On convergence of dynamics of hopping particles to a birth-and-death process in continuum

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Abstract

We show that some classes of birth-and-death processes in continuum (Glauber dynamics) may be derived as a scaling limit of a dynamics of interacting hopping particles (Kawasaki dynamics)

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1 Preliminaries

This letter deals with two classes of stochastic dynamics of infinite particle systems in continuum. Let Γ denote the space of all locally finite subsets of $\mathbb{R}^d$, $d \in \mathbb{N}$. This space is called the configuration space. Elements of Γ are called configurations, and each point of a configuration represents position of a particle. We endow Γ with the vague topology, i.e., the weakest topology in Γ with respect to which every mapping of the form $\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x\in\gamma} f(x)$, with $f \in C_0(\mathbb{R}^d)$, is continuous. Here $C_0(\mathbb{R}^d)$ is the space of all real-valued functions on $\mathbb{R}^d$ with compact support. We denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra in Γ.

A dynamics of hopping particles (Kawasaki dynamics) is a Markov process on Γ whose generator is given (on an appropriate set of functions on Γ) by

$$(L_{\mathrm{K}}F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, c(x, y, \gamma \setminus x)(F(\gamma \setminus x \cup y) - F(\gamma)).$$

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Here and below, for simplicity of notations, we just write x, y instead of $\{x\}, \{y\}$. The function $c(x, y, \gamma\backslash x)$ describes the rate at which a particle x of configuration γ jumps to y, taking into account the rest of configuration, $\gamma \setminus x$.

A birth-and-death process in continuum (Glauber dynamics) is a Markov process on Γ with generator

$$(L_{\mathrm{G}}F)(\gamma) = \sum_{x\in\gamma} d(x, \gamma \setminus x)(F(\gamma \setminus x) - F(\gamma)) + \int_{\mathbb{R}^d} dy\, b(y, \gamma)(F(\gamma \cup y) - F(\gamma)).$$

Here $d(x, \gamma \setminus x)$ describes the rate at which a particle x of configuration γ dies, whereas $b(x, \gamma)$ describes the rate at which, given configuration γ, a new particle is born at y. Fore some constructions and discussions of Glauber and Kawsaki dynamics in continuum, see [1, 3, 4, 11, 12, 13, 14, 16] and the references therein.

The aim of this letter it to show that, in many cases, a birth-and-death process may be interpreted as a limiting dynamics of hopping particles. We will restrict out attention to the case where the rate c of the Kawasaki dynamics is given by

$$c(x, y, \gamma \setminus x) = a(x - y) \exp[E^{\phi^-}(x, \gamma \setminus x) - E^{\phi^+}(y, \gamma \setminus x)].$$

Here a and $\phi^{\pm}$ are even functions on $\mathbb{R}^d$ (e.g. $a(-x) = a(x)$), a is bounded, $a \geq 0$, $\int_{\mathbb{R}^d} a(x)\, dx = 1$, and for $x \in \mathbb{R}^d$ and $\gamma \in \Gamma$,

$$E^{\phi^{\pm}}(x, \gamma) := \sum_{y\in\gamma} \phi^{\pm}(x - y),$$

provided the sum converges absolutely. Thus, $c(x, y, \gamma \setminus x)$ is a product of three terms: the term $e^{E^{\phi^-}(x,\gamma\setminus x)}$ describes the rate at which a particle $x \in \gamma$ jumps, the term $e^{-E^{\phi^+}(y,\gamma\setminus x)}$ describes the rate at which this particle lands at y, and finally the term $a(x - y)$ gives the distribution of an individual jump.

We now produce the following scaling of this dynamics. For each $\varepsilon > 0$, we define $a_\varepsilon(x) := \varepsilon^d a(\varepsilon x)$. We clearly have that $\int_{\mathbb{R}^d} a_\varepsilon(x)\, dx = 1$. Let c_ε denote the c coefficient in which function a is replaced by a_ε, and let L_ε denote the corresponding L_{K} generator. Letting $\varepsilon \to 0$, we may suggest that only jumps of infinite length will survive, i.e., jumps from a point to 'infinity', and jumps from 'infinity' to a point. Thus, we expect to arrive at a birth-and-death process. To make our suggestion more explicit, we proceed as follows.

2 Convergence of the generator of the scaled evolution of correlation functions

For simplicity, we assume, in this section, that the functions $\phi^{\pm}$ are from $C_0(\mathbb{R}^d)$. Then $E^{\phi^{\pm}}(x, \gamma)$ are well defined for each $x \in \mathbb{R}^d$ and $\gamma \in \Gamma$.

Let us briefly recall some basic facts of harmonic analysis on the configuration space, see [8, 10] for further detail. Let Γ_0 denote the space of all finite configurations in $\mathbb{R}^d$, i.e., $\Gamma_0 = \bigcup_{n=0}^{\infty} \Gamma^{(n)}$, where $\Gamma^{(n)}$ is the space of all n-point configurations in $\mathbb{R}^d$. Clearly, $\Gamma_0 \subset \Gamma$, and we define $\mathcal{B}(\Gamma_0)$ and $\mathcal{B}(\Gamma^{(n)})$ as the trace σ-algebra of Γ on Γ_0 and $\Gamma^{(n)}$, respectively. For a function $G : \Gamma_0 \to \mathbb{R}$, we define a function $(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta)$, $\gamma \in \Gamma$, provided the summation makes sense. Here $\eta \Subset \gamma$ means that η is a finite subset of γ.

