COMPARISON OF SOME SINGLE-STEP METHODS FOR THE NUMERICAL SOLUTION OF THE STRUCTURAL DYNAMIC EQUATION

by

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NUMERICAL ANALYSIS REPORT 10/84

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Submitted to Communications in Applied Numerical Methods.

SUMMARY

This paper compares the performance of the SS22 and SS23 (References 1,2) single step algorithms for the numerical solution of the second order structural dynamic equation and a related new algorithm SS32B applied to the equivalent first order system, with sine and step forcing functions. Various aspects of stability relevant to these equations are discussed.

We consider the structural dynamic system given by the N equations

$$M\ddot{\underline{x}} + C\dot{\underline{x}} + K\underline{x} = \underline{f}(t) \tag{1}$$

arising from the finite element discretization of a structure. M, C and K are the mass, damping and stiffness matrices respectively, $\underline{x}$ is the vector of displacements and $\underline{f}(t)$ is the forcing function. We make the usual assumption of Rayleigh damping (i.e. the matrix C is some linear combination of the matrices M and K), then, since the matrices M and K are symmetric and positive definite due to their finite element origin, we can use the theory of Wilkinson [3] to show that M, C and K effectively have a common complete set of eigenvectors $\underline{u}_r$, $r = 1,2,\ldots,N$. Hence we can make a modal decomposition and show that the exact solution of the system of equations (1) is

$$\underline{x} = \sum_{r=1}^{N} \alpha_r \exp(\lambda_r t)\, \underline{u}_r \tag{2}$$

where λ_r satisfies

$$m_r\lambda_r^2 + \mu_r\lambda_r + k_r = 0, \quad r = 1,2,\ldots,N \tag{3}$$

and

$$M\underline{u}_r = m_r\underline{u}_r, \quad C\underline{u}_r = \mu\underline{u}_r, \quad K\underline{u}_r = k_r\underline{u}_r \quad . \tag{4}$$

Another approach to the solution of the system of equations (1) is to reduce it to a first order system as in, for example, references [4], [5] by putting $\underline{v} = \dot{\underline{x}}$ and rewriting the equations as

$$\underline{\dot{w}} = A\underline{w} + \underline{F} \tag{5}$$

where $\underline{w} = \begin{bmatrix} \underline{v} \\ \underline{x} \end{bmatrix}$, $\underline{F} = \begin{bmatrix} \underline{f} \\ \underline{0} \end{bmatrix}$ and A is the $2N \times 2N$ matrix

$$\begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix} \tag{6}$$

The matrix A has the $2N$ eigenvalues given by the $2N$ roots of the N quadratics in equation (3).

This means that if we suppose that the damping in each mode is less than critical the matrix A has $2N$ eigenvalues in complex conjugate pairs given by

$$\lambda_r = \sqrt{\frac{kr}{m_r}} \;\exp\,[i(\pi \pm \alpha_r)], \quad i = \sqrt{-1} \tag{7}$$

where $\tan \alpha_r = \dfrac{\sqrt{4k_r m_r - \mu_r^2}}{\mu_r} = \dfrac{\sqrt{1 - \nu_r^2}}{\nu_r}$

and

$$\nu_r = \frac{\mu_r}{2\sqrt{k_r m_r}} \tag{8}$$

i.e. ν_r gives the damping as a fraction of the critical damping $2\sqrt{k_r m_r}$.

These $2N$ eigenvalues of the matrix A lie in the Argand diagram shown in Figure 1 in the wedge given by $\arg(-\lambda) \leqq \alpha$ where $\alpha = \max_r\{\alpha_r\}$.

Equation (1) can be integrated numerically by the single step method described in references [1], [2]. Here the function $\underline{x}(t)$ is approximated as a polynomial of degree p in time t, $p \geqq 2$, then $\underline{x}(t)$, $\underline{\dot{x}}(t)$ and $\underline{\ddot{x}}(t)$ are substituted into the Weighted Residual equation

$$\int_t^{t+\Delta t} W(t)[M\underline{\ddot{x}} + C\underline{\dot{x}} + K\underline{x} - \underline{f}]dt = \underline{0} \tag{9}$$

We write

$$\int_t^{t+\Delta t} W(t)t^q dt = \theta_q \Delta t^q \int_t^{t+\Delta t} W(t)dt, \quad q = 1,2,\dots,p \tag{10}$$

Thus an algorithm is formed with p parameters θ_q which can be chosen to give various stability and accuracy properties. This algorithm is described in detail in references [1] and [2].

We here introduce the related single step algorithm SSp2B which is the result of the same approach applied to the first order system of equations (5). The notation $\overset{p}{X}$ is used for the pth. derivative with respect to time as in reference [1].

SSp2B

(i)

$$\tilde{\underline{X}}_n = \underline{X}_n + \dot{\underline{X}}_n \theta_1 t + \dots + \overset{p-1}{\underline{X}}_n \theta_{p-1} \frac{\Delta t^{p-1}}{(p-1)!}$$

$$\tilde{\dot{\underline{X}}}_n = \dot{\underline{X}}_n + \underline{X}_n \theta_1 \Delta t + \dots + \overset{p-1}{\underline{X}}_n \theta_{p-2} \frac{\Delta t^{p-2}}{(p-2)!}, \quad p > 1; \quad \text{if } p = 1 \ \hat{\dot{\underline{X}}} = 0$$

$$\tilde{\underline{V}}_n = \underline{V}_n + \dot{\underline{V}}_n \theta_1 \Delta t + \dots + \overset{p-1}{\underline{V}}_n \theta_{p-1} \frac{\Delta t^{p-1}}{(p-1)!}$$

$$\tilde{\dot{\underline{V}}}_n = \dot{\underline{V}}_n + \ddot{\underline{V}}_n \theta_1 \Delta t + \dots + \overset{p-1}{\underline{V}}_n \theta_{p-2} \frac{\Delta t^{p-2}}{(p-2)!}, \quad p > 1; \quad \text{if } p = 1 \ \tilde{\dot{\underline{V}}}_n = 0.$$

