Linear and Quadratic Finite Elements for a Moving Mesh Method

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Declaration

I hereby declare that this dissertation has not been accepted in substance for any degree and not being concurrently submitted for any other degree.
I confirm that this is my own work and the use of all materials from other sources has been properly and fully acknowledged.

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Abstract

In this Dissertation linear and quadratic finite elements are used to produce numerical approximation to the solutions of first order differential equations which arise in a moving mesh finite element method. The behaviour of the moving mesh velocity is investigated in detail and is compared these results with the existing exact solutions to investigate the effect of the moving boundaries and provided the error analysis in both linear and quadratic cases.

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Chapter 1

Introduction

This dissertation concerns the finite element solution of first order differential equations. It is well known that the standard finite element method is not well suited to $\frac{dy}{dx} = f(x)$ in $(0,1)$ with $y(0) = 1$, due to the insufficient boundary conditions and difficulty of inverting the element matrix, leading to various kinds of regularisations in the literature.
There are many applications in which first order equations arise, notably in steady state fluid mechanics. The motivation in this dissertation, however comes from a moving mesh method for time-dependent Partial Differential Equations (PDEs).
The strategy is to replace the first order differential equation by a second order one with an artificial boundary condition, giving ($\frac{dy}{dx} = \frac{d^2u}{dx^2} = f(x)$ with $\frac{du}{dx}(0) = 1$ and $u(1) = 0$) a problem which is well suited to the finite element approach. It then remains to recover the solution of the first order equation from the finite element solution obtained. In moving mesh applications this has to be a continous function.
The moving mesh work is new in this field and therefore, there is a limited amount of information available in existing literature. Due to the nature of this dissertation, the majority of the preliminary work is based on programming.

1.1 Moving mesh velocity equation

A moving mesh approach to solving the Partial Differential Equation (PDE)

$$\frac{\partial p}{\partial t} = L_x p$$

where L_x is a partial operator, is to use conservation of the integral of p to move the mesh,

$$\frac{d}{dt}\int_0^{\hat{x}(t)} p dx = 0$$

By Leibnitz' Integral Rule [9]

$$\frac{d}{dt}\int_0^{\hat{x}(t)} p dx = \int_0^{\hat{x}(t)} \left(\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(py)\right) dx = 0$$

where ($y = \frac{d\hat{x}}{dt}$) is the mesh velocity, giving

$$\int_0^{\hat{x}(t)} \left(L_x p + \frac{\partial}{\partial x}(py)\right) dx = 0$$

Since this is true for all $\hat{x}$ we get the first order Ordinary Differential Equation (ODE)

$$-\frac{d}{dx}(py) = L_x p$$

1.2 Weak Form

We want to solve this equation for y by finite elements. The weak form is

$$-\int_{a(t)}^{b(t)} \omega \frac{d}{dx}(py) dx = \int_{a(t)}^{b(t)} \omega L_x p dx$$

giving, after integrating by parts, where ω is a test function

$$-\omega py|_a^b + \int_a^b p\frac{d\omega}{dx} y dx = \int_a^b \omega L_x p dx$$

The finite element form (with $\omega \approx \phi_i$ and $y \approx Y = \sum_j Y_j\phi_j$) is

$$-\phi_i pY|_a^b + \int_a^b p\frac{d\phi_i}{dx}\left(\sum_j Y_j\phi_j\right) dx = \int_a^b \phi_i L_x p dx$$

giving (apart from the boundary term) a matrix equation

$$B\underline{Y} = \underline{f}$$

for the vector of velocities Y, where B is a matrix with entries

$$B_{ij} = \int_a^b p \frac{d\phi_i}{dx} \phi_j dx$$

This is an unsymmetric matrix, similar to an anti-symmetric matrix, and difficult to invert.

1.3 Alternative approach

An alternative approach is to write $y = \frac{du}{dx}$ where u is a velocity potential, giving the second order equation

$$-\frac{d}{dx}(p\frac{du}{dx}) = L_x p$$

with weak form

$$-\int_a^b \omega \frac{d}{dx} \left(p \frac{du}{dx} \right) = \int_a^b \omega L_x p dx$$

giving, after integration by parts

$$-\phi_i p \frac{dU}{dx}|_a^b + p\frac{d\phi_i}{dx} \left(\sum_j U_j \frac{d\phi_j}{dx} \right) dx = \int_a^b \omega L_x p dx$$

leading to (apart form the boundary term)

$$K\underline{U} = \underline{f}$$

where K is the stiffness matrix with entries

$$K_{ij} = \int_a^b p \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx$$

This is a symmetric matrix, easy to invert after imposing a boundary condition, giving U. The function Y can be found (in principle) from $Y = \frac{dU}{dx}$. The problem is that Y (being a velocity) needs to be continuous. We shall investigate this second approach using a test problem.

1.4 Finite elements for a first order Differential Equations

Consider solving the first order differential equation problem

$$\frac{dy}{dx} = f(x), \qquad y(0) = 1 \tag{1.1}$$

in (0,1) by the Finite Element method. The weak form is

$$\int_0^1 \omega_i \frac{dy}{dx} = f(x)$$

Replace $\omega_i = \phi_i(x)$ by piecewise linear or quadratic basis functions and expand

$$y \approx Y = \sum Y_j \phi_j$$

which gives us

$$\sum_{j=0}^{1} Y_j \left(\int_0^1 \phi_i \frac{d\phi_j}{dx} dx \right) = \int_0^1 \phi_i f(x) dx \tag{1.2}$$

To solve this system of equation, we can write equation (1.2) in the matrix form as follows

$$B\underline{Y} = \underline{f}$$

where

$$B_{ij} = \int_0^1 \phi_i \frac{d\phi_j}{dx} dx$$

and $\underline{f}$ is the load vector

$$f_i = \int_0^1 \phi_i f(x) dx$$

$\underline{Y_j}$ are unknowns and B is Matrix, which is not a stiffness matrix, it looks like this (on a regular grid)

$$B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & & \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0.5 \end{pmatrix}$$

B is badly conditioned, anti-symmetric and difficult to invert.
In the approach used in this dissertation, put

$$y = \frac{du}{dx}$$

and instead of solving (1.1), solve the second order equation

$$\frac{d^2u}{dx^2} = f(x) \tag{1.3}$$

where $\frac{du}{dx} = 1$ at $x = 0$ and we impose (arbitrarily) the artificial boundary condition $u = 0$ at $x = 1$. From the weak form of (1.3), we have

$$\omega_i \frac{du}{dx}|_0^1 - \int_0^1 \frac{d\omega_i}{dx}\frac{du}{dx}y = \int_0^1 \omega f(x)dx$$

Replacing ω_i by finite element basis functions $\phi_i(x)$ and approximating u by

$$U = \sum_{j=0}^{N-1} U_j \phi_j$$

and applying boundary conditions it leads to the weak form

$$-e_0 - \sum_{j=0}^{N-1} U_j \left(\int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx \right) = \int_0^1 \phi_i f(x)dx$$

where $e_0 = \delta_{0i}$ or

$$\sum_{j=0}^{N-1} U_j \left(\int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx \right) = -\int_0^1 \phi_i f(x)dx - e_0 \tag{1.4}$$

We can write (1.4) in matrix form as follows

$$K\underline{U} = -(\underline{f} + \underline{e}_0)$$

where K is stiffness matrix, $\underline{U}$ is a vector of unknowns and $\underline{f}$ is a load vector and where $\underline{e}_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$.

1.5 Outline of the Dissertation

Chapter Two, is based on theory of finite elements for second order differential equations. This chapter investigate the method of Linear Finite elements and deficencies in this method for our purpose and provides an alternative Quadratic elements method to find the numerical solution.

Chapter Three, provides the Linear and Quadratic approaches to solve the first order differential equations as well as the Sturn-Liouiville type differential equations. In this chapter we solve test problems to investigate the numerical results.

Chapter Four, introduces the results for moving boundary and discusses the possible behaviour that can be arise as the boundary moves. We also discuss the numerical results of the test problem and compare them with the exact solutions to investigate the errors.

Chapter Five contains discussion and conclusions.

Chapter 2

Finite Elements for Second order Equations

There are many ways to solve Partial Differentail Equations (PDEs) numerically with advantages and disadvantages. The Finite Element Method (FEM) is a good choice for solving PDEs over complex domains, when a domain changes (as during a solid state reaction with a moving boundary), when the desired precision varies over the entire domain, or when the solution lacks smoothness. For instance, in simulations of the weather patterns on Earth, it is more important to have accurate predictions over land than over the open sea, a demand that is achievable using the finite element method.

2.1 Basic Finite Elements

2.1.1 Weak Forms

Because of limited differentiability of discrete solutions when substituting into a PDE, instead of substituting a piecewise linear representation we can substitute it into a 'weak' form. Consider a second order differential equation to illustrate the weak form of PDE.

$$-\frac{d^2u}{dx^2} = f(x), (a < x < b) \tag{2.1}$$

where $u \in C^2(a,b)$, is at least twice differentiable in domain (a,b).
Multiply both sides of 2.1 by $\omega_i(x)$, where $\omega_i(x)$ belongs to a set of test functions which are

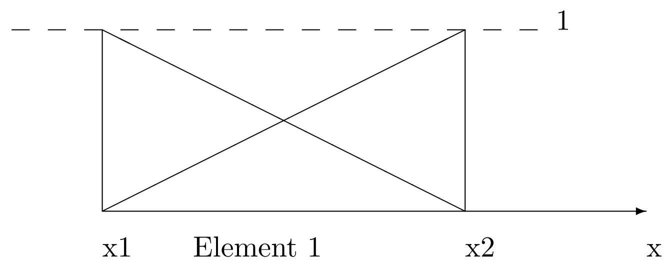


Figure 2.1: Linear finite elements for one element

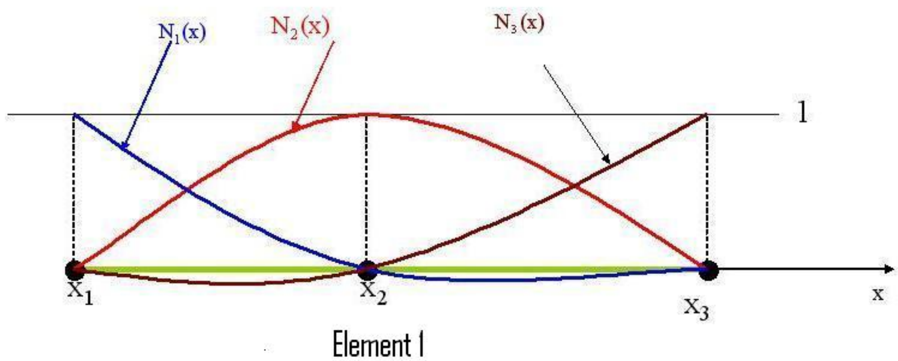


Figure 2.2: Quadratic finite elements for one element

C^1 and square integrable in (a,b) and integrate it, giving

$$-\int_a^b \omega_i(x)\frac{d^2u}{dx^2}dx = \int_a^b \omega_i(x)f(x)dx \tag{2.2}$$

Now integrate left hand side of (2.2) by parts, giving

$$-\omega_i(x)\frac{du}{dx}|_a^b + \int_a^b \frac{d\omega_i(x)}{dx}\frac{du}{dx}dx = \int_a^b \omega_i(x)f(x)dx \tag{2.3}$$

This is known as the weak form of the differential equation. In equation (2.3), we only require that $u, \omega_i \in H^1(a,b)$, once differentiable. So linear representation of such functions is allowed. Also since the integrals can be broken into subintervals, piecewise functions are allowed.