Let μ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. Then there exists a unique measure ρ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ satisfying

$$\int_\Gamma (KG)(\gamma)\, \mu(d\gamma) = \int_{\Gamma_0} G(\eta)\, \rho(d\eta)$$

for each measurable function $G : \Gamma_0 \to [0, \infty)$. The measure ρ is called the correlation measure of μ. Further, denote by λ the Lebesgue–Poisson measure on Γ_0, i.e.,

$$\lambda = \delta_\varnothing + \sum_{n=1}^{\infty} \frac{1}{n!}\, dx_1 \cdots dx_n.$$

Here $\delta_\varnothing$ is the Dirac measure with mass at $\varnothing$, and $dx_1 \cdots dx_n$ is the Lebesgue measure on $\Gamma^{(n)}$, which is naturally defined on this space. Assume that the correlation measure ρ of μ is absolutely continuous with respect to λ. Then $k := \frac{d\rho}{d\lambda}$ is called the correlation functional of μ. For a given correlation functional k, the corresponding Ursell functional $u : \Gamma_0 \to \mathbb{R}$ is defined through the formula $k(\eta) = \sum_{\pi \in \mathcal{P}(\eta)} u_\pi(\eta)$,where $\mathcal{P}(\eta)$ denotes the set of all partitions of η, and given a partition $\pi = \{\eta_1, \dots, \eta_k\}$ of η, $u_\pi(\eta) := u(\eta_1) \cdots u(\eta_k)$. Recall also that a function $G : \Gamma_0 \to \mathbb{R}$ is called translation invariant if, for each $x \in \mathbb{R}^d$, $G(\eta_x) = G(\eta)$ for all $\eta \in \Gamma_0$, where η_x denotes the configuration η shifted by vector x, i.e., $\eta_x := \{y + x \mid y \in \eta\}$. Clearly, the correlation functional k is translation invariant if and only if the corresponding Ursell functional u is translation invariant.

If k is the correlational functional of a probability measure μ on Γ, we denote

$$k^{(n)}(x_1, \dots, x_n) := k(\{x_1, \dots, x_n\}), \quad n \in \mathbb{N},$$

and analogously we define $u^{(n)}$. The $(k^{(n)})_{n=1}^{\infty}$ and $(u^{(n)})_{n=1}^{\infty}$ are called the correlation and Ursell functions of μ, respectively. Note that, if k is translation invariant, then $k^{(1)} = u^{(1)}$ is a constant.

For a function $f : \mathbb{R}^d \to \mathbb{R}$, we define $e_\lambda(f, \eta) := \prod_{x \in \eta} f(x)$, $\eta \in \Gamma_0$, where $\prod_{x \in \varnothing} f(x) := 1$. Further, let $\varphi : \mathbb{R}^d \to \mathbb{R}$. Then

$$(Ke_\lambda(e^\varphi - 1, \cdot))(\gamma) = e^{\langle \varphi, \gamma \rangle},$$

so that

$$\int_\Gamma e^{\langle \varphi, \gamma \rangle}\, \mu(d\gamma) = \int_{\Gamma_0} e_\lambda(e^\varphi - 1, \eta) k(\eta)\, \lambda(d\eta), \tag{1}$$

under some proper conditions on φ and k, see e.g. [10].

Assume that L is a Markov generator on Γ. Denote $\hat{L} := K^{-1}LK$, i.e., $\hat{L}$ is the operator acting on functions on Γ_0 which satisfies $K\hat{L}G = LKG$. Denote by $\hat{L}^*$ the dual operator of $\hat{L}$ with respect to the Lebesgue–Poisson measure λ:

$$\int_{\Gamma_0} (\hat{L}G)(\eta)k(\eta)\,\lambda(d\eta) = \int_{\Gamma_0} G(\eta)(\hat{L}^*k)(\eta)\,\lambda(d\eta).$$

Assume now that a Markov process on Γ with generator L has initial distribution μ_0. Denote by μ_t the distribution of this process at time $t > 0$. Assume that, for each $t \geq 0$, μ_t has correlation functional k_t. Then, at least at an informal level, one sees that the evolution of k_t is described by the equation $\partial k_t/\partial t = \hat{L}^* k_t$, so that $\hat{L}^*$ is the generator of evolution of correlation functionals.