(ii) Calculate $\underline{\beta}_n^p$ from

$$\left[\frac{\theta_{p-1}}{(p-1)!}\Delta t^{p-1}\underline{M} + \theta_p \frac{\Delta t^p}{p!}\underline{C} + \frac{\theta_p^2 \Delta t^{p+1}}{\theta_{p-1} p \cdot p!}\underline{K}\right]\underline{\beta}_n^p$$

$$= -\underline{M}\,\tilde{\dot{\underline{V}}}_n - \underline{C}\,\tilde{\underline{V}}_n - \underline{K}\left[\tilde{\underline{X}}_n + \frac{\theta_p \Delta t}{p\theta_{p-1}}(\tilde{\underline{V}}_n - \tilde{\dot{\underline{X}}}_n)\right] + \underline{F}.$$

(iii) Calculate $\underline{\alpha}_n^p$ from

$$\underline{\alpha}_n^p = \left[\frac{\theta_{p-1}\Delta t^{p-1}}{(p-1)!}\right]^{-1}\left[\underline{\beta}_n^p \frac{\theta_p \Delta t^p}{p!} + (\tilde{\underline{V}}_n - \tilde{\dot{\underline{X}}}_n)\right].$$

(iv) Find $\underline{X}_{n+1}$, $\dot{\underline{X}}_{n+1}$, $\underline{V}_{n+1}$, $\dot{\underline{V}}_{n+1}$ by substituting $\underline{\alpha}_n^p$, $\underline{\beta}_n^p$ into:-

$$\underline{X}_{n+1} = \underline{X}_n + \dot{\underline{X}}_n \Delta t + \ldots + \overset{p-1}{\underline{X}_n} \frac{\Delta t^{p-1}}{(p-1)!} + \underline{\alpha}_n^p \frac{\Delta t^p}{p!}$$

$$\dot{\underline{X}}_{n+1} = \dot{\underline{X}}_n + \ddot{\underline{X}}_n \Delta t + \ldots + \underline{\alpha}_n^p \frac{\Delta t^{p-1}}{(p-1)!}$$

$$\underline{V}_{n+1} = \underline{V}_n + \dot{\underline{V}}_n \Delta t + \ldots + \overset{p-1}{\underline{V}_n} \frac{\Delta t^{p-1}}{(p-1)!} + \underline{\beta}_n^p \frac{\Delta t^p}{p!}$$

$$\dot{\underline{V}}_{n+1} = \dot{\underline{V}}_n + \ddot{\underline{V}}_n \Delta t + \ldots + \underline{\beta}_n^p \frac{\Delta t^{p-1}}{(p-1)!} \quad .$$

(v) Repeat for next time step.

We note that the matrix to be inverted in step (ii) is of the same order as that in SSp2 and the extra equations in step (iii) are scalar.

In order to explain the motivation for introducing the algorithm SSp2B we list some different kinds of stability associated with the numerical solution of equations (1) or (5).

(i) A method is A_0 stable [6] if it is stable when these equations have all eigenvalues real and negative i.e. when all roots of equation (3) are real and negative. This would mean all modes of the structure over-damped. In the Argand diagram in Figure 1 all the eigenvalues would then lie on the negative real axis.

(ii) A method is A-stable [7] if it is stable when the equations have eigenvalues with negative real parts. This means that the modes can be under-damped and it does not matter how low the damping is. In Figure 1 then the eigenvalues lie in the left hand half plane.

(iii) A method is $A(\alpha)$ stable [8] if it is stable for a system of equations where the eigenvalues now lie in the wedge of angle 2α shown in Figure 1.

Another way at looking at $A(\alpha)$ stability is in relation to the stability boundary locus [9]. For example, Figure 1 shows the Argand diagram with the stability boundary locus for the Gear 3-step method [10]. This method is stable when used for equation (5) when the eigenvalues of the matrix A are outside the stability boundary. Since this boundary locus encroaches on the left hand half plane this method is clearly not A-stable but as shown in the Appendix by calculating the tangents to this curve from the origin we can obtain the equivalent $A(\alpha)$ stability region. Here we have $\tan\alpha = 14.42$ i.e. $\alpha \approx 86^{0}$. Hence for $\nu > 7\%$ damping the Gear 3-step method for equation (5) is absolutely stable.

The single step algorithms SSp2 and SSp2B are equivalent to p-step methods and with the same starting values will give exactly the same results. Thus we can apply to SSp2 and SSp2B the following results due to Dahlquist [7]:

(i) An explicit linear multi-step method cannot be A-stable.

(ii) The order of the error of an A-stable implicit method cannot exceed two.

(iii) The A-stable linear multistep method with second order accuracy and the smallest error constant [9] is given by the Trapezium Rule. This corresponds to SS22 with $\theta_1 = \theta_2 = 0.5$. [2].

Now we are looking for a single step method where we can freely change the size of time step according to some error criterion, hence we want unconditional stability. We see from the Dahlquist results that SSp2 is limited to $O(\Delta t^2)$ error when it is A-stable. Widlund [8] gives conditions which can be translated into conditions on the parameters θ_q in SS32B dependent on the angle α, such that we have unconditional stability and error $O(\Delta t^3)$. Hence the motivation for introducing the algorithm SSp2B and comparing its performance with that of SSp2.

It was intended to use the Widlund conditions but in fact as shown later they do not seem to apply.

The algorithms SS22, SS32 and SS32B are applied to the numerical solution of the representative single degree of freedom equation

$$m\ddot{x} + \mu\dot{x} + kx = f(t) \tag{11}$$

with $m = k = 1$, $\mu = 2\nu$ i.e. ν = fraction of critical damping and

(i) Step function

$$\begin{aligned} f(t) &= 0, \quad t \leqq 0 \\ f(t) &= 1, \quad 0 < t < 25 \\ f(t) &= -1, \quad t \geqq 25 \end{aligned} \tag{12}$$

(ii) $f(t) = \sin(\pi t/20)$.

The exact solutions can be obtained using the Green's function [11].

