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SHOCK BEHAVIOUR AND DIFFUSION

PETER C. SAMUELS

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U N I V E R S I T Y   O F   R E A D I N G

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Department of Mathematics

University of Reading

P O Box 220

Reading RG6 2AX

UK

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Abstract

The report concerns an attempt to gain understanding into the formation, shape and propagation of shock waves ('thick' and 'thin', theoretical and numerical) as opposed to modelling their internal structure; although it has not been possible to totally dissociate the former from the latter.

The first four sections provide a coherent progression of ideas. In the first, canonical forms of hyperbolic systems of conservation laws are provided. As the systems become more complex, the methods become more difficult and the canonical forms become weaker. Four different methods are given.

In the second section, the breakdown of smooth solutions is investigated for a single characteristic equation and a pair of Riemann invariant equations. A method is given for obtaining information at the breaking point. This is then used to transform the manifold equation into the standard form of the cusp catastrophe. Furthermore, an asymptotic analysis is provided for the flow in this region.

In the third chapter, extensions to the Cole-Hopf transformation for Burgers' equation are obtained. A considerable amount of theory is derived to show that the generalisations given really are the best that can be done analytically. Nonlinearity is discussed and individual cases and examples are provided. Implications for numerical schemes are conjectured.

The fourth chapter represents a synthesis of the previous three as we here consider the solution to model equations with a limitingly small amount of diffusion. The Cole-Hopf transformation given in the previous

section is asymptotically expanded. This leads to the idea of incorporating diffusion into the method of characteristics. Next it is shown how the characteristic unfolding function is related to the solution of Burgers' equation near the 'diffusive breaking point'. The methods of the section seem to be similar to the philosophy of shock fitting, but in a diffusive setting. These similarities are discussed along with their numerical applications.

The fifth section is separate from the proceeding argument. In it, an improvement to the current theory behind front tracking is given. This involves deriving a second type of Lagrangian position variable (and hence velocity). From this derivation it is possible to obtain shock velocities, for two-dimensional flow, in a sensible way.

Contents

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1. Transformation Methods for Hyperbolic Systems

1.0 Introduction

This section is limited largely to systems of two independent variables. These have been identified with space and time for consistency. However, there is no reason why the theory may not apply to systems modelling two-dimensional steady flow.

The basic method of this section is the canonical transformation of systems of equations into a form lending itself to simple analysis and solution. For a single equation, no transformation is required and the method of solution is called the method of characteristics. For several equations, a canonical transformation is required. The new canonical system will retain some of the properties of the single equation.

1.1 The Method of Characteristics

Let us consider a single equation

$$u_t + f(u)_x = 0 \, , \tag{1.1}$$

with

$$u(x,0) = u_o(x) \, . \tag{1.2}$$

Let

$$\lambda(u) = \frac{df(u)}{du} \, , \tag{1.3}$$

hence

$$u_t + \lambda(u)\, u_x = 0 \, . \tag{1.4}$$

Now, we wish to solve equation (1.4) over the half-space $\{(x,t) : x \in \mathbb{R},\ t > 0\}$. This may be achieved by transforming from

Eulerian co-ordinates to Lagrangian co-ordinates.

To this end, let $\{\Gamma(\xi) = \xi \in \mathbb{R}\}$ be the set of curves defined as follows:

$$\left.\begin{aligned} \Gamma(\xi):\quad \frac{dx}{dt} &= \lambda(u(x(\xi,t),t)) \, , \\ \text{with} \qquad x(\xi,0) &= \xi \end{aligned}\right\} \tag{1.5}$$

But, from equation (1.4), u is constant along $\Gamma(\xi)$. Thus we may re-define the slope of $\Gamma(\xi)$ by

$$\frac{dx}{dt} = \lambda(u_o(\xi)) \tag{1.6}$$

This may now be integrated to give the characteristic equation

$$x = \xi + \lambda(u_o(\xi))t \; . \tag{1.7}$$

Thus, the solution may be obtained by geometrically tracing the characteristic (see figure 1) for $t > 0$ by equation (1.7) and using the condition derived above that

$$u = u_o(\xi) \quad \text{on} \quad \Gamma(\xi) \tag{1.8}$$

This method of solution is called the method of characteristics (see [1]).