2.1.2 The Finite Element Method (general)

Leave out the first term of the left hand side of (2.3) for the moment and write the other terms as a sum over a set of given intervals k such as

$$\sum_{k=1}^{N+1} \int_{x_{k-1}}^{x_k} \frac{d\omega_i}{dx} \frac{du}{dx} dx \tag{2.4}$$

By replacing ω_i with piecewise test functions $\phi_i \in H^1, (i = 0, 1, \ldots, N+1)$ and u with $U \in H^1$, where

$$\frac{dU}{dx} = \sum_{j=0}^{N+1} U_j \frac{d\phi_j}{dx} \tag{2.5}$$

Equation (2.3) becomes

$$\sum_{j=0}^{N+1} U_j \int_a^b \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx = \int_a^b \phi_i(x) f(x) dx \tag{2.6}$$

which is known as finite element equation. There are N+2 equations and N+2 unknowns. We know that basis functions $\phi_i(x)$ are non-zero only on the two intervals either side of node i, so the system (2.6) for $i = 1, \ldots, N+1$ becomes

$$\sum_{j=i-1}^{j=i+1} U_j \int_{x_{i-1}}^{x_{i+1}} \frac{d\phi_i}{dx} \frac{\phi_j}{dx} dx = \int_a^b \phi_i(x) f(x) dx, (i = 1, 2, \ldots, N) \tag{2.7}$$

This set of equations can be written in matrix form as follows

$$K\underline{U} = \underline{f} \tag{2.8}$$

where U is a vector of unknowns U_j, K is a stiffness matrix consisting of elements

$$K_{ij} = \int_{x_{j-1}}^{x_{j+1}} \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx \tag{2.9}$$

$j = 0, N+1$ and f is load vector consisting of

$$f_i = \int_{x_{j-1}}^{x_{j+1}} \phi_i(x) f(x) dx \tag{2.10}$$

The first and last equations are special with basis functions which are "half hats" and are

$$\sum_{j=0}^{1} U_j \int_{x_0}^{x_1} \frac{d\phi_0}{dx} \frac{\phi_j}{dx} dx = \int_{x_0}^{x_1} \phi_0(x) f(x) dx, (i = 0) \tag{2.11}$$

and

$$\sum_{j=N}^{N+1} U_j \int_{x_N}^{x_{N+1}} \frac{d\phi_N}{dx} \frac{\phi_j}{dx} dx = \int_{x_N}^{x_{N+1}} \phi_N(x) f(x) dx, (i = N+1) \tag{2.12}$$

These are easily incorporated into the stiffness matrix and load vector. Boundary conditions can easily be applied to (2.8)

2.1.3 Evaluation of Stiffness Matrix K and Load Vector f for Linears

We define $\phi_i(x)$ as follows

$$\phi_i = \begin{cases} \frac{x - x_i}{x_i - x_{i-1}} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \leq x \leq x_{i+1} \\ 0 & otherwise \end{cases} \tag{2.13}$$

By differentiating equation (2.13), we get

$$\frac{d\phi_i}{dx} = \begin{cases} \frac{1}{x_i - x_{i-1}} & x_{i-1} \leq x \leq x_i \\ \frac{-1}{x_{i+1} - x_i} & x_i \leq x \leq x_{i+1} \\ 0 & otherwise \end{cases} \tag{2.14}$$

where $(i = 1, \ldots, N)$ are peicewise constant functions, with subsets of (2.14) for the "half hats."

The equations (2.7) reduces to

$$\frac{U_i - U_{i-1}}{x_i - x_{i-1}} - \frac{U_{i+1} - U_i}{x_{i+1} - x_i} = \int_{x_{i-1}}^{x_{i+1}} \phi_i(x) f(x) dx, (i = 1, \ldots, N) \tag{2.15}$$

First and last equations reduced to

$$-\frac{U_1 - U_0}{x_1 - x_0} = \int_{x_0}^{x^1} \phi_0(x) f(x) dx, (i = 0) \tag{2.16}$$

and

$$\frac{U_{N+1} - U_N}{x_{N+1} - x_N} = \int_{x_N}^{x^{N+1}} \phi_N(x) f(x) dx, (i = N + 1) \tag{2.17}$$

The right hand side integrals can be evaluated by numerical integration.
The stiffness matrix K is singular, which follows from the fact that the $\phi_i(x)$ form a partition of unity, so

$$\sum_{i=0}^{N+1} \phi_i(x) = 1 \Rightarrow \sum_{i=0}^{N+1} \frac{d\phi_i}{dx} = 0 \tag{2.18}$$

It means all column sums of the determinant of Matrix K are zero. By applying boundary conditions, it is possible to invert this matrix.

2.1.4 Evaluation of Stiffness matrix K and Load Vector f for Quadratics

In this section we explain how can we construct solution by adding an extra node in each element as p-refinement.
There are deficiencies in linear finite elements formulation in convection-diffusion problems. In such situations to examine the numerical solution of 1D convection-diffusion problem discretise with quadratic shape functions as shown in figure (2.2).
First of all we establish a matrix equation for the given problem, as for linear finite elements, then the discrete solution of the problem is analysed.
As shown in figure (2.2), we consider a generic element with nodes 1, 2 and 3, where node 2 is a mid-side. With reference to the condition $0 < x < 1$, the shape function of the element are
$N_1(x) = 2(x - \frac{1}{2})(x - 1)$, $N_2(x) = -4x(x - 1)$, and $N_3(x) = 2x(x - \frac{1}{2})$.
We can establish an element stiffness matrix K^e of the quadratic elements [1] as follows

$$K^e = \int_\Omega \begin{pmatrix} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x} \\ \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} \\ \frac{\partial N_3}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x} \end{pmatrix} dx$$

and load vector $\underline{f}$ as follows

$$f_i = \int_\Omega f(x) N_i(x) dx$$

where $i = (1, 2, 3)$.

2.2 A More General first order Differential Equation

Let us consider a more general differential equation of Sturm-Liouiville type

$$-\frac{d}{dx}\left(p(x)y\right) + q(x)u = f(x) \tag{2.19}$$

in $(0, 1)$ where $q(x) = 0$, with boundary conditions, at $x = 0$ and at $x = 1$.

2.2.1 Linear Finite Elements

To obtain the weak form multiply (2.19) by the test function $\omega \in H_0^1$ and integrate from 0 to 1, apply finite elements and finally we get

$$\int_0^1 p(x) \frac{d\phi_i}{dx} \left(\sum_{j=0}^{N+1} U_j \frac{d\phi_j}{dx} \right) dx = \int_0^1 \phi_i(x) f(x) dx \tag{2.20}$$

which can be written in the form

$$K\underline{U} = \underline{f}$$

where

$$K_{ij} = \int_0^1 p(x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx,$$

U_j is a vector of unknowns and

$$f_i = \int_0^1 \phi_i(x) f(x) dx$$

is a load vector.

2.2.2 Quadratic Finite Elements

Similarly from (2.19), apply quadratic finite elements and after applying all the calculations, we get the matrix equation

$$K\underline{U} = \underline{f}$$

where K is the assembly of element matrices

$$K_{ij}^{e} = \int_{\Omega} p(x) \begin{pmatrix} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x} \\ \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} \\ \frac{\partial N_3}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x} \end{pmatrix} dx$$

and

$$f_i = \int_0^1 f(x) N_i(x) dx$$

is a load vector.

Chapter 3

A Test Problem

In this chapter we solve some test problems by using the methods discussed in chapter 2. First we take a simple example and solve it by linear finite element and then by the quadratic elements method. We assemble element stiffness matrix for all elements as shown in figure (3.1) and by using Gaussian inversion method, we will invert stiffness matrix K to solve the problem. Consider the problem

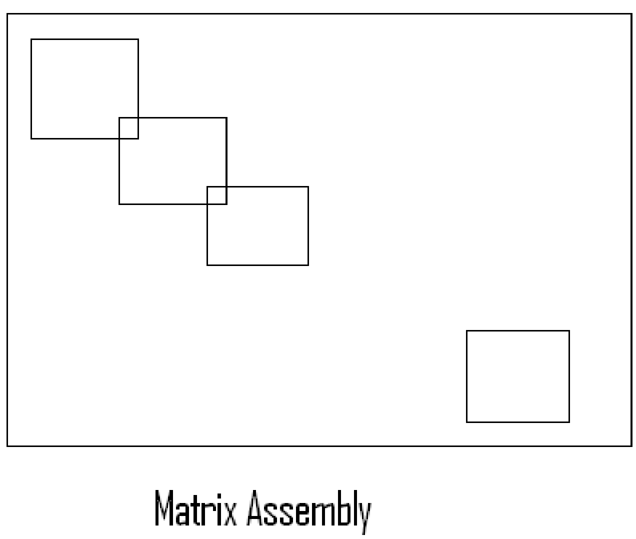


Figure 3.1: Shows matrices assembly for more elements.

$$\frac{dy}{dx} = \frac{d^2u}{dx^2} = f(x) \tag{3.1}$$

in $(0, 1)$, with $\frac{du}{dx} = 1$ at $x = 0$ and $u = 0$ at $x = 1$.

3.1 Linear Finite Elements

Let us consider finite elements for equation (3.1). We know from chapter 2, matrix equation for (3.1)is

$$K\underline{U} = -(\underline{f} + e_0) \tag{3.2}$$

where

$$K_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx,$$

$\underline{U}$ are unknowns,

$$f_i = \int \phi_i(x)f(x)dx$$

and $e_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$
We can solve this problem by dividing the interval $(0, 1)$ into 1, 2, 4, 8, and 16 elements. Stiffness matrix entries in each case are (for equal spacing h)

$$K_{ij} = \int_{x_{i-1}}^{xi+1} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- **Stiffness Matrix and load vector for One element**
  Consider the following figure (3.2) and apply the method mentioned in section (3.1), we can find the stiffness matirx as follows

$$K^e = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

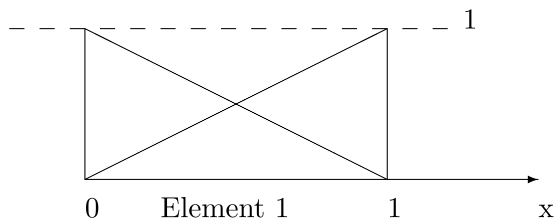


Figure 3.2: linear finite elements for one element

and load vectors for $f(x) = x^2$ are

$$f_0 = \int_0^1 (1-x)x^2 dx = \frac{1}{12}$$

$$f_1 = \int_0^1 x^3 dx = \frac{1}{4}$$

By solving the test problem, we get approximate value of velocity potential U shown in table (3.1).

Table 3.1: Results for equation $\frac{d^2u}{dx^2} = x$ and x^2 for one Element.

x value	Exact f(x)=x	LFE f(x)=x	Error (u-U)	Exact f(x)=x^2	LFE f(x)=x^2	error (u-U)
0	-1.33333	-1.16667	-0.166667	-1.08333	-1.5	0.416667
1	0	0	0	0	0	0

- **Stiffness Matrix and load vector for Two element**

  The stiffness matrix for two elements in figure (3.3) can be found by getting element stiffness matrix for each element and then assemble them as shown

$$K = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

  and load vectors are

$$f_0 = \int_0^{\frac{1}{2}} (1-2x)x^2 dx = \frac{1}{96}$$

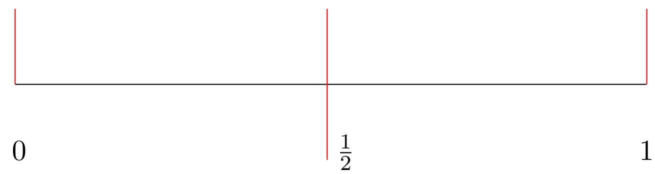


Figure 3.3: Linear finite element for Two elements.