RESULTS AND CONCLUSIONS

We present results for some particular examples of SS22, SS32 and SS32B. Values of the parameters θ_q are chosen so that the algorithms are single step equivalents of some well known p-step methods. Two methods of measuring the accuracy are used depending on the form of the forcing function. The time $t = 5$ is chosen as a typical point and the error measured here as $|x_n - x(5)|$ where $x(5)$ is the exact solution at $t = 5$ and x_n is the numerical solution at $t = n\Delta t = 5$. This error is also given as a percentage of the true solution. When the step function $f(t)$ is used we also give the "overshoot". With a forcing function which has a sudden jump the numerical solution tends to be least accurate just after the jump (See Figure 2). The numerical solution may under-estimate or overestimate the true solution. This error is measured as shown in Figures 2 and 3 and it is also given as a percentage of the true solution at the turning point (i.e. the minimum value of $x(t)$ after the jump as shown in Figures 2 and 3).

The initial conditions used are : $x(0) = \dot{x}(0) = 0$. The method of approximation of $\overline{F}$ is that suggested in Reference 1. : $\overline{F} = \theta_1 F_{n+1} + (1 - \theta_1)F_n$

1. SS22

It is shown in [2] that this is equivalent to the Newmark algorithm [12] with $\alpha = 2\beta$ where $\alpha = \theta_1$ and $2\beta = \theta_2$ i.e. $\theta_1 = \theta_2$

(a) With $\theta_1 = \theta_2 = 0.5$, SS22 is equivalent to using the trapezium rule [2], hence by the Dahlquist theorem it is the most accurate second order method. Tables 1 and 2 show the results for the step and since forcing functions. The "overshoot" as well as the error at $t = 5$ shows the $O(\Delta t^2)$ effect (error divided approximately by 4 when the time step is halved). The approximation to $\overline{F}$ is evidently quite sufficient. It is interesting that at low damping, $\nu = 0.1$, the overshoot error is less than that at $t = 5$.

(b) Although the SS22 algorithm with $\theta_1 = \theta_2 = 0.5$ is second order accurate the absence of numerical damping can be a disadvantage when we want to eliminate inaccurate high frequency modal components. The accuracy dies away gradually as θ_1 and θ_2 get further from $\theta_1 = \theta_2 = 0.5$. Hence $\theta_1 = 0.6$, $\theta_2 = 0.605$ are chosen to give a small amount of numerical damping and retain unconditional stability. Tables 3 and 4 show these results which are now first order accurate.

2. SS32

The SS32 algorithm needs three starting values, displacement, velocity and acceleration. The differential equation is used to generate $\ddot{x}(0) = f(0) - 2\nu\dot{x}(0) - x(0)$.

(a) $\theta_1 = 2$, $\theta_2 = 11/3$, $\theta_3 = 6$. This is the single step equivalent to Houbolt [13]; it is unconditionally stable and second order.

Tables 5 and 6 show the results for the step and sine functions respectively.

(b) $\theta_1 = 1.4$, $\theta_2 = 1.96$, $\theta_3 = 2.744$. This is the single step equivalent to Wilson-θ = 1.4 [14]; it is unconditionally stable and second order. Tables 7 and 8 show the results.

(c) $\theta_1 = 1.05$, $\theta_2 = 1.1$, $\theta_3 = 1.15$. This is the single step equivalent of the Bossak-Newmark method [15]. This method is only conditionally stable; for $\nu = .1$ it is stable for $\Delta t < 1.5$ approximately. It is interesting to see how well this performs compared with the SS22 (a) as well as with the Houbolt and Wilson-θ equivalents.

3. SS32B

It was originally intended to run SS32B using the conditions given by Widlund for $A(\alpha)$ stability [8]. With

$$a = \theta_1 - \tfrac{1}{2}, \quad b = \theta_2 - \theta_1 - 1/6, \quad c = \theta_3 - 3\theta_2/2 + 1/4 \tag{14}$$

the conditions given by Widlund for $A(\alpha)$ stability are equivalent to

$$c > 0, \quad 12a^2 > 1 \tag{15a}$$

and

$$b > \max\left\{4ac, \; \frac{(2c + a)^3 \tan^2\alpha}{3c(12a^2 - 1)}\right\} \tag{15b}$$

Conditions (15a) are almost the same as those obtained for the A-stability of SS32 in Reference 2 but there equality is possible. For $a = 1.5$, $c = 0.75$ the Widlund inequality (15b) gives $b > 4.5$ for $\nu = 0.5$ and $b > 45.7$ for $\nu = 0.1$. The error constant [9] which gives a measure of the accuracy of the method is numerically equal to $(2c + a)/24$. This error constant multiplies the Δt^3 term in the error but it is clear that when $\tan\alpha$ is large the large value of b implied by inquality (15b) can magnify the effect of later terms in the error. A separate investigation of the link between the Gear and Widlund approaches to the stability limits led to the realisation that it is easy to obtain the angle α for the Gear 3-step method [10] (See Appendix). Hence SS32B results are included here for the parameters which give the single step equivalent of the Gear 3-step namely $a = 1.5$, $b = 1.5$, $c = 0.75$. As shown in the Appendix these values give $A(\alpha)$ stability for $\tan \alpha = 14.42$ i.e. $\nu = 0.069$ approximately. Experiment has verified that this method remains stable with $\nu = 0.1$ and large values of Δt so that the large value of b implied by (15b) is not necessary.

The SS32B algorithm requires four starting values: $x(0)$, $\dot{x}(0)$, $\ddot{x}(0)$ and $\dddot{x}(0)$. $\ddot{x}(0)$ is generated as in the SS32 algorithm and $\dddot{x}(0)$ is generated by supposing that we can differentiate the differential equation to obtain $\dddot{x}(0) = \dot{f}(0) - 2\nu\ddot{x}(0) - \dot{x}(0)$.

