_n})_i \quad \leq \quad 2(\mathbf{b_0})_i,$$

or

$$|(\mathbf{b_n})_i| \leq 2|(\mathbf{b_0})_i|.$$

Therefore, since

$$\|\mathbf{b_n}\|_p^p = \sum_i |(\mathbf{b_n})_i|^p,$$

and it is known that

$$|(\mathbf{b_n})_i|^p \leq 2|(\mathbf{b_0})_i|^p \qquad \forall \ \ i \ and \ n,$$

we can conclude that

$$\sum_i |(\mathbf{b_n})_i|^p \leq \sum_i 2|(\mathbf{b_0})_i|^p = 2\|\mathbf{b_0}\|_p^p,$$

Which is equivalent to saying that

$$\|\mathbf{b_n}\| \quad \leq \quad 2 \ \|\mathbf{b_0}\| \qquad \forall \ \ n,$$

for a general p norm.

The errors for each realisation in expression(7.1.32) thus satisfy

$$\|\mathbf{e_n}\| \leq \frac{2 \ \delta^3 \||A|\|^3 \|A^{-1}\|^4 \|\mathbf{b_0}\|}{1 - \delta \||A|\| \|A^{-1}\|} \qquad \forall \ \ n. \tag{7.1.39}$$

This expression may be very useful when considering the error obtained when truncating the series, for the mean value of the numerical solution.

7.2 Probabilistic Approach

One method of approach to the full statistical problem might be to take the mean value over all possible realisations of the numerical scheme in equation(7.1.1)

$$-\langle p_{i,j+1}k_{i,j+\frac{1}{2}}\rangle - \langle p_{i+1,j}k_{i+\frac{1}{2},j}\rangle + \langle p_{i,j}(k_{i+\frac{1}{2},j} + k_{i-\frac{1}{2},j} + k_{i,j+\frac{1}{2}} + k_{i,j-\frac{1}{2}})\rangle$$
$$- \langle p_{i-1,j}k_{i-\frac{1}{2},j}\rangle - \langle p_{i,j-1}k_{i,j-\frac{1}{2}}\rangle = 0 \quad (7.2.1)$$

Whilst this equation is precisely true, it contains no useful information, as the discretised pressure function cannot now be separated out from the cross terms, which contain it. Indeed, equation(7.2.1) leads nowhere.

If we return to equation(7.1.28)

$$\mathbf{p_{s_n}} = A^{-1}(\mathbf{b_0} + \mathbf{b_{d_n}}) - A^{-1}D_nA^{-1}(\mathbf{b_0} + \mathbf{b_{d_n}}) + A^{-1}D_nA^{-1}D_nA^{-1}\mathbf{b_0},$$

taking the mean value, over all realisations, element-by-element, of both sides gives

$$\langle \mathbf{p_s}\rangle = A^{-1}\mathbf{b_0} + A^{-1}\langle \mathbf{b_d}\rangle - A^{-1}\langle DA^{-1}\rangle \mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle + A^{-1}\langle DA^{-1}D\rangle A^{-1}\mathbf{b_0}, \quad (7.2.2)$$

where the convention is that

$$(\langle Vector\rangle)_i = \langle (Vector)_i\rangle,$$

and

$$(\langle Matrix\rangle)_{ij} = \langle (Matrix)_{ij}\rangle.$$

These uncertain matrices and vectors obey all the usual rules of matrix algebra, including

$$\begin{aligned}(\langle AB\rangle)_{ij} &= \langle (AB)_{ij}\rangle \\ &= \langle \sum_k A_{ik}B_{kj} \rangle \\ &= \sum_k \langle A_{ik}B_{kj}\rangle.\end{aligned}$$

Also if A is known exactly, and B is uncertain, then

$$\begin{aligned}(\langle AB\rangle)_{ij} &= \sum_k \langle A_{ik}B_{kj}\rangle \\ &= \sum_k A_{ik}\langle B_{kj}\rangle \\ &= \sum_k A_{ik}(\langle B\rangle)_{kj},\end{aligned}$$

so that $\langle AB\rangle = A\langle B\rangle$ under matrix algebra rules. Similar rules apply for vector operations also,

$$\begin{aligned}(\langle A\mathbf{v}\rangle)_i &= \langle (A\mathbf{v})_i\rangle \\ &= \langle \sum_j A_{ij}v_j \rangle \\ &= \sum_j \langle A_{ij}v_j\rangle,\end{aligned}$$

and if, for example, A is known and $\mathbf{v}$ is uncertain, then,

$$(\langle A\mathbf{v}\rangle)_i = \sum_j \langle A_{ij}v_j\rangle = \sum_j A_{ij}\langle v_j\rangle \quad ,$$

so that

$$\langle A\mathbf{v}\rangle = A\langle \mathbf{v}\rangle.$$

It is fairly trivial to prove all the other algebraic rules.

The vector $\mathbf{b_d}$ and the matrix D only contain linear terms in the perturbations; and since we have already assumed that the mean value of all the perturbations, over all realisations, is zero, taking the mean value of any first order terms in $\{d_{ij}\}$ gives zero.

Equation(7.2.2) can therefore be re-written,

$$\langle \mathbf{p_s}\rangle = A^{-1}\mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle + A^{-1}\langle DA^{-1}D\rangle A^{-1}\mathbf{b_0}. \qquad (7.2.3)$$

If we were tackling this problem in a deterministic way, we might make the intuitive assumption that a good approximation to the mean value of the numerical

solution would be obtained by solving the problem using just the mean value of the permeability field as data. This is equivalent to solving the problem,

$$A\mathbf{p_A} = \mathbf{b_0}, \tag{7.2.4}$$

with solution

$$\mathbf{p_A} = A^{-1}\mathbf{b_0}. \tag{7.2.5}$$

It can be seen, by comparison of (7.2.5) with (7.2.3), that this deterministic solution is, in effect, a first order approximation to the exact mean value of the numerical solution. This means that $\mathbf{p_A}$ can be thought of as an approximation to $\langle \mathbf{p} \rangle$ that effectively contains information about the mean value of the permeability field. The information about the second order terms in $\{d_{ij}\}$ has been discarded, which means that two aspects of the statistical information have been lost,

1. variance of the field .

   Terms like $\langle d_{i,j}^2 \rangle$ represent the variance of the permeability field. For a single variable, qualitatively, the variance represents the spread of possible values that it can take, away from the mean value. For this reason, if this information is excluded, it makes the mean value approximation meaningless in a statistical sense, i.e. equation(7.2.5) could be an approximation to the solution for either a field with zero dispersion (that is, one which is known precisely), or for a field with an arbitrarily large dispersion (one that is highly uncertain).

2. correlation of the field .

   Terms like $\langle d_{i,j} d_{i',j'} \rangle$ as seen in equation(7.1.7), are equal to the discretised correlation function for the separated points (i, j), and (i', j'), and give a measure of how similar the statistical properties at the two

points are. It is very important to take this measure into account, as was mentioned in Chapter 1, because this is a fundamental property of the uncertainty in the permeability field. Leaving out these terms would give the result for a 'static' uncertain field, that is, one where the permeability is uncertain at all sample points with the values at each point being unrelated statistically to each other.

So the approximation from (7.2.3) for the mean value of the numerical solution, when written

$$\langle \mathbf{p_s} \rangle = A^{-1}\mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle + A^{-1}\langle DA^{-1}D\rangle A^{-1}\mathbf{b_0}, \tag{7.2.6}$$

contains information concerning the mean, variance, and autocorrelation function of the permeability field, which is, essentially, what is required from the original problem.

Clearly, the error introduced by making the approximation is given by

$$\langle \mathbf{e} \rangle = A^{-1} \sum_{m=3}^{\infty} \langle (-DA^{-1})^m \mathbf{b} \rangle, \tag{7.2.7}$$

i.e. the error between the approximation (7.2.6) and the exact mean value of the numerical solution, as in equation(7.2.3).

We consider a general element of this error vector

$$(\langle \mathbf{e} \rangle)_i = \langle (\mathbf{e_n})_i \rangle, \tag{7.2.8}$$

where $\mathbf{e_n}$ is as defined in (7.1.33).

Now we consider the expected value of a general functional of a number of uncertain statistical variables, assuming the functional f, and the set of all possible realisations R_m have suitable properties. Then

$$\langle f(x_1, x_2, x_3, \ldots, x_m)\rangle = \int_{R_m} f(x_1, x_2, x_3, \ldots, x_m)\rho(x_1, x_2, x_3, \ldots, x_m)d^m\mathbf{x}, \tag{7.2.9}$$

where $\rho(x_1, x_2, x_3, \ldots, x_m)$ is the (joint) multivariate distribution for the variables, and

$$\begin{aligned}
&\mid \langle f(x_1, x_2, x_3, \ldots, x_m)\rangle \mid \\
&= \mid \int_{R_m} f(x_1, x_2, x_3, \ldots, x_m)\rho(x_1, x_2, x_3, \ldots, x_m)d^m\mathbf{x} \mid \\
&\leq \int_{R_m} \mid f(x_1, x_2, x_3, \ldots, x_m)\rho(x_1, x_2, x_3, \ldots, x_m) \mid d^m\mathbf{x} \\
&\leq \int_{R_m} \mid f(x_1, x_2, x_3, \ldots, x_m) \mid \cdot \mid \rho(x_1, x_2, x_3, \ldots, x_m) \mid d^m\mathbf{x} \\
&\leq \max_{R_m} \mid f(x_1, x_2, x_3, \ldots, x_m) \mid \cdot \int_{R_m} \mid \rho(x_1, x_2, x_3, \ldots, x_m) \mid d^m\mathbf{x}.
\end{aligned}$$

Since $\rho(x_1, x_2, x_3, \ldots, x_m)$ is the m-dimensional multivariate distribution function, it is, by convention, always positive, and is assumed to be normalised, so that

$$\int_{R_m} \mid \rho(x_1, x_2, x_3, \ldots, x_m) \mid d^m\mathbf{x} = 1. \tag{7.2.10}$$

Therefore, substituting (7.2.10) into the inequality expression above gives,

$$\mid \langle f(x_1, x_2, x_3, \ldots, x_m)\rangle \mid \quad \leq \quad \max_{R_m} \mid f(x_1, x_2, x_3, \ldots, x_m) \mid . \tag{7.2.11}$$

It can therefore be deduced that the mean value of any function or functional of m statistical variables, irrespective of whether they are correlated, must lie within the extremal values of that function or functional over its admissible space. This is a completely general result and no assumptions, such as linearity, have been made here.

Also,

$$\begin{aligned}
|\langle \mathbf{v}(x_1, \ldots, x_m)\rangle| &= |\int_{R_m} \mathbf{v}(x_1, \ldots, x_m) \ \ \rho(x_1, \ldots, x_m)d^m\mathbf{x}| \\
&\leq \int_{R_m} |\mathbf{v}(x_1, \ldots, x_m)| \ \ |\rho(x_1, \ldots, x_m)|d^m\mathbf{x}
\end{aligned}$$

and since $\rho \geq 0$ over R_m,

$$|\langle \mathbf{v}(x_1, \ldots, x_m) \rangle| \leq \int_{R_m} |\mathbf{v}(x_1, \ldots, x_m)| \;\; \rho(x_1, \ldots, x_m) d^m \mathbf{x}.$$

Hence for any L_p norm $\| \cdot \|$,

$$|\langle \mathbf{v}(x_1, \ldots, x_m) \rangle| \leq \langle |\mathbf{v}(x_1, \ldots, x_m)| \rangle, \tag{7.2.12}$$

irrespective of the distribution function for the variables $\{x_j\}$.

We can therefore establish a bound on the error that has been introduced when using expression(7.2.6) as an approximation to the mean value of the numerical solution, $\langle \mathbf{e} \rangle$.

First, substituting $\langle \mathbf{e} \rangle$ into expression(7.2.12) gives

$$\| \langle \mathbf{e} \rangle \| \quad \leq \quad \langle \| \mathbf{e} \| \rangle. \tag{7.2.13}$$

Now, since

$$\mathbf{e_n} = A^{-1} \sum_{m=3}^{\infty} (-D_n A^{-1})^m (\mathbf{b_0} + \mathbf{b_n}),$$

$\mathbf{e_n}$ is a vector function, in some specified way, of all the $\{d_{ij}\}$ terms, which are themselves uncertain statistical variables. Therefore, the process of taking the norm of a general vector $\mathbf{e_n}$ can be thought of as a functional of $\mathbf{e_n}$ - which means it is itself a function of the $\{d_{ij}\}$ type variables. So, the function $\|\mathbf{e}\|$ must satisfy the expression(7.2.11).