In the case where $L = L_\varepsilon$, we proceed as follows, First we write $L_\varepsilon = L_\varepsilon^- + L_\varepsilon^+$, where

$$(L_\varepsilon^- F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y) r(x,y,\gamma\setminus x)(F(\gamma\setminus x) - F(\gamma)),$$

$$(L_\varepsilon^+)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y) r(x,y,\gamma\setminus x)(F(\gamma\setminus x\cup y) - F(\gamma\setminus x)).$$

Here, $r(x,y,\gamma\setminus x) := \exp[E^{\phi^-}(x,\gamma\setminus x) - E^{\phi^+}(y,\gamma\setminus x)]$. We also set

$$(L_0^- F)(\gamma) = \sum_{x\in\gamma} \exp[E^{\phi^-}(x,\gamma\setminus x)](F(\gamma\setminus x) - F(\gamma)),$$

$$(L_0^+ F)(\gamma) = \int_{\mathbb{R}^d} dy\, \exp[-E^{\phi^+}(y,\gamma)](F(\gamma\cup y) - F(\gamma)).$$

Theorem 1. *Let k be the correlation functional of a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$, and let u be the corresponding Ursell functional. Assume that the following conditions are satisfied:*

i) *k fulfills the bound $k(\eta) \leq (|\eta|!)^s C^{|\eta|}$, $\eta \in \Gamma_0$, for some $0 \leq s < 1$ and $C > 0$. Here $|\eta|$ denotes the cardinality of set η.*

ii) *k is translation invariant.*

iii) *The measure μ has a decay of correlations in the sense that, for any $n, m \in \mathbb{N}$, $a \in \mathbb{R}^d$, $a \neq 0$, and $\{x_1, \dots, x_{n+m}\} \in \Gamma^{(n+m)}$,*

$$u(\{x_1,\dots,x_n,x_{n+1}+(a/\varepsilon),\dots,x_{n+m}+(a/\varepsilon)\}) \to 0 \quad \textit{as } \varepsilon \to 0.$$

Then, for each $\eta \in \Gamma_0$,

$$(\hat{L}_\varepsilon^{-*}k)(\eta) \to c^-(k)(\hat{L}_0^{-*}k)(\eta), \quad (\hat{L}_\varepsilon^{+*}k)(\eta) \to c^+(k)(\hat{L}_0^{+*}k)(\eta),$$

where

$$c^-(k) := \int_{\Gamma_0} \lambda(d\xi)\, e_\lambda(e^{-\phi^+} - 1, \xi)k(\xi),$$

$$c^+(k) := \int_{\Gamma_0} \lambda(d\xi)\, e_\lambda(e^{\phi^-} - 1, \xi)k(\xi\cup 0). \tag{2}$$

Proof. A straightforward calculation (see [8]) shows that

$$\begin{aligned}
(\hat{L}_\varepsilon^{-*}k)(\eta) &= -\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y)r(x,y,\eta\setminus x)\\
&\quad\times\int_{\Gamma_0}\lambda(d\xi)\,k(\xi\cup\eta)e_\lambda(e^{\phi^-(x-\cdot)-\phi^+(y-\cdot)}-1,\xi), \qquad (3)\\
(\hat{L}_\varepsilon^{+*}k)(\eta) &= \sum_{y\in\eta}\int_{\mathbb{R}^d} dx\, a_\varepsilon(x-y)r(x,y,\eta\setminus y)\\
&\quad\times\int_{\Gamma_0}\lambda(d\xi)\,k(\xi\cup(\eta\setminus y)\cup x)e_\lambda(e^{\phi^-(x-\cdot)-\phi^+(y-\cdot)}-1,\xi),\\
(\hat{L}_0^{-*}k)(\eta) &= -\sum_{x\in\eta}\exp[E^{\phi^-}(x,\eta\setminus x)]\\
&\quad\times\int_{\Gamma_0}\lambda(d\xi)\,e_\lambda(e^{\phi^-(x-\cdot)}-1,\xi)k(\eta\cup\xi),\\
(\hat{L}_0^{+*}k)(\eta) &= \sum_{y\in\eta}\exp[-E^{\phi^-}(y,\eta\setminus y)]\\
&\quad\times\int_{\Gamma_0}\lambda(d\xi)\,e_\lambda(e^{-\phi^+(y-\cdot)}-1,\xi)k((\eta\setminus y)\cup\xi).
\end{aligned}$$

We will now briefly explain the convergence of $(\hat{L}_\varepsilon^{-*}k)(\eta)$ (the case of $(\hat{L}_\varepsilon^{+*}k)(\eta)$ can be dealt with analogously). From (3) and the definition of λ, by making a change of variable, we easily have:

$$\begin{aligned}
&(\hat{L}_\varepsilon^{-*}k)(\eta) = -\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, a(y)r(x,(y/\varepsilon)+x,\eta\setminus x)\sum_{n=0}^\infty\frac{1}{n!}\sum_{k=0}^n\binom{n}{k}\\
&\times\int_{(\mathbb{R}^d)^n}du_1\cdots du_n\prod_{i=1}^k\left(e^{-\phi^+((y/\varepsilon)+x-u_i)}(e^{\phi^-(x-u_i)}-1)\right)\\
&\times\prod_{j=k+1}^n(e^{-\phi^+((y/\varepsilon)+x-u_j)}-1)k(\xi\cup\{u_1,\dots,u_n\})\\
&= -\sum_{x\in\eta}\int_{\mathbb{R}^d}dy\,a(y)r(x,(y/\varepsilon)+x,\eta\setminus x)\sum_{n=0}^\infty\sum_{k=0}^n\frac{1}{k!(n-k)!}\\
&\times\int_{(\mathbb{R}^d)^n}du_1\cdots du_n\prod_{i=1}^k\left(e^{-\phi^+((y/\varepsilon)-u_i)}(e^{\phi^-(u_i)}-1)\right)\prod_{j=k+1}^n(e^{-\phi^+(u_j)}-1)\\
&\times k(\xi\cup\{u_1+x,\dots,u_k+x,u_{k+1}+x+(y/\varepsilon),\dots,u_n+x+(y/\varepsilon)\}).
\end{aligned}$$