The parameters $\theta_1 = 2$, $\theta_2 = 11/3$, $\theta_3 = 6$ (the same as in SS32(a) Houbolt) make SS32B equivalent to the Gear 3-step. The algorithm was initially run with zero forcing function to check that it gave third order accuracy. Tables 11 and 12 show the results with the step and sine functions. In Table 11 the error at $t = 5$ is approximately $O(\Delta t^3)$ as Δt gets smaller but the "overshoot" is not. The results in Table 12 make it clear that the approximation for F is only sufficient to give $O(\Delta t^2)$ accuracy. However, for both values of ν this version of SS32B gives overshoot values which compare favourably with SS32(a) Houbolt and for the higher damping it is better than SS32(b) Wilson-θ. The "overshoot" for both values of ν is an underestimate of the true response.

Thus the SS32B algorithm with the parameters used here is unconditionally stable for the damping ratio $\nu \geqq 7\%$ and we have $O(\Delta t^3)$ error provided the forcing function is approximated appropriately. Dynamic tests on a structure done by Galambos and Mayes [16], however, have shown that the damping can be as low as 4% which would necessitate $A(\alpha)$ stability with $\alpha \simeq 88^{\circ}$. With this kind of structure it would seem advisable to use SS32 with values of the parameters to give A-stability which corresponds to $\alpha = 90^{\circ}$. With SS32 we can have $O(\Delta t^2)$ error and choose parameters to damp out the inaccurate higher frequencies which is not possible with the second order SS22(a).

We note that the algorithm SS32 with $6(\theta_2 - \theta_1) + 1 = 0$ gives $O(\Delta t^3)$ accuracy [2], but it is not possible to find an α such that this is $A(\alpha)$ stable.

TABLE 1

SCHEME	SS22
PARAMETERS	$\theta_1 = 0.5$
	$\theta_2 = 0.5$
FORCING TERM	$F = 0 \quad t \leqq 0$
	$F = +1 \quad 0 < t \leqq 25$
	$F = -1 \quad t > 25$
DAMPING	$\nu = 0.5$ (overestimate in overshoot)

Δt	OVERSHOOT (% ERROR)		ERROR AT t = 5.0 (% ERROR)	
0.5	2.34×10^{-2}	(1.77)	8.23×10^{-3}	(7.66×10^{-1})
0.25	6.33×10^{-3}	(4.78×10^{-1})	1.97×10^{-3}	(1.83×10^{-1})
0.125	1.54×10^{-3}	(1.16×10^{-1})	4.87×10^{-4}	(4.54×10^{-2})
0.0625	3.85×10^{-4}	(2.90×10^{-2})	1.22×10^{-4}	(1.13×10^{-2})
0.03125	9.62×10^{-5}	(7.25×10^{-3})	3.04×10^{-5}	(2.83×10^{-3})
0.015625	2.41×10^{-5}	(1.81×10^{-3})	7.59×10^{-6}	(7.06×10^{-4})

DAMPING $\nu = 0.1$ (underestimate in overshoot)

Δt	OVERSHOOT (% ERROR)		ERROR AT t = 5.0 (% ERROR)	
0.5	1.83×10^{-2}	(7.68×10^{-1})	5.70×10^{-2}	(6.33)
0.25	7.14×10^{-3}	(2.98×10^{-1})	1.42×10^{-2}	(1.58)
0.125	1.22×10^{-3}	(5.10×10^{-2})	3.55×10^{-3}	(3.94×10^{-1})
0.0625	3.01×10^{-4}	(1.25×10^{-2})	8.88×10^{-4}	(9.85×10^{-2})
0.03125	8.27×10^{-5}	(3.44×10^{-3})	2.22×10^{-4}	(2.46×10^{-2})
0.015625	1.97×10^{-5}	(8.20×10^{-4})	5.55×10^{-5}	(6.16×10^{-3})

TABLE 2

SCHEME	SS22
PARAMETERS	$\theta_1 = 0.5$
	$\theta_2 = 0.5$
FORCING TERM	$F = \sin(\pi t/20)$
DAMPING	$\nu = 0.5$

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	7.83×10^{-5}	1.32×10^{-2}
0.25	2.86×10^{-5}	4.82×10^{-3}
0.125	7.69×10^{-6}	1.29×10^{-3}
0.0625	1.96×10^{-6}	3.29×10^{-4}
0.03125	4.91×10^{-7}	8.25×10^{-5}
0.015625	1.23×10^{-7}	2.07×10^{-5}

DAMPING $\nu = 0.1$

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	3.04×10^{-3}	3.80×10^{-1}
0.25	8.71×10^{-4}	1.09×10^{-1}
0.125	2.25×10^{-4}	2.81×10^{-2}
0.0625	5.67×10^{-5}	7.09×10^{-3}
0.03125	1.42×10^{-5}	1.78×10^{-3}
0.015625	3.55×10^{-6}	4.44×10^{-4}

TABLE 3

SCHEME	SS22
PARAMETERS	$\theta_1 = 0.6$
	$\theta_2 = 0.605$
FORCING TERM	$F = 0 \quad t \leqq 0$
	$F = +1 \quad 0 < t \leqq 25$
	$F = -1 \quad t > 25$
DAMPING	$\nu = 0.5$ (underestimate in overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	1.62×10^{-2}	(1.24)	1.70×10^{-2}	(1.6)
0.25	7.01×10^{-3}	(5.37×10^{-1})	6.69×10^{-3}	(6.23×10^{-1})
0.125	3.55×10^{-3}	(2.68×10^{-1})	2.90×10^{-3}	(2.70×10^{-1})
0.0625	2.12×10^{-3}	(1.60×10^{-1})	1.34×10^{-3}	(1.25×10^{-1})
0.03125	1.15×10^{-3}	(8.66×10^{-2})	6.46×10^{-4}	(6.01×10^{-2})
0.015625	5.968×10^{-4}	(4.50×10^{-2})	3.16×10^{-4}	(2.94×10^{-2})