1.2 The Theory of Riemann Invariants

The theory of Riemann Invariants is not applicable to a single equation (it can be thought of as being equivalent to the method of characteristics in this case). We therefore begin this subsection by introducing some notation for a system of two equations

Let the system be defined by

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + A(u,v) \begin{bmatrix} u \\ v \end{bmatrix}_x = \underline{0} . \tag{1.9}$$

where $A(u,v)$ is a 2×2 matrix. Suppose the initial data is

$$\left.\begin{aligned} u(x,0) &= u_o(x) \\ v(x,0) &= v_o(x) \end{aligned}\right\} . \tag{1.10}$$

Now let the eigenvalues of A be $\lambda(u,v)$, $\mu(u,v)$; the left eigenvectors be $\underline{l}(u,v)$, $\underline{m}(u,v)$; and the right eigenvectors be $\underline{r}(u,v)$, $\underline{s}(u,v)$. Thus we obtain the relations:

$$\begin{bmatrix} \underline{l}^T \\ \underline{m}^T \end{bmatrix} A = \begin{bmatrix} \lambda \underline{l}^T \\ \mu \underline{m}^T \end{bmatrix} \tag{1.11}$$

$$A[\underline{r},\underline{s}] = [\lambda \underline{r},\mu \underline{s}] . \tag{1.12}$$

For simplicity, we shall assume that all the quantities are real, a sufficient condition for which is that A is a real symmetric matrix. Let M be the matrix of right eigenvectors and Λ the diagonal matrix of eigenvalues, i.e.

$$M = [\underline{r},\underline{s}] . \tag{1.13}$$

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \tag{1.14}$$

Equation (1.12) may now be rewritten in the form

$$AM = M\Lambda \ . \tag{1.15}$$

Pre-multiplying by M^{-1} gives

$$M^{-1}AM = \Lambda \ . \tag{1.16}$$

Post-multiplying by M^{-1} gives

$$M^{-1}A = \Lambda M^{-1} \ . \qquad (1.16)$$

Hence $\underline{l}$ and $\underline{m}$ may be defined such that

$$M^{-1} = \begin{bmatrix} \underline{l}^T \\ \underline{m}^T \end{bmatrix} , \tag{1.18}$$

as this is consistent with equations (1.11) and (1.17). Finally, the following identity is noted

$$M^{-1}M = I \Rightarrow \begin{bmatrix} \underline{l}^T \\ \underline{m}^T \end{bmatrix} [\underline{r},\underline{s}] = I ,$$

i.e.

$$\begin{bmatrix} \underline{l}^T\underline{r} & \underline{l}^T\underline{s} \\ \underline{m}^T\underline{r} & \underline{m}^T\underline{s} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{1.19}$$

This is the well-known biorthogonality property of matrices.

Now, consider a transformation of variables

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} \theta \\ \phi \end{bmatrix}, \quad \text{such that}$$

$$\left.\begin{aligned} \frac{\partial\theta}{\partial \underline{u}} \underline{s} &= 0 \\ \frac{\partial\phi}{\partial \underline{u}} \underline{r} &= 0 \end{aligned}\right\} . \qquad (1.20)$$

where $\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix}$.

It can be shown ([2], p.95) that this transformation is well-defined at least locally.

From equations (1.19) and (1.20) we infer that

$$\left.\begin{aligned} \frac{\partial\theta}{\partial \underline{u}} &= \alpha(\underline{u})\underline{l}^T \\ \frac{\partial\phi}{\partial \underline{u}} &= \beta(\underline{u})\underline{m}^T \end{aligned}\right\} . \qquad (1.21)$$

for some functions α, β .