Therefore,

$$\langle \| \mathbf{e} \| \rangle \quad \leq \quad \max_{all \;\; realisations, n} \{ \| \mathbf{e_n} \| \}. \tag{7.2.14}$$

Then, by combining (7.2.13) with (7.2.15), we can see that

$$\| \langle \mathbf{e} \rangle \| \quad \leq \quad \max_{all \;\; realisations, n} \{ \| \mathbf{e_n} \| \}, \tag{7.2.15}$$

which is a bound for the norm on the error introduced, when expression(7.2.6) is used to approximate the mean value of the numerical solution.

Therefore, combining equation(7.2.15), with equation(7.1.39), gives,

$$\|\langle \mathbf{e} \rangle\| \leq \frac{2\ \delta^3 \||A|\|^3 \|A^{-1}\|^4 \|\mathbf{b_0}\|}{1 - \delta \||A|\| \|A^{-1}\|}, \tag{7.2.16}$$

the required bound on the approximation to the mean value of the numerical solution.

7.3 Numerical Approach to the Problem

At first sight, the statistical terms in equation (7.2.6)

$$\langle \mathbf{p_s} \rangle = A^{-1}\mathbf{b_0} - A^{-1}\langle DA^{-1}\mathbf{b_d} \rangle + A^{-1}\langle DA^{-1}D \rangle A^{-1}\mathbf{b_0},$$

appear very awkward to evaluate, because, due to the presence of A^{-1} between the two matrices in each term, the terms $\langle DA^{-1}\mathbf{b_d} \rangle$, and $\langle DA^{-1}D \rangle A^{-1}\mathbf{b_0}$ involve very complicated linear combinations of the correlation terms, $\{\langle d_{i,j} d_{i',j'} \rangle\}$, dependent on the inverse of A. This problem can be resolved by considering the structure of the D_n matrices, and noticing that each term like $\{d_{i,j}\}$ occurs either twice, if it is next to a boundary, or four times if it is on an internal half grid-point.

Suppose (i, j) is an internal half grid-point lying between grid-points labelled y, and z. It can be seen from equations (7.1.21) to (7.1.24) that the $d_{i,j}$ term only occurs at the matrix positions (y, y), (y, z), (z, y), and (z, z) in the following way,

$$
\begin{array}{c} \\ y \\ \\ z \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & y & & z & & \end{array} \\
\left[\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & d_{i,j} & \ldots & -d_{i,j} & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & -d_{i,j} & \ldots & d_{i,j} & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & &
\end{array}\right]
\end{array}
. \qquad (7.3.1)
$$

This contribution can be thought of as $d_{i,j}E^{ij}$, where

$$
E^{ij} = \left[\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & 1 & \ldots & -1 & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
\ldots & \ldots & -1 & \ldots & 1 & \ldots & \ldots \\
 & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & &
\end{array}\right]
. \qquad (7.3.2)
$$

The full matrix D_n can then be written as a weighted sum of simple elemental matrices like E^{ij} that contain either only four or two non-zero elements,

$$
D_n = \sum_{halfgrid-points} d_{i,j}^n E^{ij} \qquad (7.3.3)
$$

If a sum over all half grid-points is written as $\sum_{i,j}$ (each half grid-point being labelled (i,j)), then the expression (3.2.5) can be written,

$$
\langle DA^{-1}D \rangle = \langle \sum_{i,j} d_{i,j}E^{ij}A^{-1} \sum_{i',j'} d_{i',j'}E^{i'j'} \rangle, \qquad (7.3.4)
$$

where the mean is taken over the joint multivariate distribution function for all the values of $d_{i,j}$ at half grid-points. Consider a general element of this matrix,

$$\begin{aligned}
\langle DA^{-1}D\rangle_{kl} &= \langle\sum_{i,j} d_{i,j}E^{ij}A^{-1}\sum_{i',j'} d_{i',j'}E^{i'j'}\rangle_{kl} \\
&= \langle(\sum_{i,j} d_{i,j}E^{ij}A^{-1}\sum_{i',j'} d_{i',j'}E^{i'j'})_{kl}\rangle \\
&= \langle(\sum_{i,j} d_{i,j}E^{ij})_{km}A^{-1}_{mn}(\sum_{i',j'} d_{i',j'}E^{i'j'})_{nl}\rangle \\
&= \langle\sum_{i,j} d_{i,j}E^{ij}_{km}A^{-1}_{mn}\sum_{i',j'} d_{i',j'}E^{i'j'}_{nl}\rangle \\
&= \langle\sum_{i,j}\sum_{i',j'}(d_{i,j}E^{ij}_{km})A^{-1}_{mn}(d_{i',j'}E^{i'j'}_{nl})\rangle \\
&= \langle\sum_{i,j}\sum_{i',j'}(d_{i,j}d_{i',j'}E^{ij}_{km}A^{-1}_{mn}E^{i'j'}_{nl})\rangle \\
&= \sum_{i,j}\sum_{i',j'}(\langle d_{i,j}d_{i',j'}\rangle E^{ij}_{km}A^{-1}_{mn}E^{i'j'}_{nl}) \\
&= \sum_{i,j,i',j'}(\langle d_{i,j}d_{i',j'}\rangle E^{ij}_{km}A^{-1}_{mn}E^{i'j'}_{nl}), \qquad (7.3.5)
\end{aligned}$$

where $\sum_{i,j,i',j'}$ has the meaning of a sum over all possible ways that each half grid-point can correlate with each other half grid-point. Of course, if there are N of these half grid-points, then there will be $(N-1)*N$ possible ways that they can correlate with each other.

A similar procedure can be applied to the second term in equation (7.2.6), $A^{-1}\langle DA^{-1}\mathbf{b_d}\rangle$ to obtain a similar series term which involves a weighted sum. In this case the sum is performed over half grid-points that are adjacent to the boundary only, correlating with all other half grid-points within the region.

The $\langle d_{i,j}d_{i',j'}\rangle$ terms are now just equal to the normalised discretised correlation function between points (i,j) and (i',j'), as seen in equation (7.1.7), and is, in the most general case, a vector function of the separation of the two points.

An isotropic assumption can be made about the correlation, in which case, the terms $\langle d_{i,j}d_{i',j'}\rangle$ are just a function of the absolute distance of separation of the points (i,j) and (i',j'). A more interesting case is one in which the correlation function has a different form in different directions. There might be a strong

correlation in the properties of the rocks in a horizontal direction, and much a weaker correlation in the vertical direction, representing the usual layered structure of oil bearing strata. A further development in an anisotropic correlation function is periodicity in the vertical direction, which represents a repetition in the layered structure of the rock properties. This introduces an extra parameter into the correlation function, in addition to the two correlation lengths, which is the spatial periodicity of the rock layering.

Typical possibilities for models for the spatial correlation function are an exponential or a Gaussian-type decay. An isotropic exponential decay has the form

$$\rho(\mathbf{r}-\mathbf{r}') = \sigma^2 \exp(-|\mathbf{r}-\mathbf{r}'|/\lambda), \tag{7.3.6}$$

where λ is the (characteristic) correlation length.

For a general correlation function, the correlation length, λ_α, associated with direction α is defined to be

$$\lambda_\alpha = \int_0^\infty \rho_\alpha(x)dx, \tag{7.3.7}$$

[29]. So a Gaussian-type correlation function, takes the form

$$\rho(\mathbf{r},\mathbf{r}') = \sigma^2 \exp(-\frac{\pi|\mathbf{r}-\mathbf{r}'|^2}{4\lambda^2}), \tag{7.3.8}$$

where λ is the (in this case, isotropic) correlation length.

A Gaussian-type form for the correlation function is more useful mathematically when considering anisotropic correlations, because the distance squared terms can be separated out straightforwardly - for example, in two dimensions,

$$|\mathbf{r}-\mathbf{r}'|^2 = |(x-x')^2+(y-y')^2| = (x-x')^2+(y-y')^2,$$

and so the correlation function becomes a simple product

$$\rho(\mathbf{r}-\mathbf{r}') = \sigma^2 \exp(-\frac{\pi(x-x')^2}{4\lambda_x^2}) \exp(-\frac{\pi(y-y')^2}{4\lambda_y^2}). \tag{7.3.9}$$

An introduction of periodicity can also be made by an extra product. For example, if there is periodicity assumed in the x-direction, then

$$\rho(\mathbf{r}-\mathbf{r}') = \sigma^2 \exp\{-\frac{\pi(x-x')^2}{4\lambda_x^2}\} \exp\{-\frac{\pi(y-y')^2}{4\lambda_y^2}\} \cos(\omega(x-x')), \qquad (7.3.10)$$

where $\frac{2\pi}{\omega}$ is the model for the spatial periodicity, an approximate length equivalent to the repetition distance in the rock structure.

7.4 Results

The following section shows results for different types of imposed boundary conditions. In each case, the mean value for permeability is normalised to one.

Figures 7.4.1(a)-(e) have enforced flow at one end, no flow at the opposite end, and zero pressure conditions along the two sides. 7.4.1(a) is the deterministic solution with homogeneous permeability values; 7.4.1(b) is the result with variance equal to 0.1; 7.4.1(c) has variance 0.2; 7.4.1(d) has variance 0.4; and for 7.4.1(e) the variance is equal to 1.0.

Figures 7.4.2(a)-(e) have no flow conditions at either end and a pressure difference of 1.0 across the region horizontally. Again, figures (a), (b), (c), (d), and (e) are the results for increasing covariance values 0.0 (that is the deterministic solution), 0.1, 0.2, 0.4, and 1.0.

Figures 7.4.3(a)-(d) show the effects of increasing correlation length in relation to grid size, and overall scale of the region, with the same boundary conditions as 5.1. Figure (a) shows results for a correlation length of 0.01, which is approximately one third of the grid size; figures (b), (c), and (d) are results for correlation lengths 0.1, 0.5, and 10.0, respectively, which are all larger than the grid size. The variance for all these results is 0.2.

The final figures 7.4.4(a)-(b) show results for anisotropic correlation lengths.

The boundary conditions are, again the same as those in 7.1. Figure (a) has correlation length 0.1 in the x direction, and 1.0 in the y direction, and figure (b) has correlation length 1.0 in the x, and 0.1 in the y direction. Variance is 0.2 throughout.

Figure 7.4.1 Plots of pressure verses position, with mean value for permeability equal to 1.0, and correlation function of Gaussian form.

Figure 7.4.1(a) deterministic pressure solution, $\sigma^2 = 0.0$

Figure 7.4.1(b) mean pressure for $\sigma^2 = 0.1$.

Figure 7.4.1(c) mean pressure for $\sigma^2 = 0.2$

Figure 7.4.1(d) mean pressure for $\sigma^2 = 0.4$

Figure 7.4.1(e) mean pressure for $\sigma^2 = 1.0$

Figure 7.4.2 Plots of pressure verses position, $\langle k \rangle = 1.0$, correlation function Gaussian form.

Figure 7.4.2(a) deterministic pressure solution for $\sigma^2 = 0.0$

Figure 7.4.2(b) mean pressure for $\sigma^2 = 0.1$.

Figure 7.4.2(c) mean pressure for $\sigma^2 = 0.2$

Figure 7.4.2(d) mean pressure for $\sigma^2 = 0.4$

Figure 7.4.2(e) mean pressure for $\sigma^2 = 1.0$

Figure 7.4.3 Plots of pressure verses position, showing effects of increasing the (anisotropic) correlation length. Correlation function is Gaussian form, with $\sigma^2 = 0.2$, and $\langle k \rangle = 1.0$.

Figure 7.4.3(a) mean pressure for $\lambda_x = \lambda_y = 0.01$

Figure 7.4.3(b) mean pressure for $\lambda_x = \lambda_y = 0.1$

Figure 7.4.3(c) mean pressure for $\lambda_x = \lambda_y = 0.5$

Figure 7.4.3(d) mean pressure for $\lambda_x = \lambda_y = 10.0$

Figure 7.4.4 Plots of pressure verses position, showing effects of anisotropic correlation lengths. Correlation function is Gaussian form, with $\sigma^2 = 0.2$, and $\langle k \rangle = 1.0$.

Figure 7.4.4(a) mean pressure for $\lambda_x = 0.1$, $\lambda_y = 1.0$

Figure 7.4.4(b) mean pressure for $\lambda_x = 1.0$, $\lambda_y = 0.1$

Figures 7.4.1(a) to 7.4.1(e) show how increasing the variance of the permeability field increases the difference between the second order approximation to the mean value of the pressure field, and its deterministic solution. Particular features, such as gradients, seem to be distorted, and exaggerated.