DAMPING $\nu = 0.1$ (overestimate in overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	6.89×10^{-2}	(2.88)	9.54×10^{-2}	(10.59)
0.25	2.85×10^{-2}	(1.19)	3.68×10^{-2}	(4.09)
0.125	1.74×10^{-2}	(7.23×10^{-1})	1.54×10^{-2}	(1.71)
0.0625	8.86×10^{-3}	(3.69×10^{-1})	6.91×10^{-3}	(7.67×10^{-1})
0.03125	4.40×10^{-3}	(1.83×10^{-1})	3.25×10^{-3}	(3.61×10^{-1})
0.015625	2.23×10^{-3}	(9.28×10^{-1})	1.58×10^{-3}	(1.75×10^{-1})

TABLE 4

SCHEME	SS22
PARAMETERS	$\theta_1 = 0.6$
	$\theta_2 = 0.605$
FORCING TERM	$F = \sin(\pi t/20)$
DAMPING	$\nu = 0.5$

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	2.14×10^{-3}	3.60×10^{-1}
0.25	1.04×10^{-3}	1.75×10^{-1}
0.125	5.24×10^{-4}	8.81×10^{-2}
0.0625	2.64×10^{-4}	4.43×10^{-2}
0.03125	1.32×10^{-4}	2.23×10^{-2}
0.015625	6.63×10^{-5}	1.11×10^{-2}

DAMPING $\nu = 0.1$

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	7.74×10^{-3}	9.68×10^{-1}
0.25	4.67×10^{-3}	5.85×10^{-1}
0.125	2.59×10^{-3}	3.24×10^{-1}
0.0625	1.36×10^{-3}	1.71×10^{-1}
0.03125	6.98×10^{-4}	8.74×10^{-2}
0.015625	3.54×10^{-4}	4.43×10^{-2}

TABLE 5

SCHEME	SS32	
PARAMETERS	$\theta_1 = 2.0$ $\theta_2 = 11/3$ $\theta_3 = 6.0$	Single step Houbolt equivalent
FORCING TERM	$F = 0 \quad t \leqq 0$ $F = +1 \quad 0 < t \leqq 25$ $F = -1 \quad t > 25$	
DAMPING	$\nu = 0.5$ (overestimate in overshoot)	

Δt	OVERSHOOT	(% ERROR)	ERROR AT $t = 5.0$	(% ERROR)
0.5	1.80×10^{-2}	(1.35)	5.81×10^{-2}	(5.41)
0.25	1.71×10^{-2}	(1.28)	7.87×10^{-3}	(7.32×10^{-1})
0.125	5.10×10^{-3}	(3.83×10^{-1})	1.20×10^{-3}	(1.12×10^{-1})
0.0625	1.35×10^{-3}	(1.01×10^{-1})	2.22×10^{-4}	(2.07×10^{-2})
0.03125	3.46×10^{-4}	(2.29×10^{-2})	4.69×10^{-5}	(4.36×10^{-3})
0.015625	8.99×10^{-5}	(6.76×10^{-3})	1.07×10^{-5}	(9.96×10^{-4})

DAMPING $\nu = 0.1$ (underestimate in overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT $t = 5.0$	(% ERROR)
0.5	1.98×10^{-1}	(8.31)	2.26×10^{-1}	(25.1)
0.25	7.14×10^{-2}	(3.10)	6.93×10^{-2}	(7.69)
0.125	2.03×10^{-2}	(8.52×10^{-1})	1.79×10^{-2}	(1.98)
0.0625	5.20×10^{-3}	(2.18×10^{-1})	4.43×10^{-3}	(4.92×10^{-1})
0.03125	1.68×10^{-3}	(7.05×10^{-2})	1.10×10^{-3}	(1.22×10^{-1})
0.015625	5.76×10^{-4}	(2.42×10^{-2})	2.72×10^{-4}	(3.02×10^{-2})

TABLE 6

SCHEME	SS32	
PARAMETERS	$\theta_1 = 2.0$	Single step Houbolt equivalent
	$\theta_2 = 11/3$	
	$\theta_3 = 6.0$	
FORCING TERM	$F = \sin(\pi t/20)$	
DAMPING	$\nu = 0.5$	

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	2.48×10^{-3}	4.18×10^{-1}
0.25	1.94×10^{-3}	3.26×10^{-1}
0.125	5.84×10^{-4}	9.82×10^{-2}
0.0625	1.54×10^{-4}	2.59×10^{-2}
0.03125	3.92×10^{-5}	6.59×10^{-3}
0.015625	9.89×10^{-6}	1.66×10^{-3}

DAMPING $\nu = 0.1$

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	5.97×10^{-3}	7.47×10^{-1}
0.25	5.57×10^{-3}	6.97×10^{-1}
0.125	1.99×10^{-3}	2.49×10^{-1}
0.0625	5.66×10^{-4}	7.09×10^{-2}
0.03125	1.49×10^{-4}	1.87×10^{-2}
0.015625	3.82×10^{-5}	4.78×10^{-3}

TABLE 7

SCHEME SS32

PARAMETERS $\theta_1 = 1.4$, $\theta_2 = 1.96$, $\theta_3 = 2.744$ } Single step Wilson $\Theta = 1.4$ equivalent

FORCING TERM
$F = 0 \quad t \leq 0$
$F = +1 \quad 0 < t \leq 25$
$F = -1 \quad t > 25$

DAMPING $\nu = 0.5$ (overestimate at overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	3.04×10^{-2}	(2.30)	1.28×10^{-2}	(1.19)
0.25	1.06×10^{-2}	(8.02×10^{-1})	1.87×10^{-3}	(1.74×10^{-1})
0.125	2.80×10^{-3}	(2.12×10^{-1})	3.44×10^{-4}	(3.20×10^{-2})
0.0625	7.07×10^{-4}	(5.35×10^{-2})	7.31×10^{-5}	(6.80×10^{-3})
0.03125	1.78×10^{-4}	(1.35×10^{-2})	1.68×10^{-5}	(1.56×10^{-3})
0.015625	4.52×10^{-5}	(3.42×10^{-3})	4.02×10^{-6}	(3.74×10^{-4})

