A TWO-DIMENSIONAL COMPRESSIBLE SOLVER INCORPORATING BODY-FITTED COORDINATES

P. GLAISTER

Numerical Analysis Report No. 11/87

DEPARTMENT OF MATHEMATICS,
P.O. BOX 220,
UNIVERSITY OF READING,
WHITEKNIGHTS,
READING, RG6 2AX

This work forms part of the research programe for the Institute for Computational Fluid Dynamics at the Universities of Oxford and Reading and was funded by A.W.R.E. Aldermaston under contract No. NSN/13B/2A88719.

SUMMARY

A finite difference scheme based on flux difference splitting is presented for the solution of the two dimensional Euler equations of gas dynamics in a generalised coordinate system. The scheme is based on numerical characteristic decomposition and solves locally linearised Riemann problems using upwind differencing. The decomposition is for a generalised coordinate system and a convex equation of state. This ensures good shock capturing properties when incorporated with operator splitting and the advantage of using body-fitted coordinates. The resulting scheme is applied to supersonic flow of 'real air' past a circular cylinder.

KEYWORDS

Euler Equations, Riemann Solver, Generalised Coordinates.

1. INTRODUCTION

The approximate (linearised) Riemann solver of Roe[1] has proved to be successful in its application to the compressible flow of an ideal gas in one dimension[2], and in two dimensions when incorporated with operator splitting[3]. A similar Riemann solver was proposed by Glaister[4] for compressible flows in one dimension and general convex equations of state. This scheme was extended[5] to the two dimensional Euler equations in Cartesian coordinates and general convex equations of state using operator splitting.

In this paper we seek to extend further the analysis of Glaister to a generalised coordinate system. The resulting scheme can be used with non-Cartesian, body-fitted meshes in two dimensions. The area of generating body fitted meshes is one that is increasing in importance[6,7].

In §2 we consider the Jacobian matrix of one of the flux functions for the Euler equations in a generalised coordinate system, and in §3 derive an approximate Riemann solver for the solution of these equations. Finally, in §4 we describe a two dimensional test problem and display the numerical results achieved using the scheme of §3.

2. EULER EQUATIONS

In this section we state the equations of motion for an inviscid, compressible fluid in two dimensions in terms of two generalised space coordinates. We also give the eigenvalues and eigenvectors of the Jacobian of one of the corresponding flux functions.

2.1 Equations of flow

The two dimensional Euler equations for the flow of an inviscid, compressible fluid can be written in generalised coordinates ξ,η as (see Appendix A)

$$(J\underset{\sim}{w})_t + \underset{\sim}{F}_\xi + \underset{\sim}{G}_\eta = \underset{\sim}{0} , \tag{1}$$

where

$$\underset{\sim}{w} = (\rho,\rho u,\rho v,e)^T , \tag{2}$$

$$\underset{\sim}{F}(\underset{\sim}{w}) = (\rho U, y_\eta p + \rho u U, - x_\eta p + \rho v U, U(e+p))^T , \tag{3}$$

$$\underset{\sim}{G}(\underset{\sim}{w}) = (\rho V, - y_\xi p + \rho u V, x_\xi p + \rho v V, V(e+p))^T , \tag{4}$$

$$e = \rho i + \tfrac{1}{2}\rho(u^2 + v^2) \tag{5}$$

and

$$U = y_\eta u - x_\eta v , \quad V = x_\xi v - y_\xi u . \tag{6a-b}$$

The Jacobian of the grid transformation $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ from Cartesian coordinates x,y to generalised coordinates ξ,η is given by

$$J = x_\xi y_\eta - x_\eta y_\xi . \tag{7}$$

The quantities $\rho(\xi,\eta,t)$, $u(\xi,\eta,t)$, $v(\xi,\eta,t)$, $p(\xi,\eta,t)$ $i = i(\xi,\eta,t)$ and $e(\xi,\eta,t)$ represent the density, velocity in x and y coordinate directions, pressure, specific internal energy and total energy, respectively, at a general position ξ,η in space and at time t . In addition, we have an equation of state of the form

$$p = p(\rho, i) . \tag{8}$$

In the case of an ideal gas, equation (8) takes the form

$$p = (\gamma - 1)\rho i , \tag{9}$$

where γ is the ratio of specific heat capacities of the fluid.

2.2 Structure of the Jacobian

We now give the Jacobian of the flux function $\underset{\sim}{F}(\underset{\sim}{w})$, and its eigenvalues and eigenvectors, since this information, together with similar information for the Jacobian of $\underset{\sim}{G}(\underset{\sim}{w})$ will form the basis for the approximate Riemann solver.

The Jacobian $\underset{\sim}{A} = \frac{\partial \underset{\sim}{F}}{\partial \underset{\sim}{w}}$ of the flux function $\underset{\sim}{F}(\underset{\sim}{w})$ is given by

$$\underset{\sim}{A} = \begin{bmatrix} 0 & y_\eta & -x_\eta & 0 \\ y_\eta\left[a^2-\frac{p_i}{\rho}(H-q^2)\right]-uU & U+y_\eta u\left[1-\frac{p_i}{\rho}\right] & -x_\eta u-y_\eta v\frac{p_i}{\rho} & y_\eta\frac{p_i}{\rho} \\ -x_\eta\left[a^2-\frac{p_i}{\rho}(H-q^2)\right]-uU & y_\eta v+x_\eta u\frac{p_i}{\rho} & U-x_\eta v\left[1-\frac{p_i}{\rho}\right] & -x_\eta\frac{p_i}{\rho} \\ U\left[a^2-H-\frac{p_i}{\rho}(H-q^2)\right] & y_\eta H-uU\frac{p_i}{\rho} & -x_\eta H-vU\frac{p_i}{\rho} & U\left[1+\frac{p_i}{\rho}\right] \end{bmatrix} \tag{10}$$

where the fluid speed q , enthalpy H and sound speed a are given by

$$q^2 = u^2 + v^2 \ , \tag{11}$$

$$H = p/\rho + i + \tfrac{1}{2}q^2 \tag{12}$$

and

$$a^2 = pp_i/\rho^2 + p_\rho \ , \tag{13}$$

and the quantities p_i, p_ρ denote the derivatives $\frac{\partial p}{\partial i}(\rho,i)\Big|_\rho$, $\frac{\partial p}{\partial \rho}(\rho,i)\Big|_i$, respectively. The eigenvalues λ_i of $\underset{\sim}{A}$ are given by

$$\lambda_{1,2,3,4} = U \pm a\sqrt{x_\eta^2 + y_\eta^2}, \ U, U \tag{14a-d}$$

with corresponding eigenvectors

$$\underset{\sim}{e}_{1,2} = \left[1, \ u \pm \frac{ay_\eta}{\sqrt{x_\eta^2+y_\eta^2}}, \ v \mp \frac{ax_\eta}{\sqrt{x_\eta^2+y_\eta^2}}, \ H \pm \frac{aU}{\sqrt{x_\eta^2+y_\eta^2}}\right]^T \tag{15a-b}$$

$$\underset{\sim}{e}_3 = \left[1, u, v, \ i + \tfrac{1}{2}(u^2+v^2) - \frac{\rho p_\rho}{p_i}\right]^T \tag{15c}$$

and

$$\underset{\sim}{e}_4 = \left[0, \ x_\eta, \ y_\eta, \ x_\eta u + y_\eta v\right]^T . \tag{15d}$$

Similar results hold for the Jacobian of $\underset{\sim}{G}(\underset{\sim}{w})$.

In the next section we develop an approximate Riemann solver based on the results of this section.

3. APPROXIMATE RIEMANN SOLVER

In this section we derive an approximate Riemann solver for the solution of equations (1)-(8).

We propose solving equations (1)-(8) using operator splitting, i.e. we solve successively

$$\tfrac{1}{2}(J\underset{\sim}{w})_t + \underset{\sim}{F}_\xi = \underset{\sim}{0} \qquad (16a)$$

and

$$\tfrac{1}{2}(J\underset{\sim}{w})_t + \underset{\sim}{G}_\eta = \underset{\sim}{0} \qquad (16b)$$

along ξ and η coordinate lines, respectively. We describe the scheme for solving equation (16a) and the solution of equation (16b) will follow in a similar way.

3.1 Linearised Riemann problem

If the solution of equation (16a) is sought along a ξ coordinate line given by $\eta = \eta_o$, a constant, using a finite difference method, then the solution is known at a set of discrete mesh points $(\xi,\eta,t) = (\xi_j,\eta_0,t_n)$ at any time t_n . Following Godunov[8] the approximate solution $\underset{\sim}{w}_j^n$ to $\underset{\sim}{w}$ at $(\xi_j\eta_0,t_n)$ can be considered as a set of piecewise constants $\underset{\sim}{w} = \underset{\sim}{w}_j^n$ for $\xi \in (\xi_j - \frac{\Delta\xi}{2}, \ \xi_j + \frac{\Delta\xi}{2})$ at time t_n where $\Delta\xi = \xi_j - \xi_{j-1}$ is a constant mesh spacing. A Riemann problem is now present at each interface $\xi_{j-\frac{1}{2}} = \tfrac{1}{2}(\xi_{j-1} + \xi_j)$ separating adjacent states $\underset{\sim}{w}_{j-1}^n, \underset{\sim}{w}_j^n$. We consider solving the linearised Riemann problem

$$\tfrac{1}{2}(J\underset{\sim}{w})_t + \tilde{\underset{\sim}{A}}(\underset{\sim}{w}_{j-1}^n,\underset{\sim}{w}_j^n)\underset{\sim}{w}_\xi = \underset{\sim}{0} \qquad (17)$$

where $\tilde{\underset{\sim}{A}}_{j-\frac{1}{2}} = \tilde{\underset{\sim}{A}}(\underset{\sim}{w}^n_{j-1}, \underset{\sim}{w}^n_j)$ is an approximation to the Jacobian $\underset{\sim}{A}$ and is a constant matrix depending on the states either side of $\xi_{j-\frac{1}{2}}$. The matrix $\tilde{\underset{\sim}{A}}_{j-\frac{1}{2}}$ will be required to satisfy the following three properties

(i) $\tilde{\underset{\sim}{A}}_{j-\frac{1}{2}}(\underset{\sim}{w}^n_{j-1}, \underset{\sim}{w}^n_j) \rightarrow \underset{\sim}{A}(\underset{\sim}{w})$ as $\underset{\sim}{w}^n_{j-1} \rightarrow \underset{\sim}{w}^n_j \rightarrow \underset{\sim}{w}$,

(ii) $\tilde{\underset{\sim}{A}}_{j-\frac{1}{2}}$ has four linearly independent eigenvectors

and

(iii) $\Delta \underset{\sim}{F} = \tilde{\underset{\sim}{A}}_{j-\frac{1}{2}} \Delta \underset{\sim}{w}$.