Figures 7.4.2(a) to 7.4.2(e) show a similar effect on a different set of boundary conditions, with no flow at either ends, and an induced pressure difference across the region. The deterministic case just results in a simple constant flow across the pressure difference. When the non-deterministic case is considered for small variance, the flow (as in the gradient of pressure) is induced to increase, but because of the necessity of satisfying the boundary conditions, the gradient is forced to decrease close to the higher boundary condition. With increasing variance, this effect becomes more pronounced, with the decrease in gradient becoming more sharp, as the boundary conditions always have to be satisfied. Eventually, this effect becomes so great that in 7.4.2(d), the maximum principle for pressure is violated, and this roughly corresponds with the series expansion (7.1.29) becoming invalid due to large stochastic perturbation in permeability.

Figures 7.4.3(a) to 7.4.3(d) show effects of changing the correlation length, relative to the grid size. For the first case, the correlation length is considerably less than the grid size, meaning that the statistical properties at each grid-point have virtually no correlation with each other. The values are therefore virtually the same as the equivalent deterministic case figure 7.4.1(a). In figure 7.4.3(b), the correlation length is roughly three times the grid size, meaning that the properties of nearby grid-points are correlated much more strongly than between arbitrary grid-points. This manifests itself as a slight distortion in the shape of the solution, with respect to the deterministic solution. For figure (c) the correlation length is 0.5, meaning that the statistical properties of the pressure at the grid-points

are correlated over roughly half the region, producing slightly more distortion. In the last figure, the correlation length is ten times the dimensions of the region, meaning that all grid-points are roughly equally correlated with each other. This results in a large distortion compared to the deterministic case, as expected.

Figures 7.4.4(a) and 7.4.4(b) are supposed to show differences in cases where the correlation function is spatially anisotropic. However, the results for several tests of this type were not very interesting; and tended to give results similar to the isotropic cases with the smaller correlation length. That is, the shorter correlation length always tended to dominate the observed behaviour, and this does not seem to lead to very significant results. The reason for this could be that the more important behaviour is contained in the higher order moment term, and it will be particularly interesting to see how these results behave in time.

7.5 Summary

We have developed a method for estimating mean values for numerical solutions to systems of steady-state partial differential equations which contain a spatially varying uncertain parameter, that takes into account terms up to second order in the multivariate distribution function. This gives a significant insight into the behaviour of the mean when dispersion of the parameter values is taken into account, as opposed to using just the deterministic solution, which only includes information about the mean of the uncertain parameter, and consequently has no real statistical information. Bounds on the accuracy of this approach have been found and could be developed further, by evaluation and comparison for different multivariate distribution functions.

The problem with many of these results is the non-physicality of the model

equations, and the need to impose artificial boundary conditions, which do not allow a proper development of the flow behaviour. It is, however always interesting to observe the behaviour of the solutions with respect to the deterministic solution, and this is where the bulk of the work has been performed.

We make an additional note that it is also possible to use this perturbation method to obtain a second order accurate expression for the variance, and covariance, of the numerical pressure. Let us refer to the equation for the numerical solution for the pressure equation for a general realisation, equation (7.1.29),

$$\mathbf{p_n} = A^{-1} \sum_{i=0}^{\infty} (-D_n A^{-1})^i \mathbf{b_n}.$$

The covariance matrix is defined as

$$Cov(\mathbf{p}) = (\mathbf{p} - \langle \mathbf{p} \rangle)(\mathbf{p} - \langle \mathbf{p} \rangle)^T, \tag{7.5.1}$$

where the i^{th} diagonal term represents the variance of the pressure at the i^{th} gridpoint. It is fairly easy to see that, by using similar arguments as before, we can write down the second order approximation to the covariance matrix for the general realisation n,

$$Cov(\mathbf{p}) = \left(A^{-1}\mathbf{b_d} - A^{-1}DA^{-1}\mathbf{b_0} \right) \left(A^{-1}\mathbf{b_d} - A^{-1}DA^{-1}\mathbf{b_0} \right)^T. \tag{7.5.2}$$

So, the second order approximation to the full covariance matrix of the probabilistic problem would be

$$\begin{aligned} Cov(\mathbf{p}) = & \\ & A^{-1}\langle \mathbf{b_d} \mathbf{b_d^T} \rangle A^{-T} - A^{-1}\langle DA^{-1}\mathbf{b_0} \mathbf{b_d^T} \rangle A^{-T} \\ - \; & A^{-1}\langle \mathbf{b_d} \mathbf{b_0^T} A^{-T} D^T \rangle A^{-T} + A^{-1}\langle DA^{-1}\mathbf{b_0} \mathbf{b_0^T} A^{-T} D^T \rangle A^{-T}. \end{aligned} \tag{7.5.3}$$

In principle, this can be evaluated by splitting the matrix elements up into the weighted sums of elemental matrices of equations (7.3.2) and (7.3.3). The variance

of the pressure could then be calculated, as it would be found as the diagonal terms of the matrix in equation (7.5.3). We decided not to evaluate these terms numerically, but variance terms for the pressure are evaluated explicitly in the following chapter when we consider the time-dependent model equation.

The equivalent results to those shown here, for time-varying systems of equations, are investigated in the next chapter. In this chapter we observe both the time-varying mean value to the numerical solution, and its time-varying variance. The effects of using more physical (that is, time-varying) boundary conditions are also explored, and we extend the perturbation methods to more practical lognormal distribution functions.

Chapter 8

Two-Dimensional Dynamic Model

In this chapter we consider the dynamic model equation (2.2.1), in two dimensions (with the assumption that most results are able to be generalised to three dimensions), from Darcy's law plus the continuity equation for single-phase flow,

$$\gamma \frac{\partial p}{\partial t} - \nabla(k \nabla p) = f(\mathbf{r}, t). \tag{8.0.1}$$

We assume that the boundary conditions are such that the flow, $-k\nabla p$, is known around the edges of the rectangular region under consideration.

8.1 Hierarchical Equations

We first develop the hierarchical equations for a general admissible realisation. By developing these systems of equations as far as possible, before taking mean values on either side, we can obtain the equations required.

8.1.1 Standard Form

For a permeability distribution function that is symmetric about the mean value, a simple linear perturbation about the mean can be considered. We therefore treat the two-dimensional permeability field for a single realisation as a perturbation about some pre-defined mean value field,

$$k = k_0 + \alpha k_1. \tag{8.1.1}$$

We assume that $k_0 = \langle k \rangle$ is a deterministic mean, knowledge of which is available. Note that in this chapter we are using the formulation that includes α in the analysis, in contrast to the form $k = k_0 + k_1$ which was used in chapter 7. No particular significance should be deduced from this change and all analytical results obtained can be easily converted into the other form. The advantage of this slight change is that it makes it clearer when equating the equations in successive powers of α. This is more appropriate for the case where the equations do not naturally split into a perturbation series as in equation (7.1.29).

Equation (8.0.1) can then be written

$$\gamma \frac{\partial p}{\partial t} - \nabla \left((k_0 + \alpha k_1) \nabla p \right) = f_0(\mathbf{r}, t) + \alpha f_1(\mathbf{r}, t), \tag{8.1.2}$$

where p is the pressure solution for the specific realisation under consideration.

As in much work by Dagan, [34], and Dupuy and Schwydler, [21], we assume the pressure solution can be expressed in the form

$$p(\mathbf{r}, t) = p_0(\mathbf{r}, t) + \alpha p_1(\mathbf{r}, t) + \alpha^2 p_2(\mathbf{r}, t) + \alpha^3 R_3(\mathbf{r}, t),$$

where R_3 is some residue term due to the enforced lack of accuracy when this series is truncated at second order in α.

Then, equation (8.1.2) can be re-written

$$\gamma \frac{\partial}{\partial t}(p_0 + \alpha p_1 + \alpha^2 p_2 + \alpha^3 R_3)$$

$$- \quad \nabla \left((k_0 + \alpha k_1) \nabla (p_0 + \alpha p_1 + \alpha^2 p_2 + \alpha^3 p_3) \right) = f_0(\mathbf{r}, t) + \alpha f_1(\mathbf{r}, t). \tag{8.1.3}$$

If p_0 is defined to be the solution of the mean value problem, or equivalently of the deterministic problem,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \left(k_0 \nabla p_0 \right) = f_0, \tag{8.1.4}$$

then, by equating successive powers of α, equation (8.1.3) splits up into the system of hierarchical equations,

$$\gamma \frac{\partial p_1}{\partial t} - \nabla \left(k_0 \nabla p_1 \right) - \nabla \left(k_1 \nabla p_0 \right) = f_1, \tag{8.1.5}$$

$$\gamma \frac{\partial p_2}{\partial t} - \nabla \left(k_0 \nabla p_2 \right) - \nabla \left(k_1 \nabla p_1 \right) = 0, \tag{8.1.6}$$

$$\gamma \frac{\partial R_3}{\partial t} - \nabla \left((k_0 + \alpha k_1) \nabla R_3 \right) - \nabla \left(k_1 \nabla p_2 \right) = 0. \tag{8.1.7}$$

This represents a set of coupled p.d.e.s for each admissible realisation. Truncating this series at some point, of course, means imposing a level of accuracy on the possible solutions. We are not able to solve the third equation (8.1.7), and so these equations are of second order accuracy. It may, of course, be possible to obtain bounds on the size of these residue terms over all admissible realisations. This would effectively give a measure of the accuracy of the hierarchical approximation.

Theoretically, a higher N^{th} order accuracy can be obtained by taking

$$k = k_0 + \alpha k_1,$$

and

$$p = \sum_{m=0}^{N} \alpha^m p_m + R_{N+1},$$

where R_{N+1} is the residue due to truncating the series for N^{th} order accuracy.

This leads to the $N + 1$ set of hierarchical equations, where the obtainable accuracy is N^{th} order,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \left(k_0 \nabla p_0 \right) = f_0, \tag{8.1.8}$$

$$\gamma\frac{\partial p_1}{\partial t} - \nabla\left(k_0\nabla p_1\right) - \nabla\left(k_1\nabla p_0\right) = f_1, \tag{8.1.9}$$

$$\vdots$$

$$\gamma\frac{\partial p_m}{\partial t} - \nabla\left(k_0\nabla p_m\right) - \nabla\left(k_1\nabla p_{m-1}\right) = 0, \tag{8.1.10}$$

$$\vdots$$

$$\gamma\frac{\partial p_N}{\partial t} - \nabla\left(k_0\nabla p_N\right) - \nabla\left(k_1\nabla p_{N-1}\right) = 0, \tag{8.1.11}$$

$$\gamma\frac{\partial R_{N+1}}{\partial t} - \nabla\left((k_0+k_1)\nabla\alpha R_{N+1}\right) - \nabla\left(k_1\nabla p_N\right) = 0. \tag{8.1.12}$$

8.1.2 Lognormal Distribution

If a lognormal distribution function is assumed for the permeability, the expansion must be done about the geometric mean, [16]. This is equivalent to a linear expansion about the log of the permeability.

$$ln(k) = y = y_0 + \beta y_1,$$

where, $y_0 = \langle y \rangle$. So,

$$\begin{aligned} k &= e^{y_0} + \beta y_1 e^{y_0} + \frac{\beta^2 y_1^2}{2} e^{y_0} + \cdots \\ &= \kappa_g + \beta\kappa_1 + \beta^2\kappa_2 + \cdots = \kappa_g + \sum_{m=1}^{\infty} \beta^m \kappa_m, \end{aligned}$$

where κ_g is the geometric mean.

Performing the same procedure, assuming the pressure has the form

$$p = \sum_{m=0}^{N} \beta^m p_m + S_{N+1},$$

and substituting for pressure and permeability into equation (8.0.1) gives

$$\gamma\frac{\partial}{\partial t}\left(\sum_{m=0}^{N} \beta^m p_m + S_{N+1}\right) - \nabla\left((\kappa_g + \sum_{m=1}^{\infty} \beta^m \kappa_m)\nabla\left(\sum_{m=0}^{N} \beta^m p_m + S_{N+1}\right)\right) = f(\mathbf{r}, t). \tag{8.1.13}$$

Again, by equating powers of β we obtain the system of hierarchical equations

$$\gamma\frac{\partial p_0}{\partial t} - \nabla\left(\kappa_g \nabla p_0\right) = f_0 \tag{8.1.14}$$

$$\gamma\frac{\partial p_1}{\partial t} - \nabla\left(\kappa_g \nabla p_1\right) - \nabla\left(\kappa_1 \nabla p_0\right) = 0 \tag{8.1.15}$$

$$\gamma\frac{\partial p_2}{\partial t} - \nabla\left(\kappa_0 \nabla p_2\right) - \nabla\left(\kappa_1 \nabla p_1\right) - \nabla\left(\kappa_2 \nabla p_0\right) = 0 \tag{8.1.16}$$

$$\vdots$$

$$\gamma\frac{\partial p_m}{\partial t} - \nabla\left(\kappa_g \nabla p_m\right) - \sum_{i=1}^{m} \nabla\left(\kappa_i \nabla p_{m-i}\right) = 0 \tag{8.1.17}$$

$$\vdots$$

$$\gamma\frac{\partial p_N}{\partial t} - \nabla\left(\kappa_g \nabla p_N\right) - \sum_{i=1}^{N} \nabla\left(\kappa_i \nabla p_{N-i}\right) = 0 \tag{8.1.18}$$

$$\gamma\frac{\partial S_{N+1}}{\partial t} - \nabla\left(\kappa_g \nabla S_{N+1}\right) - \nabla\left(\left(\sum_{1}^{\infty} \beta^m \kappa_m\right)\nabla S_{N+1}\right) - \sum_{i=1}^{N+1} \nabla\left(\kappa_i \nabla p_{(N+1-i)}\right) = 0 \tag{8.1.19}$$

8.2 Statistical Properties of Analytical Equations

To progress further, we must consider the statistical properties of the solutions to all of the above equations, by taking mean values on either side.