Next, represent the correlation functionals in the above expression through a sum of Ursell functionals. Using the dominated convergence theorem and conditions i) and iii), we see that, in the limit, all the Ursell functionals containing at least one point from $\xi\cup\{u_1+x,\dots,u_k+x\}$ and at least one point from

$\{u_{k+1}+x+(y/\varepsilon),\dots,u_n+x+(y/\varepsilon)\}$ will vanish, and by virtue of ii), we conclude that $(\hat{L}_\varepsilon^{-*}k)(\eta)$ converges to

$$-\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\, a(y)\exp[E^{\phi^-}(x,\eta\setminus x)]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}$$
$$\times\int_{(\mathbb{R}^d)^n} du_1\cdots du_n\prod_{i=1}^{k}(e^{\phi^-(x-u_i)}-1)\prod_{j=k+1}^{n}(e^{-\phi^+(u_j)}-1)$$
$$\times\, k(\xi\cup\{u_1,\dots,u_k\})k(\{u_{k+1},\dots,u_n\}),$$

from where the statement follows. □

From Theorem 1, we can make the following conclusion. Assume that a dynamics of hopping particle with Markov generator L_{K} has initial distribution μ_0. Let μ_t be the distribution of this process at time $t>0$. Assume that, for each $t\ge 0$, μ_t has correlation functional k_t which satisfies conditions i)–iii) of Theorem 1. Further assume that $c^\pm(k_t)$, $t\ge 0$, given through (2) remain constant. Then, we can expect that the scaled dynamics of hopping particles converges to a birth-and-death process with generator $L_0:=c^-(k_0)L_0^-+c^+(k_0)L_0^+$ and initial distribution μ_0. We will discuss below two cases where this statement can be proven rigorously (at least in the sense of convergence of the generators).

3 Convergence of non-equilibrium free dynamics

This case has been discussed in [14], so here we will explain its connection with Theorem 1.

Let $\Theta\in\mathcal{B}(\Gamma)$ be the set of those configurations $\gamma\in\Gamma$ for which there exist $\alpha\ge d$ and $K>0$ such that

$$|\gamma\cap B(n)|\le Kn^\alpha,\quad \text{for all } n\in\mathbb{N}, \tag{4}$$

where $B(n)$ denotes the ball in $\mathbb{R}^d$ centered at 0 and of radius n. Note that the estimate (4) controls the growth of the number of particles of γ at infinity.

Let $a\in S(\mathbb{R}^d)$ (the Schwartz space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$). Consider a random walk in $\mathbb{R}^d$ with transition kernel $Q(x,dy):=a(x-y)\,dy$. This is a Markov process in $\mathbb{R}^d$ with generator

$$(L^{(1)}f)(x)=\int_{\mathbb{R}^d}(f(y)-f(x))a(x-y)\,dy.$$

The corresponding Markov semigroup on $L^2(\mathbb{R}^d,dx)$ is then given by

$$(p_tf)(x)=e^{-t}f(x)+\int_{\mathbb{R}^d}G(x-y)f(y)\,dy, \tag{5}$$

where G is the inverse Fourier transform of $e^{-t}(\exp[t(2\pi)^{d/2}\hat{a}]-1)$, where $\hat{a}$ is the Fourier transform of a. (Note that we have normalized the direct and inverse

Fourier transforms so that they are unitary operators in $L^2(\mathbb{R}^d \to \mathbb{C}, dx)$.) For any $\gamma \in \Theta$, consider a dynamics of independent particles which starts at γ and such that each separate particle moves according to the semigroup p_t (i.e., independent random walks in $\mathbb{R}^d$). Then, this process has *cádlág* paths on Γ and a.s. it never leaves Θ, cf. [14]. The generator of the obtained Markov process on Θ is then given by

$$(L_{\mathrm{K}}F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y)(F(\gamma\setminus x \cup y) - F(\gamma)), \tag{6}$$

so that now $\phi^{\pm} = 0$.

Proposition 1. *Let μ_0 be a probability measure on Γ whose correlation functional k_0 satisfies conditions i)–iii) of Theorem 1, and $\mu_0(\Theta) = 1$. Consider the Markov process on Θ with the generator L_{K} given by (6) and with the initial distribution μ_0. Denote by μ_t the distribution of this process at time $t > 0$. Then, for each $t > 0$, μ_t has correlation functional k_t which satisfies conditions i)–iii) of Theorem 1, and furthermore $c^-(k_t) = 1$ and $c^+(k_t) = k_0^{(1)}$, $t \geq 0$.*