These properties were shown by Roe[1] in the ideal gas case in Cartesian coordinates to guarantee conservation and have good one-dimensional shock-capturing properties.

3.2 Numerical scheme

Once such a matrix has been constructed equation (17) can be solved approximately as

$$J_{j-\frac{1}{2}} \frac{(\underset{\sim}{w}^{n+1}_k - \underset{\sim}{w}^n_k)}{2\Delta t} + \tilde{\underset{\sim}{A}}_{j-\frac{1}{2}} \frac{(\underset{\sim}{w}^n_j - \underset{\sim}{w}^n_{j-1})}{\Delta \xi} = \underset{\sim}{0} \tag{18}$$

where k can be $j-1$ or j, $\Delta t = t_{n+1} - t_n$ is a constant time step and $J_{j-\frac{1}{2}}$ is an approximation to the grid Jacobian at $(\xi, \eta) = (\xi_{j-\frac{1}{2}}, \eta_0)$. If we project

$$\Delta \underset{\sim}{w} = \underset{\sim}{w}^n_j - \underset{\sim}{w}^n_{j-1} = \sum_{i=1}^{4} \tilde{\alpha}_i \tilde{\underset{\sim}{e}}_i \tag{19}$$

where $\tilde{\underset{\sim}{e}}_i$ are the eigenvectors of $\tilde{\underset{\sim}{A}}_{j-\frac{1}{2}}$ then equation (18) can be written as

$$J_{j-\frac{1}{2}} \frac{(\underset{\sim}{w}^{n+1}_k - \underset{\sim}{w}^n_k)}{2\Delta t} + \frac{\sum_{i=1}^{4} \tilde{\lambda}_i \tilde{\alpha}_i \tilde{\underset{\sim}{e}}_i}{\Delta \xi} = \underset{\sim}{0} . \tag{20}$$

where $\tilde{\lambda}_j$ are the eigenvalues of $\tilde{A}_{j-1/2}$. Equation (20) now gives rise to the following first order upwind algorithm

$$\underset{\sim}{w}_{j-1}^{n+1} = \underset{\sim}{w}_{j-1}^{n} - \frac{2\Delta t}{J_{j-1/2}\Delta\xi} \tilde{\lambda}_i \tilde{\alpha}_i \tilde{\underset{\sim}{e}}_i \quad \text{if} \quad \tilde{\lambda}_i < 0 \tag{21a}$$

and

$$\underset{\sim}{w}_{j}^{n+1} = \underset{\sim}{w}_{j}^{n} - \frac{2\Delta t}{J_{j-1/2}\Delta\xi} \tilde{\lambda}_i \tilde{\alpha}_i \tilde{\underset{\sim}{e}}_i \quad \text{if} \quad \tilde{\lambda}_i > 0 \ . \tag{21b}$$

Extensions of this first order algorithm to second order can be made[9] and to non-uniform grids[10,11].

3.3 Grid generation and grid Jacobian

The purpose of this paper is to present an approximate Riemann solver for use with non-cartesian body fitted coordinates. The mapping from physical (x-y) space to computational (ξ-η) space can be given analytically, or constructed numerically[12]. In the case where the mapping $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ is known analytically we can approximate $J_{j-1/2}$ in equation (18) as

$$J_{j-1/2} = (x_\xi y_\eta - x_\eta y_\xi)(\xi_{j-1/2},\eta_0) \ ; \tag{21}$$

alternatively, $J_{j-1/2}$ can be approximated using central differences. In addition, we will need a suitable approximation $x_\eta^{j-1/2}$ for x_η at $(\xi_{j-1/2},\eta_0)$ and in the analytic case we take

$$x_\eta^{j-1/2} = x_\eta(\xi_{j-1/2},\eta_0) \tag{22}$$

as in equation (21); otherwise we set $x_\eta^{j-1/2}$ to be the arithmetic mean of central difference approximations to x_η at (ξ_{j-1},η_0) and (ξ_j,η_0) . Similar approximations hold for x_ξ, y_η and y_ξ (see Appendix B).

3.4 Construction of $\tilde{A}_{j-1/2}$

Consider a ξ coordinate line given by $\eta = \eta_0$, a constant, and denote points ξ_{j-1}, ξ_j on this line by ξ_L, ξ_R, respectively. In addition, we denote $w^n_{j-1} = w_L$, $w^n_j = w_R$, and assume that $X = x_\eta^{j-1/2}$, $Y = y_\eta^{j-1/2}$ denote approximations to x_η, y_ξ that are constant in the interval (ξ_L, ξ_R). Our aim is to construct a matrix $\tilde{A}_{j-1/2} = \tilde{A}(w_L, w_R)$ satisfying properties (i)-(iii) of §3.2. Equivalently, we could find average eigenvalues $\tilde{\lambda}_i$ and average eigenvectors $\tilde{e}_i$ of the Jacobian A at ξ_L, ξ_R given by equations (14a-15d) such that

$$\Delta w = \sum_{i=1}^{4} \tilde{\alpha}_i \tilde{e}_i \tag{23a-d}$$

and

$$\Delta F = \sum_{i=1}^{4} \tilde{\lambda}_i \tilde{\alpha}_i \tilde{e}_i \tag{24a-d}$$

for some wavestrengths $\tilde{\alpha}_i$, where

$$\Delta(\cdot) = (\cdot)_R - (\cdot)_L \, . \tag{25}$$

This yields the following approximate Jacobian

$$\tilde{A}_{j-1/2} = \tilde{M}_{j-1/2} \tilde{D}_{j-1/2} \tilde{M}^{-1}_{j-1/2} \tag{26}$$

with the required properties, where $\tilde{M}_{j-1/2} = [\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4]$ and $\tilde{D}_{j-1/2} = \mathrm{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4)$. The choice of wavestrengths in equations (23a-24d) is made by initially considering states w_L and w_R that are close to some average state w as follows.

We seek $\alpha_1, \alpha_2, \alpha_3$ and α_4 such that

$$\Delta w = \sum_{i=1}^{4} \alpha_i e_i \tag{27a-d}$$

to within $O(\Delta^2)$ where $\underset{\sim}{e}_i$ are given in §2 and $\underset{\sim}{w}_L, \underset{\sim}{w}_R$ are close to some average state $\underset{\sim}{w}$. After some manipulation (see Appendix C) we find that equation (27a-d) yields the following expressions for α_i

$$\alpha_{1,2} = \frac{1}{2a^2}\left[\Delta p \pm \rho \frac{a(Y\Delta u - X\Delta v)}{\sqrt{X^2+Y^2}}\right] \tag{28a-b}$$

$$\alpha_3 = \Delta\rho - \frac{\Delta p}{a^2} \tag{28c}$$

$$\alpha_4 = \rho\frac{(X\Delta u + Y\Delta v)}{X^2+Y^2} \tag{28d}$$

where we have made the assumption that to within $O(\Delta^2)$

$$\Delta(\rho Z) = Z\Delta\rho + \rho\Delta Z \;, \quad Z = u, v, H \text{ and } i \;, \tag{29a-d}$$

$$\Delta(\rho Z^2) = Z^2\Delta\rho + 2\rho Z\Delta Z \;, \; Z = u \text{ or } v \;, \tag{30a-b}$$

and

$$\Delta p = p_\rho \Delta\rho + p_i \Delta i \;. \tag{31}$$

With the expressions given by equations (28a-d) it is possible to show that

$$\Delta\underset{\sim}{F} = \sum_{i=1}^{4} \lambda_i \alpha_i \underset{\sim}{e}_i \tag{32}$$

to within $O(\Delta^2)$.

We now return to the general case, i.e. consider two states $\underset{\sim}{w}_L, \underset{\sim}{w}_R$ not necessarily close such that equations (23a-24d) are satisfied exactly, where

$$\tilde{\lambda}_i = \tilde{U} \pm \tilde{a}\sqrt{X^2+Y^2} \quad \tilde{U}, \tilde{U} \tag{33a-d}$$

$$\underset{\sim}{\tilde{e}}_{1,2} = \left[1, \tilde{u} \pm \frac{\tilde{a}Y}{\sqrt{X^2+Y^2}}, \tilde{v} \mp \frac{\tilde{a}X}{\sqrt{X^2+Y^2}}, \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + ½\tilde{u}^2 + ½\tilde{v}^2 \pm \frac{\tilde{a}\tilde{U}}{\sqrt{X^2+Y^2}}\right]^T \tag{34a-b}$$

$$\tilde{\underset{\sim}{e}}_3 = \left[1, \tilde{u}, \tilde{v}, \tilde{i} + \tfrac{1}{2}(\tilde{u}^2+\tilde{v}^2) - \frac{\widetilde{\rho p_\rho}}{\tilde{p}_i}\right]^T \qquad (34c)$$

$$\tilde{\underset{\sim}{e}}_4 = (0, X, Y, X\tilde{u} + Y\tilde{v})^T \qquad (34d)$$

$$\tilde{\alpha}_{1,2} = \frac{1}{2\tilde{a}^2}\left[\Delta p \pm \frac{\widetilde{\rho a}(Y\Delta u - X\Delta v)}{\sqrt{X^2+Y^2}}\right] \qquad (35a\text{-}b)$$

$$\tilde{\alpha}_3 = \Delta\rho - \frac{\Delta p}{\tilde{a}^2} \qquad (35c)$$

$$\tilde{\alpha}_4 = \tilde{\rho}\frac{(X\Delta u + Y\Delta v)}{X^2+Y^2} \qquad (35d)$$

$$\tilde{U} = Y\tilde{u} - X\tilde{v} \qquad (36)$$

and

$$\tilde{a}^2 = \tilde{p}_\rho + \frac{\widetilde{pp_i}}{\tilde{\rho}^2} \quad . \qquad (37)$$