8.2.1 Standard Form

Firstly, we just consider the second order approximations for symmetric, or standard-form, permeability distribution functions.

Taking mean values on either sides of equations (8.1.4) to (8.1.7), and assuming k_1 is a perturbation about the absolute mean, so that $\langle k_1 \rangle = 0$, we obtain

$$\gamma\frac{\partial p_0}{\partial t} - \nabla\left(k_0 \nabla p_0\right) = f_0, \tag{8.2.1}$$

$$\gamma\frac{\partial\langle p_1\rangle}{\partial t} - \nabla\left(k_0\nabla\langle p_1\rangle\right) = \langle f_1\rangle, \tag{8.2.2}$$

$$\gamma\frac{\partial\langle p_2\rangle}{\partial t} - \nabla\left(k_0\nabla\langle p_2\rangle\right) - \nabla\langle k_1\nabla p_1\rangle = 0, \tag{8.2.3}$$

and,

$$\gamma\frac{\partial\langle R_3\rangle}{\partial t} - \nabla\left(k_0\nabla\langle R_3\rangle\right) + \nabla\langle\alpha k_1\nabla R_3\rangle - \nabla\langle k_1\nabla p_2\rangle = 0. \tag{8.2.4}$$

As they stand, these equations are not solvable, even just up to second order, due to the presence of the cross-correlation term, $\nabla\langle k_1\nabla p_1\rangle$. For this to be possible, a method to evaluate the correlation function, $\langle k_1\nabla p_1\rangle$ is needed.

Consider multiplying k_1 by the grad of equation (8.1.5) to give an extra p.d.e. The result of this is to give higher order cross-correlation terms, such as $\langle k_1\nabla^2\left(k_1\nabla p_0\right)\rangle$ to evaluate, which would involve introducing subsequently higher order cross-correlation terms. This process, of course, is only feasible if a closure can be imposed on the system of equations, under consideration. As they stand, this is not possible.

8.2.2 Lognormal Distribution

The same procedure on the set of equations for the lognormal permeability distribution function, equations (8.1.14) to (8.1.16), gives the similar, but adapted equations,

$$\gamma\frac{\partial p_0}{\partial t} - \nabla\left(\kappa_g\nabla p_0\right) = f_0, \tag{8.2.5}$$

$$\gamma\frac{\partial\langle p_1\rangle}{\partial t} - \nabla\left(\kappa_g\nabla\langle p_1\rangle\right) = \langle f_1\rangle, \tag{8.2.6}$$

$$\gamma\frac{\partial\langle p_2\rangle}{\partial t} - \nabla\left(\kappa_g\nabla\langle p_2\rangle\right) - \nabla\langle\kappa_1\nabla p_1\rangle - \nabla\left(\langle\kappa_2\rangle\nabla p_0\right) = 0. \tag{8.2.7}$$

The difference here is the presence of the third term, $\nabla\langle\kappa_2\rangle\nabla p_0$ in equation (8.2.7), but this term just links in the first equation in the series, with an extra moment of the distribution, $\langle\kappa_2\rangle$ which is a known property of the distribution.

However, the basic problem is the same, that is the presence of $\langle \kappa_1 \nabla p_1 \rangle$, for which a method for solving simultaneously must be obtained, for example in [10] and [16].

8.2.3 Variance

A second order approximation to the covariance can be obtained in a similar way to [35], by considering $\gamma \frac{\partial p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t)}{\partial t}$, the values of the perturbation, at two distinct points,

$$\gamma \frac{\partial}{\partial t} \left(p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t) \right) = p_1(\mathbf{r_1},t) \gamma \frac{\partial p_1(\mathbf{r_2},t)}{\partial t} + p_1(\mathbf{r_2},t) \gamma \frac{\partial p_1(\mathbf{r_1},t)}{\partial t} \tag{8.2.8}$$

and substituting for $\gamma \frac{\partial p_1}{\partial t}$, etc. from (8.1.5),

$$\begin{aligned} & \gamma \frac{\partial}{\partial t} \left(p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t) \right) \\ & - \nabla_2 \left(k_0(\mathbf{r_2}) \nabla_2 p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t) \right) - \nabla_2 \left(k_1(\mathbf{r_2}) p_1(\mathbf{r_1},t) \nabla_2 p_0(\mathbf{r_2},t) \right) \\ & - \nabla_1 \left(k_0(\mathbf{r_1}) \nabla_1 p_1(\mathbf{r_2},t) p_1(\mathbf{r_1},t) \right) - \nabla_1 \left(k_1(\mathbf{r_1}) p_1(\mathbf{r_2},t) \nabla_1 p_0(\mathbf{r_1},t) \right) = 0. \end{aligned} \tag{8.2.9}$$

Taking the mean value on either side of this equation results in an equation for the behaviour of the covariance of the solution,

$$\begin{aligned} & \gamma \frac{\partial}{\partial t} \left(\langle p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t) \rangle \right) \\ & - \nabla_2 \left(k_0(\mathbf{r_2},t) \nabla_2 \langle p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t) \rangle \right) - \nabla_2 \left(\langle k_1(\mathbf{r_2}) p_1(\mathbf{r_1},t) \rangle \nabla_2 p_0(\mathbf{r_2},t) \right) \\ & - \nabla_1 \left(k_0(\mathbf{r_1}) \nabla_1 \langle p_1(\mathbf{r_2},t) p_1(\mathbf{r_1},t) \rangle \right) - \nabla_1 \left(\langle k_1(\mathbf{r_1}) p_1(\mathbf{r_2},t) \rangle \nabla_1 p_0(\mathbf{r_1},t) \right) = 0. \end{aligned} \tag{8.2.10}$$

If the covariance, at time t, between pressure values at two points $\mathbf{r_1}$, and $\mathbf{r_2}$ is denoted by $C(\mathbf{r_1},\mathbf{r_2},t)$, then these equations are

$$\gamma \frac{\partial}{\partial t} \left(C(\mathbf{r_1},\mathbf{r_2},t) \right)$$

$$- \quad \nabla_2 \left(k_0(\mathbf{r_2}, t) \nabla_2 C(\mathbf{r_1}, \mathbf{r_2}, t) \right) - \nabla_2 \left(\langle k_1(\mathbf{r_2}) p_1(\mathbf{r_1}, t) \rangle \nabla_2 p_0(\mathbf{r_2}, t) \right)$$

$$- \quad \nabla_1 \left(k_0(\mathbf{r_1}) \nabla_1 C(\mathbf{r_2}, \mathbf{r_1}, t) \right) - \nabla_1 \left(\langle k_1(\mathbf{r_1}) p_1(\mathbf{r_2}, t) \rangle \nabla_1 p_0(\mathbf{r_1}, t) \right) = 0. \tag{8.2.11}$$

Evaluation of the terms in the expression is again rendered impossible, if no method for solving the cross-correlation term is available.

So, we have found that developing a method to solve equations for the lowest moments of the distribution function of the solution to equation (2.2.1), in this case second order accurate approximations to mean and variance, requires some method of solving or evaluating the cross-correlation terms $\langle k_1 \nabla p_1 \rangle$, for values of spatial separation and time. Finding a solvable equation for these terms has proved problematic, but it has been found that we may obtain closure in this system of equations if we consider them in a discretised form.

8.3 Discretisation

We now show that the problem of providing a solution for $\langle k_1 \nabla p_1 \rangle$, or $\langle \kappa_1 \nabla p_1 \rangle$, may be overcome by consideration of the discretised versions of these equations.

8.3.1 Standard Form

We consider a discretisation of the equations (8.1.4) to (8.1.6), with a simple explicit time scheme, and a general (unspecified) spatial discretisation,

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h \left(k_{ij}^0 \nabla_h p_{0\ ij}^{n} \right) = f_{0\ ij}^{n}, \tag{8.3.1}$$

$$\frac{\gamma p_{1\ ij}^{n+1} - \gamma p_{1\ ij}^{n}}{\Delta t} - \nabla_h \left(k_{ij}^0 \nabla_h p_{1\ ij}^{n} \right) - \nabla_h \left(k_{ij}^1 \nabla_h p_{0\ ij}^{n} \right) = f_{1\ ij}^{n}, \tag{8.3.2}$$

and,

$$\frac{\gamma p_{2\ ij}^{n+1} - \gamma p_{2\ ij}^{n}}{\Delta t} - \nabla_h \left(k_{ij}^0 \nabla_h p_{2\ ij}^{n} \right) - \nabla_h \left(k_{ij}^1 \nabla_h p_{1\ ij}^{n} \right) = 0, \tag{8.3.3}$$

where the (i,j) indices refer to spatial points $(i\Delta x, j\Delta y)$ in cartesian co-ordinates, and $p_{z\ ij}^{n}$ refers to the numerical solution for $p_z(\mathbf{r}, n\Delta t)$, where $\mathbf{r}$ is also in Cartesian co-ordinates.

Now let us denote a general value of the perturbation k_1 at a discrete point $(i\Delta x, j\Delta y)$ by $k^1_{i'j'}$, and consider the value at a second reference point, (i', j'). Multiplying this into equation (8.3.2), and taking the mean values throughout the resultant, together with equations (8.3.1) and (8.3.3)), gives

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h \left(k^0_{ij} \nabla_h p_{0\ ij}^{n} \right) = f_{0\ ij}^{n}, \tag{8.3.4}$$

$$\frac{\gamma \langle k^1_{i'j'} p_{1\ ij}^{n+1} \rangle - \gamma \langle k^1_{i'j'} p_{1\ ij}^{n} \rangle}{\Delta t}$$

$$- \langle k^1_{i'j'} \nabla_h \left(k^0_{ij} \nabla_h p_{1\ ij}^{n} \right) \rangle - \langle k^1_{i'j'} \nabla_h \left(k^1_{ij} \nabla_h p_{0\ ij}^{n} \right) \rangle = \langle k^1_{i'j'} f_{1\ ij}^{n} \rangle, \tag{8.3.5}$$

$$\frac{\gamma \langle p_{2\ ij}^{n+1} \rangle - \gamma \langle p_{2\ ij}^{n} \rangle}{\Delta t} - \nabla_h \left(k^0_{ij} \nabla_h \langle p_{2\ ij}^{n} \rangle \right) - \langle \nabla_h \left(k^1_{ij} \nabla_h p_{1\ ij}^{n} \right) \rangle = 0. \tag{8.3.6}$$

This is now a complete set of coupled (numerical) p.d.e.s that can be solved. When these equations are being solved, simultaneously, the cross-correlation function is found, from equation (8.3.5), and then substituted into equation (8.3.6). In this form, it is a function of two (discretised) spatial points. The discretised autocorrelation function of the permeability field occurs in the $\langle k^1_{i'j'} \nabla_h (k^1_{ij} \nabla_h p_{0\ ij}^{n}) \rangle$ terms. These are basically just linear combinations of the autocorrelation parameters, with coefficients specifically dependent on the particular spatially-discretised scheme under consideration. The boundary conditions have been incorporated into the right hand side terms of the equations.