DAMPING $\nu = 0.1$ (underestimate at overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	7.80×10^{-2}	(3.27)	1.06×10^{-1}	(11.8)
0.25	2.06×10^{-2}	(8.65×10^{-1})	2.79×10^{-2}	(3.10)
0.125	6.04×10^{-3}	(2.53×10^{-1})	6.97×10^{-3}	(7.74×10^{-1})
0.0625	1.81×10^{-3}	(7.60×10^{-2})	1.73×10^{-3}	(1.92×10^{-1})
0.03125	6.99×10^{-4}	(2.93×10^{-2})	4.31×10^{-3}	(4.78×10^{-2})
0.015625	2.73×10^{-4}	(1.14×10^{-2})	1.19×10^{-2}	(1.32×10^{-2})

TABLE 8

SCHEME	SS32	
PARAMETERS	$\theta_1 = 1.4$	Single step Wilson-θ = 1.4 equivalent
	$\theta_2 = 1.96$	
	$\theta_3 = 2.744$	
FORCING TERM	$F = \sin(\pi t/20)$	
DAMPING	$\nu = 0.5$	

Δt	ERROR AT t = 5.0	% ERROR
0.5	2.90×10^{-3}	5.01×10^{-1}
0.25	9.01×10^{-4}	1.52×10^{-1}
0.125	2.35×10^{-4}	3.95×10^{-2}
0.0625	6.26×10^{-5}	1.05×10^{-2}
0.03125	1.50×10^{-5}	2.52×10^{-3}
0.015625	3.76×10^{-6}	6.32×10^{-4}

DAMPING $\nu = 0.1$

Δt	ERROR AT t = 5.0	% ERROR
0.5	8.68×10^{-3}	1.09
0.25	3.15×10^{-3}	3.94×10^{-1}
0.125	8.92×10^{-4}	1.12×10^{-1}
0.0625	2.34×10^{-4}	2.93×10^{-2}
0.03125	5.98×10^{-5}	7.48×10^{-3}
0.015625	1.51×10^{-5}	1.89×10^{-3}

TABLE 9

SCHEME	SS32
PARAMETERS	$\theta_1 = 1.05$
	$\theta_2 = 1.1$ — Bossak-Newmark equivalent
	$\theta_3 = 1.15$
FORCING TERM	$F = 0 \quad t \leqq 0$
	$F = +1 \quad 0 < t \leqq 25$
	$F = -1 \quad t > 25$
DAMPING	$\nu = 0.5$ (overestimate in overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	1.72×10^{-3}	(1.30×10^{-1})	1.55×10^{-3}	(1.44×10^{-1})
0.25	6.23×10^{-3}	(4.71×10^{-1})	3.36×10^{-4}	(3.13×10^{-2})
0.125	5.39×10^{-4}	(4.08×10^{-2})	7.86×10^{-5}	(7.31×10^{-3})
0.0625	1.44×10^{-4}	(1.09×10^{-2})	1.90×10^{-5}	(1.77×10^{-3})
0.03125	4.17×10^{-5}	(3.15×10^{-3})	4.67×10^{-6}	(4.35×10^{-2})
0.015625	1.34×10^{-5}	(1.01×10^{-3})	1.16×10^{-6}	(1.08×10^{-4})

DAMPING $\nu = 0.1$ (underestimate in overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	8.65×10^{-4}	(3.63×10^{-2})	3.28×10^{-2}	(3.64)
0.25	1.57×10^{-3}	(6.59×10^{-2})	8.23×10^{-3}	(9.13×10^{-1})
0.125	1.84×10^{-3}	(7.72×10^{-2})	2.05×10^{-3}	(2.28×10^{-1})
0.0625	3.40×10^{-4}	(1.43×10^{-2})	5.12×10^{-4}	(5.68×10^{-2})
0.03125	2.78×10^{-4}	(1.17×10^{-2})	1.28×10^{-4}	(1.42×10^{-2})
0.015625	1.19×10^{-4}	(4.99×10^{-3})	3.19×10^{-5}	(3.54×10^{-3})

TABLE 10

SCHEME	SS32	
PARAMETERS	$\theta_1 = 1.05$	Bossak-Newmark equivalent
	$\theta_2 = 1.1$	
	$\theta_3 = 1.15$	
FORCING TERM	$F = \sin(\pi t/20)$	
DAMPING	$\nu = 0.5$	

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	9.23×10^{-4}	1.55×10^{-1}
0.25	2.32×10^{-4}	3.91×10^{-2}
0.125	5.83×10^{-5}	9.30×10^{-3}
0.0625	1.46×10^{-5}	2.46×10^{-3}
0.03125	4.54×10^{-6}	7.63×10^{-4}
0.015625	9.14×10^{-7}	1.54×10^{-4}

DAMPING $\nu = 0.1$

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	3.87×10^{-3}	4.85×10^{-1}
0.25	1.04×10^{-3}	1.31×10^{-1}
0.125	2.67×10^{-4}	3.35×10^{-2}
0.0625	6.76×10^{-5}	8.46×10^{-3}
0.03125	1.70×10^{-5}	2.12×10^{-3}
0.015625	4.25×10^{-6}	5.32×10^{-4}

TABLE 11

SCHEME: SS32B

PARAMETERS:
$\theta_1 = 2.0$
$\theta_2 = 11/3$
$\theta_3 = 6.0$
Single step equivalent to Gear 3-step

FORCING TERM:
$F = 0 \quad t \leq 0$
$F = +1 \quad 0 < t \leq 25$
$F = -1 \quad t > 25$

DAMPING: $\nu = 0.5$ (underestimate in overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	3.75×10^{-2}	(2.83)	1.06×10^{-2}	(9.85×10^{-1})
0.25	4.95×10^{-3}	(3.74×10^{-1})	1.62×10^{-3}	(1.51×10^{-1})
0.125	6.42×10^{-4}	(4.86×10^{-2})	2.11×10^{-4}	(1.96×10^{-2})
0.0625	2.23×10^{-4}	(1.69×10^{-2})	2.67×10^{-5}	(2.48×10^{-3})
0.03125	5.80×10^{-5}	(4.39×10^{-3})	3.35×10^{-6}	(3.11×10^{-4})
0.015625	1.23×10^{-5}	(9.30×10^{-4})	4.19×10^{-7}	(3.90×10^{-5})