Proof. For each $f \in C_0(\mathbb{R}^d)$ and $t > 0$, we have, by (1) and the construction of the process:

$$\begin{aligned}
&\int_\Theta \mu_t(d\gamma) e^{\langle f,\gamma\rangle} = \int_\Theta \mu_0(d\gamma) \prod_{x\in\gamma} (p_t e^f)(x) \\
&= \int_{\Gamma_0} \lambda(d\eta) k_0(\eta) \prod_{x\in\eta} (p_t(e^f - 1))(x) \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n\, k^{(n)}(x_1,\dots,x_n) \prod_{i=1}^{n} (p_t(e^f-1))(x_i) \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n\, (p_t^{\otimes n} k^{(n)})(x_1,\dots,x_n) \prod_{i=1}^{n} (e^{f(x_i)} - 1).
\end{aligned}$$

Therefore, μ_t has correlation functional k_t, and furthermore $k_t^{(n)} = p_t^{\otimes n} k_0^{(n)}$. The latter equality, in turn, implies that $u_t^{(n)} = p_t^{\otimes n} u_0^{(n)}$. From here it easily follows that, for each $t > 0$, μ_t satisfies assumptions i)–iii) of Theorem 1. Furthermore, by (2),

$$\begin{aligned}
c^-(k_t) &= k_t(\varnothing) = 1, \\
c^+(k_t) &= k_t(\{0\}) = k_t^{(1)} = p_t k_0^{(1)} = k_0^{(1)}. \quad \square
\end{aligned}$$

Thus, according to Section 2, we expect that the scaled free dynamics with initial distribution μ_0 converges to the birth-and-death process with generator

$$(L_0F)(\gamma) = \sum_{x\in\gamma} (F(\gamma\setminus x) - F(\gamma)) + k_0^{(1)} \int_{\mathbb{R}^d} dy\, (F(\gamma \cup y) - F(\gamma)) \tag{7}$$

and initial distribution μ_0. This dynamics can be constructed as follows, cf. [14, 19]. For each $\gamma \in \Theta$, denote by P_γ the law of a process on Θ which is at γ at time zero, and after this, points of γ randomly die, independently of each other, so that the probability that at time $t > 0$ a particle $x \in \gamma$ is still alive is equal to e^{-t}. Next, let π denote the Poisson point process in $\mathbb{R}^d \times (0, \infty)$ with the intensity measure $k_0^{(1)}\, dx\, dt$. The measure π is concentrated on configurations $\widehat{\gamma} = \{(x_n, t_n)\}_{n=1}^\infty$ in $\mathbb{R}^d \times (0, \infty)$ such that $\{x_n\}_{n=1}^\infty \in \Theta$, $0 < t_1 < t_2 < \cdots$, and $t_n \to \infty$ as $n \to \infty$. For any such configuration, we denote by $P_{\widehat{\gamma}}$ the law of a process on Θ such that at time $t = 0$, the configuration is empty, and then at each time t_n, $n \in \mathbb{N}$, a new particle is born at x_n, and after time t_n this particle randomly dies, independently of the other particles, so that at time $s > t_n$ the probability that the particle is still alive is $e^{-(s-t_n)}$. Finally, the law of the process with generator (7) and initial distribution μ_0 is given by

$$\int \mu_0(d\gamma) P_\gamma * \int \pi(d\widehat{\gamma}) P_{\widehat{\gamma}}.$$

Here $*$ stays for convolution of measures, see [14] for details.

We will use $\ddot{\Gamma}$ to denote the space of multiple configurations over $\mathbb{R}^d$ equipped with the vague topology, see e.g. [9] for details. Note that $\Gamma \subset \ddot{\Gamma}$, and the trace σ-algebra of $\mathcal{B}(\ddot{\Gamma})$ on Γ is $\mathcal{B}(\Gamma)$.

Theorem 2 ([14])**.** *Consider the stochastic process from Proposition 1 as taking values in $\ddot{\Gamma}$. Then, after scaling, this process converges, in the sense of weak convergence of finite-dimensional distributions, to the Markov process with the generator L_0 given by (7) and with the initial distribution μ_0.*

Note that the limiting process also lives in Θ, and we used the $\ddot{\Gamma}$ space only to identify the type of convergence.

For reader's convenience, let us explain the idea of the proof of Theorem 2. Fix arbitrary $0 = t_0 < t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$, and denote by $\mu^\varepsilon_{t_0,t_1,\dots,t_n}$, $\varepsilon \geq 0$, the corresponding finite-dimensional distribution of the initial process scaled by $\varepsilon > 0$, and that of the limiting process if $\varepsilon = 0$, respectively. Then, by [9], the statement of the theorem is equivalent to staying that, for any non-positive $f_0, f_1, \dots, f_n \in C_0(\mathbb{R}^d)$,

$$\int_{\Theta^n} \exp\left[\sum_{i=0}^n \langle f_i, \gamma\rangle\right] d\mu^\varepsilon_{t_0,t_1,\dots,t_n}(\gamma_0, \gamma_1, \dots, \gamma_n) \\ \to \int_{\Theta^n} \exp\left[\sum_{i=0}^n \langle f_i, \gamma\rangle\right] d\mu^0_{t_0,t_1,\dots,t_n}(\gamma_0, \gamma_1, \dots, \gamma_n) \quad \text{as } \varepsilon \to 0. \quad (8)$$