8.3.2 Lognormal Form

Performing the expansion for a lognormal distribution function, about the geometric mean, results in an extra term in the second order equation, as seen in

equation (8.1.13). In discretised form, the set of coupled numerical equations becomes

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h \left(k_{ij}^0 \nabla_h p_{0\ ij}^n \right) = f_{0\ ij}^n, \tag{8.3.7}$$

$$\begin{aligned} &\frac{\gamma \langle k_{i'j'}^1 p_{1\ ij}^{n+1} \rangle - \gamma \langle k_{i'j'}^1 p_{1\ ij}^{n} \rangle}{\Delta t} \\ &- \langle k_{i'j'}^1 \nabla_h \left(k_{ij}^0 \nabla_h p_{1\ ij}^n \right) \rangle - \langle k_{i'j'}^1 \nabla_h \left(k_{ij}^1 \nabla_h p_{0\ ij}^n \right) \rangle = \langle k_{i'j'}^1 f_{1\ ij}^n \rangle, \end{aligned} \tag{8.3.8}$$

$$\frac{\gamma \langle p_{2\ ij}^{n+1} \rangle - \gamma \langle p_{2\ ij}^{n} \rangle}{\Delta t} - \nabla_h \left(k_{ij}^0 \nabla_h \langle p_{2\ ij}^n \rangle \right) - \langle \nabla_h \left(k_{ij}^1 \nabla_h p_{1\ ij}^n \right) \rangle - \nabla_h \left(\langle k_{ij}^2 \rangle \nabla_h p_{0\ ij}^n \right) = 0. \tag{8.3.9}$$

8.3.3 Variance Equations

The same discretisation performed on the covariance equations (8.2.11) (which have the same form in the linear case and lognormal distributions) results in the following equations,

$$\begin{aligned} &\frac{\gamma C_{i'j'ij}^{n+1} - \gamma C_{i'j'ij}^{n}}{\Delta t} \\ &- \nabla_h \left(k_{ij}^0 \nabla_h C_{i'j'ij}^n \right) - \nabla_h \left(\langle k^1 p_1 \rangle_{i'j'ij}^n \nabla_h p_{0\ ij}^n \right) \\ &- \nabla_h \left(k_{i'j'}^0 \nabla_h C_{ijij'j'}^n \right) - \nabla_h \left(\langle k^1 p_1 \rangle_{iji'j'}^n \nabla_h p_{0\ i'j'}^n \right) = 0. \end{aligned} \tag{8.3.10}$$

The quantity of particular interest is the variance of the pressure distribution, an important characterisation of the complete distribution function. In discretised form, the variance for time level $n\Delta t$, at spatial position $(i\Delta x, j\Delta y)$ is the value of C_{ijij}^n. Unfortunately, in the process of solving for this value, the correlation values for distinct points, $C_{i'j'ij}^n$ must also be solved and stored for each time-level. These can be considered as a bonus to the required information, having an academic, rather than practical point of interest, although an idea for the correlation length of the solution variable is now clearly available through this technique.

8.3.4 Summary

The result of the manipulation of the hierarchical equations (8.1.8) to (8.1.12) gives us a set of coupled numerical p.d.e.s for the first two moments that characterise the probability distribution function of the pressure solution. They can be solved at each successive time-level to follow their progression in time. This results in an approximate idea of the time development of the distribution function.

To summarise, these equations are

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h \left(k_{ij}^0 \nabla_h p_{0\ ij}^{n} \right) = f_{0\ ij}^{n}, \tag{8.3.11}$$

$$\begin{aligned} &\frac{\gamma \langle k^1 p_1 \rangle_{i'j'ij}^{n+1} - \gamma \langle k^1 p_1 \rangle_{i'j'ij}^{n}}{\Delta t} \\ &- \langle k_{i'j'}^1 \nabla_h \left(k_{ij}^0 \nabla_h p_{1\ ij}^{n} \right) \rangle - \langle k_{i'j'}^1 \nabla_h \left(k_{ij}^1 \nabla_h p_{0\ ij}^{n} \right) \rangle = \langle k_{i'j'}^1 f_{1\ ij}^{n} \rangle, \end{aligned} \tag{8.3.12}$$

$$\frac{\gamma \langle p_{2\ ij}^{n+1} \rangle - \gamma \langle p_{2\ ij}^{n} \rangle}{\Delta t} - \nabla_h \left(k_{ij}^0 \nabla_h \langle p_{2\ ij}^{n} \rangle \right) - \langle \nabla_h \left(k_{ij}^1 \nabla_h p_{1\ ij}^{n} \right) \rangle = 0, \tag{8.3.13}$$

and,

$$\begin{aligned} &\frac{\gamma C_{i'j'ij}^{n+1} - \gamma C_{i'j'ij}^{n}}{\Delta t} \\ &- \nabla_h \left(k_{ij}^0 \nabla_h C_{i'j'ij}^{n} \right) - \nabla_{2h} \left(\langle k^1 p_1 \rangle_{i'j'ij}^{n} \nabla_{2h} p_{0\ ij}^{n} \right) \\ &- \nabla_h \left(k_{i'j'}^0 \nabla_h C_{ijij'}^{n} \right) - \nabla_h \left(\langle k^1 p_1 \rangle_{ijij'}^{n} \nabla_h p_{0\ i'j'}^{n} \right) = 0. \end{aligned} \tag{8.3.14}$$

For an assumed lognormal distribution function, the coupled equations take the slightly adapted form

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h \left(k_{ij}^g \nabla_h p_{0\ ij}^{n} \right) = f_{0\ ij}^{n}, \tag{8.3.15}$$

$$\begin{aligned} &\frac{\gamma \langle k^1 p_1 \rangle_{i'j'ij}^{n+1} - \gamma \langle k^1 p_1 \rangle_{i'j'ij}^{n}}{\Delta t} \\ &- \langle k_{i'j'}^1 \nabla_h \left(k_{ij}^g \nabla_h p_{1\ ij}^{n} \right) \rangle - \langle k_{i'j'}^1 \nabla_h \left(k_{ij}^1 \nabla_h p_{0\ ij}^{n} \right) \rangle = \langle k_{i'j'}^1 f_{1\ ij}^{n} \rangle, \end{aligned} \tag{8.3.16}$$

$$\frac{\gamma\langle p_2^{n+1}{}_{ij}\rangle - \gamma\langle p_2^{n}{}_{ij}\rangle}{\Delta t}$$
$$- \quad \nabla_h\left(k_{ij}^g \nabla_h \langle p_2^n{}_{ij}\rangle\right) - \langle \nabla_h\left(k_{ij}^1 \nabla_h p_1^n{}_{ij}\right)\rangle - \nabla_h\left(\langle k_{ij}^2\rangle \nabla_h p_0^n{}_{ij}\right) = 0, \tag{8.3.17}$$

and,

$$\frac{\gamma C_{i'j'ij}^{n+1} - \gamma C_{i'j'ij}^{n}}{\Delta t}$$
$$- \quad \nabla_h\left(k_{ij}^g \nabla_h C_{i'j'ij}^n\right) - \nabla_h\left(\langle k^1 p_1\rangle_{i'j'ij}^n \nabla_h p_0^n{}_{ij}\right)$$
$$- \quad \nabla_h\left(k_{i'j'}^g \nabla_h C_{iji'j'}^n\right) - \nabla_h\left(\langle k^1 p_1\rangle_{iji'j'}^n \nabla_h p_0^n{}_{i'j'}\right) = 0. \tag{8.3.18}$$

8.4 Application

We now apply this technique to a specific example of a discretisation.

Consider a simple five-point difference scheme, where the value of the permeability at points halfway between adjacent gridpoints (i,j) and $(i \pm 1, j)$ or $(i, j \pm 1)$ is always approximated by an average of the two values at the grid-points. Equation (8.3.1) in this case becomes,

$$\frac{\gamma p_0^{n+1}{}_{ij} - \gamma p_0^n{}_{ij}}{\Delta t}$$
$$+ \quad \frac{(k_{i+1j}^0 + k_{ij}^0)}{2\Delta x^2} p_0{}_{i+1j} + \frac{(k_{i-1j}^0 + k_{ij}^0)}{2\Delta x^2} p_{i-1j}^0$$
$$+ \quad \frac{(k_{ij+1}^0 + k_{ij}^0)}{2\Delta y^2} p_0{}_{ij+1} + \frac{(k_{ij-1}^0 + k_{ij}^0)}{2\Delta y^2} p_0{}_{ij-1}$$
$$- \quad \left\{ \frac{(k_{i+1j}^0 + k_{i-1j}^0 + 2k_{ij}^0)}{2\Delta x^2} + \frac{(k_{ij+1}^0 + k_{ij-1}^0 + 2k_{ij}^0)}{2\Delta y^2} \right\} p_0{}_{ij} = f_0{}_{ij}. \tag{8.4.1}$$

8.4.1 Standard Form

Making use of the similarity in structure of equations (8.3.11) to (8.3.14), the remaining system of equations can be written down immediately,

$$\frac{\gamma\langle k_{i'j'}^1 p_1^{n+1}{}_{ij}\rangle - \gamma\langle k_{i'j'}^1 p_1^n{}_{ij}\rangle}{\Delta t}$$

$$
\begin{aligned}
&+ \quad \frac{(k^0_{i+1j}+k^0_{ij})}{2\Delta x^2}\langle k^1 p_1\rangle_{i'j'i+1j} + \frac{(k^0_{i-1j}+k^0_{ij})}{2\Delta x^2}\langle k^1 p_1\rangle_{i'j'i-1j} \\
&+ \quad \frac{(k^0_{ij+1}+k^0_{ij})}{2\Delta y^2}\langle k^1 p_1\rangle_{i'j'ij+1} + \frac{(k^0_{ij-1}+k^0_{ij})}{2\Delta y^2}\langle k^1 p_1\rangle_{i'j'ij-1} \\
&- \quad \left\{\frac{(k^0_{i+1j}+k^0_{i-1j}+2k^0_{ij})}{2\Delta x^2} + \frac{(k^0_{ij+1}+k^0_{ij-1}+2k^0_{ij})}{2\Delta y^2}\right\}\langle k^1 p_1\rangle_{i'j'ij} \\
&+ \quad \frac{(\langle k^1 k^1\rangle_{i'j'i+1j}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta x^2} p_{0\ i+1j} + \frac{(\langle k^1 k^1\rangle_{i'j'i-1j}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta x^2} p^0_{i-1j} \\
&+ \quad \frac{(\langle k^1 k^1\rangle_{i'j'ij+1}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta y^2} p_{0\ ij+1} + \frac{(\langle k^1 k^1\rangle_{i'j'ij-1}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta y^2} p_{0\ ij-1} \\
&- \quad \left\{\frac{(\langle k^1 k^1\rangle_{i'j'i+1j}+\langle k^1 k^1\rangle_{i'j'i-1j}+2\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta x^2}\right. \\
&\quad \left. + \frac{(\langle k^1 k^1\rangle_{i'j'ij+1}+\langle k^1 k^1\rangle_{i'j'ij-1}+2\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta y^2}\right\} p_{0\ ij} = \langle k^1 f_1\rangle_{i'j'ij},
\end{aligned}
\tag{8.4.2}
$$

$$
\begin{aligned}
&\frac{\gamma p_{2\ ij}^{n+1} - \gamma p_{2\ ij}^{n}}{\Delta t} \\
&\quad + \quad \frac{(k^0_{i+1j}+k^0_{ij})}{2\Delta x^2} p_{2\ i+1j} + \frac{(k^0_{i-1j}+k^0_{ij})}{2\Delta x^2} p_{2\ i-1j} \\
&\quad + \quad \frac{(k^0_{ij+1}+k^0_{ij})}{2\Delta y^2} p_{2\ ij+1} + \frac{(k^0_{ij-1}+k^0_{ij})}{2\Delta y^2} p_{2\ ij-1} \\
&\quad - \quad \left\{\frac{(k^0_{i+1j}+k^0_{i-1j}+2k^0_{ij})}{2\Delta x^2} + \frac{(k^0_{ij+1}+k^0_{ij-1}+2k^0_{ij})}{2\Delta y^2}\right\} p_{2\ ij} \\
&\quad - \quad \frac{(\langle k^1 p_1\rangle_{i+1ji+1j}+\langle k^1 p_1\rangle_{iji+1j}+\langle k^1 p_1\rangle_{i-1ji-1j}+\langle k^1 p_1\rangle_{iji-1j})}{2\Delta x^2} \\
&\quad - \quad \frac{(\langle k^1 p_1\rangle_{ij+1ij+1}+\langle k^1 p_1\rangle_{ijij+1}+\langle k^1 p_1\rangle_{ij-1ij-1}+\langle k^1 p_1\rangle_{ijij-1})}{2\Delta y^2} \\
&\quad + \quad \frac{(\langle k^1 p_1\rangle_{i+1jij}+\langle k^1 p_1\rangle_{i-1jij}+2\langle k^1 p_1\rangle_{ijij})}{2\Delta x^2} \\
&\quad + \quad \frac{(\langle k^1 p_1\rangle_{ij+1ij}+\langle k^1 p_1\rangle_{ij-1ij}+2\langle k^1 p_1\rangle_{ijij})}{2\Delta y^2} = 0,
\end{aligned}
\tag{8.4.3}
$$