,

$$f_1 = \int_0^{\frac{1}{2}} (2x)x^2 dx + \int_{\frac{1}{2}}^1 (2-2x)x^2 dx = \frac{7}{48}$$

and

$$f_2 = \int_{\frac{1}{2}}^1 (2x-1)x^2 dx = \frac{17}{96}$$

Similarly if $f(x) = x$ the load vector is

$$f_i = \begin{pmatrix} \frac{1}{24} & \frac{1}{4} & \frac{5}{24} \end{pmatrix}^T$$

The solution for U is shown in table (3.2)

Table 3.2: Results for equation $\frac{d^2u}{dx^2} = x$ and x^2 of Two Element.

x value	Exact f(x)=x	LFE f(x)=x	Error (u-U)	Exact f(x)=x^2	LFE f(x)=x^2	error (u-U)
0	-1.33333	-1.16667	-0.166667	-1.08333	-1.08333	2.22045e-016
0.5	-0.791667	-0.645833	-0.145833	-0.578125	-0.578125	1.11022e-016
1	0	0	0	0	0	0

- **Stiffness Matrix and load vector for Four element**

  The assembled stiffness matrix for four elements is

$$K = \begin{pmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{pmatrix}$$

and the load vector is

$$f_0 = \int_0^{\frac{1}{4}} (1-4x)x^2 dx = \frac{1}{768}$$

,

$$f_1 = \int_0^{\frac{1}{4}} (4x)x^2 dx + \int_{\frac{1}{4}}^{\frac{1}{2}(2-4x)x^2 dx = \frac{7}{384}}$$

$$f_2 = \int_{\frac{1}{4}}^{\frac{1}{2}} (4x-1)x^2 dx + \int_{\frac{1}{2}}^{\frac{3}{4}(3-4x)x^2 dx = \frac{25}{384}}$$

$$f_3 = \int_{\frac{1}{2}}^{\frac{3}{4}} (4x-2)x^2 dx + \int_{\frac{3}{4}}^{1} (4-4x)x^2 dx = \frac{55}{384}$$

and

$$f_4 = \int_{\frac{3}{4}}^{1} (4x-3)x^2 dx = \frac{27}{256}$$

The load vector f is when $f(x) = x$ is

$$f_i = \begin{pmatrix} \frac{1}{96} & \frac{1}{16} & \frac{1}{8} & \frac{3}{8} & \frac{11}{96} \end{pmatrix}^T$$

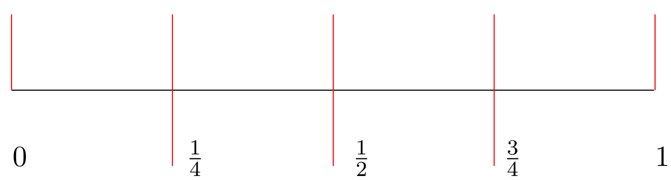


Figure 3.4: Linear finite element for Four elements.

The solution is shown in table (3.3)

Table 3.3: Results for equation $\frac{d^2u}{dx^2} = x$ and x^2 of Four Element.

x value	Exact f(x)=x	LFE f(x)=x	Error (u-U)	Exact f(x)=x^2	LFE f(x)=x^2	error (u-U)
0	-1.33333	-1.16667	-0.166667	-1.08333	-1.08333	0
0.25	-1.07813	-0.914063	-0.164063	-0.833008	-0.833008	1.11022e-016
0.5	-0.791667	-0.645833	-0.145833	-0.578125	-0.578125	1.11022e-016
0.75	-0.442708	-0.346354	-0.0963542	-0.306966	-0.306966	1.11022e-016
1	0	0	0	0	0	0

- **Solutions for eight and sixteen elements**
  Similarly we can find the stiffness matrix and load vectors for eight and sixteen and solve them to get results shown in table (3.4) and (3.5).

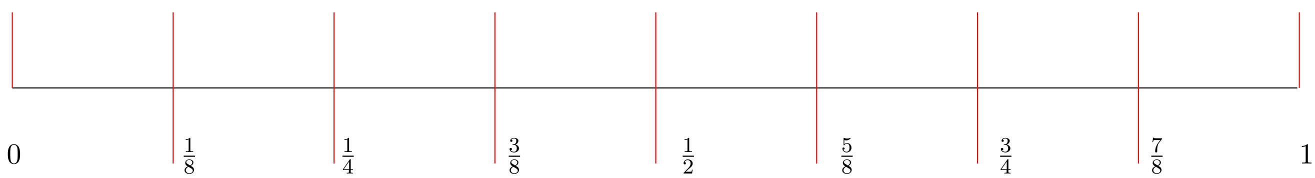


Figure 3.5: Linear finite element for eight elements.

Table 3.4: Results for equation $\frac{d^2u}{dx^2} = x^2$ of Eight Element.

x value	Exact f(x)=x	LFE f(x)=x	Error (u-U)	Exact f(x)=x^2	LFE f(x)=x^2	error (u-U)
0	-1.33333	-1.16667	-0.166667	-1.08333	-1.08333	2.22045e-016
0.125	-1.20768	-1.04134	-0.166341	-0.958313	-0.958313	1.11022e-016
0.25	-1.07813	-0.914063	-0.164063	-0.833008	-0.833008	1.11022e-016
0.375	-0.940755	-0.782878	-0.157878	-0.706685	-0.706685	2.22045e-016
0.5	-0.791667	-0.645833	-0.145833	-0.578125	-0.578125	0
0.625	-0.626953	-0.500977	-0.125977	-0.445618	-0.445618	5.55112e-017
0.75	-0.442708	-0.346354	-0.0963542	-0.306966	-0.306966	0
0.875	-0.235026	-0.180013	-0.055013	-0.159485	-0.159485	2.77556e-017
1	0	0	0	0	0	0

Solution for sixteen elements

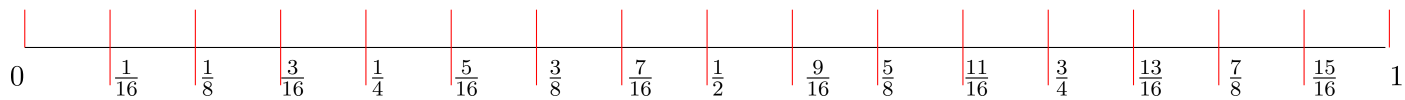


Figure 3.6: Linear finite element for sixteen elements.

Table 3.5: Results for equation $\frac{d^2u}{dx^2} = x^2$ of Sixteen Element.

x value	Exact f(x)=x	LFE f(x)=x	Error (u-U)	Exact f(x)=x^2	LFE f(x)=x^2	error (u-U)
0	-1.33333	-1.16667	-0.166667	-1.08333	-1.08333	0
0.0625	-1.27075	-1.10413	-0.166626	-1.02083	-1.02083	0
0.125	-1.20768	-1.04134	-0.166341	-0.958313	-0.958313	1.11022e-016
0.1875	-1.14364	-0.978068	-0.165568	-0.89573	-0.89573	-1.11022e-016
0.25	-1.07813	-0.914063	-0.164063	-0.833008	-0.833008	1.11022e-016
0.3125	-1.01066	-0.84908	-0.16158	-0.770039	-0.770039	2.22045e-016
0.375	-0.940755	-0.782878	-0.157878	-0.706685	-0.706685	-2.22045e-016
0.4375	-0.86792	-0.71521	-0.15271	-0.64278	-0.64278	1.11022e-016
0.5	-0.791667	-0.645833	-0.145833	-0.578125	-0.578125	3.33067e-016
0.5625	-0.711507	-0.574504	-0.137004	-0.512491	-0.512491	2.22045e-016
0.625	-0.626953	-0.500977	-0.125977	-0.445618	-0.445618	5.55112e-017
0.6875	-0.537516	-0.425008	-0.112508	-0.377216	-0.377216	-1.66533e-016
0.75	-0.442708	-0.346354	-0.0963542	-0.306966	-0.306966	2.77556e-016
0.8125	-0.342041	-0.264771	-0.0772705	-0.234516	-0.234516	8.32667e-017
0.875	-0.235026	-0.180013	-0.055013	-0.159485	-0.159485	1.94289e-016
0.9375	-0.121175	-0.0918376	-0.0293376	-0.0814603	-0.0814603	1.38778e-016
1	0	0	0	0	0	0

3.2 Quadratic Finite Element Method

In this section we explain, how can we construct solution by adding an extra node in each element. So the quadratic solution for (3.1) is as follows

3.2.1 Solution for Two Elements

Let us consider the figure (3.7). First of all we need to find the node values to get stiffness matrix and load vector. The node values are as follows for two elements.

1. First Element
   $N_1(x) = 8(x - \frac{1}{4})(x - \frac{1}{2})$,
   $N_2(x) = -16x(x - \frac{1}{2})$,
   $N_3(x) = 8x(x - \frac{1}{4})$

2. Second Element

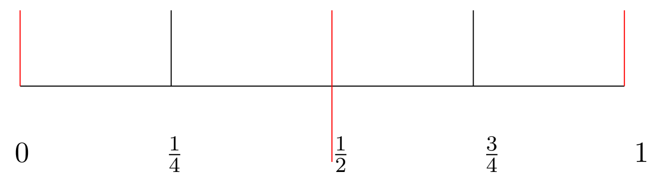


Figure 3.7: Quadratic finite element for Two elements.

$$N_1(x) = 8(x - \tfrac{3}{4})(x - 1),$$
$$N_2(x) = -16(x - \tfrac{1}{2})(x - 1),$$
$$N_3(x) = 8(x - \tfrac{1}{2})(x - \tfrac{3}{4})$$

For each element, the stiffness matrix is

$$K^e = \begin{pmatrix} \frac{14}{3} & \frac{-16}{3} & \frac{2}{3} \\ \frac{-16}{3} & \frac{32}{3} & \frac{-16}{3} \\ \frac{2}{3} & \frac{-16}{3} & \frac{14}{3} \end{pmatrix}$$

so we need to assemble this to get stiffness matrix for two elemts.
The load vector of function $f(x) = x$ is

$$f_i^1 = \begin{bmatrix} 0 & \frac{1}{12} & \frac{1}{24} & \frac{1}{4} & \frac{1}{12} \end{bmatrix}^T$$

and for $f(x) = x^2$ is

$$f_i^2 = \begin{bmatrix} \frac{-1}{480} & \frac{1}{40} & \frac{3}{60} & \frac{23}{120} & \frac{13}{160} \end{bmatrix}^T$$

The matrix after assembly is

$$K = \begin{pmatrix} \frac{14}{3} & \frac{-16}{3} & \frac{2}{3} & 0 & 0 \\ \frac{-16}{3} & \frac{32}{3} & \frac{-16}{3} & 0 & 0 \\ \frac{2}{3} & \frac{-16}{3} & \frac{28}{3} & \frac{-16}{3} & \frac{2}{3} \\ 0 & 0 & \frac{-16}{3} & \frac{32}{3} & \frac{-16}{3} \\ 0 & 0 & \frac{2}{3} & \frac{-16}{3} & \frac{14}{3} \end{pmatrix}$$

By solving the above matrix system we get the result as Table (3.6) shows the U values of equation 3.1.

Table 3.6: Results for equation $\frac{d^2u}{dx^2} = x$ and x^2 for Two Element.

x	Exact (x)	QFE (x)	Error (x)	Exact (x^2)	QFE (x^2)	Error (x^2)
0	-1.33333	-1.14583	-0.1875	-1.08333	-1.07396	-0.009375
0.25	-1.07813	-0.893229	-0.184896	-0.833008	-0.823698	-0.0093099
0.5	-0.791667	-0.625	-0.166667	-0.578125	-0.56875	-0.009375
0.75	-0.442708	-0.335938	-0.106771	-0.306966	-0.302344	-0.0046224
1	0	0	0	0	0	0

3.2.2 Solution for Four Elements

Let us consider the figure (3.8) The node values are as follows

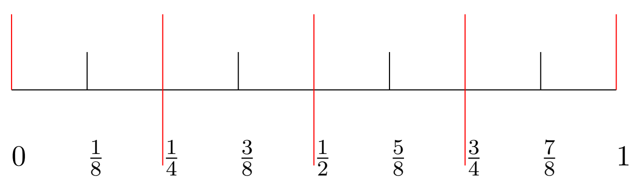


Figure 3.8: Quadratic finite element for Four elements.