Thus, we have to determine averages $\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p}_i, \tilde{p}_\rho, \tilde{p}$ and $\tilde{i}$ such that equations (23a-24d) are satisfied subject to equations (33a-37). This problem has a solution and can be determined in a similar way to that of Glaister[5] for the Cartesian case (see Appendix D). The required averages are

$$\tilde{Z} = \frac{\sqrt{\rho_L}\, Z_L + \sqrt{\rho_R}\, Z_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} , \quad Z = u, v, i \text{ or } H , \qquad (38a\text{-}d)$$

$$\tilde{\rho} = \sqrt{\rho_L \rho_R} \qquad (39)$$

and

$$\tilde{p} = \tilde{\rho}(\tilde{H} - \tilde{i} - \tfrac{1}{2}\tilde{u}^2 - \tfrac{1}{2}\tilde{v}^2) \ . \qquad (40)$$

In addition,

$$\Delta p = \tilde{p}_\rho \Delta\rho + \tilde{p}_i \Delta i \ , \tag{41}$$

and suitable approximations satisfying equation (41) are

$$\tilde{p}_\rho = \tfrac{1}{2}\left[\frac{p(\rho_R, i_R) - p(\rho_L, i_R)}{\Delta\rho} + \frac{p(\rho_R, i_L) - p(\rho_L, i_L)}{\Delta\rho}\right], \rho_L \neq \rho_R \ , \tag{42a}$$

$$\tilde{p}_\rho = \tfrac{1}{2}\left[p_\rho(\rho, i_R) + p_\rho(\rho, i_L)\right] \ , \quad \rho_L = \rho_R = \rho \ , \tag{42b}$$

$$\tilde{p}_i = \tfrac{1}{2}\left[\frac{p(\rho_R, i_R) - p(\rho_R, i_L)}{\Delta i} + \frac{p(\rho_L, i_R) - p(\rho_L, i_L)}{\Delta i}\right], i_L \neq i_L , \tag{43a}$$

and

$$\tilde{p}_i = \tfrac{1}{2}\left[p_i(\rho_R, i) + p_i(\rho_L, i)\right] \ , \quad i_L = i_R = i \ . \tag{43b}$$

(In practice we would replace the conditions $\Delta\rho = 0$, $\Delta i = 0$ with $|\Delta\rho| \leq 10^{-m}$, $|\Delta i| \leq 10^{-m}$, where the integer m is machine dependent.) In the case of an ideal gas equations (42a-43b) yield

$$\tilde{p}_\rho = (\gamma - 1)\tfrac{1}{2}(i_L + i_R) \tag{44a}$$

$$\tilde{p}_i = (\gamma - 1)\tfrac{1}{2}(\rho_L + \rho_R) \ . \tag{44b}$$

The approximate Riemann solver presented here can now be implemented as in equations (21a-b) where the required wavespeeds $\tilde{\lambda}_i$, wavestrengths $\tilde{\alpha}_i$ and associated directions $\tilde{\underset{\sim}{e}}_i$ are given by equations (33a-43b). In particular, the required approximate Jacobian is given by

$$\tilde{\underset{\sim}{A}}_{j-\frac{1}{2}} = \begin{bmatrix} 0 & Y & -X & 0 \\ Y\left[\tilde{a}^2-\frac{\tilde{p}_i}{\tilde{\rho}}(\tilde{H}-\tilde{u}^2-\tilde{v}^2)\right]-\tilde{u}\tilde{U} & \tilde{U}+Y\tilde{u}\left[1-\frac{\tilde{p}_i}{\tilde{\rho}}\right] & -X\tilde{u}-Y\tilde{v}\frac{\tilde{p}_i}{\tilde{\rho}} & Y\frac{\tilde{p}_i}{\tilde{\rho}} \\ -X\left[\tilde{a}^2-\frac{\tilde{p}_i}{\tilde{\rho}}(\tilde{H}-\tilde{u}^2-\tilde{v}^2)\right]-\tilde{v}\tilde{U} & Y\tilde{v}+X\tilde{u}\frac{\tilde{p}_i}{\tilde{\rho}} & \tilde{U}-X\tilde{v}\left[1-\frac{\tilde{p}_i}{\tilde{\rho}}\right] & -X\frac{\tilde{p}_i}{\tilde{\rho}} \\ \tilde{U}\left[\tilde{a}^2-\tilde{H}-\frac{\tilde{p}_i}{\tilde{\rho}}(\tilde{H}-\tilde{u}^2-\tilde{v}^2)\right] & Y\tilde{H}-\tilde{u}\tilde{U}\frac{\tilde{p}_i}{\tilde{\rho}} & -X\tilde{H}-\tilde{v}\tilde{U}\frac{\tilde{p}_i}{\tilde{\rho}} & \tilde{U}\left[1+\frac{\tilde{p}_i}{\tilde{\rho}}\right] \end{bmatrix}.$$

In the next section we describe a test problem used to test the algorithm of this section.

4. TEST PROBLEM AND NUMERICAL RESULTS

In this section we describe a standard test problem in two dimensional gas dynamics, and give the numerical results achieved for this problem using the Riemann solver described in §3.

The problem is that of uniform flow of 'real air' past a circular cylinder. The equation of state used can be written as

$$p = (\gamma(\rho, i) - 1)\rho i$$

where the form of $\gamma(\rho, i)$ is determined via curve fits to experimental data[13]. The radius of the cylinder is 0.5 and the initial conditions chosen are $\rho = 1.4$, $u = 8.0$, $v = 0$ and $p = 1$, corresponding to Mach 8 flow. An O-type computational mesh is used and thus the grid transformation is from (x,y) physical space to $(\xi,\eta) \equiv (R,\phi)$ computational space, where R,ϕ are standard plane polar coordinates (see Appendix F). Because of the line of symmetry along $\phi = \pi$ and the supersonic conditions along $\phi = \pi/2$, the region of computation considered is $(R,\phi) \in [0.5, R_{max}] \times [\pi/2, \pi]$. (The exterior of the boundary is taken as $R_{max} = 3$ for the computations shown here.) The grid spacing in the ϕ-direction is uniform with 32 grid lines given by $\phi_j = (j-\frac{1}{2})\frac{\pi}{64} + \frac{\pi}{2}$, $j = 1,\ldots 32$. In the R-direction two types of grid spacing are chosen. The first type of grid spacing is uniform with 33 grid lines given by $R_j = (j-\frac{1}{2})\frac{5}{66} + 0.5$, $j = 1,\ldots 33$. The second grid spacing is of a non-uniform, geometric type with 34 grid lines given by

$R_1 = 0.5 + \frac{1}{2}k$, $R_j = R_{j-1} + k\mu^{j-2}$, $j = 2,\ldots 34$ where $k = \frac{\pi}{1280}$ and $\mu = 1.1648336$. Along $\phi = \pi$ a symmetry boundary condition is applied and along $\phi = \pi/2$ supersonic boundary conditions are applied. Reflecting boundary conditions are applied along the surface of the cylinder $R = 0.5$ (see Appendix G) and inflow conditions are applied along $R = R_{max}$. The scalar scheme used is first order[7], however, a second order TVD (total variation diminishing) scheme could be used.

Figures 1,2 and 3 display the density contours at $t = 0.2$, 0.4 and 0.6, respectively, for the mesh with uniform spacing in the R-direction. Corresponding results for the case where the mesh spacing is non-uniform in the R-direction are shown in figures 4,5 and 6. In both cases the shock has been captured over at most three cells.