$$
\begin{aligned}
&\frac{\gamma C^{n+1}_{i'j'ij} - \gamma C^{n}_{i'j'ij}}{\Delta t} \\
&- \quad \frac{(k^0_{i+1j}+k^0_{ij})}{2\Delta x^2} C^n_{i'j'i+1j} - \frac{(k^0_{i-1j}+k^0_{ij})}{2\Delta x^2} C^n_{i'j'i-1j} \\
&- \quad \frac{(k^0_{ij+1}+k^0_{ij})}{2\Delta y^2} C^n_{i'j'ij+1} - \frac{(k^0_{ij-1}+k^0_{ij})}{2\Delta y^2} C^n_{i'j'ij-1} \\
&+ \quad \left\{\frac{(k^0_{i+1j}+k^0_{i-1j}+2k^0_{ij})}{2\Delta x^2} + \frac{(k^0_{ij+1}+k^0_{ij-1}+2k^0_{ij})}{2\Delta y^2}\right\} C^n_{i'j'ij} \\
&+ \quad \frac{(\langle k^1 p_1\rangle^n_{i'j'i+1j}+\langle k^1 p_1\rangle^n_{i'j'ij})}{2\Delta x^2} p^n_{0\ i+1j} + \frac{(\langle k^1 p_1\rangle^n_{i'j'i-1j}+\langle k^1 p_1\rangle^n_{i'j'ij})}{2\Delta x^2} p^n_{0\ i-1j}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{(\langle k^1 p_1\rangle^n_{i'j'ij+1} + \langle k^1 p_1\rangle^n_{i'j'ij})}{2\Delta y^2} p^n_{0\ ij+1} + \frac{(k^0_{ij-1} + k^0_{ij})}{2\Delta y^2} p_{0\ ij-1} \\
&- \quad \Bigg\{ \frac{(\langle k^1 p_1\rangle^n_{i'j'i+1j} + \langle k^1 p_1\rangle^n_{i'j'i-1j} + 2\langle k^1 p_1\rangle^n_{i'j'ij})}{2\Delta x^2} \\
&+ \quad \frac{(\langle k^1 p_1\rangle^n_{i'j'ij+1} + \langle k^1 p_1\rangle^n_{i'j'ij-1} + 2\langle k^1 p_1\rangle^n_{i'j'ij})}{2\Delta y^2} \Bigg\} p^n_{0\ ij} \\
&- \quad \frac{(k^0_{i'+1j'} + k^0_{i'j'})}{2\Delta x^2} C^n_{iji'+1j'} - \frac{(k^0_{i'-1j'} + k^0_{i'j'})}{2\Delta x^2} C^n_{iji'-1j'} \\
&- \quad \frac{(k^0_{i'j'+1} + k^0_{i'j'})}{2\Delta y^2} C^n_{iji'j'+1} - \frac{(k^0_{i'j'-1} + k^0_{i'j'})}{2\Delta y^2} C^n_{iji'j'-1} \\
&+ \quad \Bigg\{ \frac{(k^0_{i'+1j'} + k^0_{i'-1j'} + 2k^0_{i'j'})}{2\Delta x^2} + \frac{(k^0_{i'j'+1} + k^0_{i'j'-1} + 2k^0_{i'j'})}{2\Delta y^2} \Bigg\} C^n_{iji'j'} \\
&+ \quad \frac{(\langle k^1 p_1\rangle^n_{iji'+1j'} + \langle k^1 p_1\rangle^n_{iji'j'})}{2\Delta x^2} p^n_{0\ i'+1j'} + \frac{(\langle k^1 p_1\rangle^n_{iji'-1j'} + \langle k^1 p_1\rangle^n_{iji'j'})}{2\Delta x^2} p^n_{0\ i'-1j'} \\
&+ \quad \frac{(\langle k^1 p_1\rangle^n_{iji'j'+1} + \langle k^1 p_1\rangle^n_{iji'j'})}{2\Delta y^2} p^n_{0\ i'j'+1} + \frac{(k^0_{i'j'-1} + k^0_{i'j'})}{2\Delta y^2} p_{0\ i'j'-1} \\
&- \quad \Bigg\{ \frac{(\langle k^1 p_1\rangle^n_{iji'+1j'} + \langle k^1 p_1\rangle^n_{iji'-1j'} + 2\langle k^1 p_1\rangle^n_{iji'j'})}{2\Delta x^2} \\
&+ \quad \frac{(\langle k^1 p_1\rangle^n_{iji'j'+1} + \langle k^1 p_1\rangle^n_{iji'j'-1} + 2\langle k^1 p_1\rangle^n_{iji'j'})}{2\Delta y^2} \Bigg\} p^n_{0\ i'j'} = 0 \qquad (8.4.4)
\end{aligned}
$$

8.4.2 Lognormal Form

In the example used to illustrate the numerical technique here, a lognormal distribution is assumed. The deterministic equation takes a similar form to equation (8.4.1),

$$
\begin{aligned}
&\frac{\gamma p^{n+1}_{0\ ij} - \gamma p^n_{0\ ij}}{\Delta t} \\
&+ \quad \frac{(k^g_{i+1j} + k^g_{ij})}{2\Delta x^2} p_{0\ i+1j} + \frac{(k^g_{i-1j} + k^g_{ij})}{2\Delta x^2} p^0_{i-1j} \\
&+ \quad \frac{(k^g_{ij+1} + k^g_{ij})}{2\Delta y^2} p_{0\ ij+1} + \frac{(k^g_{ij-1} + k^g_{ij})}{2\Delta y^2} p_{0\ ij-1} \\
&- \quad \Bigg\{ \frac{(k^g_{i+1j} + k^g_{i-1j} + 2k^g_{ij})}{2\Delta x^2} + \frac{(k^g_{ij+1} + k^g_{ij-1} + 2k^g_{ij})}{2\Delta y^2} \Bigg\} p_{0\ ij} = f_{0\ ij}. \qquad (8.4.5)
\end{aligned}
$$

Equations (8.3.16) to (8.3.18) then can be re-written

$$
\begin{aligned}
&\frac{\gamma\langle k^1_{i'j'} p^{n+1}_1{}_{ij}\rangle - \gamma\langle k^1_{i'j'} p^n_1{}_{ij}\rangle}{\Delta t} \\
&+ \quad \frac{(k^g_{i+1j} + k^g_{ij})}{2\Delta x^2} \langle k^1 p_1\rangle_{i'j'i+1j} + \frac{(k^g_{i-1j} + k^g_{ij})}{2\Delta x^2} \langle k^1 p_1\rangle_{i'j'i-1j}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{(k^g_{ij+1}+k^g_{ij})}{2\Delta y^2}\langle k^1 p_1\rangle_{i'j'ij+1} + \frac{(k^g_{ij-1}+k^g_{ij})}{2\Delta y^2}\langle k^1 p_1\rangle_{i'j'ij-1} \\
&- \quad \left\{\frac{(k^g_{i+1j}+k^g_{i-1j}+2k^g_{ij})}{2\Delta x^2} + \frac{(k^g_{ij+1}+k^g_{ij-1}+2k^g_{ij})}{2\Delta y^2}\right\}\langle k^1 p_1\rangle_{i'j'ij} \\
&+ \quad \frac{(\langle k^1 k^1\rangle_{i'j'i+1j}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta x^2} p_{0\ i+1j} + \frac{(\langle k^1 k^1\rangle_{i'j'i-1j}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta x^2} p^0_{i-1j} \\
&+ \quad \frac{(\langle k^1 k^1\rangle_{i'j'ij+1}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta y^2} p_{0\ ij+1} + \frac{(\langle k^1 k^1\rangle_{i'j'ij-1}+\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta y^2} p_{0\ ij-1} \\
&- \quad \left\{\frac{(\langle k^1 k^1\rangle_{i'j'i+1j}+\langle k^1 k^1\rangle_{i'j'i-1j}+2\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta x^2}\right. \\
&\qquad \left. + \frac{(\langle k^1 k^1\rangle_{i'j'ij+1}+\langle k^1 k^1\rangle_{i'j'ij-1}+2\langle k^1 k^1\rangle_{i'j'ij})}{2\Delta y^2}\right\} p_{0\ ij} = \langle k^1 f_1\rangle_{i'j'ij},
\end{aligned}
\tag{8.4.6}
$$

$$
\begin{aligned}
&\frac{\gamma p_{2\ ij}^{n+1} - \gamma p_{2\ ij}^{n}}{\Delta t} \\
&+ \quad \frac{(k^g_{i+1j}+k^g_{ij})}{2\Delta x^2} p_{2\ i+1j} + \frac{(k^g_{i-1j}+k^g_{ij})}{2\Delta x^2} p_{2\ i-1j} \\
&+ \quad \frac{(k^g_{ij+1}+k^g_{ij})}{2\Delta y^2} p_{2\ ij+1} + \frac{(k^g_{ij-1}+k^g_{ij})}{2\Delta y^2} p_{2\ ij-1} \\
&- \quad \left\{\frac{(k^g_{i+1j}+k^g_{i-1j}+2k^g_{ij})}{2\Delta x^2} + \frac{(k^g_{ij+1}+k^g_{ij-1}+2k^g_{ij})}{2\Delta y^2}\right\} p_{2\ ij} \\
&- \quad \frac{(\langle k^1 p_1\rangle_{i+1ji+1j}+\langle k^1 p_1\rangle_{iji+1j}+\langle k^1 p_1\rangle_{i-1ji-1j}+\langle k^1 p_1\rangle_{iji-1j})}{2\Delta x^2} \\
&- \quad \frac{(\langle k^1 p_1\rangle_{ij+1ij+1}+\langle k^1 p_1\rangle_{ijij+1}+\langle k^1 p_1\rangle_{ij-1ij-1}+\langle k^1 p_1\rangle_{ijij-1})}{2\Delta y^2} \\
&+ \quad \frac{(\langle k^1 p_1\rangle_{i+1jij}+\langle k^1 p_1\rangle_{i-1jij}+2\langle k^1 p_1\rangle_{ijij})}{2\Delta x^2} \\
&+ \quad \frac{(\langle k^1 p_1\rangle_{ij+1ij}+\langle k^1 p_1\rangle_{ij-1ij}+2\langle k^1 p_1\rangle_{ijij})}{2\Delta y^2} \\
&+ \quad \frac{(\langle k^2_{i+1j}\rangle+\langle k^2_{ij}\rangle)}{2\Delta x^2} p_{0\ i+1j} + \frac{(\langle k^2_{i-1j}\rangle+\langle k^2_{ij}\rangle)}{2\Delta x^2} p_{0\ i-1j} \\
&+ \quad \frac{(\langle k^2_{ij+1}\rangle+\langle k^2_{ij}\rangle)}{2\Delta y^2} p_{0\ ij+1} + \frac{(\langle k^2_{ij-1}\rangle+\langle k^2_{ij}\rangle)}{2\Delta y^2} p_{0\ ij-1} \\
&- \quad \left\{\frac{(\langle k^2_{i+1j}\rangle+\langle k^2_{i-1j}\rangle+2\langle k^2_{ij}\rangle)}{2\Delta x^2} + \frac{(\langle k^2_{ij+1}\rangle+\langle k^2_{ij-1}\rangle+2\langle k^2_{ij}\rangle)}{2\Delta y^2}\right\} p_{0\ ij} = 0,
\end{aligned}
\tag{8.4.7}
$$