1. First Element
   $N_1(x) = 32(x - \frac{1}{8})(x - \frac{1}{4})$,
   $N_2(x) = -64x(x - \frac{1}{4})$,
   $N_3(x) = 32x(x - \frac{1}{8})$

2. Second Element
   $N_1(x) = 32(x - \frac{3}{8})(x - \frac{1}{2})$,
   $N_2(x) = -64(x - \frac{1}{4})(x - \frac{1}{2})$,
   $N_3(x) = 32(x - \frac{1}{4})(x - \frac{3}{8})$

3. Third Element
   $N_1(x) = 32(x - \frac{5}{8})(x - \frac{3}{4})$,

$N_2(x) = -64(x - \frac{1}{2})(x - \frac{3}{4})$,
$N_3(x) = 32(x - \frac{1}{2})(x - \frac{5}{8})$

4. Fourth Element
$N_1(x) = 32(x - \frac{7}{8})(x - 1)$,
$N_2(x) = -64(x - \frac{3}{4})(x - 1)$,
$N_3(x) = 32(x - \frac{3}{4})(x - \frac{7}{8})$

For each element, the stiffness matrix is

$$K^e = \begin{pmatrix} \frac{28}{3} & \frac{-32}{3} & \frac{4}{3} \\ \frac{-32}{3} & \frac{64}{3} & \frac{-32}{3} \\ \frac{4}{3} & \frac{-32}{3} & \frac{28}{3} \end{pmatrix}$$

and load vectors for $f(x) = x$ and $f(x) = x^2$ are

$$f_i^1 = \begin{bmatrix} 0 & \frac{1}{48} & \frac{1}{96} & \frac{1}{16} & \frac{1}{48} & \frac{5}{48} & \frac{1}{32} & \frac{7}{48} & \frac{1}{24} \end{bmatrix}^T$$

and

$$f_i^2 = \begin{bmatrix} \frac{-1}{3840} & \frac{1}{320} & \frac{3}{1280} & \frac{23}{960} & \frac{21}{320} & \frac{89}{3540} & \frac{41}{320} & \frac{53}{1280} \end{bmatrix}^T$$

The matrix after assembly is

$$K = \frac{1}{3} \begin{pmatrix} 28 & -32 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 64 & -32 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -32 & 56 & -32 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -32 & 64 & -32 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -32 & 56 & -32 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -32 & 64 & -32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -32 & 56 & -32 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -32 & 64 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -32 & 28 \end{pmatrix}$$

Table (3.7) gives the solution for four elements

Table 3.7: Results for equation $\frac{d^2u}{dx^2} = x$ and x^2 for Four Element.

x	Exact (x)	QFE (x)	Error (x)	Exact (x^2)	QFE (x^2)	Error (x^2)
0	-1.33333	-1.15625	-0.177083	-1.08333	-1.06953	-0.0138021
0.125	-1.20768	-1.03092	-0.176758	-0.958313	-0.944515	-0.013798
0.25	-1.07813	-0.903646	-0.174479	-0.833008	-0.819206	-0.0138021
0.375	-0.940755	-0.773763	-0.166992	-0.706685	-0.693376	-0.0133097
0.5	-0.791667	-0.638021	-0.153646	-0.578125	-0.565299	-0.0128255
0.625	-0.626953	-0.49707	-0.129883	-0.445618	-0.434554	-0.0110636
0.75	-0.442708	-0.346354	-0.0963542	-0.306966	-0.297656	-0.0093099
0.875	-0.235026	-0.17513	-0.0598958	-0.159485	-0.154834	-0.00465088
1	0	0	0	0	0	0

3.2.3 Soluiton for Eight and sixteen Elements

Similarly we can construct solution for eight and sixteen elements. The element stiffness matrix is given for each case, assemble it acoordingly, also load vectors are shown. So to solve these systems, we get the results as shown in table (3.8) and (3.9). For each element,

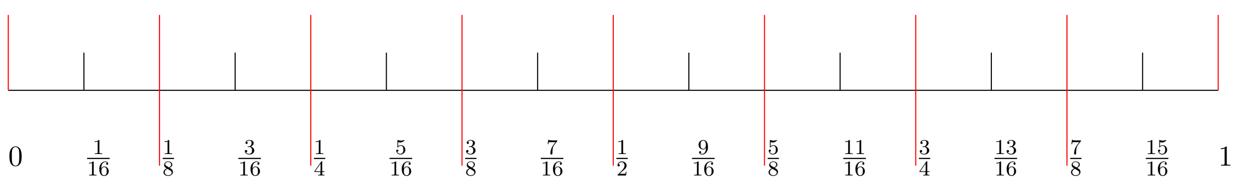


Figure 3.9: Quadratic finite element for eight elements.

the stiffness matrix is

$$K^e = \begin{pmatrix} \frac{56}{3} & \frac{-64}{3} & \frac{8}{3} \\ \frac{-64}{3} & \frac{128}{3} & \frac{-64}{3} \\ \frac{8}{3} & \frac{-64}{3} & \frac{128}{3} \end{pmatrix}$$

and load vectors for (5.3) and (5.5) are

$$f_i^1 = \begin{bmatrix} 0 & \frac{1}{192} & \frac{1}{384} & \frac{1}{64} & \frac{1}{384} & \frac{5}{194} & \frac{1}{128} & \frac{7}{192} & \frac{1}{96} & \frac{3}{64} & \frac{5}{384} & \frac{11}{192} & \frac{1}{64} & \frac{13}{192} & \frac{7}{384} & \frac{5}{64} & \frac{1}{48} \end{bmatrix}^T$$

and

$$f_i^2 = \begin{bmatrix} \frac{-1}{30720} & \frac{1}{2560} & \frac{3}{10240} & \frac{23}{7680} & \frac{3}{10240} & \frac{21}{2560} & \frac{89}{30720} & \frac{41}{2560} & \frac{53}{10240} & \frac{203}{7680} & \frac{83}{10240} \\ \frac{101}{2560} & \frac{359}{30720} & \frac{141}{2560} & \frac{163}{10240} & \frac{563}{7680} & \frac{213}{10240} \end{bmatrix}^T$$

Table 3.8: Results for equation $\frac{d^2u}{dx^2} = x$ and x^2 for eight Element.

x	Exact (x)	QFE (x)	Error (x)	Exact (x^2)	QFE (x^2)	Error (x^2)
0	-1.33333	-1.13688	-0.196452	-1.08333	-1.06904	-0.0142904
0.0625	-1.27075	-1.07434	-0.196411	-1.02083	-1.00654	-0.0142901
0.125	-1.20768	-1.01156	-0.196126	-0.958313	-0.944023	-0.0142904
0.1875	-1.14364	-0.948446	-0.19519	-0.89573	-0.881459	-0.0142718
0.25	-1.07813	-0.884603	-0.193522	-0.833008	-0.818754	-0.0142537
0.3125	-1.01066	-0.820109	-0.190552	-0.770039	-0.755944	-0.0140948
0.375	-0.940755	-0.754395	-0.186361	-0.706685	-0.692749	-0.0139364
0.4375	-0.86792	-0.687703	-0.180216	-0.64278	-0.629184	-0.0135963
0.5	-0.791667	-0.619303	-0.172363	-0.578125	-0.564868	-0.0132568
0.5625	-0.711507	-0.549601	-0.161906	-0.512491	-0.499897	-0.0125933
0.625	-0.626953	-0.477702	-0.149251	-0.445618	-0.433687	-0.0119303
0.6875	-0.537516	-0.404175	-0.133341	-0.377216	-0.366456	-0.0107602
0.75	-0.442708	-0.327962	-0.114746	-0.306966	-0.297375	-0.00959066
0.8125	-0.342041	-0.250061	-0.09198	-0.234516	-0.226826	-0.00769018
0.875	-0.235026	-0.170573	-0.0644531	-0.159485	-0.153695	-0.0057902
0.9375	-0.121175	-0.0857747	-0.0354004	-0.0814603	-0.0785655	-0.00289485
1	0	0	0	0	0	0

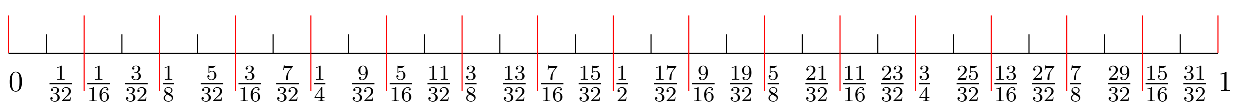


Figure 3.10: Quadratic finite element for sixteen elements.

For each element, the stiffness matrix is

$$K^e = \begin{pmatrix} \frac{112}{3} & \frac{-128}{3} & \frac{16}{3} \\ \frac{-128}{3} & \frac{256}{3} & \frac{-128}{3} \\ \frac{16}{3} & \frac{-128}{3} & \frac{112}{3} \end{pmatrix}$$

and load vectors for $f(x) = x$ and $f(x) = x^2$ are

$$f_i^1 = \begin{bmatrix} 0 & \frac{1}{768} & \frac{1}{1536} & \frac{1}{256} & \frac{1}{768} & \frac{5}{768} & \frac{1}{512} & \frac{7}{768} & \frac{1}{384} & \frac{3}{256} & \frac{5}{1536} & \frac{11}{768} & \frac{5}{256} \\ \frac{13}{1768} & \frac{7}{1536} & \frac{5}{256} & \frac{1}{192} & \frac{17}{768} & \frac{3}{512} & \frac{19}{768} & \frac{5}{768} & \frac{7}{256} & \frac{11}{1536} & \frac{23}{768} & \frac{1}{128} & \frac{25}{768} \\ \frac{13}{1536} & \frac{9}{256} & \frac{7}{768} & \frac{29}{768} & \frac{31}{768} & \frac{1}{96} \end{bmatrix}^T$$

and

$$f_i^2 = \begin{bmatrix} \frac{-1}{245760} & \frac{1}{20480} & \frac{3}{81920} & \frac{23}{61440} & \frac{13}{81920} & \frac{21}{20480} & \frac{89}{245760} & \frac{41}{20480} & \frac{51}{81920} & \frac{203}{61440} & \frac{83}{81920} \\ \frac{101}{20480} & \frac{359}{245760} & \frac{141}{20480} & \frac{163}{81920} & \frac{563}{61440} & \frac{213}{81920} & \frac{241}{20440} & \frac{809}{245760} & \frac{301}{20480} & \frac{333}{81920} & \frac{1103}{61440} \\ \frac{403}{81920} & \frac{441}{20480} & \frac{1439}{245760} & \frac{521}{20480} & \frac{563}{81920} & \frac{1823}{61440} & \frac{653}{81920} & \frac{701}{20480} & \frac{2249}{245760} & \frac{801}{20480} & \frac{853}{81920} \end{bmatrix}^T$$

Table 3.9: Results for equation $\frac{d^2u}{dx^2} = x$ and x^2 for sixteen Element.