Figure 1

Figure 2

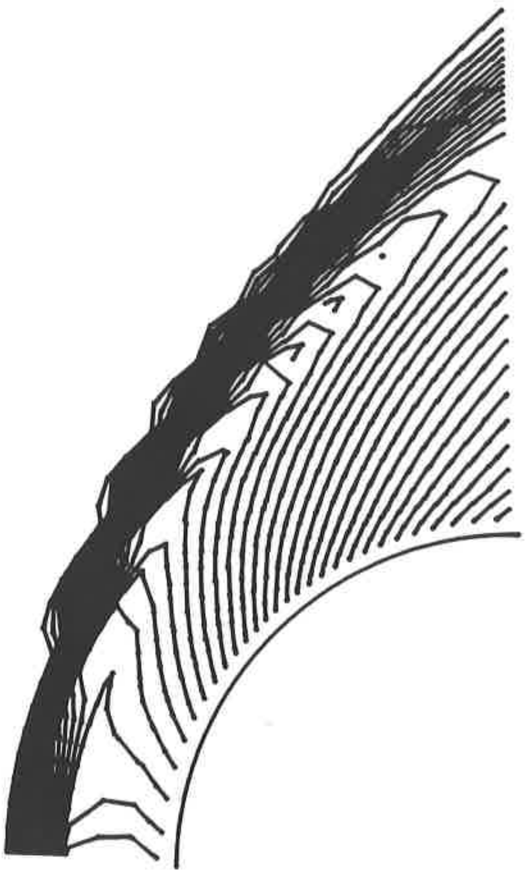

Figure 3

Figure 4

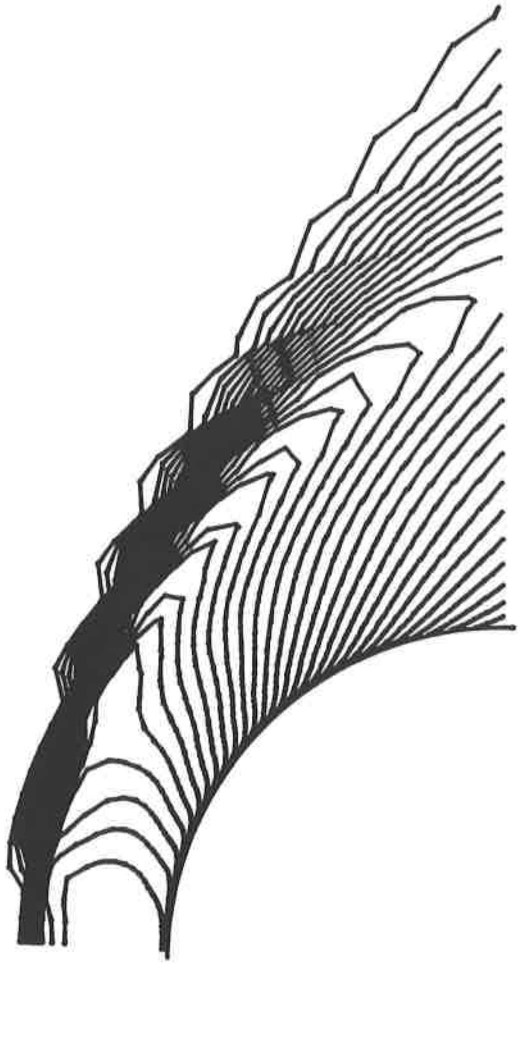

Figure 5

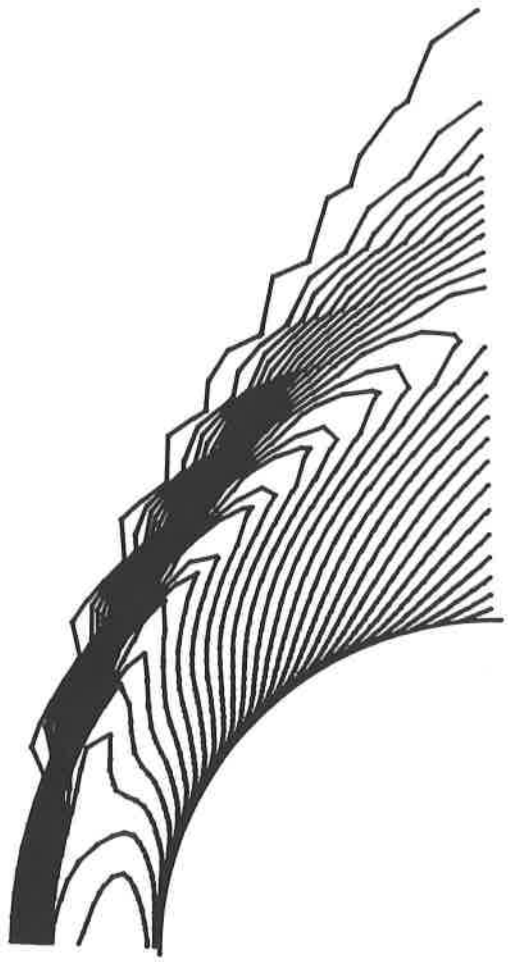

Figure 6

5. CONCLUSIONS

We have presented an approximate linearised Riemann solver for two-dimensional compressible flows using body fitted coordinates. The resulting scheme has been applied to supersonic flow of a real gas past a circular cylinder. The numerical results achieved show that the shock has been captured over only a few cells. Furthermore, the scheme developed applies to any convex equation of state and to any regular, body-fitted mesh.

ACKNOWLEDGEMENTS

I acknowledge the financial support of A.W.R.E., Aldermaston.

REFERENCES

1. P.L. ROE, 'Approximate Riemann Solvers, Parameter Vectors and Difference Schemes' J.Comput.Phys. 49, 357 (1983).

2. P.L. ROE and J. PIKE, 'Efficient Construction and Utilisation of Approximate Riemann Solutions', Computing Methods in Applied Science and Engineering VI, 499 (1984).

3. P.L. ROE, 'Some Contributions to the Modelling of Discontinuous Flows, Proc. AMS/SIAM Seminar, San Diego, (1983).

4. P. GLAISTER, 'An Approximate Linearised Riemann Solver for the Euler equations in One Dimension for Real Gases', J.Comput.Phys., (to appear).

5. P. GLAISTER, 'Approximate Solutions of the Two Dimensional Euler Equations for Real Gases using Operator Splitting', UCINA Report Series 87/1, Oxford University Computing Laboratory, 1987.

6. J.F. THOMPSON, 'Numerical Grid Generation', Proc. of Int. Conf., Nashville, USA, North-Holland, (1982).

7. J. HAÜSER and C. TAYLOR, 'Numerical Grid Generation in Computational Fluid Dynamics', Proc. of Int. Conf., Landshut, West Germany, Pineridge Press, (1986)

8. S.K. GODUNOV, 'A Difference Method for the Numerical Computation of Continuous Solutions of Hydrodynamic Equations', Mat.Sbornik, 47, no. 271, (1959).

9. P.K. SWEBY 'High Resolution Schemes using Flux Limiters for Hyperbolic Conservation Laws', SIAM J.Numer.Anal. 21, 995, (1984).

10. P. GLAISTER 'Shock Capturing on Irregular Grids', Numerical Analysis Report 4-86, University of Reading, (1986).

11. J. PIKE, 'Grid Adaptive Algorithms for the Solution of the Euler Equations on Irregular Grids', J.Comput.Phys. 71, 194, (1987).

12. J.F. THOMPSON, Z.U.A. WARSI and C.W. MASTIN, 'Numerical Grid Generation-Foundations and Applications', North-Holland, (1985).

13. S. SRINIVASAN, J.C. TANNEHILL and K.J. WEILMUENSTER, 'Simplified Curve Fits for the Thermodynamic Properties of Equilibrium Air', Iowa State University Engineering Research Institute Project 1626, (1986).

APPENDIX A

In this appendix we give the derivation of equations (1)-(7).

The Euler equations for compressible flow in two dimensions are

$$\underset{\sim}{w}_t + \underset{\sim}{f}_x + \underset{\sim}{g}_y = \underset{\sim}{0} \tag{A1}$$

where x,y are Cartesian coordinates, $\underset{\sim}{w}$ is given by equation (2) and

$$\underset{\sim}{f} = (\rho u,\ p+\rho u^2,\ \rho uv,\ u(e+p))^T \tag{A2}$$

$$\underset{\sim}{g} = (\rho v,\ \rho uv,\ p+\rho v^2,\ v(e+p))^T . \tag{A3}$$

Defining an invertible, twice continuously differentiable mapping from (x,y) space to (ξ,η) space by $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ then we have

$$\frac{\partial}{\partial\xi} = x_\xi \frac{\partial}{\partial x} + y_\xi \frac{\partial}{\partial y} \tag{A4}$$

and

$$\frac{\partial}{\partial\eta} = x_\eta \frac{\partial}{\partial x} + y_\eta \frac{\partial}{\partial y} \tag{A5}$$

using the chain rule for partial derivatives. Combining equations (A4) and (A5) yields

$$\frac{\partial}{\partial x} = \frac{1}{J}\left[y_\eta \frac{\partial}{\partial\eta} - y_\xi \frac{\partial}{\partial\eta}\right] \tag{A6}$$

and

$$\frac{\partial}{\partial y} = \frac{1}{J}\left[x_\xi \frac{\partial}{\partial\eta} - x_\eta \frac{\partial}{\partial\xi}\right] . \tag{A7}$$

where

$$J = x_\xi y_\eta - y_\xi x_\eta . \tag{A8}$$

Using the expressions given by equations (A6) and (A7), equation (A1) becomes

$$J\underset{\sim}{w}_t + y_\eta \underset{\sim}{f}_\xi - y_\xi \underset{\sim}{f}_\eta + x_\xi \underset{\sim}{g}_\eta - x_\eta \underset{\sim}{g}_\xi = \underset{\sim}{0} \ . \qquad \text{(A9)}$$

If we note, however, that

$$\frac{\partial}{\partial\xi}\left[y_\eta \underset{\sim}{f} - x_\eta \underset{\sim}{g}\right] + \frac{\partial}{\partial\eta}\left[x_\xi \underset{\sim}{g} - y_\xi \underset{\sim}{f}\right]$$

$$= y_\eta \underset{\sim}{f}_\xi + y_{\eta\xi} \underset{\sim}{f} - x_\eta \underset{\sim}{g}_\xi - x_{\eta\xi} \underset{\sim}{g}$$

$$+ x_\xi \underset{\sim}{g}_\eta + x_{\xi\eta} \underset{\sim}{g} - y_\xi \underset{\sim}{f}_\eta - y_{\xi\eta} \underset{\sim}{f}$$

$$= y_\xi \underset{\sim}{f}_\eta - y_\xi \underset{\sim}{f}_\eta + x_\xi \underset{\sim}{g}_\eta - x_\eta \underset{\sim}{g}_\xi \ ,$$

(since $x_{\xi\eta} = x_{\eta\xi}$ and $y_{\xi\eta} = y_{\eta\xi}$), and that $J\underset{\sim}{w}_t = (J\underset{\sim}{w})_t$, then equation (A9) becomes

$$(J\underset{\sim}{w})_t + \underset{\sim}{F}_\xi + \underset{\sim}{G}_\eta = \underset{\sim}{0} \ . \qquad \text{(A10)}$$

where

$$\underset{\sim}{F} = y_\eta \underset{\sim}{f} - x_\eta \underset{\sim}{g}$$

$$= (\rho U, \ y_\eta p + \rho u U, \ - x\eta p + \rho v U, \ U(e+p))^T$$

$$\underset{\sim}{G} = x_\xi \underset{\sim}{g} - y_\xi \underset{\sim}{f}$$

$$= (\rho V, \ - y_\xi p + \rho u V, \ x_\xi p + \rho v V, \ V(e+p))^T$$

and U,V are given by

$$U = y_\eta u = x_\eta v \ , \quad V = x_\xi v - y_\xi u \ .$$

APPENDIX B

In this appendix we give suitable central difference approximations for x_η, y_η, x_ξ and y_ξ, when the transformation $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ is generated numerically.