and,

$$
\frac{\gamma C^{n+1}_{i'j'ij} - \gamma C^{n}_{i'j'ij}}{\Delta t}
$$

$$
\begin{aligned}
- & \frac{(k^g_{i+1j}+k^g_{ij})}{2\Delta x^2}C^n_{i'j'i+1j}-\frac{(k^g_{i-1j}+k^g_{ij})}{2\Delta x^2}C^n_{i'j'i-1j}\\
- & \frac{(k^g_{ij+1}+k^g_{ij})}{2\Delta y^2}C^n_{i'j'ij+1}-\frac{(k^g_{ij-1}+k^g_{ij})}{2\Delta y^2}C^n_{i'j'ij-1}\\
+ & \left\{\frac{(k^g_{i+1j}+k^g_{i-1j}+2k^g_{ij})}{2\Delta x^2}+\frac{(k^g_{ij+1}+k^g_{ij-1}+2k^g_{ij})}{2\Delta y^2}\right\}C^n_{i'j'ij}\\
+ & \frac{(\langle k^1p_1\rangle^n_{i'j'i+1j}+\langle k^1p_1\rangle^n_{i'j'ij})}{2\Delta x^2}p^n_{0\ i+1j}+\frac{(\langle k^1p_1\rangle^n_{i'j'i-1j}+\langle k^1p_1\rangle^n_{i'j'ij})}{2\Delta x^2}p^n_{0\ i-1j}\\
+ & \frac{(\langle k^1p_1\rangle^n_{i'j'ij+1}+\langle k^1p_1\rangle^n_{i'j'ij})}{2\Delta y^2}p^n_{0\ ij+1}+\frac{(k^0_{ij-1}+k^0_{ij})}{2\Delta y^2}p_{0\ ij-1}\\
- & \left\{\frac{(\langle k^1p_1\rangle^n_{i'j'i+1j}+\langle k^1p_1\rangle^n_{i'j'i-1j}+2\langle k^1p_1\rangle^n_{i'j'ij})}{2\Delta x^2}\right.\\
+ & \left.\frac{(\langle k^1p_1\rangle^n_{i'j'ij+1}+\langle k^1p_1\rangle^n_{i'j'ij-1}+2\langle k^1p_1\rangle^n_{i'j'ij})}{2\Delta y^2}\right\}p^n_{0\ ij}\\
- & \frac{(k^g_{i'+1j'}+k^g_{i'j'})}{2\Delta x^2}C^n_{iji'+1j'}-\frac{(k^g_{i'-1j'}+k^g_{i'j'})}{2\Delta x^2}C^n_{iji'-1j'}\\
- & \frac{(k^g_{i'j'+1}+k^g_{i'j'})}{2\Delta y^2}C^n_{iji'j'+1}-\frac{(k^g_{i'j'-1}+k^g_{i'j'})}{2\Delta y^2}C^n_{iji'j'-1}\\
+ & \left\{\frac{(k^g_{i'+1j'}+k^g_{i'-1j'}+2k^g_{i'j'})}{2\Delta x^2}+\frac{(k^g_{i'j'+1}+k^g_{i'j'-1}+2k^g_{i'j'})}{2\Delta y^2}\right\}C^n_{iji'j'}\\
+ & \frac{(\langle k^1p_1\rangle^n_{iji'+1j'}+\langle k^1p_1\rangle^n_{iji'j'})}{2\Delta x^2}p^n_{0\ i'+1j'}+\frac{(\langle k^1p_1\rangle^n_{iji'-1j'}+\langle k^1p_1\rangle^n_{iji'j'})}{2\Delta x^2}p^n_{0\ i'-1j'}\\
+ & \frac{(\langle k^1p_1\rangle^n_{iji'j'+1}+\langle k^1p_1\rangle^n_{iji'j'})}{2\Delta y^2}p^n_{0\ i'j'+1}+\frac{(k^0_{i'j'-1}+k^0_{i'j'})}{2\Delta y^2}p_{0\ i'j'-1}\\
- & \left\{\frac{(\langle k^1p_1\rangle^n_{iji'+1j'}+\langle k^1p_1\rangle^n_{iji'-1j'}+2\langle k^1p_1\rangle^n_{iji'j'})}{2\Delta x^2}\right.\\
+ & \left.\frac{(\langle k^1p_1\rangle^n_{iji'j'+1}+\langle k^1p_1\rangle^n_{iji'j'-1}+2\langle k^1p_1\rangle^n_{iji'j'})}{2\Delta y^2}\right\}p^n_{0\ i'j'}=0. \qquad (8.4.8)
\end{aligned}
$$

8.5 Results

In this section we present some illustrative samples of the type of results that we have obtained using this method to solve the full statistical problem.

In each case we consider a single Fourier mode as the initial condition, with no flow conditions around the boundary, and zero forcing function. The region under investigation is square with unit length. All lengths and times are normalised for the purposes of this research.

Using a single Fourier mode as the initial condition means that in the case of a homogeneous mean value for the permeability, the solution to the p.d.e. under consideration, equation (8.0.1), may be expressed as the Fourier mode with an exponentially decaying amplitude,

$$p(x, y, t) = e^{-\pi^2 \frac{k}{\gamma} t} cos(\pi x). \tag{8.5.1}$$

It is fairly trivial to show by substitution that this is a solution to the model equation, satisfying the zero boundary conditions. We choose this test function as it is a straightforward solution whose deterministic behaviour is well-known. We restrict the step-sizes to 0.05 in each illustration. The distributions are, in each case, assumed to be lognormal.

8.5.1 Figures 8.5.1

In the first example, we have the case where the homogeneous mean value is 0.2, and the variance $\sigma^2 = 0.05$. Correlation lengths in both the x- and y-directions are the same, equal to 1.0, the size of the region under investigation. In Figure 8.5.1(a) we show the initial condition for the deterministic solution, a one-dimensional Fourier mode, given by equation (8.5.1) at $t = 0$,

$$p(x, y) = cos(\pi x). \tag{8.5.2}$$

The numerical amplitude at time $t = 1.0$ is 0.140 compared to the analytic value of $e^{-\pi^2 \times 0.2} = 0.139$. In Figures 8.5.1(b), 8.5.1(c), 8.5.1(d), and 8.5.1(e) we show three dimensional plots of the variances throughout the region. The initial value of the variance is taken to be zero throughout the region (equivalent to a deterministic initial condition), and the Figures show how the variance function changes over time intervals of 0.3, starting at $t = 0.1$, and then at 0.4, 0.7, and 1.0 for respective Figures 8.5.1(b) to 8.5.1(e). Numerically, the covariances have

maximum values 5.637×10^{-5} at $t = 0.1$, 2.515×10^{-4} at $t = 0.4$, 2.381×10^{-4} at $t = 0.7$, and 1.629×10^{-4} at $t = 1.0$. Figures 8.5.1(f), 8.5.1(g), 8.5.1(h), and 8.5.1(i) show the second order correction to the mean value with respect to the deterministic solution, $\langle p_2 \rangle$, at time intervals $t = 0.1$, 0.4, 0.7, and 1.0 respectively. The values have the following maxima, 4.640×10^{-4} for 8.5.1(f), 2.253×10^{-3} 8.5.1(g), 4.010×10^{-3} 8.5.1(h), and 5.434×10^{-3} for Figure 8.5.1(i).

8.5.2 Figures 8.5.2

In this second example we consider a similar case to Figures 8.5.1, but with a lower value for the homogeneous mean, $\langle k \rangle = 0.1$. The statistical variables are the same as above, with the variance $\sigma^2 = 0.05$, and correlation lengths 1.0 in both directions. The values for the deterministic solution are similar, but with a slower decay rate, the numerical amplitude being 0.375 after one unit of time, compared with the analytic value of 0.373. In Figure 8.5.2(a) we show a three dimensional plot of the variance function throughout the region of interest, plotted at time $t = 1.0$. The maximum numerical value for the function is 2.630×10^{-4}. Figure 8.5.2(b) shows the second order correction to the mean value, at time $t = 1.0$ also. The maximum value is 2.841×10^{-3}.

8.5.3 Figures 8.5.3

In Figures 8.5.3(a) and 8.5.3(b) we show the effect of increasing the variance of the permeability. The general data is the same as for Figure 8.5.1, but with variance $\sigma^2 = 0.1$. Figure 8.5.3(a) shows the variance plotted after time $t = 1.0$, with maximum value 6.517×10^{-4}. Figure 8.5.3(b) shows the second order correction to the mean, with maximum value 2.173×10^{-2}.

8.5.4 Figures 8.5.4

In the Figures 8.5.4(a) to 8.5.4(f) we show the differing types of behaviour seen when using an anisotropic correlation function, after a time interval $t = 1.0$. The mean and variance for permeability are the same in each Figure, $\langle k \rangle = 0.2$ and $\sigma^2 = 0.1$ respectively. In 8.5.4(a) and 8.5.4(b) we see the case where the correlation length is comparatively short in the x-direction, $\lambda_x = 0.1$, and long in the y-direction, $\lambda_y = 1.0$. 8.5.4(a) shows the variance, with maximum value 2.515×10^{-4}, and 8.5.4(b) the correction to the mean, with maximum 1.641×10^{-2}.

In 8.5.4(c) and 8.5.4(d), the correlation lengths are reversed, compared to 8.5.4(a) and 8.5.4(b), with $\lambda_x = 1.0$, and $\lambda_y = 0.1$. 8.5.4(c) shows variance, with maximum value 1.184×10^{-4}, and 8.5.4(d) shows correction, with maximum value 8.449×10^{-3}.

In 8.5.4(e) and 8.5.4(f) we see the effect of a short isotropic correlation length in both directions, $\lambda_x = 0.1$, and $\lambda_y = 0.1$. 8.5.4(e) shows variance, with maximum value 3.558×10^{-5}, and 8.5.4(f) shows correction, with maximum value 8.553×10^{-3}.

8.5.5 Figures 8.5.5

The final Figures show an example of the type of behaviour seen when a spatially varying mean value for permeability is assumed. In this case, the mean value linearly decreases in the x-direction, from 0.2 at $x = 0$, to 0.1 at $x = 1.0$. The variance of the permeability is $\sigma^2 = 0.05$, and the correlation lengths are the same for both directions, $\lambda_x = 1.0$, and $\lambda_y = 1.0$. Figure 8.5.5(a) shows a three-dimensional plot of the deterministic solution throughout the region; 8.5.5(b) is a plot of the variance, with maximum value 3.135×10^{-4}, and 8.5.5(c) shows the correction term, with maximum value 4.037×10^{-3}.

Figures 8.5.1 Plots of pressure and pressure variance verses position, at time intervals of 0.3 seconds. $\langle \kappa \rangle = 0.2$, $\sigma^2 = 0.05$

Figure 8.5.1(a) Initial condition for deterministic pressure solution

Figure 8.5.1(b) Pressure variance at $t = 0.1$

Figure 8.5.1(c) Pressure variance at $t = 0.4$

Figure 8.5.1(d) Pressure variance at $t = 0.7$

Figure 8.5.1(e) Pressure variance at $t = 1.0$

Figure 8.5.1(f) Second order correction to mean value, $\langle p_2 \rangle$, at $t = 0.1$

Figure 8.5.1(g) $\langle p_2 \rangle$ at $t = 0.4$

Figure 8.5.1(h) $\langle p_2 \rangle$ at $t = 0.7$

Figure 8.5.1(i) $\langle p_2 \rangle$ at $t = 1.0$

Figures 8.5.2 Plots of pressure variance and second order correction to the mean pressure verses position, after time interval of 1.0 seconds. $\langle \kappa \rangle = 0.1$, $\sigma^2 = 0.05$

Figure 8.5.2(a) pressure variance at $t = 1.0$

Figure 8.5.2(b) $\langle p_2 \rangle$ at $t = 1.0$

Figures 8.5.3 Plots of pressure variance and second order correction to the mean pressure verses position, after time of $t = 1.0$; $\langle \kappa \rangle = 0.2$, $\sigma^2 = 0.1$

Figure 8.5.3(a) pressure variance at $t = 1.0$

Figure 8.5.3(b) $\langle p_2 \rangle$ at $t = 1.0$

Figures 8.5.4 Plots of pressure variance and second order correction to mean pressure after time $t = 1.0$, with differences of anisotropy in the correlation lengths; with $\langle\kappa\rangle = 0.2$ and $\sigma^2 = 0.1$.

Figure 8.5.4(a) pressure variance at $t = 1.0$; $\lambda_x = 0.1$, $\lambda_y = 1.0$

Figure 8.5.4(b) $\langle p_2 \rangle$ at $t = 1.0$; $\lambda_x = 0.1$, $\lambda_y = 1.0$

Figure 8.5.4(c) pressure variance at $t = 1.0$; $\lambda_x = 1.0$, $\lambda_y = 0.1$

Figure 8.5.4(d) $\langle p_2 \rangle$ at $t = 1.0$; $\lambda_x = 1.0$, $\lambda_y = 0.1$

Figure 8.5.4(e) pressure variance at $t = 1.0$; $\lambda_x = 0.1$, $\lambda_y = 0.1$

Figure 8.5.4(f) $\langle p_2 \rangle$ at $t = 1.0$; $\lambda_x = 0.1$, $\lambda_y = 0.1$

Figures 8.5.5 Plots of pressure, pressure variance and second order correction to the mean value pressure after $t = 1.0$, with spatially-varying mean value for permeability field. $\sigma^2 = 0.05$ and $\lambda_x = 1.0$ and $\lambda_y = 1.0$.