DAMPING: $\nu = 0.1$ (underestimate in overshoot)

Δt	OVERSHOOT	(% ERROR)	ERROR AT t = 5.0	(% ERROR)
0.5	5.30×10^{-2}	(2.22)	9.56×10^{-3}	(1.06)
0.25	2.16×10^{-2}	(9.06×10^{-1})	2.87×10^{-3}	(3.18×10^{-1})
0.125	3.68×10^{-3}	(1.54×10^{-1})	5.87×10^{-4}	(6.51×10^{-2})
0.0625	1.44×10^{-3}	(6.04×10^{-2})	8.66×10^{-5}	(9.60×10^{-3})
0.03125	8.11×10^{-4}	(3.40×10^{-2})	1.16×10^{-5}	(1.29×10^{-3})
0.015625	3.76×10^{-4}	(1.58×10^{-2})	1.50×10^{-6}	(1.67×10^{-4})

TABLE 12

SCHEME	SS32B	
PARAMETERS	$\theta_1 = 2.0$	Single step equivalent to Gear 3-step
	$\theta_2 = 11/3$	
	$\theta_3 = 6.0$	
FORCING TERM	$F = \sin(\pi t/20)$	
DAMPING	$\nu = 0.5$	

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	5.53×10^{-3}	9.30×10^{-1}
0.25	1.01×10^{-3}	1.70×10^{-1}
0.125	2.18×10^{-4}	3.66×10^{-2}
0.0625	5.09×10^{-5}	8.55×10^{-3}
0.03125	1.23×10^{-5}	2.07×10^{-3}
0.015625	3.03×10^{-6}	5.10×10^{-4}

DAMPING $\nu = 0.1$

Δt	ERROR AT $t = 5.0$	% ERROR
0.5	1.69×10^{-2}	2.12
0.25	2.81×10^{-3}	3.52×10^{-1}
0.125	4.76×10^{-4}	5.96×10^{-2}
0.0625	9.11×10^{-5}	1.14×10^{-2}
0.03125	1.94×10^{-5}	2.42×10^{-3}
0.015625	4.42×10^{-6}	5.54×10^{-4}

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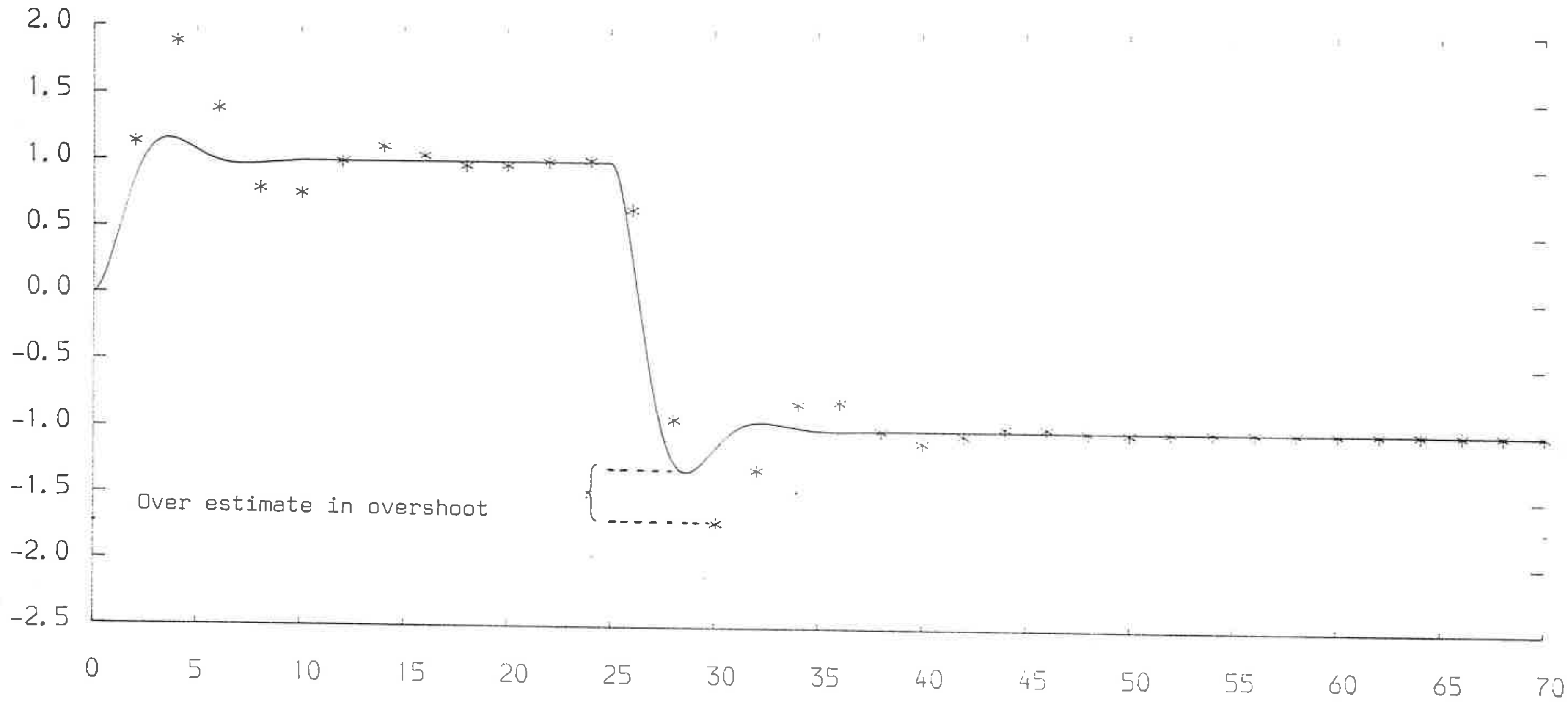