When we are performing an update in the η-direction, along the line $\eta = \eta_i$ we require approximations $x_\eta^{j-\frac{1}{2}}$, $y_\eta^{j-\frac{1}{2}}$, $x_\xi^{j-\frac{1}{2}}$ and $y_\xi^{j-\frac{1}{2}}$ to x_η, y_η, x_ξ and y_ξ, respectively, in the interval (ξ_{j-1},ξ_j). This gives rise to approximations

$$x_\eta^{j-\frac{1}{2}} = \tfrac{1}{2}\left[\frac{x(\xi_j,\eta_{i+1}) - x(\xi_j,\eta_{i-1})}{\eta_{i+1} - \eta_{i-1}} + \frac{x(\xi_{j-1},\eta_{i+1}) - x(\xi_{j-1},\eta_{i-1})}{\eta_{i+1} - \eta_{i-1}}\right] \quad \text{(B1)}$$

$$x_\xi^{j-\frac{1}{2}} = \tfrac{1}{2}\left[\frac{x(\xi_j,\eta_{i+1}) - x(\xi_{j-1},\eta_{i+1})}{\xi_j - \xi_{i-1}} + \frac{x(\xi_j,\eta_{i-1}) - x(\xi_{j-1},\eta_{i-1})}{\xi_j - \xi_{j-1}}\right] \quad \text{(B2)}$$

with similar expressions for $y_\eta^{j-\frac{1}{2}}$ and $y_\xi^{j-\frac{1}{2}}$. If we expand the expressions on the right-hand sides of equations (B1) and (B2) as a Taylor series about the point $(\xi_{j-\frac{1}{2}},\eta_i) = (\frac{1}{2}(\xi_j+\xi_{j-1}),\eta_i)$ we obtain

$$x_\eta^{j-\frac{1}{2}} = x_\eta(\xi_{j-\frac{1}{2}},\eta_i) + \tfrac{1}{2}x_{\eta\eta}(\xi_{j+\frac{1}{2}},\eta_i)\delta^2\eta_i + O(\Delta^2)$$

and

$$x_\xi^{j-\frac{1}{2}} = x_\xi(\xi_{j-\frac{1}{2}},\eta_i) + \tfrac{1}{2}x_{\xi\eta}(\xi_{j-\frac{1}{2}},\eta_i)\delta^2\eta_i + O(\Delta^2) ,$$

where

$$\delta^2\eta_i = \eta_{i+1} - 2\eta_i + \eta_{i-1} ,$$

representing first order approximations on non-uniform grids. If the grid in the η-direction is uniform then the approximations given by equations (B1) and (B2) are second order accurate. In addition, we now have an approximation $J_{j-\frac{1}{2}}$ for the grid Jacobian J at $(\xi_{j-\frac{1}{2}}, \eta_i)$ given by

$$J_{j-\frac{1}{2}} = x_\xi^{j-\frac{1}{2}} y_\eta^{j-\frac{1}{2}} - x_\eta^{j-\frac{1}{2}} y_\xi^{j-\frac{1}{2}} .$$

APPENDIX C

In this appendix we derive the expressions given in equations (28a-d) for the wavestrengths α_i so that equations (27a-d) are satisfied to within $O(\Delta^2)$. Writing equations (281-d) out in full we have

$$\Delta\rho = \alpha_1 + \alpha_2 + \alpha_3 \tag{C1}$$

$$\Delta(\rho u) = \alpha_1(u + \frac{aY}{d}) + \alpha_2(u - \frac{aY}{d}) + \alpha_3 u + \alpha_4 X \tag{C2}$$

$$\Delta(\rho v) = \alpha_1(v - \frac{aX}{d}) + \alpha_2(v + \frac{aX}{d}) + \alpha_3 v + \alpha_4 Y \tag{C3}$$

$$\Delta e = \alpha_1(H + \frac{aU}{d}) + \alpha_2(H - \frac{aU}{d}) + \alpha_3(H - \frac{\rho a^2}{p_i}) + \alpha_4(Xu + Yv) \ , \tag{C4}$$

where

$$d = \sqrt{X^2 + Y^2} \ . \tag{C5}$$

From equations (C1-C3) we have

$$\Delta(\rho u) - u\Delta\rho = \frac{aY}{d}(\alpha_1 - \alpha_2) + \alpha_4 X \tag{C6a}$$

$$\Delta(\rho v) - v\Delta\rho = -\frac{aX}{d}(\alpha_1 - \alpha_2) + \alpha_4 Y \tag{C6b}$$

and combining equations (C6a) and (C6b) we obtain

$$d^2\alpha_4 = X(\Delta(\rho u) - u\Delta\rho) + Y(\Delta(\rho v) - v\Delta\rho) \tag{C7}$$

and

$$\Delta(\rho U) - U\Delta\rho = a(\alpha_1 - \alpha_2)d \ , \tag{C8}$$

since $U = Yu - Xv$ and X, Y are constant. Since $e = \rho H - p$, equation (C1) and (C4) yield

$$\Delta(\rho H) - H\Delta\rho - \Delta p \;=\; (\alpha_1 - \alpha_2)\,\frac{aU}{d} - \rho\,\frac{a^2}{p_i}\,\alpha_3 + \alpha_4(Xu + Yv)\;. \qquad \text{(C9)}$$

If we substitute for $\alpha_4 X$ from equation (C6a) and $\alpha_4 Y$ from equation (C6b) into equation (C9) we get the following expression for α_3

$$\frac{\rho a^2}{p_i}\,\alpha_3 \;=\; u(\Delta(\rho u) - u\Delta\rho) + v(\Delta(\rho v) - v\Delta\rho)$$

$$+\;\Delta p + H\Delta\rho - \Delta(\rho H)\;. \qquad \text{(C10)}$$

Now, to within $O(\Delta^2)$ we have

$$\Delta(\rho Z) \;=\; \rho\Delta Z + Z\Delta\rho\;, \qquad Z \;=\; u, v, H \text{ and } U \qquad \text{(C11a-d)}$$

so equation (C10) becomes

$$\frac{\rho a^2}{p_i}\,\alpha_3 \;=\; \rho u\Delta u + \rho v\Delta v + \Delta p - \rho\Delta H\;. \qquad \text{(C12)}$$

Also,

$$\Delta H \;=\; \Delta(p/\rho + i + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2)$$

$$= \frac{\Delta p}{\rho} - \frac{p}{\rho^2}\,\Delta\rho + \Delta i + u\Delta u + v\Delta v \qquad \text{(C13)}$$

to within $O(\Delta^2)$ so that equation (C12) gives

$$\alpha_3 \;=\; \Delta\rho - \frac{p_\rho\Delta\rho + p_i\Delta i}{a^2} \qquad \text{(C14)}$$

where we have used

$$p \;=\; \frac{\rho^2}{p_i}\,(a^2 - p_\rho)\;. \qquad \text{(C15)}$$

Finally, since

$$\Delta p \;=\; p_i\Delta i + p_\rho\Delta\rho \qquad \text{(C16)}$$

to within $O(\Delta^2)$, equation gives

$$\alpha_3 = \Delta\rho - \frac{\Delta p}{a^2} . \tag{C17}$$

This now gives

$$\alpha_1 + \alpha_2 = \Delta\rho - \alpha_3 = \frac{\Delta p}{a^2} \tag{C18}$$

from equation (C1), and combining equations (C8) and (C18) gives

$$\alpha_{1,2} = \frac{1}{2a^2} (\Delta p \pm \rho \frac{a}{d} \Delta U)$$

$$= \frac{1}{2a^2} \left[\Delta p \pm \rho \frac{a}{d} (Y\Delta u - X\Delta v)\right] , \tag{C19}$$

where we have used equation (C11d). The remaining wavestrength α_4 is now given by equation (C7) as

$$\alpha_4 = \frac{\rho(X\Delta u + Y\Delta v)}{d^2} \tag{C20}$$

where equations (C11a-b) have been used.

APPENDIX D

In this appendix we derive the averages that make equations (23a)-(24d) satisfied.

Writing equations (23a)-(24d) out in full we have

$$\Delta\rho = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 \tag{D1}$$

$$\Delta(\rho u) = \tilde{\alpha}_1\left[\tilde{u} + \frac{\widetilde{aY}}{d}\right] + \tilde{\alpha}_2\left[\tilde{u} - \frac{\widetilde{aY}}{d}\right] + \tilde{\alpha}_3\tilde{u} + \tilde{\alpha}_4 X \tag{D2}$$

$$\Delta(\rho v) = \tilde{\alpha}_1\left[\tilde{v} - \frac{\widetilde{aX}}{d}\right] + \tilde{\alpha}_2\left[\tilde{v} + \frac{\widetilde{aX}}{d}\right] + \tilde{\alpha}_3\tilde{v} + \tilde{\alpha}_4 Y \tag{D3}$$

$$\begin{aligned}\Delta e &= \Delta(\rho i) + \Delta(\rho\,\frac{u^2}{2}) + \Delta(\rho\,\frac{v^2}{2}) \\ &= \tilde{\alpha}_1\left[\widetilde{p/\rho} + \tilde{i} + \widetilde{{\textstyle\frac{1}{2}}u^2} + \widetilde{{\textstyle\frac{1}{2}}v^2} + \frac{\widetilde{aU}}{d}\right] + \tilde{\alpha}_2\left[\widetilde{p/\rho} + \tilde{i} + \widetilde{{\textstyle\frac{1}{2}}u^2} + \widetilde{{\textstyle\frac{1}{2}}v^2} - \frac{\widetilde{\widetilde{aU}}}{d}\right] \\ &\quad + \tilde{\alpha}_3\left[\tilde{i} + \widetilde{{\textstyle\frac{1}{2}}u^2} + \widetilde{{\textstyle\frac{1}{2}}v^2} - \rho\,\frac{\tilde{p}_\rho}{\tilde{p}_i}\right] + \tilde{\alpha}_4(X\tilde{u} + Y\tilde{v})\end{aligned} \tag{D4}$$

$$\Delta(\rho U) = \tilde{\alpha}_1(\tilde{U} + \tilde{ad}) + \tilde{\alpha}_2(\tilde{U} - \tilde{ad}) + \tilde{\alpha}_3\tilde{U} \tag{D5}$$