Figure 8.5.5(a) deterministic solution at time $t = 1.0$

Figure 8.5.5(b) pressure variance at $t = 1.0$

Figure 8.5.5(c) $\langle p_2 \rangle$ at $t = 1.0$

8.6 Summary

The examples we see plotted in this chapter are basically a selection of illustrative examples of the general type of behaviour that we have observed using this method of evaluation. We employed a very simple explicit numerical discretisation scheme, which turned out to be severely limiting on the examples we were able to solve effectively. We note here that the stability condition for the deterministic scheme we use, equation (8.4.1) is

$$\frac{4\Delta t k}{\gamma h^2} < 1. \tag{8.6.1}$$

We found that the scheme would generally become unstable in cases where there was a significant probability that admissible realisations would lie outside the general stability range of the scheme. Experiments on the specific point at which instabilities start to occur have yet to be done, but it has been observed that they can certainly be shown to occur when $\langle k \rangle + 3\sigma$ lies outside the stability range for our scheme.

In Figures 8.5.1(a) to 8.5.1(i) we see the time-dependent behaviour for a single Fourier mode, where the mean of the permeability is homogeneous, and the variance comparatively low, so that the results lie well within the stability range. The deterministic solution (shown only at one time value) behaves as expected, decaying exponentially, whilst retaining the basic shape of the (one-dimensional) mode. The basic shape of the three-dimensional plot of the variance remains the same throughout the time region under investigation, with maxima at the two edges of the region given by $x = 0.0$, and $x = 1.0$. The maximum variance was seen to reach a maximum at around $t = 0.5$, thereafter gradually decreasing, with the maximum variance concentrating in the corners whilst it decays. The second order correction to the mean begins by taking a similar shape to the deterministic

solution, on a much smaller scale, of course. This value is much more subject to instabilities than the variance and deterministic approximations, and we see large increases for large time values.

In Figures 8.5.2(a) and 8.5.2(b) we can compare the previous behaviour with that for a lower mean value for (still) homogeneous permeability. Consequently, the deterministic solution has a correspondingly lower decay rate. The general shape assumed by the variance and second order approximations after one time unit are the same. The numerical value of the variance is, however, higher due to a greater relative spread in admissible realisations. There is a lower numerical value for $\langle p_2 \rangle$ after the time interval. This may be due to the fact that $\langle p_2 \rangle$ is related to the decay of the Fourier mode.

In Figures 8.5.3(a) and 8.5.3(b) we show the equivalent data to 8.5.1, but with a larger assumed variance. As expected, both variance and correction term have larger numerical values, whilst assuming a similar general shape.

The next figures show data for anisotropic correlation lengths. In the case of strong correlation in the y-direction, and much less correlation in the x-direction, Figures 8.5.4(a) and 8.5.4(b), we see that the statistical properties throughout the region are more homogeneous in themselves than in 8.5.4(c) and 8.5.4(d) where the situation is reversed and there are much higher variance figures concentrated in the corners. This seems to be due partly to the numerical process in solution of the stochastic p.d.e. which from earlier figures, seems to favour correlated properties in the y-direction. The third case, where correlation lengths in both directions are small compared to the entire scale of the region, Figures 8.5.4(e) and 8.5.4(f) shows similar concentration of variance in the corners, with numerical values of one order of magnitude lower, which is the sort of behaviour we would expect if the statistical properties are weakly correlated.

We include the final couple of figures, 8.5.5(a) and 8.5.5(b), just as of a matter of interest. The mean of the variance in this case was not homogeneous, decreasing linearly from 0.2 at $x = 0.0$, to 0.1 at $x = 1.0$. This inhomogeneity does not seem to have been different enough throughout the region to have induced any particularly interesting numerical results, so we do not comment further.

The results presented here should only be considered as an introduction to this approach of studying uncertain p.d.e.s in oil reservoir modelling. We have shown some of the early results that this method provides us with, but feel that much more research can be done in this specific area. Some further areas of potential research may include

- Investigation of the differing effect of other schemes on this method, especially implicit methods;
- A full investigation of the effect of grid-size, in relation to correlation length, and, in particular, how their ratio effects results;
- Improving the efficiency of the method, or reducing the computational burden;
- Investigation of the numerical correlation length of the numerical results obtained;
- Investigation of the convergence and consistency of the schemes.

8.6.1 Alternative Schemes

The results obtained in Sections 1 to 4 of this chapter, in particular, equations (8.4.5), (8.4.6), (8.4.7), and (8.4.8), can be generalised to any scheme with the same time discretisation. Some degree of extra stability might be gained from

using different, more accurate spatial discretisations. However, we feel that it is probably more important to generalise the time discretisation also, to include implicit schemes also, to enable a more practical application of the method. We do not envisage any problems in doing this.

8.6.2 Investigation of the effect of grid-size

The relative lengths of grid-size and correlation length may be of vital importance when applying this method in any practical case. This is because there may be cases where the correlation length is less than the size of the grid, which would imply no statistical correlation of the numerical results. This would probably not be a bad model of the analytic problem, which ought to retain some of the correlation due to the continuum. It is envisaged that a careful comparison of the ratio of correlation length to grid-size in all directions would be necessary before employing this method in a practical situation.

8.6.3 Efficiency of method/Computational burden

As these results stand we need to calculate all the cross-correlation terms in equation (8.4.6) for a full application of this method. This represents a potentially restrictive amount of computational burden for this numerical technique. For example, if there are $N = M^2$ gridpoints, equations like (8.3.12) and (8.3.14) represent N^2 operations at each successive time level, when solving for auto-correlation terms of the solution and cross-correlation terms successively. This compares with $\sim N$ operations for the deterministic equation (8.3.11), and may represent a large computational burden compared with, say, a Monte-Carlo simulation, probably meaning that in a practical sense, implementation of this method would be severely limited.

However, we firmly believe that it is feasible to develop criteria for disregarding many of these correlation terms when they represent those for two points separated by a large distance, particularly compared to correlation length of the permeability autocorrelation function. For this to be possible, an analytic model of the covariance function of the pressure solution would be required. This would allow us to develop a quantitative criterion for deciding which of the cross-correlation terms are negligible. Qualitatively, we would expect the solution pressures to be much more strongly correlated along the direction of flow than in other directions, such as perpendicular to the flow. Research done where the effort is made to evaluate the solution covariance function, such as that by Gelhar and Vomvoris, [28], shows that solution covariance functions appear to be highly anisotropic in many case. Large correlation lengths are observed along the direction of flow, and significantly smaller lengths in other directions. This would suggest that the cross-correlation terms aligned to the flow are the most significant ones, and will dominate those for most other directions. Making full use of this dominance would mean that the number of operations would be of the order of $N \times M$, or $N \times N^{\frac{1}{2}}$, assuming flow roughly parallel to the boundaries of the region. This would be further reduced if the largest correlation length aligned to the flow was small compared to the dimensions of the entire region. There might still be some correlations perpendicular to the flow to consider, but these would be only of the order of $L \times N$, where L is a low-valued integer, possible of the order of 8, depending on the type of numerical scheme under consideration. The total number of operations would then be drastically reduced from $\sim N^2$ to $\sim N^{\frac{3}{2}} + LN$, with corresponding reduction on computation time.

8.6.4 Correlation Length of solution

We are interested, in an academic sense, in the types of statistical correlation between numerical solutions at differing grid-points. Because evaluation of this method involves calculation of all correlation terms for pressure at different points, $C^n_{i'j'ij}$, or $\langle p^n_{1\ i',j'} p^n_{1\ i,j} \rangle$, in equation (8.4.8), we already have this information available. Due to lack of time, numerical experiments have not yet been done on these quantities, but we would be greatly interested to calculate relative correlation lengths for pressure, in relation to those for permeability. The ratio of these two quantities may be in some way connected with the variance reduction factor formulation of Schwydler and Dupuy, [21], and we will investigate in the future. Also, a potential comparison of numerically obtained correlation lengths and the analytic models mentioned in the previous subsection would be of great interest.

8.6.5 Convergence and consistency of schemes

We must further investigate the convergence of the numerical/statistical schemes we are using. It is envisaged that an approach similar to that in chapter 7 may be used, whereby the assumed series form of the pressure,

$$p = \sum_{m=0}^{N} \beta^m p_m + S_{N+1}, \tag{8.6.2}$$

is considered for each admissible realisation. If we are able to show some form of convergence for a specific realisation, we may be able to show convergence in the general case. This would suggest that some of the statistical results of chapter 7 that show how functionals of multivariate statistical variables are bounded over all admissible realisations, equation (7.2.11), ought to provide convergence for the series (8.6.2) under certain conditions. These conditions have not yet been formulated.

For the consistency of these numerical techniques, it is best to consider conditions that can be applied to each admissible realisation. If consistency can be shown for all these cases, then by similar arguments to those in chapter 7, it ought to be able to be shown to apply for the full probabilistic problem.

Chapter 9

Conclusions

In this thesis we have developed methods to analyse various types of systems of differential equations that contain uncertainty in some of their governing parameters, due to the heterogeneity of rock formations in underground reservoirs. The new research done here has generally been written about in the order in which it was actually done. Hopefully, this has given both a flavour of what we feel we have achieved that is new, and also serves as a good indication of how the research proceeded step-by-step.

We began by introducing the specific problem under consideration and developing mathematical models that we would be able to analyse in detail. All the techniques in formulating the model equations were taken from standard mathematical texts and literature. At all points in the thesis we referred our study to the equations that might be used as a "classical" solution to the mathematical problems. That is, the solutions obtained by making an assumption that substituting the mean value into the model would provide a valid approximation to the mean value of the solutions. This was referred to as the deterministic solution throughout the thesis.

Some statistical concepts, necessary for a complete understanding of all the

techniques used and developed throughout the thesis, were introduced, with particular emphasis on the types of statistical parameters that were deemed to be especially important in the research. Specific types of distribution functions were also discussed.

Some of the literature in this, and adjacent areas of research, was reviewed with comments on which areas were considered the most useful to pursue further.

The main part of the original research in this thesis then began with an extension of an analytical technique that had already been developed. Although the approach we followed in attempting to extend the work of P. King was unsuccessful in providing any useful practical results, we did gain some interesting theoretical insight into the problem. It is felt that this may be readily extendible in the future, and ought to give some useful practical results.

We went on to introduce the specifically numerical parts of the research. The main object of this research was to develop numerical methods to evaluate uncertainty in our model equations, with particular emphasis on how existing numerical techniques might be adapted to analyse the general behaviour of the statistical problem, particularly the mean and variance behaviour. We feel this has been achieved in that simple techniques for numerical discretisations have been adapted to give approximations, at least, for the mean and the variance of the numerical solutions. We have always taken a careful approach to assuming that the deterministic solution is a valid, or approximate, model for the full statistical problem. In the case of our steady-state model we were able to develop an approach, based on straightforward matrix algebra, that permitted us to obtain mean values of numerical solutions. Because this problem could be put in this form, we were able to assess rigorously the validity and accuracy of these approximations, when used in conjunction with various simple results for multivariate distribution func-

tions. We were able to evaluate the results formulated using standard fortran programming, and some of the experimental findings proved of interest.

The last stage of research involved development of methods to analyse the dynamic behaviour of our model equations. The techniques we employed were similar to those in the previous chapter, although the problem was approached in a more general sense, with particular interest in the effects of making assumptions of lognormal distribution functions for the permeability. Experimental results were obtained for simple discretisations under investigation. We were able to gain insight into what type of numerical schemes might best be used when employing this method in future research, by this practical application. Further suggestions as to how to proceed in this research were also discussed.

The limitations on this type of numerical approach were discussed, particularly at the end of chapter 8. Because of similarities of the techniques used in chapters 7 and 8, these limitations broadly apply in both time-dependent and steady-state equations in a numerical context. In either case, we are implicitly disregarding the tails of the input permeability distributions, and assuming that the remaining admissible realisations satisfy the appropriate numerical stability condition. In some cases this may seem to be a rather extreme assumption to make, but in some research in this area, such as that by Bellin et al, [9], this may be valid, particularly when considering inhomogeneous mean value functions for the permeability field. In any statistical technique of this type, that needs to include probabilistic considerations such as Monte-Carlo simulations, it is inevitable that the distribution is implicitly truncated, even when the original distribution is assumed to be infinite. This is especially important when problems due to numerical schemes that are introduced must be taken into account. In this research we have obtained plausible approximations for at least the lower order

moments of the solution distributions and this is considered to be an important achievement in the context of this research project.

The implementation of all the techniques we developed in this research has provided insight into how it may be extended further to provide practical results, specifically in the area of oil reservoir modelling, and, in more general problems that involve uncertainty of one form or another. We found that the specific ways in which the uncertainties occurred in this problem were different to those that have been previously studied in standard stochastic differential equations. It is hoped that more research will be pursued in this area, and more direct techniques for the evaluation of uncertain differential problems will be developed in this interesting area of mathematics.

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