x	Exact (x)	QFE (x)	Error (x)	Exact (x^2)	QFE (x^2)	Error (x^2)
0	-1.33333	-1.15159	-0.18174	-1.08333	-1.06953	-0.0138041
0.03125	-1.30207	-1.12034	-0.181735	-1.05208	-1.03828	-0.0138041
0.0625	-1.27075	-1.08905	-0.181699	-1.02083	-1.00703	-0.0138041
0.09375	-1.23931	-1.05773	-0.181582	-0.989577	-0.975774	-0.013803
0.125	-1.20768	-1.02631	-0.181373	-0.958313	-0.944511	-0.0138018
0.15625	-1.17581	-0.99481	-0.181002	-0.927034	-0.913238	-0.0137957
0.1875	-1.14364	-0.963158	-0.180478	-0.89573	-0.881941	-0.0137896
0.21875	-1.11109	-0.931384	-0.17971	-0.864393	-0.85062	-0.0137722
0.25	-1.07813	-0.899396	-0.178729	-0.833008	-0.819253	-0.0137548
0.28125	-1.04467	-0.867246	-0.177421	-0.801562	-0.787845	-0.0137171
0.3125	-1.01066	-0.834821	-0.17584	-0.770039	-0.756359	-0.0136795
0.34375	-0.976044	-0.802193	-0.173851	-0.73842	-0.72481	-0.0136102
0.375	-0.940755	-0.769229	-0.171526	-0.706685	-0.693144	-0.0135409
0.40625	-0.904734	-0.735532	-0.169202	-0.674814	-0.661388	-0.0134259
0.4375	-0.86792	-0.701439	-0.166481	-0.64278	-0.629469	-0.013311
0.46875	-0.830251	-0.667061	-0.16319	-0.61056	-0.597426	-0.0131339
0.5	-0.791667	-0.632225	-0.159442	-0.578125	-0.565168	-0.0129567
0.53125	-0.752106	-0.596887	-0.155219	-0.545446	-0.532747	-0.0126983
0.5625	-0.711507	-0.56103	-0.150477	-0.512491	-0.500051	-0.01244
0.59375	-0.66981	-0.524807	-0.145003	-0.479226	-0.467148	-0.0120787
0.625	-0.626953	-0.488003	-0.13895	-0.445618	-0.4339	-0.0117175
0.65625	-0.582876	-0.450794	-0.132082	-0.411627	-0.400398	-0.0112292
0.6875	-0.537516	-0.412943	-0.124574	-0.377216	-0.366475	-0.0107409
0.71875	-0.490814	-0.374644	-0.11617	-0.342344	-0.332245	-0.0100989
0.75	-0.442708	-0.335644	-0.107065	-0.306966	-0.297509	-0.00945689
0.78125	-0.393138	-0.296155	-0.0969824	-0.271039	-0.262407	-0.00863188
0.8125	-0.342041	-0.255904	-0.0861374	-0.234516	-0.226709	-0.00780691
0.84375	-0.289358	-0.215123	-0.0742345	-0.197348	-0.190581	-0.00676713
0.875	-0.235026	-0.173519	-0.0615075	-0.159485	-0.153757	-0.00572739
0.90625	-0.178986	-0.131344	-0.0476413	-0.120874	-0.116435	-0.00443851
0.9375	-0.121175	-0.0882851	-0.03289	-0.0814603	-0.0783106	-0.00314967
0.96875	-0.0615336	-0.0446156	-0.016918	-0.0411885	-0.0396137	-0.00157482
1	0	0	0	0	0	0

3.3 A More General 1-D Differential Equation

Now consider the problem

$$-\frac{d}{dx}\left((2x+1)\frac{du}{dx}\right) = x^2 \tag{3.3}$$

in $(0,1)$ with artificial boundary conditions $\frac{du}{dx} = 1$ at $x = 0$ and $u = 0$ at $x = 1$. This is representative of the moving mesh movement equation.

3.3.1 Linear Finite Elements

So by using the approach discussed in chapter 2, we can find stiffness and load vector as shown below.

- **Solution for One element**

  Stiffness matrix for one elemnt with $h = 1$ and $p(x) = 2x + 1$ is

$$K = \frac{1}{h^2}\int_0^1 p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

  and load vectors are

$$f_0 = \int_0^1 (1-x)x^2 dx = \frac{1}{12}$$

$$f_1 = \int_0^1 x^3 dx = \frac{1}{4}$$

  We have element stiffness matrix and load vector, solve this to get the result, which is shown in table (3.10)

Table 3.10: Results for equation $-\frac{d}{dx}\left((2x+1)\frac{du}{dx}\right) = x^2$ for One Element.

x values	Exact values (u)	Linar Finite values (U)	Error (u-U)
0	-0.516638	-0.458333	-0.0583047
1	0	0	0

- **Solution for Two elements**

  We require two stiffness element matrices for two elements, hence $h = 2$

$$K_1^e = \frac{1}{h^2}\int_0^{\frac{1}{2}} p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

$$K_2^e = \frac{1}{h^2}\int_{\frac{1}{2}}^{1} p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}$$

and load vectors are

$$f_0 = \int_0^{\frac{1}{2}} (1-2x)x^2 dx = \frac{1}{96}$$

$$f_1 = \int_0^{\frac{1}{2}} 2x^3 dx + \int_{\frac{1}{2}}^{1} (2-2x)x^2 dx = \frac{7}{48}$$

$$f_2 = \int_{\frac{1}{2}}^{1} (2x-1)x^2 dx = \frac{7}{48}$$

After assemling above stiffness element matrices,we get

$$K = \begin{pmatrix} 3 & -3 & 0 \\ -3 & 8 & -5 \\ 0 & -5 & 5 \end{pmatrix}$$

To solve this system we get the result shown in table (3.11)

Table 3.11: Results for equation $-\frac{d}{dx}\left((2x+1)\frac{du}{dx}\right) = x^2$ for Two Element.

x values	Exact values (u)	Linar Finite values (U)	Error (u-U)
0	-0.516638	-0.498611	-0.0180269
0.5	-0.172985	-0.16875	-0.00423495
1	0	0	0

- **Solution for Four elements**

  In this section we need four stiffness element matrices with $h = 4$

$$K_1^e = \frac{1}{h^2}\int_0^{\frac{1}{4}} p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}$$

$$K_2^e = \frac{1}{h^2}\int_{\frac{1}{4}}^{\frac{1}{2}} p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \begin{pmatrix} 7 & -7 \\ -7 & 7 \end{pmatrix}$$

$$K_3^e = \frac{1}{h^2}\int_{\frac{1}{2}}^{\frac{3}{4}} p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$$

$$K_4^e = \frac{1}{h^2}\int_{\frac{3}{4}}^{1} p(x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \begin{pmatrix} 11 & -11 \\ -11 & 11 \end{pmatrix}$$

and load vectors are

$$f_0 = \int_0^{\frac{1}{4}}(1-4x)x^2dx = \frac{1}{768}$$

$$f_1 = \int_0^{\frac{1}{4}} 4x^3dx + \int_{\frac{1}{4}}^{\frac{1}{2}}(2-4x)x^2dx = \frac{7}{384}$$

$$f_2 = \int_{\frac{1}{4}}^{\frac{1}{2}}(4x-1)x^2dx + \int_{\frac{1}{2}}^{\frac{3}{4}}(3-4x)x^2dx = \frac{25}{384}$$

$$f_3 = \int_{\frac{1}{2}}^{\frac{3}{4}}(4x-2)x^2dx + \int_{\frac{3}{4}}^{1}(4-4x)x^2dx = \frac{55}{384}$$

$$f_4 = \int_{\frac{3}{4}}^{1}(4x-3)x^2dx = \frac{27}{256}$$

The stiffness matrix for four elements is

$$K = \begin{pmatrix} 5 & -5 & 0 & 0 & 0 \\ -5 & 12 & -7 & 0 & 0 \\ 0 & -7 & 16 & -9 & 0 \\ 0 & 0 & -9 & 20 & -11 \\ 0 & 0 & 0 & -11 & 11 \end{pmatrix}$$

Table 3.12: Results for equation $-\frac{d}{dx}\left((2x+1)\frac{du}{dx}\right) = x^2$ for Four Element.

x values	Exact values (u)	Linar Finite values (U)	Error (u-U)
0	-0.516638	-0.511708	-0.00493015
0.25	-0.314139	-0.311968	-0.00217054
0.5	-0.172985	-0.171901	-0.00108365
0.75	-0.0706532	-0.0701941	-0.000459115
1	0	0	0

Similarly we can find solutions for eight and sixteen elements.

3.3.2 Quadratic Finite Elements

The solution for equation (3.3) can be found by using the same technique used for first test problem but for stiffness element matrix, we need to multiply the integral of each entry of the stiffness element matrix before integrating.

- **Solution for Two Elements**

  So the stiffness matrix for two elements is

$$K = \begin{pmatrix} 5 & -5 & 0 & 0 & 0 \\ -5 & 12 & -7 & 0 & 0 \\ 0 & -7 & 16 & -9 & 0 \\ 0 & 0 & -9 & 20 & -11 \\ 0 & 0 & 0 & -11 & 11 \end{pmatrix}$$

  So we use the same load vector for $f(x) = x^2$ in quadratic finite elements as used before in this chapter. The solution is shown in Table (3.13)

Table 3.13: Results for equation $-\frac{d}{dx}\left((2x+1)\frac{du}{dx}\right) = x^2$ for Two Element.

x values	Exact values (u)	Linar Finite values (U)	Error (u-U)
0	-0.516638	-0.520043	0.00340488
0.25	-0.314139	-0.31824	0.00410118
0.5	-0.172985	-0.176774	0.00378869
0.75	-0.0706532	-0.0723606	0.0017074
1	0	0	0

- **Solution for Four Elements**

  The stiffness matrix for four elements is

$$K = \begin{pmatrix} 31 & -36 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & 80 & -44 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -44 & 84 & -52 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -52 & 112 & -60 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -60 & 112 & -68 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & -68 & 144 & -76 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & -76 & 140 & -84 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & -84 & 176 & -92 \\ 0 & 0 & 0 & 0 & 0 & 0 & 11 & -92 & 81 \end{pmatrix}$$

We get the solution as shown in Table (3.14)

Table 3.14: Results for equation $-\frac{d}{dx}\left((2x+1)\frac{du}{dx}\right) = x^2$ for Four Element.

x values	Exact values (u)	Linar Finite values (U)	Error (u-U)
0	-0.516638	-0.521588	0.00495024
0.125	-0.405083	-0.410112	0.00502911
0.25	-0.314139	-0.319118	0.00497899
0.375	-0.237867	-0.24268	0.00481295
0.5	-0.172985	-0.177631	0.00464633
0.625	-0.117608	-0.12152	0.00391259
0.75	-0.0706532	-0.0739058	0.00325257
0.875	-0.0315377	-0.0330893	0.00155158
1	0	0	0

Similarly we can find the solution for eight and sixteen elements.

Chapter 4

Recovery of the Solution to the first order Problem

In this chapter we describe how we can accomplish the solution of first order differential equation i.e. recovery of velocity. In previous chapter we mentioned the approach that replaced the velocity (y) with the velocity potential (u) as $y = \frac{du}{dx}$ and solved it for U. Now we describe the approach that gives us Y from U which is an approximation to the exact solution.

4.1 Discontinuous Solution (by differentiation)

- **Linears**
  We can apply different approaches to find the apprximated velocity vector (Y). One way is to get the Y values from U for each element as follows

$$Y_i = \frac{du}{dx} = \frac{U_{i+1} - U_i}{x_{i+1} - x_i}$$

  Here U is piecewise linear and the Y function is piecewise constant. So Y is not continuous. We need to look for another way to go from U to Y (=dU$_{\overline{dx)}}$ which gives us a continuous function which we also discuss in the next section.

- **Quadratics**
  In this case recovery of Y from U by differentiation also gives us discontinuous func-

tion. So we need to adapt these values for each node to get continuous function which we discuss in next section.

4.2 Continuous Solution

Consider the following method to find the continuous solution for linear finite elements,

4.2.1 Least Squares for Linears

Let's consider the following least square approach

$$\left\| y - \frac{dU}{dx} \right\|^2 = \int_0^1 \left(Y - \frac{dU}{dx} \right)^2 dx$$

Minimise it over Y, where $Y = \sum \bar{y}_j \phi_j$ is continuous, requiring minimization of

$$\int_0^1 \left(\sum_{j=0}^{1} \bar{y}_j \phi_j - \frac{dU}{dx} \right)^2 dx$$

. Minimise over $\bar{y}$ values,

$$\frac{d}{d\bar{y}_i} \left(\int_0^1 \sum_{j=0}^{1} \bar{y}_j \phi_j - \frac{dU}{dx} \right)^2 dx = 0$$

and generally, it is

$$\int_0^1 \left(\sum_j \bar{y}_j \phi_j - \frac{dU}{dx} \right) \phi_i dx = 0$$

The above equation can be written in the form

$$\sum_j \left(\int_0^1 \phi_i \phi_j dx \right) \bar{y}_j - \int_0^1 \phi_i \frac{dU}{dx} dx = 0$$

and in matrix form as

$$M \underline{\bar{y}} = \underline{g}$$

where M is the element mass matrix. For one element $M^e = h \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}$, $\underline{\bar{y}}$ is a velocity vector and

$$g_i^e = \frac{dU}{dx}\int_0^1 \phi_i dx = \frac{1}{2}h\frac{dU}{dx} = \begin{pmatrix} \frac{1}{2}(\frac{U_1-U_0}{x_1-x_0}) \\ \frac{1}{2}(\frac{U_{i+1}-U_i}{x_{i+1}-x_i}) \\ \dots \\ \frac{1}{2}(\frac{U_n-U_{n-1}}{x_n-x_{n-1}}) \end{pmatrix}$$

We can get an approximation to the velocity vector by applying the approach discussed above. There are some solutions in section (4.3) for the test problems discussed in chapter 3.