$$\begin{aligned}\Delta(Yp + \rho uU) &= \tilde{\alpha}_1(\tilde{U} + \tilde{ad})\left[\tilde{u} + \frac{\widetilde{aY}}{d}\right] + \tilde{\alpha}_2(\tilde{U} - \tilde{ad})\left[\tilde{u} - \frac{\widetilde{aY}}{d}\right] \\ &\quad + \tilde{\alpha}_3\widetilde{Uu} + \tilde{\alpha}_4\tilde{U}X\end{aligned} \tag{D6}$$

$$\begin{aligned}\Delta(-Xp + \rho vU) &= \tilde{\alpha}_1(\tilde{U} + \tilde{ad})\left[\tilde{v} - \frac{\widetilde{aX}}{d}\right] + \tilde{\alpha}_2(\tilde{U} - \tilde{ad})\left[\tilde{v} + \frac{\widetilde{aX}}{d}\right] \\ &\quad + \tilde{\alpha}_3\widetilde{Uv} + \tilde{\alpha}_4\tilde{U}Y\end{aligned} \tag{D7}$$

and

$$\Delta\left[U(e+p)\right] = \Delta\left[U(\rho i + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + p)\right]$$

$$= \tilde{\alpha}_1(\tilde{U} + \tilde{a}d)\left[\tilde{p}/\tilde{\rho} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2 + \frac{\widetilde{aU}}{d}\right]$$

$$+ \tilde{\alpha}_2(\tilde{U} - \tilde{a}d)\left[\tilde{p}/\tilde{\rho} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}v^2 - \frac{\widetilde{aU}}{d}\right]$$

$$+ \tilde{\alpha}_3 U\left[\tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2 - \rho\,\frac{\tilde{p}_\rho}{\tilde{p}_i}\right]$$

$$+ \tilde{\alpha}_4\tilde{U}(X\tilde{u} + Y\tilde{v}) \quad , \qquad \text{(D8)}$$

where

$$U = Yu - Xv \quad , \qquad \text{(D9a)}$$

$$\tilde{a}^2 = \frac{\widetilde{pp_i}}{\tilde{\rho}^2} + \tilde{p}_\rho \quad , \qquad \text{(D9b)}$$

$$d = \sqrt{X^2 + Y^2} \quad , \qquad \text{(D10)}$$

X,Y are constant and $\tilde{\alpha}_i$, $i = 1,\ldots 4$ are given by

$$\tilde{\alpha}_{1,2} = \frac{1}{2\tilde{a}^2}\left[\Delta p \pm \frac{\widetilde{\rho a}}{d}\,\Delta U\right] \qquad \text{(D11a-b)}$$

$$\tilde{\alpha}_3 = \Delta\rho - \frac{\Delta p}{\tilde{a}^2} \qquad \text{(D11c)}$$

and

$$\tilde{\alpha}_4 = \rho\,\frac{(X\Delta u + Y\Delta v)}{d^2} \quad . \qquad \text{(D11d)}$$

Firstly, equation (D1) is satisfied by any average, and equations (D1-D3) yield

$$\Delta(\rho u) = \tilde{\rho}\Delta u + \tilde{u}\Delta\rho \qquad \text{(D12)}$$

$$\Delta(\rho v) = \tilde{\rho}\Delta v + \tilde{v}\Delta\rho \quad , \qquad \text{(D13)}$$

whilst equation (D5) gives

$$\Delta(\rho U) = \tilde{\rho}\Delta U + \tilde{U}\Delta\rho \quad . \qquad \text{(D14)}$$

From equations (D6) and (D7) we obtain, respectively,

$$\Delta(\rho u U) \;=\; \widetilde{uU}\Delta\rho + \widetilde{u\rho}\Delta U + \widetilde{\rho U}\Delta u \qquad \text{(D15)}$$

and

$$\Delta(\rho v U) \;=\; \widetilde{vU}\Delta\rho + \widetilde{v\rho}\Delta U + \widetilde{\rho U}\Delta v \;, \qquad \text{(D16)}$$

which combine to give

$$\Delta(\rho U^2) \;=\; \tilde{U}^2\Delta\rho + 2\widetilde{\rho U}\Delta U \quad . \qquad \text{(D17)}$$

Substituting for $\tilde{\rho}$ from equation (D14) into equation (D17) yields the following quadratic equation for $\tilde{U}$

$$\Delta\rho\tilde{U}^2 - 2\tilde{U}\Delta(\rho U) + \Delta(\rho U^2) \;=\; 0 \;. \qquad \text{(D18)}$$

Only one solution of equation (D18) is productive, namely

$$\tilde{U} \;=\; \frac{\Delta(\rho U) - \sqrt{(\Delta(\rho U))^2 - \Delta\rho\Delta(\rho U^2)}}{\Delta\rho}$$

i.e.

$$\tilde{U} \;=\; \frac{\sqrt{\rho_L}\,U_L + \sqrt{\rho_R}\,U_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \;, \qquad \text{(D19)}$$

and equation (D14) now gives

$$\tilde{\rho} \;=\; \frac{\Delta(\rho U) - \tilde{U}\Delta\rho}{\Delta U} \;=\; \sqrt{\rho_L\rho_R} \;. \qquad \text{(D20)}$$

Equations (D12) and (D13) give

$$\tilde{u} \;=\; \frac{\Delta(\rho u) - \tilde{\rho}\Delta u}{\Delta\rho} \;=\; \frac{\sqrt{\rho_L}\,u_L + \sqrt{\rho_R}\,u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad \text{(D21)}$$

and

$$\tilde{v} \;=\; \frac{\Delta(\rho v) - \tilde{\rho}\Delta v}{\Delta\rho} \;=\; \frac{\sqrt{\rho_L}\,v_L + \sqrt{\rho_R}\,v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \;. \qquad \text{(D22)}$$

We note that equations (D9a), (D19), (D21) and (D22) imply that

$$\tilde{U} = Y\tilde{u} - X\tilde{v} .$$

Two equations now remain, namely (D4) and (D8). If we employ equation (D9b), equation (D4) yields

$$\Delta(\rho i) - \tilde{i}\Delta\rho + \frac{\widetilde{\widetilde{\rho p_\rho}}}{\tilde{p}_i}\Delta\rho - \frac{\tilde{\rho}}{\tilde{p}_i}\Delta p = \tfrac{1}{2}\tilde{u}^2\Delta\rho + \tfrac{1}{2}\tilde{v}^2\Delta\rho + \widetilde{\widetilde{\rho u}}\Delta u + \widetilde{\widetilde{\rho v}}\Delta v - \Delta\left[\frac{\rho u^2}{2}\right] - \Delta\left[\frac{\rho v^2}{2}\right] , \quad \text{(D23)}$$

and equations (D20)-(D22) imply

$$\Delta(\rho u^2) = \tilde{u}^2\Delta\rho + 2\widetilde{\widetilde{u\rho}}\Delta u \quad \text{(D24)}$$

$$\Delta(\rho v^2) = \tilde{v}^2\Delta\rho + 2\widetilde{\widetilde{v\rho}}\Delta v \quad \text{(D25)}$$

so that equation (D23) becomes

$$\Delta(\rho i) - \tilde{i}\Delta\rho + \frac{\widetilde{\widetilde{\rho p_\rho}}}{\tilde{p}_i}\Delta\rho - \frac{\tilde{\rho}\Delta p}{\tilde{p}_i} = 0 . \quad \text{(D26)}$$

In addition, we note the following identities

$$\Delta(Up) = \tilde{U}\Delta p + \tilde{p}\Delta U\left[\frac{\sqrt{\rho_L}\,(p_L/\rho_L) + \sqrt{\rho_R}\,(p_R/\rho_R)}{\sqrt{\rho_L} + \sqrt{\rho_R}}\right] \quad \text{(D27)}$$

$$\Delta(\rho U z^2) = \frac{\widetilde{\widetilde{Uz^2}}}{2}\Delta\rho + \tilde{\rho}\,\tilde{U}\,\tilde{z}\Delta z + \frac{\tilde{\rho}\Delta U}{2}\left[\frac{\sqrt{\rho_L}\,z_L^2 + \sqrt{\rho_R}\,z_R^2}{\sqrt{\rho_L} + \sqrt{\rho_R}}\right] , \quad z = u \text{ or } v , \quad \text{(D28a-b)}$$

so that after using equation (D26) we find that equation (D8) yields

$$\Delta(\rho Ui) + \tilde{\rho}\Delta U\left[\frac{\sqrt{\rho_L}\left[p_L/\rho_L + \frac{1}{2}u_L^2 + \frac{1}{2}v_L^2\right] + \sqrt{\rho_R}\left[p_R/\rho_R + \frac{1}{2}u_R^2 + \frac{1}{2}v_R^2\right]}{\sqrt{\rho_L} + \sqrt{\rho_R}}\right]$$

$$- \tilde{U}\Delta(\rho i) \;=\; \tilde{\rho}\left[\tilde{p}/\tilde{\rho} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2\right]\Delta U \quad . \tag{D29}$$

Now

$$\Delta(\rho Ui) - \tilde{U}\Delta(\rho i) \;=\; \tilde{\rho}\Delta U\left[\frac{\sqrt{\rho_L}\, i_L + \sqrt{\rho_R}\, i_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}\right] \tag{D30}$$

and thus equation (D29) gives a mean enthalpy $\tilde{H}$ given by

$$\tilde{H} \;=\; \tilde{p}/\tilde{\rho} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2 \tag{D31}$$

where

$$\tilde{H} \;=\; \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \quad . \tag{D32}$$

Furthermore, since $H = p/\rho + i + \frac{1}{2}u^2 + \frac{1}{2}v^2$ then H and i are related linearly and in view of equation (D31) this suggests that i is averaged in the same way as H , i.e. we choose

$$\tilde{i} \;=\; \frac{\sqrt{\rho_L}\, i_L + \sqrt{\rho_R}\, i_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \quad . \tag{D33}$$

Finally, following equation (D33) we find that

$$\Delta(\rho i) - \tilde{i}\Delta\rho \;=\; \tilde{\rho}\Delta i$$

so that equation (D26) becomes

$$\Delta p \;=\; \tilde{p}_\rho \Delta\rho + \tilde{p}_i \Delta i$$

giving a means of determining averages $\tilde{p}_\rho$ and $\tilde{p}_i$.