For $\varepsilon > 0$, denote by $p^\varepsilon_t(x, dy)$ the transition probability of the Markov semigroup (4) scaled by ε. Set

$$g^\varepsilon(x) := e^{f_0(x)} \int_{\mathbb{R}^d} p^\varepsilon_{t_1}(x, dx_1) \int_{\mathbb{R}^d} p^\varepsilon_{t_2-t_1}(x_1, dx_2) \times \cdots \times \int_{\mathbb{R}^d} p^\varepsilon_{t_n-t_{n-1}}(x_{n-1}, dx_n) \prod_{i=1}^n e^{f_i(x_i)}, \quad x \in \mathbb{R}^d.$$

Then, by (1) and the construction of the process, the first integral in (8) (with $\varepsilon > 0$) is equal to

$$\int_\Theta \prod_{x\in\gamma} g^\varepsilon(x)\,\mu_0(d\gamma) = 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^n (g^\varepsilon(x_i) - 1) k_0^{(n)}(x_1, \dots, x_n)\, dx_1 \cdots dx_n.$$

In the above integrals, one represents the correlation functions through the Ursell functions, makes a change of variables under the sign of integral, and after a careful analysis of the obtained expression, one takes its limit as $\varepsilon \to 0$. Finally, one shows that the obtained limit is indeed equal to the second integral in (8).

4 Convergence of equilibrium Kawasaki dynamics of interacting particles

In this section, we will consider equilibrium dynamics of interacting particles having a Gibbs measure as an equilibrium measure. Our result will extend that of [7], where just one special case of such a dynamics was considered (see also [15]). We start with a description of the class of Gibbs measures we are going to use.

A pair potential is a Borel-measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$, we define a relative energy of interaction between a particle at x and the configuration γ as $E(x, \gamma) := \sum_{y\in\gamma} \phi(x-y)$, provided that the latter sum converges absolutely, and otherwise it is set to be $= \infty$. A (grand canonical) Gibbs measure corresponding to the pair potential ϕ and activity $z > 0$ is a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ which satisfies the Georgii–Nguyen–Zessin identity:

$$\int_\Gamma \mu(d\gamma) \sum_{x\in\gamma} F(\gamma, x) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z\, dx \exp[-E(x,\gamma)] F(\gamma \cup x, x) \tag{9}$$

for any measurable function $F : \Gamma \times \mathbb{R}^d \to [0, +\infty)$. A pair potential ϕ is said to be stable if there exists $B \geq 0$ such that, for any $\eta \in \Gamma_0$,

$$\sum_{\{x,y\}\subset\eta} \phi(x-y) \geq -B|\eta|. \tag{10}$$

In particular, we then have $\phi(x) \ge -2B$, $x \in \mathbb{R}^d$. Next, we say that the condition of low activity–high temperature regime is fulfilled if

$$\int_{\mathbb{R}^d} |e^{-\phi(x)} - 1| z\, dx < (2e^{1+2B})^{-1}, \tag{11}$$

where B is as in (10). A classical result of Ruelle [17, 18] says that, under the assumption of stability and low activity–high temperature regime, there exists a Gibbs measure μ corresponding to ϕ and z, and this measure has correlation functional which satisfies conditions i)–iii) of Theorem 1, with $s = 0$ in condition i) (which is then called the Ruelle bound). Furthermore, the corresponding Ursell functions satisfy $u^{(n)}(0, \cdot, \dots, \cdot) \in L^1((\mathbb{R}^d)^{n-1}, dx_1 \cdots dx_n)$ for each $n \ge 2$. In what follows, we will assume that the potential ϕ is also bounded from above outside some finite ball in $\mathbb{R}^d$ (which is always true for any realistic potential, since it should converge to zero at infinity).

We now fix arbitrary parameters $u, v \in [0, 1]$, and assume that

$$\int_{\mathbb{R}^d} |\exp[(2(u \vee v) - 1)\phi(x)] - 1|\, dx < \infty. \tag{12}$$

It can be easily shown that, if $u, v \in [0, 1/2]$, then (12) is a corollary of (11) and the condition that ϕ be bounded outside some finite ball. Note that, even if $u \vee v \in (1/2, 1]$, condition (12) still admits potentials which have ‘weak’ singularity at zero.

We introduce the set $\mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g(\langle f_1, \gamma \rangle, \dots, \langle f_N, \gamma \rangle),$$

where $N \in \mathbb{N}$, $f_1, \dots, f_N \in C_0(\mathbb{R}^d)$, and $g \in C_b(\mathbb{R}^N)$. Here $C_b(\mathbb{R}^N)$ denotes the set of all continuous bounded functions on $\mathbb{R}^N$. For each $F \in \mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma)$, we define

$$\begin{aligned}(L_{\mathrm{K}} F)(\gamma) = \frac{1}{2} \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy\, a(x - y)\big(&\exp[uE(x, \gamma \setminus x) - (1 - v)E(y, \gamma \setminus x)] \\ &+ \exp[vE(x, \gamma \setminus x) - (1 - u)E(y, \gamma \setminus x)]\big)(F(\gamma \setminus x \cup y) - F(\gamma)). \end{aligned} \tag{13}$$