4.2.2 For Quadratics

After finding the values of $\underline{U}$ in chapter 3, we need to find the $\underline{Y}$ values i.e $y \approx Y$ for each element, see the following

$$Y_i = \sum_{j=1}^{3} U_i \frac{\partial N_j(x_i)}{\partial x}$$

where $i = (0, 1, 2)$.
Each element gives us three values of Y but after the first element we use only two values for each element till the last element because we already have a value for the first node in each element. Then we combine together to get the approximate solution for the velocity vector, which is continuous. We give some solutions for the test problems from chapter 3 in section (4.4).

4.3 Linear Elements

We show the results for linear finite elements for test problem 1 and 2.

1. **Linear Solution for First Test Problem**
   The tables (4.1), (4.2), (4.3) and (4.4) show the results for velocity y and the numerical solution for the velocity Y by using linear finite elements for 2, 4, 8 and 16 elements.

Table 4.1: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 2 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x^2)	Y=LFE(f(x)=x^2)	Error(x^2)
0	1	0.973958	0.0260417	1	0.973958	0.0260417
0.5	1.125	1.08333	0.0416667	1.04167	1.08333	-0.0416667
1	1.5	1.19271	0.307292	1.33333	1.19271	0.140625

Table 4.2: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 4 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x^2)	Y=LFE(f(x)=x^2)	Error(x^2)
0	1	0.998512	0.0014881	1	0.998512	0.0014881
0.25	1.03125	1.00688	0.0243676	1.00521	1.00688	-0.00167411
0.5	1.125	1.03646	0.0885417	1.04167	1.03646	0.00520833
0.75	1.28125	1.15978	0.121466	1.14063	1.15978	-0.0191592
1	1.5	1.2619	0.238095	1.33333	1.2619	0.0714286

Table 4.3: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 8 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x^2)	Y=LFE(f(x)=x^2)	Error(x^2)
0	1	0.999904	9.58824e-005	1	0.999904	9.58824e-005
0.125	1.00781	1.00068	0.00713245	1.00065	1.00068	-2.90044e-005
0.25	1.03125	1.00519	0.0260618	1.00521	1.00519	2.01353e-005
0.375	1.07031	1.01763	0.0526828	1.01758	1.01763	-5.15368e-005
0.5	1.125	1.04148	0.0835193	1.04167	1.04148	0.000186012
0.625	1.19531	1.08207	0.11324	1.08138	1.08207	-0.000692511
0.75	1.28125	1.13804	0.143209	1.14063	1.13804	0.00258403
0.875	1.38281	1.23295	0.149862	1.22331	1.23295	-0.00964361
1	1.5	1.29734	0.202657	1.33333	1.29734	0.0359904

Table 4.4: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 16 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x^2)	Y=LFE(f(x)=x^2)	Error(x^2)
0	1	2.366	-1.366	1	2.366	-1.366
0.0625	1.00195	0.634063	0.36789	1.00008	0.634063	0.366018
0.125	1.00781	1.09873	-0.0909128	1.00065	1.09873	-0.0980742
0.1875	1.01758	0.975918	0.0416598	1.0022	0.975918	0.0262789
0.25	1.03125	1.01225	0.0190003	1.00521	1.01225	-0.00704141
0.3125	1.04883	1.00829	0.0405423	1.01017	1.00829	0.00188673
0.375	1.07031	1.01808	0.0522289	1.01758	1.01808	-0.000505516
0.4375	1.0957	1.02778	0.067925	1.02791	1.02778	0.000135334
0.5	1.125	1.0417	0.0832975	1.04167	1.0417	-3.58178e-005
0.5625	1.1582	1.05932	0.0988849	1.05933	1.05932	7.9377e-006
0.625	1.19531	1.08138	0.113936	1.08138	1.08138	4.067e-006
0.6875	1.23633	1.10834	0.127987	1.10832	1.10834	-2.42057e-005
0.75	1.28125	1.14053	0.140718	1.14063	1.14053	9.27558e-005
0.8125	1.33008	1.17914	0.150939	1.17879	1.17914	-0.000346818
0.875	1.38281	1.22201	0.1608	1.22331	1.22201	0.00129451
0.9375	1.43945	1.27949	0.159964	1.27466	1.27949	-0.00483124
1	1.5	1.3153	0.184697	1.33333	1.3153	0.0180304

2. **Linear Solution for Second Test problem**
We fixed the error occured in finding the solution of the second test problem by applying linear finite elements for moving mesh. So the following tables (4.5), (4.6), (4.7) and (4.8) show the results for exact velocity y and approximated velocity Y recovered from the velocity ptential U. All the tables are self explanatory, showing results for 2, 4, 8, and 16 elements.

Table 4.5: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 2 elements.

x values	Exact values (y)	Linear values (Y)	Error(y-Y)
0	1	0.740278	0.259722
0.5	0.479167	0.498611	-0.0194444
1	0.222222	0.256944	-0.0347222

Table 4.6: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 4 elements.

x values	Exact values (y)	Linear values (Y)	Error(y-Y)
0	1	0.854184	0.145816
0.25	0.663194	0.688506	-0.025312
0.5	0.479167	0.469468	0.00969817
0.75	0.34375	0.334909	0.00884075
1	0.222222	0.25371	-0.0314879

Table 4.7: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 8 elements.

x values	Exact values (y)	Linear values (Y)	Error(y-Y)
0	1	0.925543	0.074457
0.125	0.799479	0.815147	-0.0156674
0.25	0.663194	0.656595	0.00659966
0.375	0.561384	0.561588	-0.000204438
0.5	0.479167	0.478101	0.00106524
0.625	0.408275	0.407409	0.000866366
0.75	0.34375	0.344444	-0.000693876
0.875	0.282434	0.277335	0.00509873
1	0.222222	0.239135	-0.0169126

Table 4.8: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 16 elements.

x values	Exact values (y)	Linear values (Y)	Error(y-Y)
0	1	0.96295	0.03705
0.0625	0.888817	0.897572	-0.00875548
0.125	0.799479	0.796498	0.00298088
0.1875	0.725675	0.726042	-0.000367152
0.25	0.663194	0.662797	0.00039698
0.3125	0.609125	0.609022	0.000102929
0.375	0.561384	0.561263	0.000120506
0.4375	0.518446	0.518373	7.33262e-005
0.5	0.479167	0.479111	5.58403e-005
0.5625	0.44267	0.44263	4.04946e-005
0.625	0.408275	0.408249	2.60083e-005
0.6875	0.375445	0.375412	3.36574e-005
0.75	0.34375	0.343778	-2.81752e-005
0.8125	0.312841	0.312661	0.000180272
0.875	0.282434	0.283046	-0.000612277
0.9375	0.252293	0.249956	0.00233675
1	0.222222	0.230896	-0.00867367

4.4 Quadratic Elements

In this section we recover velocity Y from U by using the quadratic finite elements. We are trying to solve by increasing elements each time. Let us consider the the solution as follows

1. **Quadratic Solution for First Test Problem**

   The recovery of velocity from velocity potential is shown by Table (4.9), (4.10), (4.11) and (4.12) for 2, 4, 8 and 16 elements by quadratic approach.

Table 4.9: Results by solving equation $\frac{dy}{dx}$ with $f(x) = x$ and $f(x) = x^2$ for 2 elements.

x	y=Exact(f(x)=x)	Y=QFE(f(x)=x)	Error(x)	y=Exact (f(x)=x^2)	Y=QFE(f(x)=x^2)	Error(x^2)
0	1	0.979167	0.0208333	1	0.991667	0.00833333
0.25	1.03125	1.04167	-0.0104167	1.00521	1.01042	-0.00520833
0.5	1.125	1.10417	0.0208333	1.04167	1.02917	0.0125
0.75	1.28125	1.25	0.03125	1.14063	1.1375	0.003125
1	1.5	1.4375	0.0625	1.33333	1.28125	0.0520833

Table 4.10: Results by solving equation $\frac{dy}{dx}$ with $f(x) = x$ and $f(x) = x^2$ for 4 Elements.

x	y=Exact (f(x)=x)	Y=QFE (f(x)=x)	Error (x)	y=Exact (f(x)=x^2)	Y=QFE (f(x)=x^2)	Error (x^2)
0	1	0.994792	0.00520833	1	0.998958	0.00104167
0.125	1.00781	1.01042	-0.00260417	1.00065	1.0013	-0.000651042
0.25	1.03125	1.02604	0.00520833	1.00521	1.00365	0.0015625
0.375	1.07031	1.0625	0.0078125	1.01758	1.01562	0.00195313
0.5	1.125	1.10937	0.015625	1.04167	1.03359	0.00807292
0.625	1.19531	1.16667	0.0286458	1.08138	1.07057	0.0108073
0.75	1.28125	1.24479	0.0364583	1.14063	1.11979	0.0208333
0.875	1.38281	1.38542	-0.00260417	1.22331	1.19062	0.0326823
1	1.5	1.41667	0.0833333	1.33333	1.28672	0.0466146

Table 4.11: Results by solving equation $\frac{dy}{dx} with$f(x)=xandf(x)=x^2 for 8 Elements.

x	y=Exact (f(x)=x)	Y=QFE (f(x)=x)	Error (x)	y=Exact (f(x)=x^2)	Y=QFE (f(x)=x^2)	Error (x^2)
0	1	0.998698	0.00130208	1	0.99987	0.000130208
0.0625	1.00195	1.0026	-0.000651042	1.00008	1.00016	-8.13802e-005
0.125	1.00781	1.00651	0.00130208	1.00065	1.00046	0.000195312
0.1875	1.01758	1.01562	0.00195313	1.0022	1.00215	4.88281e-005
0.25	1.03125	1.02734	0.00390625	1.00521	1.00439	0.000813802
0.3125	1.04883	1.04167	0.00716146	1.01017	1.00804	0.00213216
0.375	1.07031	1.0612	0.00911458	1.01758	1.01419	0.00338542
0.4375	1.0957	1.08073	0.014974	1.02791	1.02305	0.00486654
0.5	1.125	1.10807	0.0169271	1.04167	1.03506	0.00660807
0.5625	1.1582	1.13281	0.0253906	1.05933	1.04945	0.00987956
0.625	1.19531	1.16797	0.0273438	1.08138	1.06927	0.0121094
0.6875	1.23633	1.19792	0.0384115	1.10832	1.09049	0.0178223
0.75	1.28125	1.24089	0.0403646	1.14063	1.12008	0.0205404
0.8125	1.33008	1.25911	0.0709635	1.17879	1.14945	0.0293457
0.875	1.38281	1.28451	0.0983073	1.22331	1.19076	0.0325521
0.9375	1.43945	1.36458	0.0748698	1.27466	1.22956	0.0451009
1	1.5	1.38021	0.119792	1.33333	1.28454	0.0487956

Table 4.12: Results by solving equation $\frac{dy}{dx}$ with $f(x) = x$ and $f(x) = x^2$ for 16 Elements.