APPENDIX E

In this appendix we derive an approximate Riemann solver for the specific case of an ideal gas. In this case we generalise the Riemann solver of Roe[1] for cartesian coordinates to body-fitted coordinates using an 'intermediate' or 'parameter' vector.

If the gas is ideal then the equation of state is given by

$$p = (\gamma - 1)\rho i \tag{E1}$$

so that equations (5), (13) and (12) become, respectively,

$$e = \frac{p}{\gamma-1} + \tfrac{1}{2}\rho(u^2 + v^2) \tag{E2}$$

$$a^2 = \frac{\gamma p}{\rho} \tag{E3}$$

and

$$H = \frac{a^2}{\gamma-1} + \tfrac{1}{2}(u^2 + v^2) \ . \tag{E4}$$

Firstly, we define a 'parameter' vector

$$\underset{\sim}{u} = \rho^{\frac{1}{2}}(1, u, v, H)^T \tag{E5}$$

so that $\underset{\sim}{w}$ and $\underset{\sim}{F}$ are quadratic in $\underset{\sim}{u}$, specifically

$$\underset{\sim}{w} = \left[u_1^2, u_1u_2, u_1u_3, \frac{u_1u_4}{\gamma} + \frac{(\gamma-1)}{2\gamma}u_2^2 + \frac{(\gamma-1)}{2\gamma}u_3^2\right]^T \tag{E6}$$

$$\begin{aligned}\underset{\sim}{F} = \Big[&Yu_1u_2 - Xu_1u_3,\ Y\frac{(\gamma-1)}{\gamma}u_1u_4 + \frac{(\gamma+1)}{2\gamma}u_2^2 - \frac{(\gamma+1)}{2\gamma}Yu_2^2 - Yu_2u_3\ , \\ &- X\frac{(\gamma-1)}{\gamma}u_1u_4 - \frac{(\gamma+1)}{2\gamma}Xu_3^2 + \frac{(\gamma-1)}{2\gamma}Xu_2^2 + Yu_2u_3, Yu_2u_4 - Yu_2u_4\Big] \ .\end{aligned} \tag{E7}$$

Now since

$$\Delta(ab) = \bar{a}\Delta b + \bar{b}\Delta a \tag{E8}$$

where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$ and the overbar denotes the arithmetic mean, i.e. $\bar{\cdot} = \frac{1}{2}\left[(\cdot)_L + (\cdot)_R\right]$, then equations (E6) and (E7) give

$$\Delta \underset{\sim}{w} = B\Delta \underset{\sim}{u} \tag{E9}$$

and

$$\Delta \underset{\sim}{F} = C\Delta \underset{\sim}{u} \tag{E10}$$

where

$$B = \begin{bmatrix} 2\bar{u}_1 & 0 & 0 & 0 \\ \bar{u}_2 & \bar{u}_1 & 0 & 0 \\ \bar{u}_3 & 0 & \bar{u}_1 & 0 \\ \frac{\bar{u}_4}{\gamma} & \frac{(\gamma-1)}{\gamma}\bar{u}_2 & \frac{(\gamma-1)}{\gamma}\bar{u}_3 & \frac{\bar{u}_1}{\gamma} \end{bmatrix} \tag{E11}$$

and

$$C = \begin{bmatrix} Y\bar{u}_2 - X\bar{u}_3 & Y\bar{u}_1 & -X\bar{u}_1 & 0 \\ Y\frac{(\gamma-1)}{\gamma}\bar{u}_4 & \frac{(\gamma+1)}{\gamma}Y\bar{u}_2 - X\bar{u}_3 & -\frac{(\gamma-1)}{\gamma}Y\bar{u}_3 - X\bar{u}_2 & Y\frac{(\gamma-1)}{\gamma}\bar{u}_1 \\ -X\frac{(\gamma-1)}{\gamma}\bar{u}_4 & \frac{(\gamma-1)}{\gamma}X\bar{u}_2 + Y\bar{u}_3 & -\frac{(\gamma+1)}{\gamma}X\bar{u}_3 + Y\bar{u}_2 & -X\frac{(\gamma-1)}{\gamma}\bar{u}_1 \\ 0 & Y\bar{u}_4 & -X\bar{u}_4 & Y\bar{u}_2 - X\bar{u}_3 \end{bmatrix}. \tag{E12}$$

(N.B. X and Y are constant). From equations (E9) and (E10) we obtain

$$\Delta \underset{\sim}{F} = CB^{-1}\Delta \underset{\sim}{w} , \tag{E13}$$

and thus the required property

$$\Delta \underset{\sim}{F} = \tilde{A}\Delta \underset{\sim}{w} \tag{E14}$$

gives an approximate Jacobian $\tilde{A}$ given by

$$\tilde{A} = CB^{-1} . \tag{E15}$$

If we write

$$\tilde{u} = \frac{\bar{u}_2}{\bar{u}_1} = \frac{\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} , \tag{E16}$$

$$\tilde{v} = \frac{\bar{u}_3}{\bar{u}_1} = \frac{\sqrt{\rho_L}\, v_L + \sqrt{\rho_R}\, v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{E17}$$

and

$$\tilde{H} = \frac{\bar{u}_4}{\bar{u}_1} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{E18}$$

then the eigenvalues of $\tilde{A}$ are found to be

$$\tilde{\lambda}_i = \tilde{U} \pm \tilde{a}d,\ \tilde{U},\ \tilde{U} \tag{E19a-d}$$

with corresponding eigenvectors

$$\tilde{e}_i = \begin{bmatrix} 1 \\ \tilde{u} \pm \frac{\tilde{a}Y}{d} \\ \tilde{v} \mp \frac{\tilde{a}X}{d} \\ \tilde{H} \pm \frac{\tilde{a}\tilde{U}}{d} \end{bmatrix} , \begin{bmatrix} 1 \\ \tilde{u} \\ \tilde{v} \\ \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 \end{bmatrix} , \begin{bmatrix} 1 \\ X \\ Y \\ X\tilde{u} + Y\tilde{v} \end{bmatrix} \tag{E20a-d}$$

where

$$\tilde{U} = Y\tilde{u} - X\tilde{v} \tag{E21}$$

$$d = \sqrt{X^2 + Y^2} \tag{E22}$$

and

$$\tilde{a}^2 = (\gamma-1)(\tilde{H} - \tfrac{1}{2}\tilde{u}^2 - \tfrac{1}{2}\tilde{v}^2) \quad \text{(E23)}$$

(N.B. In the ideal case $p = (\gamma-1)\rho i$, $p_\rho = (\gamma-1)i$, $p_i = (\gamma-1)\rho$ so that the third component of the continuous eigenvector $\tilde{e}_3$ becomes $\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2$.)

Finally, to complete the Riemann solver we need to project $\Delta \underset{\sim}{w}$ onto the eigenvectors $\underset{\sim}{\tilde{e}}_i$ as

$$\Delta \underset{\sim}{w} = \sum_{i=1}^{4} \tilde{\alpha}_i \underset{\sim}{\tilde{e}}_i \quad \text{(E24)}$$

so that $\Delta \underset{\sim}{F}$ can be decomposed into the four waves as

$$\Delta \underset{\sim}{F} = \tilde{A} \Delta \underset{\sim}{w} = \sum_{i=1}^{4} \tilde{\lambda}_i \tilde{\alpha}_i \underset{\sim}{\tilde{e}}_i \ , \quad \text{(E25)}$$

since $\tilde{A}$ has eigenvalues $\tilde{\lambda}_i$ with eigenvectors $\underset{\sim}{\tilde{e}}_i$. Writing out equation (E24) we have

$$\Delta\rho = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 \quad \text{(E26)}$$

$$\Delta(\rho u) = \tilde{\alpha}_1\left[\tilde{u} + \frac{\tilde{a}Y}{d}\right] + \tilde{\alpha}_2\left[\tilde{u} - \frac{\tilde{a}Y}{d}\right] + \tilde{\alpha}_3\tilde{u} + \tilde{\alpha}_4 X \quad \text{(E27)}$$

$$\Delta(\rho v) = \tilde{\alpha}_1\left[\tilde{v} - \frac{\tilde{a}Y}{d}\right] + \tilde{\alpha}_2\left[\tilde{v} + \frac{\tilde{a}Y}{d}\right] + \tilde{\alpha}_3\tilde{v} + \tilde{\alpha}_4 X \quad \text{(E28)}$$

$$\begin{aligned}\Delta e &= \frac{\Delta p}{\gamma-1} + \Delta\left[\rho\frac{u^2}{2}\right] + \Delta\left[\rho\frac{v^2}{2}\right] \\ &= \tilde{\alpha}_1\left[\tilde{H} + \frac{\widetilde{aU}}{d}\right] + \tilde{\alpha}_2\left[\tilde{H} - \frac{\widetilde{aU}}{d}\right] + \tilde{\alpha}_3\left[\tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2\right] \\ &\quad + \tilde{\alpha}_4(X\tilde{u} + Y\tilde{v}) \ . \end{aligned} \quad \text{(E29)}$$

From equations (E26)-(E28) we have

$$\Delta(\rho u) - \tilde{u}\Delta\rho = (\tilde{\alpha}_1 - \tilde{\alpha}_2)\frac{\tilde{a}Y}{d} + \tilde{\alpha}_4 X \qquad \text{(E30a)}$$

and

$$\Delta(\rho v) - \tilde{v}\Delta\rho = (\tilde{\alpha}_1 - \tilde{\alpha}_2)\frac{\tilde{a}X}{d} + \tilde{\alpha}_4 Y \qquad \text{(E30b)}$$

which combine to give

$$\tilde{\alpha}_4 d^2 = X\left[\Delta(\rho u) - \tilde{u}\Delta\rho\right] + Y\left[\Delta(\rho v) - \tilde{v}\Delta\rho\right] \qquad \text{(E31)}$$

and

$$\Delta(\rho U) - \tilde{U}\Delta\rho = (\tilde{\alpha}_1 - \tilde{\alpha}_2)\tilde{\alpha}d\ . \qquad \text{(E32)}$$