Note that the first addend in (13) corresponds to the choice of $\phi^- = u\phi$, $\phi^+ = (1 - v)\phi$, whereas the second addend corresponds to $\phi^- = v\phi$, $\phi^+ = (1 - u)\phi$. In the special case where $u = v$, we get

$$\begin{aligned}(L_{\mathrm{K}} F)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy\, a(x - y) \exp[uE(x, \gamma \setminus x) - (1 - u)E(y, \gamma \setminus x)] \\ \times (F(\gamma \setminus x \cup y) - F(\gamma)).\end{aligned}$$

By [13], $(L_{\mathrm{K}}, \mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma))$ is a Hermitian, non-negative operator in $L^2(\Gamma, \mu)$, and we denote by $(L_{\mathrm{K}}, D(L_{\mathrm{K}}))$ its Friedrichs' extension. As shown in [13] by using the theory of Dirichlet forms, there exists a Markov process on Γ with *cádlág* paths whose generator is $(L_{\mathrm{K}}, D(L_{\mathrm{K}}))$. If we consider this process with initial distribution μ, then it is an equilibrium process, i.e., it has distribution $\mu_t = \mu$ at any moment of time $t \geq 0$. Thus, for each $t \geq 0$, $\mu_t = \mu$ has correlation function which satisfies conditions i)–iii) of Theorem 1.

Lemma 1. *Let k denote the correlation function of the Gibbs measure μ under consideration. Denote*

$$C_u := \int_\Gamma \mu(d\gamma) \exp[-(1-u)\langle \phi, \gamma \rangle].$$

Then we have:

$$\int_{\Gamma_0} \lambda(d\xi) e_\lambda(e^{-(1-u)\phi} - 1, \xi) k(\xi) = C_u, \tag{14}$$

$$\int_{\Gamma_0} \lambda(d\xi) e_\lambda(e^{u\phi} - 1, \xi) k(\xi \cup 0) = z C_u. \tag{15}$$

Proof. Equality (14) follows from (1). Next, using (1), (9), and translation invariance of k, we have, for each $f \in C_0(\mathbb{R}^d)$:

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx\, f(x) \int_\Gamma \mu(d\gamma) \exp[-(1-u)\langle \phi, \gamma \rangle] \\
&= \int_{\mathbb{R}^d} dx\, f(x) \int_\Gamma \mu(d\gamma) \exp[-(1-u) E(x, \gamma)] \\
&= z^{-1} \int_\Gamma \mu(d\gamma) \sum_{x \in \gamma} f(x) \exp[u E(x, \gamma \setminus x)] \\
&= z^{-1} \int_\Gamma \mu(d\gamma) \sum_{x \in \gamma} f(x) \sum_{\xi \Subset \gamma \setminus x} e_\lambda(e^{u\phi} - 1, \xi) \\
&= z^{-1} \int_\Gamma \mu(d\gamma) \sum_{\xi \Subset \gamma} \sum_{x \in \xi} f(x) e_\lambda(e^{u\phi} - 1, \xi \setminus x) \\
&= z^{-1} \int_{\Gamma_0} \lambda(d\xi) k(\xi) \sum_{x \in \xi} f(x) e_\lambda(e^{u\phi} - 1, \xi \setminus x) \\
&= z^{-1} \int_{\Gamma_0} \lambda(d\xi) \int_{\mathbb{R}^d} dx\, k(\xi \cup x) f(x) e_\lambda(e^{u\phi} - 1, \xi) \\
&= z^{-1} \int_{\mathbb{R}^d} f(x) \int_{\Gamma_0} \lambda(d\xi) k(\xi_{-x} \cup 0) e_\lambda(e^{u\phi} - 1, \xi) \\
&= z^{-1} \int_{\mathbb{R}^d} f(x) \int_{\Gamma_0} \lambda(d\xi) k(\xi \cup 0) e_\lambda(e^{u\phi} - 1, \xi),
\end{aligned}$$

from where equality (15) follows. □

Thus, by Lemma 1, according to Section 2, we expect that the scaled equilibrium dynamics (with initial distribution μ) converges to the birth-and-death process with generator

$$
\begin{aligned}
(L_0F)(\gamma) = \sum_{x\in\gamma} \frac{1}{2}\big(C_v \exp[uE(x,\gamma\setminus x)] + C_u \exp[vE(x,\gamma\setminus x)]\big)\\
\times (F(\gamma\setminus x) - F(\gamma))\\
+ \int_{\mathbb{R}^d} z\,dy\, \frac{1}{2}\big(C_u \exp[-(1-v)E(y,\gamma)] + C_v \exp[-(1-u)E(y,\gamma)]\big)\\
\times (F(\gamma\cup y) - F(\gamma)) \qquad (16)
\end{aligned}
$$

and the initial distribution μ. In fact, by [13], $(L_0, \mathcal{F}C_b(C_0(\mathbb{R}^d),\Gamma)))$ is a Hermitian, non-negative operator in $L^2(\Gamma,\mu)$, and its Friedrichs' extension $(L_0, D(L_0))$ is the generator of a Markov process on Γ with *cádlág* paths.