x	y=Exact (f(x)=x)	Y=QFE (f(x)=x)	Error (x)	y=Exact (f(x)=x^2)	Y=QFE (f(x)=x^2)	Error (x^2)
0	1	0.999674	0.000325521	1	0.999984	1.6276e-005
0.03125	1.00049	1.00065	-0.00016276	1.00001	1.00002	-1.01725e-005
0.0625	1.00195	1.00163	0.000325521	1.00008	1.00006	2.44141e-005
0.09375	1.00439	1.00391	0.000488281	1.00027	1.00027	6.10352e-006
0.125	1.00781	1.00684	0.000976563	1.00065	1.00055	0.000101725
0.15625	1.01221	1.01042	0.00179036	1.00127	1.00113	0.00014445
0.1875	1.01758	1.0153	0.00227865	1.0022	1.0019	0.000301107
0.21875	1.02393	1.02018	0.00374349	1.00349	1.003	0.000486247
0.25	1.03125	1.02702	0.00423177	1.00521	1.0045	0.000703939
0.28125	1.03955	1.0332	0.00634766	1.00742	1.0063	0.00111287
0.3125	1.04883	1.04199	0.00683594	1.01017	1.00878	0.0013916
0.34375	1.05908	1.04948	0.00960286	1.01354	1.01143	0.00210571
0.375	1.07031	1.06022	0.0100911	1.01758	1.01513	0.00244548
0.40625	1.08252	1.08464	-0.00211589	1.02235	1.0188	0.00354614
0.4375	1.0957	1.09733	-0.0016276	1.02791	1.02397	0.00394694
0.46875	1.10986	1.10742	0.00244141	1.03433	1.02882	0.00551554
0.5	1.125	1.12207	0.00292969	1.04167	1.03569	0.00597738
0.53125	1.14111	1.13912	0.00198851	1.04998	1.04188	0.0080953
0.5625	1.1582	1.15573	0.00247679	1.05933	1.05071	0.00861816
0.59375	1.17627	1.16842	0.00784788	1.06977	1.05841	0.0113668
0.625	1.19531	1.18698	0.00833616	1.08138	1.06943	0.0119507
0.65625	1.21533	1.20097	0.0143583	1.09421	1.0788	0.0154114
0.6875	1.23633	1.22148	0.0148466	1.10832	1.09226	0.0160563
0.71875	1.2583	1.23678	0.0215198	1.12377	1.10346	0.0203105
0.75	1.28125	1.25924	0.022008	1.14063	1.11961	0.0210164
0.78125	1.30518	1.27584	0.0293323	1.15895	1.1328	0.0261454
0.8125	1.33008	1.30026	0.0298205	1.17879	1.15188	0.0269124
0.84375	1.35596	1.31816	0.0377958	1.20023	1.16723	0.0329976
0.875	1.38281	1.34453	0.0382841	1.22331	1.18948	0.0338257
0.90625	1.41064	1.36373	0.0469104	1.2481	1.20715	0.0409485
0.9375	1.43945	1.39205	0.0473987	1.27466	1.23282	0.0418376
0.96875	1.46924	1.41256	0.056676	1.30305	1.25297	0.0500793
1	1.5	1.44284	0.0571643	1.33333	1.2823	0.0510295

2. **Quadratic Solution for Second Test Problem**

   The following tables (4.13), (4.14), (4.15) and (4.16) give us the results for 2, 4, 8 and 16 elements by using quadratic approach.

Table 4.13: Results by solving equation $-\frac{d}{dx}\left((2x+1)y\right) = x^2$ for 2 elements.

x values	Exact values (y)	Quadratic values (Y)	Error(y-Y)
0	1	0.927885	0.0721154
0.25	0.663194	0.686538	-0.023344
0.5	0.479167	0.445192	0.0339744
0.75	0.34375	0.353547	-0.0097973
1	0.222222	0.225338	-0.00311562

Table 4.14: Results by solving equation $-\frac{d}{dx}\left((2x+1)y\right) = x^2$ for 4 elements.

x values	Exact values (y)	Quadratic values (Y)	Error(y-Y)
0	1	0.973733	0.0262669
0.125	0.799479	0.809882	-0.0104026
0.25	0.663194	0.64603	0.017164
0.375	0.561384	0.565946	-0.00456213
0.5	0.479167	0.474829	0.0043379
0.625	0.408275	0.414902	-0.0066264
0.75	0.34375	0.346927	-0.00317665
0.875	0.282434	0.295623	-0.0131896
1	0.222222	0.233805	-0.011583

Table 4.15: Results by solving equation $-\frac{d}{dx}\left((2x+1)y\right) = x^2$ for 8 elements.

x values	Exact values (y)	Quadratic values (Y)	Error(y-Y)
0	1	0.991845	0.0081547
0.0625	0.888817	0.892426	-0.00360984
0.125	0.799479	0.793007	0.00647168
0.1875	0.725675	0.727765	-0.00208987
0.25	0.663194	0.659971	0.0032238
0.3125	0.609125	0.61174	-0.00261577
0.375	0.561384	0.560897	0.000486572
0.4375	0.518446	0.521957	-0.00351112
0.5	0.479167	0.480754	-0.00158723
0.5625	0.44267	0.448019	-0.00534887
0.625	0.408275	0.412336	-0.00406035
0.6875	0.375445	0.383522	-0.00807682
0.75	0.34375	0.350878	-0.007128
0.8125	0.312841	0.324515	-0.0116744
0.875	0.282434	0.293326	-0.010892
0.9375	0.252293	0.268426	-0.0161336
1	0.222222	0.237632	-0.0154098

Table 4.16: Results by solving equation $-\frac{d}{dx}\left((2x+1)y\right) = x^2$ for 16 elements.

x values	Exact values (y)	Quadratic values (Y)	Error(y-Y)
0	1	0.997705	0.00229458
0.03125	0.941167	0.942245	-0.00107789
0.0625	0.888817	0.886784	0.0020324
0.09375	0.841874	0.842661	-0.000787368
0.125	0.799479	0.798074	0.00140485
0.15625	0.760936	0.761631	-0.000695042
0.1875	0.725675	0.724777	0.000897804
0.21875	0.693225	0.694016	-0.000790709
0.25	0.663194	0.662797	0.000397904
0.28125	0.635254	0.636327	-0.00107276
0.3125	0.609125	0.609288	-0.000163062
0.34375	0.584569	0.586112	-0.00154289
0.375	0.561384	0.562212	-0.000828374
0.40625	0.539394	0.541598	-0.0022039
0.4375	0.518446	0.520073	-0.00162671
0.46875	0.498409	0.501468	-0.00305881
0.5	0.479167	0.481744	-0.00257778
0.53125	0.460617	0.464727	-0.00411046
0.5625	0.44267	0.446366	-0.00369557
0.59375	0.425246	0.430608	-0.00536141
0.625	0.408275	0.413266	-0.00499026
0.65625	0.391694	0.398508	-0.00681386
0.6875	0.375445	0.381915	-0.00646945
0.71875	0.359479	0.367949	-0.00846977
0.75	0.34375	0.351889	-0.00813891
0.78125	0.328216	0.338547	-0.0103308
0.8125	0.312841	0.322844	-0.0100031
0.84375	0.29759	0.309989	-0.0123983
0.875	0.282434	0.294499	-0.0120656
0.90625	0.267343	0.282017	-0.0146735
0.9375	0.252293	0.266622	-0.0143291
0.96875	0.23726	0.254417	-0.0171575
1	0.222222	0.239018	-0.0167959

Chapter 5

Discussion

Comparison of the results.

5.1 Exact solution for y

- **First Problem**

  To find the exact solution of

$$\frac{dy}{dx} = f(x) \tag{5.1}$$

in $(0, 1)$, with $y = 1$ at $x = 0$. Let us consider $f(x) = x$ and integrate (5.1),giving

$$y = \frac{x^2}{2} + A$$

by applying boundary conditions, giving

$$y = \frac{x^2}{2} + 1 \tag{5.2}$$

the exact solution for y.

Now let us consider $y = \frac{du}{dx}$ and $\frac{du}{dx} = 1$ at $x = 0$ and $u = 0$ at $x = 1$, giving

$$\frac{du^2}{dx^2} = x \tag{5.3}$$

By integrating (5.3) twice and applying boundary conditions, giving

$$u = \frac{x^3}{3} + x - \frac{4}{3} \tag{5.4}$$

the exact solution for u.

Let us consider $f(x) = x^2$, giving

$$\frac{du^2}{dx^2} = x^2 \tag{5.5}$$

Now by integrating (5.5), the exact solution for y is

$$y = \frac{x^3}{3} + 1 \tag{5.6}$$

and for u is

$$u = \frac{x^4}{12} + x - \frac{13}{12} \tag{5.7}$$

- **Second Example**

  To solve the following equation

$$-\frac{d}{dx}\left((2x+1)y\right) = x^2 \tag{5.8}$$

in $(0,1)$ with $y = 1$ at $x = 0$ and integrating (5.8), giving

$$-(2x+1)y = \frac{x^3}{3} + B \tag{5.9}$$

by applying boundary condition, we get

$$y = \frac{3 - x^3}{6x + 3} \tag{5.10}$$

Consider $y = \frac{du}{dx}$ in equation (5.8), giving

$$-\frac{d}{dx}\left((2x+1)\frac{du}{dx}\right) = x^2 \tag{5.11}$$

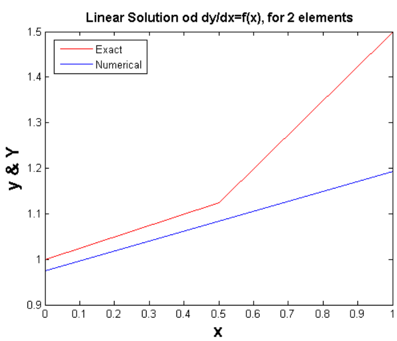
integrating (5.11) twice and applying boundary conditions, we get

$$u = -\frac{x^3}{18} + \frac{x^2}{24} + -\frac{x}{24} + \frac{25}{48} ln(2x+1) - 0.516638 \tag{5.12}$$

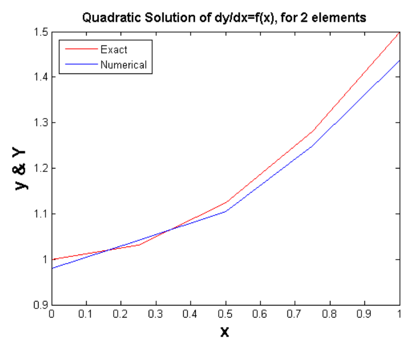
5.2 Results

5.2.1 Linear and Quadratic continuous Solution for y (Problem-1)

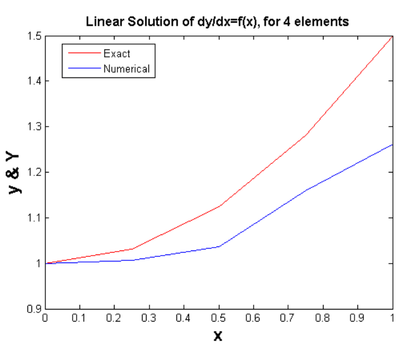
All the graphs shown below give us enough information to understand about the results. By comparison of linear and quadratic results it is clear that linear approach is not good as compared to quadratic in case of first test problem. Quadratic finite elements gives us better results.



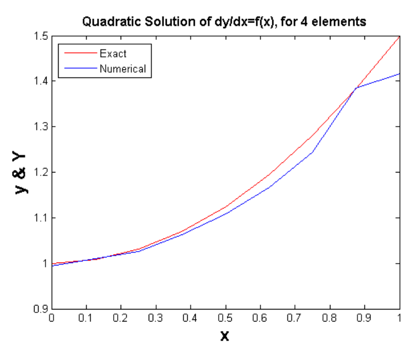
(a) Linear finite Elements for $f(x) = x$.



(b) Quadratic finite Elements for $f(x) = x$.

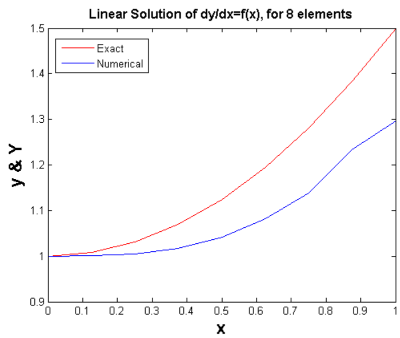


(c) Linear finite Elements for $f(x) = x$.