Using equations (E23) and (E26) equation (E29) gives

$$\frac{\Delta p}{\gamma-1} + \Delta\left[\rho\frac{u^2}{2}\right] + \Delta\left[\rho\frac{v^2}{2}\right] = (\tilde{\alpha}_1 + \tilde{\alpha}_2)\frac{\tilde{a}^2}{\gamma-1}\tfrac{1}{2}\tilde{u}^2\Delta\rho + \tfrac{1}{2}\tilde{v}^2\Delta\rho$$

$$+ (\tilde{\alpha}_1 - \tilde{\alpha}_2)\frac{\tilde{a}\tilde{U}}{d} + \tilde{\alpha}_4(X\tilde{u} + Y\tilde{v})\ , \qquad \text{E33)}$$

and on substituting for $\tilde{\alpha}_4 X$ from equation (E30a) and for $\tilde{\alpha}_4 Y$ from equation (E30b), equation (E33) becomes

$$\frac{\Delta p}{\gamma-1} + \Delta\left[\rho\frac{u^2}{2}\right] + \Delta\left[\rho\frac{v^2}{2}\right] = (\tilde{\alpha}_1 + \tilde{\alpha}_2)\frac{\tilde{a}^2}{\gamma-1} + \tilde{u}\Delta(\rho u) + \tilde{v}\Delta(\rho v)$$

$$- \tfrac{1}{2}\tilde{u}^2\Delta\rho - \tfrac{1}{2}\tilde{v}^2\Delta\rho\ . \qquad \text{(E34)}$$

Combining equations (E32) and (E34) gives

$$\tilde{\alpha}_{1,2} = \frac{1}{2\tilde{a}^2}\left[\Delta p \pm \frac{\tilde{a}(\Delta(\rho U) - \tilde{U}\Delta\rho}{d} \pm \frac{(\gamma-1)}{2}\left[\Delta(\rho u^2) + \Delta(\rho v^2) - 2\tilde{u}\Delta(\rho u) - 2\tilde{v}\Delta(\rho v) + \tilde{u}^2\Delta\rho + \tilde{v}^2\Delta\rho\right]\right]\ . \qquad \text{(E35)}$$

In view of the expressions for $\tilde{u},\tilde{v}$ and $\tilde{U}$ given by equations (E16), (E17) and (E21) we can simplify

$$\Delta(\rho u) - \tilde{u}\Delta\rho = \tilde{\rho}\Delta u \qquad \text{(E36)}$$

$$\Delta(\rho v) - \tilde{v}\Delta\rho = \tilde{\rho}\Delta v \qquad \text{(E37)}$$

and

$$\Delta(\rho U) - \tilde{U}\Delta\rho = \tilde{\rho}\Delta U \qquad \text{(E38)}$$

where we have defined

$$\tilde{\rho} = \sqrt{\rho_L \rho_R} \ , \qquad \text{(E39)}$$

and in addition we have

$$\Delta(\rho u^2) - 2\tilde{u}\Delta(\rho u) + \tilde{u}^2\Delta\rho = 0 \qquad \text{(E40)}$$

$$\Delta(\rho v^2) - 2\tilde{v}\Delta(\rho v) + \tilde{v}^2\Delta\rho = 0 \ . \qquad \text{(E41)}$$

Thus

$$\tilde{\alpha}_{1,2} = \frac{1}{2\tilde{a}^2}\left[\Delta p \pm \tilde{\rho}\frac{\tilde{a}}{d}\Delta U\right] \qquad \text{(E42)}$$

and from equation (E1)

$$\tilde{\alpha}_3 = \Delta\rho - (\tilde{\alpha}_1 + \tilde{\alpha}_2) = \Delta\rho - \frac{\Delta p}{\tilde{a}^2} \ , \qquad \text{(E43)}$$

and finally, from equations (E31), (E36) and (E37) we obtain

$$\tilde{\alpha}_4 = \frac{\tilde{\rho}(X\Delta u + Y\Delta v)}{d^2} \ . \qquad \text{(E44)}$$

APPENDIX F

In this appendix we look in detail at the analytic grid transformation used for the problem of flow past a circular cylinder.

The grid transformation used to obtain an 'O'-type computational mesh is given by

$$x = \xi \cos \eta \,, \qquad y = \xi \sin \eta \qquad \text{(F1a-b)}$$

or equivalently

$$\xi = \sqrt{x^2 + y^2} \,, \qquad \eta = \text{arc tan } (y/x) \,. \qquad \text{(F2a-b)}$$

(N.B. ξ and η correspond to plane polar coordinates R and ϕ, respectively.)

From equations (F1a-b) we have

$$x_\eta = \xi \sin \eta \,, \qquad y_\eta = \xi \cos \eta$$

and

$$x_\xi = \cos \eta \,, \qquad y_\xi = \sin \eta \,.$$

Thus, when we update along a line $\eta = \eta_0$ we take

$$x_\eta^{j-\frac{1}{2}} = -\,\xi_{j-\frac{1}{2}} \sin \xi_0 \,, \qquad y_\eta^{j-\frac{1}{2}} \cos \eta_0$$

$$x_\xi^{j-\frac{1}{2}} = \cos \eta_0 \,, \qquad y_\xi^{j-\frac{1}{2}} = \sin \eta_0$$

where $\xi_{j-\frac{1}{2}} = \frac{1}{2}(\xi_{j-1} + \xi_j)$. In particular, we have

$$(x_\eta^{j-\frac{1}{2}})^2 + (y_\eta^{j-\frac{1}{2}})^2 = \xi_{j-\frac{1}{2}}^2$$

and

$$J_{j-\frac{1}{2}} = x_\xi^{j-\frac{1}{2}} y_\eta^{j-\frac{1}{2}} - x_\eta^{j-\frac{1}{2}} y_\xi^{j-\frac{1}{2}} = \xi_{j-\frac{1}{2}} \,.$$

Similarly, when we update along a line $\xi = \xi_0$ we take

$$x_\eta^{j-\frac{1}{2}} = -\xi_0 \sin \eta_{j-\frac{1}{2}}, \qquad y_\eta^{j-\frac{1}{2}} = \xi_0 \cos \eta_{j-\frac{1}{2}}$$

and

$$x_\xi^{j-\frac{1}{2}} = \cos \eta_{j-\frac{1}{2}}, \qquad y_\xi^{j-\frac{1}{2}} = \sin \eta_{j-\frac{1}{2}}$$

where $\eta_{j-\frac{1}{2}} = \frac{1}{2}(\eta_{j-1} + \eta_j)$. In particular, we have

$$(x_\xi^{j-\frac{1}{2}})^2 + (y_\xi^{j-\frac{1}{2}})^2 = 1$$

and

$$J_{j-\frac{1}{2}} = x_\xi^{j-\frac{1}{2}} y_\eta^{j-\frac{1}{2}} - x_\eta^{j-\frac{1}{2}} y_\xi^{j-\frac{1}{2}} = \xi_0 .$$

APPENDIX G

In this appendix we describe how the boundary conditions are applied for the problem of flow past a circular cylinder.

(i) Along the boundary $\phi = \pi/2$ supersonic boundary conditions apply, i.e. all waves are leaving the region of computation and therefore no special treatment is required.

(ii) Along the boundary $\phi = 3\pi/2$ we apply symmetry boundary conditions, i.e. for updating along an arc $R = R_0$ we position a cell interface at $\phi = 3\pi/2$ and set ρ, u, v and p to have the same values on the exterior of the region as on the interior of the region. In this way $\tilde{\alpha}_i = 0$ and thus $\Delta \underset{\sim}{F} = 0$.

(iii) Along the boundary $R = R_{max}$ supersonic inflow applies, i.e. we prescribe the initial data for ρ, u, v and p .

(iv) Along the boundary $R = 0.5$ we apply rigid wall boundary conditions, i.e. for updating along a line $\phi = \phi_0$ we position a cell interface at $R = 0.5$ and set ρ, p and tangential velocity to have the same values on the exterior of the region as on the interior of the region. In addition, on the exterior of the region we set the normal velocity to have equal magnitude but opposite sign to that on the interior of the region. Specifically, if we denote by subscripts E and I to mean exterior and interior values respectively, then

$$\begin{aligned}(\text{tangential velocity})_I &= u_I \cos(\phi_0 - \pi/2) + v_I \cos(\pi - \phi_0) \\ &= u_I \sin\phi_0 - v_I \cos\phi_0 \ ,\end{aligned}$$

$$(\text{normal velocity})_I = u_I\cos(\pi - \phi_0) - v_I\cos(\phi_0 - \pi/2)$$

$$= - u_I\cos\phi_0 - v_I\sin\phi_0 \ .$$

and thus

$$(\text{tangential velocity})_E = u_E\sin\phi_0 - v_E\cos\phi_0 \ .$$

$$(\text{normal velocity})_E = - u_E\cos\phi_0 - v_E\sin\phi_0 \ .$$

We set

$$\rho_E = \rho_I \ , \qquad p_E = p_I$$

and

$$(\text{tangential velocity})_E = (\text{tangential velocity})_I$$

$$(\text{normal velocity})_E = - (\text{normal velocity})_I$$

which imply

$$u_E\sin\phi_0 - v_E\cos\phi_0 = u_I\sin\phi_0 - v_I\cos\phi_0$$

$$- u_E\cos\phi_0 - v_E\sin\phi_0 = - (- u_I\cos\phi_0 - v_I\sin\phi_0)$$

and hence

$$u_E = - u_I\cos2\phi_0 - v_I\sin2\phi_0$$

$$v_E = - u_I\sin2\phi_0 + v_I\cos2\phi_0 \ .$$

In particular,

$$e_E = \frac{p_E}{\gamma-1} + \tfrac{1}{2}\rho_E(u_E^2 + v_E^2)$$

$$= \frac{p_I}{\gamma-1} + \tfrac{1}{2}\rho_I\Big[(- u_I\cos2\phi_0 - v_I\sin2\phi_0)^2$$

$$+ (- u_I\sin2\phi_0 + v_I\cos2\phi_0)^2\Big]$$

$$= \frac{p_I}{\gamma-1} + \tfrac{1}{2}\rho_I(u_I^2 + v_I^2) = e_I \ .$$