We recall that L_ε denotes the L_K generator (given by (13)) scaled by ε. The following theorem states that, at least on an appropriate set of test functions, the operator L_ε converges to L_0 in the L^2-norm.

Theorem 3. *For each $f \in C_0(\mathbb{R}^d)$, we have $e^{\langle f,\cdot\rangle} \in D(L_\varepsilon)$ for all $\varepsilon \geq 0$, and*

$$L_\varepsilon e^{\langle f,\cdot\rangle} \to L_0 e^{\langle f,\cdot\rangle} \quad \text{in } L^2(\Gamma,\mu) \text{ as } \varepsilon\to 0.$$

Proof. We will only sketch the proof of the theorem. Let $f \in C_0(\mathbb{R}^d)$. By approximation, one easily shows that, for each $\varepsilon \geq 0$, the function $F(\gamma) = e^{\langle f,\gamma\rangle}$ belongs to $D(L_\varepsilon)$, and that the action of L_ε onto F is given, for $\varepsilon > 0$ by the right hand side of (13) in which a is replaced by a_ε, and for $\varepsilon = 0$ by (16), respectively.

Denote

$$
\begin{aligned}
(\mathcal{L}_\varepsilon^- F)(\gamma) &= \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y)\\
&\times \exp[uE(x,\gamma\setminus x) - (1-v)E(y,\gamma\setminus x)] e^{\langle f,\gamma\setminus x\rangle}(1-e^{f(x)}),\\
(\mathcal{L}_\varepsilon^+ F)(\gamma) &= \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y)\\
&\times \exp[uE(x,\gamma\setminus x) - (1-v)E(y,\gamma\setminus x)] e^{\langle f,\gamma\setminus x\rangle}(e^{f(y)}-1),\\
(\mathcal{L}_0^- F)(\gamma) &= C_v \sum_{x\in\gamma} \exp[uE(x,\gamma\setminus x)] e^{\langle f,\gamma\setminus x\rangle}(1-e^{f(x)}),\\
(\mathcal{L}_0^+ F)(\gamma) &= C_u z \int_{\mathbb{R}^d} dy\, \exp[-(1-v)E(y,\gamma)] e^{\langle f,\gamma\rangle}(e^{f(y)}-1).
\end{aligned}
$$

To prove the theorem, it suffices to show that

$$
\begin{aligned}
&\|\mathcal{L}_\varepsilon^\pm F\|^2_{L^2(\Gamma,\mu)} \to \|\mathcal{L}_0^\pm F\|^2_{L^2(\Gamma,\mu)},\\
&(\mathcal{L}_\varepsilon^\pm F, \mathcal{L}_0^\pm F)_{L^2(\Gamma,\mu)} \to \|\mathcal{L}_0^\pm F\|^2_{L^2(\Gamma,\mu)} \qquad (17)
\end{aligned}
$$

as $\varepsilon \to 0$. To this end, one proceeds as follows. By using (9), one represents each of the expressions appearing in (17) in terms of integrals over Γ with respect to μ, as well as integrals over $\mathbb{R}^d$ with respect to Lebesgue measure. As a result one gets rid of all summations $\sum_{x\in\gamma}$. Then, one makes a change of variables, so that instead of $a_\varepsilon(x-y)$ one gets $a(x)$, and in y variable one gets a function which is dominated by an integrable function of y. Next, one replaces integration $\int_\Gamma \mu(d\gamma)\cdots$ by corresponding integration $\int_{\Gamma_0} \lambda(d\eta)k(\eta)\cdots$. In the obtained expression, one represents the correlation functional through a sum of Ursell functionals. Finally, one takes the limit as $\varepsilon \to 0$ by analogy with the final part of the proof of Theorem 1. □

By using the well-known result of the theory of semigroups (see e.g. [2]), we get the following corollary of Theorem 3.

Corollary 1. *Assume that the set of finite linear combinations of exponential functions $e^{\langle f,\cdot\rangle}$, $f \in C_0(\mathbb{R}^d)$, is a core for the limiting generator $(L_0, D(L_0))$. Then, we have the weak convergence of finite-dimensional distributions of the scaled Markov process in Γ with the generator $(L_\varepsilon, D(L_\varepsilon))$ and with the initial distribution μ to the Markov process in Γ with the generator $(L_0, D(L_0))$ and with the initial distribution μ. In particular, if additionally $\phi \geq 0$, then this kind of convergence holds when $u = v = 0$.*

We note that the final statement of Corollary 1 holds due to a result of [12] on essential self-adjointness of the generator of Glauber dynamics in the case $\phi \geq 0$ and $u = v = 0$ (see also [7]). In the latter case, we even expect that the *weak convergence of laws* holds. To this end, one needs to consider all processes as taking values in a negative Sobolev space. The tightness of the laws of scaled processes may be proven by analogy with the proof of [5, Theorem 7.1]. Next, one shows that this set of laws has, in fact, a unique limiting point—the law of the Markov process with generator $(L_0, D(L_0))$ and initial distribution μ_0. This is done by identifying the limit via the martingale problem, and using convergence of the generators (compare with the proof of [6, Theorem 6.7] and that of [5, Theorem 7.5]).

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