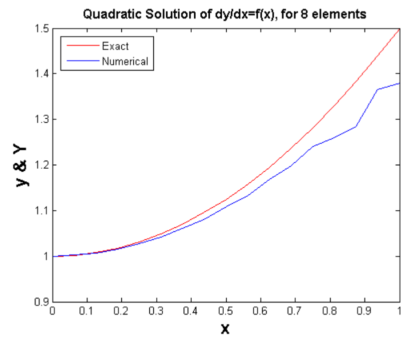


(d) Quadratic finite Elements for $f(x) = x$.

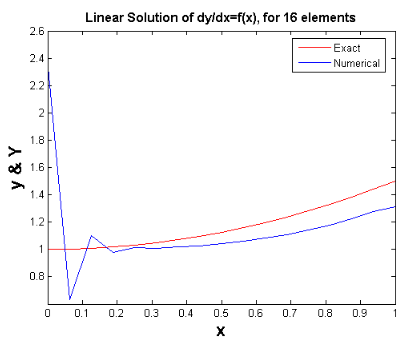
Figure 5.1: Graphs showing the results for linear and quadratic finite elements of first test problem for 2 and 4 elemets.



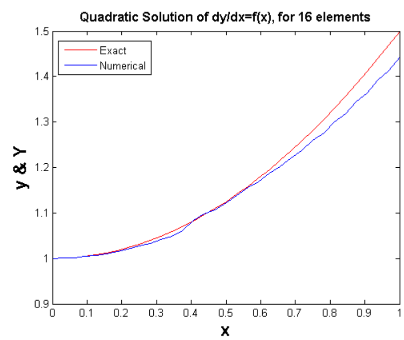
(a) Linear finite Elements for $f(x) = x$.



(b) Quadratic finite Elements for $f(x) = x$.

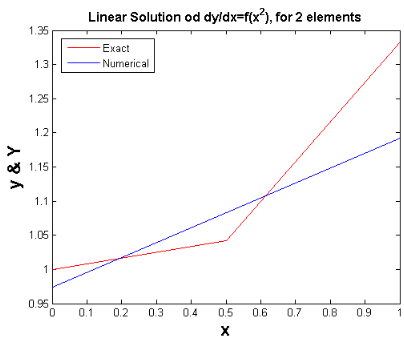


(c) Linear finite Elements for $f(x) = x$.

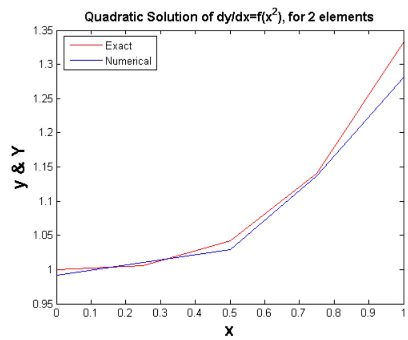


(d) Quadratic finite Elements for $f(x) = x$.

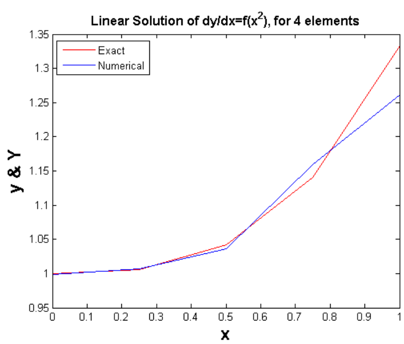
Figure 5.2: Graphs showing the results for linear and quadratic finite elements of first test problem for 8 and 16 elemets.



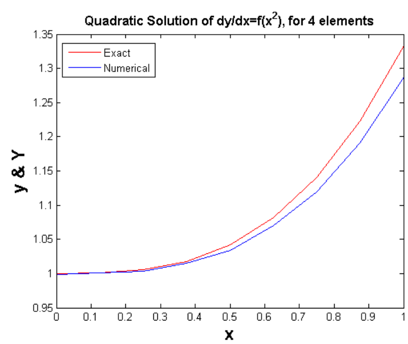
(a) Linear finite Elements for $f(x) = x^2$.



(b) Quadratic finite Elements for $f(x) = x^2$.

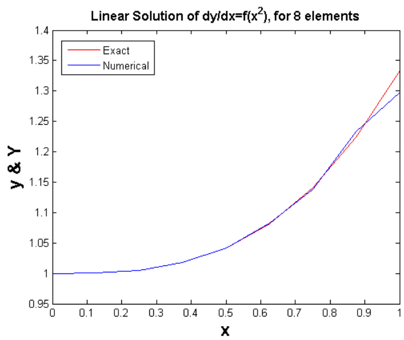


(c) Linear finite Elements for $f(x) = x^2$.

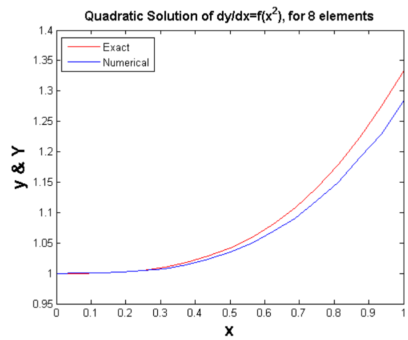


(d) Quadratic finite Elements for $f(x) = x^2$.

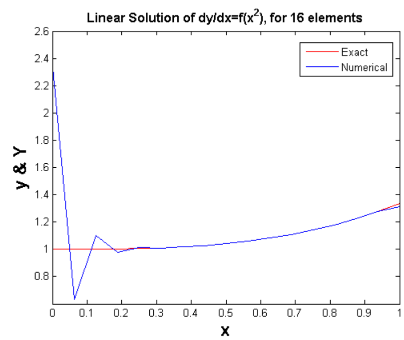
Figure 5.3: Graphs showing the results for linear and quadratic finite elements of first test problem for 2 and 4 elemets.



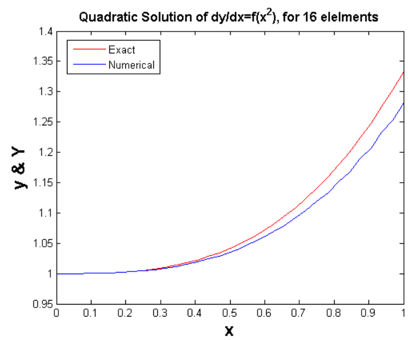
(a) Linear finite Elements for $f(x) = x^2$.



(b) Quadratic finite Elements for $f(x) = x^2$.



(c) Linear finite Elements for $f(x) = x^2$.



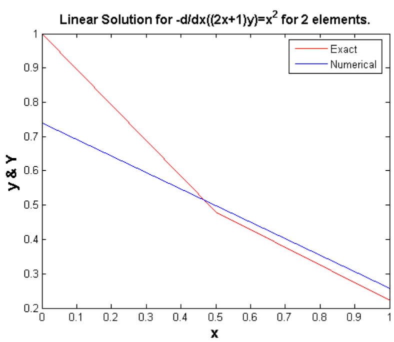
(d) Quadratic finite Elements for $f(x) = x^2$.

Figure 5.4: Graphs showing the results for linear and quadratic finite elements of first test problem for 8 and 16 elemets.

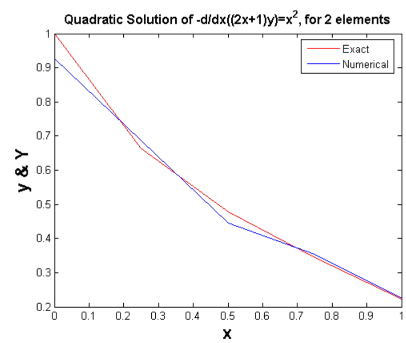
5.2.2 Linear and Quadratic continuous Solution for y (Problem-2)

The comparison of the results for linear and quadratic finite elements of second test problem tells us

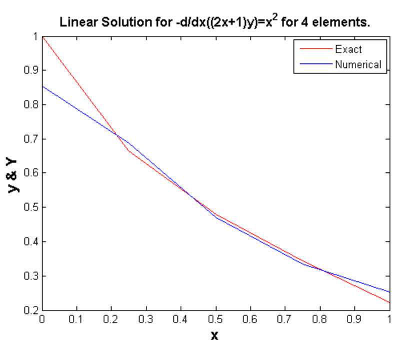
- Graphs show that numerical values for Y get better as we increase the number of elements.
- Linear results are really good except for end values for 16 elements.
- Quadratic results are better at the start of any number of elements.



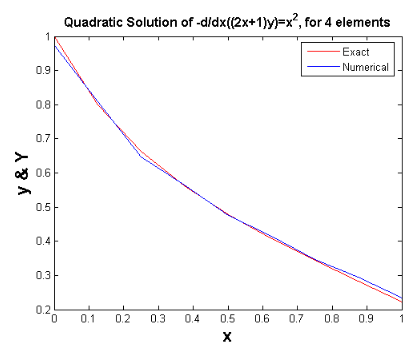
(a) Linear finite Elements.



(b) Quadratic finite Elements.

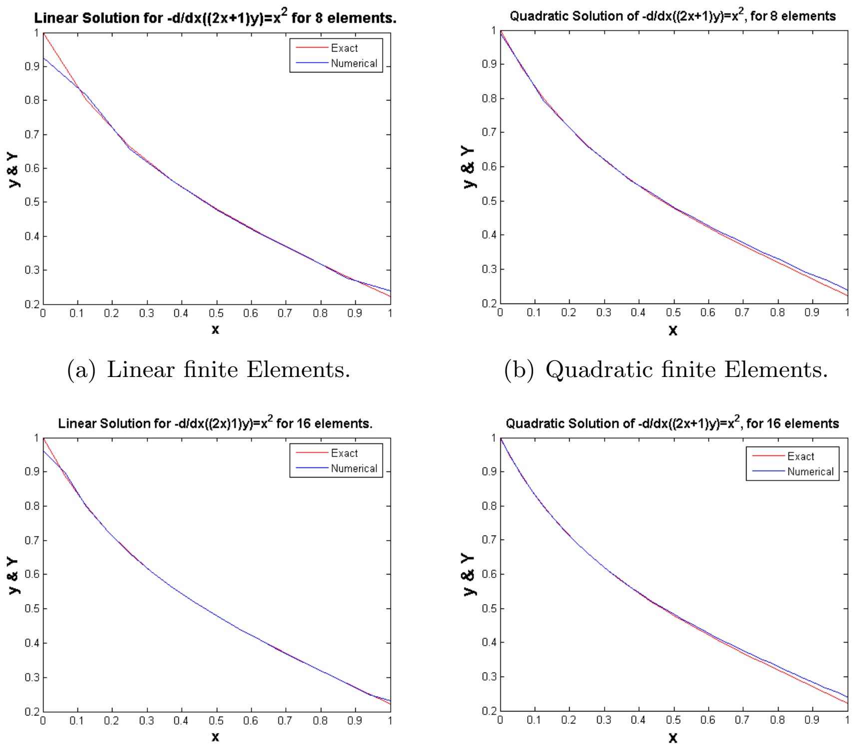


(c) Linear finite Elements.



(d) Quadratic finite Elements.

Figure 5.5: Graphs showing the results for linear and quadratic finite elements of second test problem for 2 and 4 elemets.



(a) Linear finite Elements. (b) Quadratic finite Elements.

(c) Linear finite Elements. (d) Quadratic finite Elements.

Figure 5.6: Graphs showing the results for linear and quadratic finite elements of second test problem for 8 and 16 elemets.

5.3 Conclusion

We have showed that when linear elements approach does not work very well to recover the values of velocity (Y) from potential velocity (U), then we need to apply quadratic finite elements approach to get better accuracy.

Chapter two gave the theory of finite elements for second order differential equations. This chapter investigated the method of Linear Finite elements and deficencies in this method for our purpose and provided an alternative Quadratic elements method to find the numerical solution.

Chapter Three provided the Linear and Quadratic approaches to solve the first order differential equations as well as the Sturn-Liouiville type differential equations. In this chapter we solved test problems to investigate the numerical results.

Chapter Four introduced the results for moving boundary and discussed the possible behaviour that can arise as the boundary moves. We also discussed the numerical results of the test problem and compared them with the exact solutions to investigate the errors.

5.4 Future Work

Our next target is to find the solutions for higher order differential equations.

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