Displacement   Step forcing term

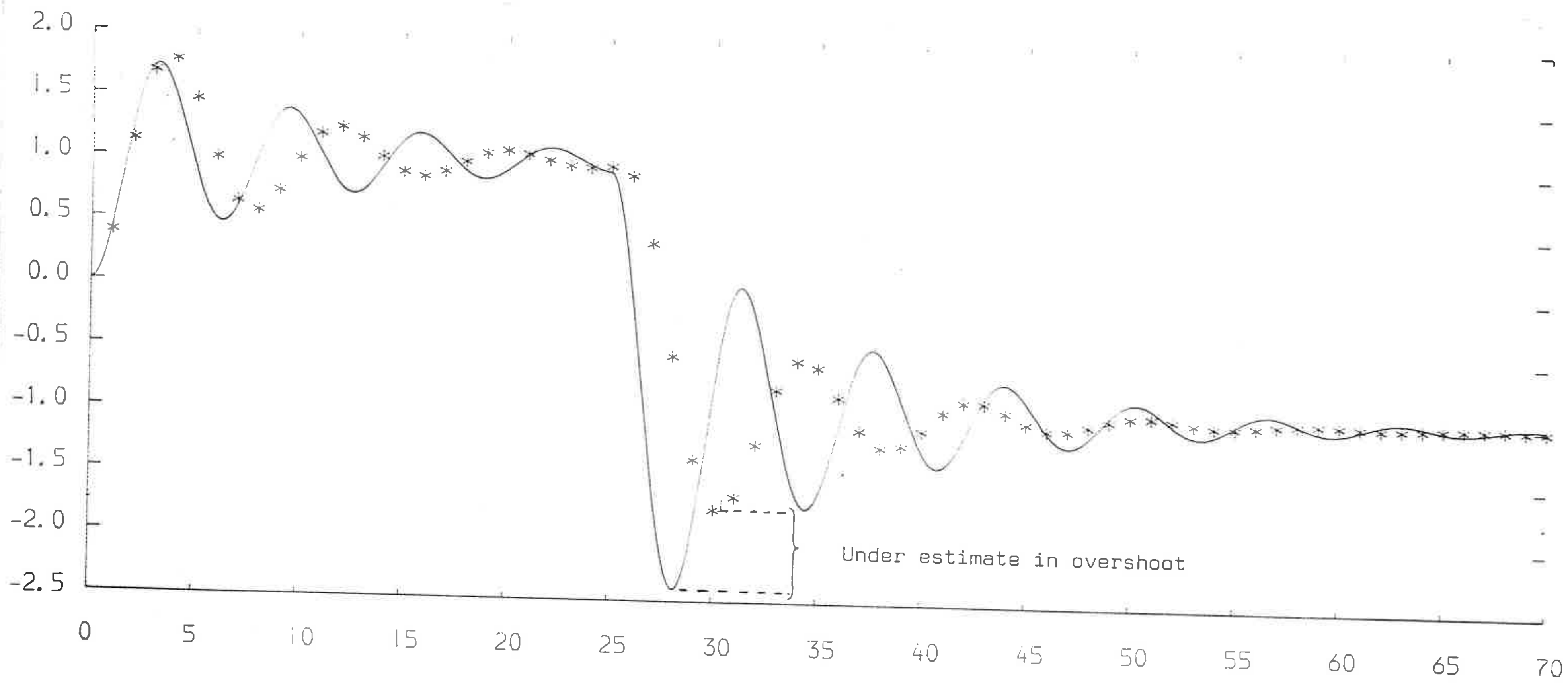
SSPj = SS 32

TH1 = 1.40000   TH2 = 1.96000   TH3 = 2.74400

H = 2.0000

NU = 0.5000

FIG. (2)



Displacement   Step forcing term

SSPj = SS 32

TH1 = 2.00000 TH2 = 3.66667 TH3 = 6.00000

H = 1.0000

NU = 0.1000

FIG. (3)

APPENDIX

Given a multistep method whose stability polynomial [9] is

$$\sum_{j=0}^{k} \gamma_j r^j + \lambda\Delta t \sum_{j=0}^{k} \beta_j r^j = 0 \tag{A1}$$

the stability boundary locus in the Argand diagram with axes OX, OY is given by

$$\Delta t\lambda(\theta) = - \frac{\sum_{j=0}^{k} \gamma_j e^{ji\theta}}{\sum_{j=0}^{k} \beta_j e^{ji\theta}} = X + iY \tag{A2}$$

i.e. for λ on one side of this boundary the method is stable and for λ on the other side it is unstable. (Figure 1).

For the Gear 3-step method [10] the stability boundary locus is given by

$$\begin{aligned} X(\theta) &= \frac{11}{6} - 3\cos\theta + \frac{3}{2}\cos 2\theta - \frac{1}{3}\cos 3\theta \\ Y(\theta) &= 3\sin\theta - \frac{3}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta \end{aligned} \tag{A3}$$

We have $X'(\theta) = 0$ when $\theta = 0, \pi$ or $\pm \frac{\pi}{3}$. Hence the left hand bound of the locus is where

$$X = X\left(\frac{\pi}{3}\right) = -1/12 \tag{A4}$$

Also if $Y(\theta) + X(\theta)\tan\alpha = 0$ is a tangent through the origin to the locus then $Y'(\theta) + X'(\theta)\tan\alpha = 0$ simultaneously. Eliminating $\tan\alpha$ then gives

$$(c - 1)^2(22c - 13) = 0 \tag{A5}$$

where $c = \cos\theta$.

Thus the Y axis has fourth order contact with the locus at the origin as expected with a third order method and the other tangent meets the locus where $c = 13/22 = \cos\theta_1$ say. Then $\tan\alpha = -Y(\theta_1)/X(\theta_1) = 14.42$ which corresponds to $\nu = 0.069$.

We can say that there will be stability if

$$R_e(\lambda\Delta t) < -1/12 \quad \text{i.e.} \quad \frac{\mu\Delta t}{2m} > 1/12 \tag{A6}$$

or we can say that we have $A(\alpha)$ stability if $\nu \geqq 7\%$.

We note that looking at the diagrams given by Gear [10] for the Gear 4, 5 and 6 methods it is evident that these have smaller values of α .

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