Equation (1.9) may be written in the simplified form

$$\underline{u}_t + A\underline{u}_x = \underline{0} . \qquad (1.22)$$

Now, consider pre-multiplying equation (1.22) by $\frac{\partial\theta}{\partial \underline{u}}$:

$$\frac{\partial\theta}{\partial \underline{u}} \underline{u}_t + \frac{\partial\theta}{\partial \underline{u}} A\underline{u}_x = 0 . \qquad (1.23)$$

Using equation (1.21), this gives

$$\frac{\partial \theta}{\partial \underline{u}} \underline{u}_t + \alpha \underline{l}^T A \underline{u}_x = 0 \ . \qquad (1.24)$$

Using equation (1.11) we obtain

$$\frac{\partial \theta}{\partial \underline{u}} \underline{u}_t + \alpha \lambda \underline{l}^T \underline{u}_x = 0 \ . \qquad (1.25)$$

Substituting back from equation (1.21) gives

$$\frac{\partial \theta}{\partial \underline{u}} \underline{u}_t + \lambda \frac{\partial \theta}{\partial \underline{u}} \underline{u}_x = 0 \ . \qquad (1.26)$$

Finally, using the chain rule,

$$\theta_t + \lambda \theta_x = 0 \qquad (1.27)$$

is obtained.

An exactly analogous argument may be used for ϕ to obtain

$$\phi_t + \mu \phi_x = 0 \ . \qquad (1.28)$$

θ and ϕ are called the Riemann invariants.

It can be seen that equations (1.27) and (1.28) are very similar in structure to equation (1.4) for the method of characteristics. This similarity may be exploited to an extent in an analogous transformation from Eulerian to Lagrangian co-ordinates. For simplicity, only the first equation is transformed.

Let $\{\Gamma(\xi) : \xi \in \mathbb{R}\}$ be defined as follows:

$$\left.\begin{aligned} \Gamma(\xi) \ : \ \frac{dx}{dt} &= \lambda(\theta(x(\xi,t),t) \ , \ \phi(x(\xi,t),t)), \\ \text{with} \qquad x(\xi,0) &= \xi \end{aligned}\right\} \tag{1.29}$$

Again in an analogous way to §1.1, equation (1.27) shows that θ is constant on $\Gamma(\xi)$. We write

$$\theta(x(\xi,t),t) = \theta(\xi,0) = \theta_o(\xi) \ . \tag{1.30}$$

Thus, the gradiant equation in (1.29) may be simplified to

$$\frac{dx}{dt} = \lambda(\theta_o(\xi) \ , \ \phi(x(\xi,t),t)) \ . \tag{1.31}$$

Unfortunately, this analogy between the method of characteristics and the theory of Riemann invariants does not extend isomorphically to systems of more than two equations.

Riemann invariants may, however, be defined for these systems. To this end, more notation is introduced.

Let a general system of n hyperbolic equations be given by

$$\underline{u}_t + A(\underline{u})\underline{u}_x = \underline{0} \ ; \tag{1.32}$$

where $A(u)$ is again taken to be a real symmetric matrix for simplicity. Let the eigenvalues of A be $\lambda_1(\underline{u}),\ldots, \lambda_n(\underline{u})$. Let the left eigenvectors of A be $l_1(\underline{u}),\ldots, l_n(\underline{u})$. Let the right eigenvectors of A be $\underline{r}_1(\underline{u}),\ldots, \underline{r}_n(\underline{u})$. Then it is easy to show that we have relations analogous to before, namely

if

$$M = [\underline{r}_1, \ldots, \underline{r}_n] \tag{1.33}$$

and

$$\Lambda = \mathrm{diag}\ \{\lambda_1, \ldots, \lambda_n\}\ , \tag{1.34}$$

then

$$M^{-1}AM = \Lambda \tag{1.35}$$

and $\underline{l}_1, \ldots, \underline{l}_n$ may be chosen such that

$$M^{-1} = \begin{bmatrix} \underline{l}_1^T \\ \underline{l}_n^T \end{bmatrix} . \tag{1.36}$$

Finally we have the generalised biorthogonality condition

$$\underline{l}_i^T\ \underline{r}_j = \delta_{ij} \qquad\qquad \forall i,j \leq n\ . \tag{1.37}$$

However, this is as far as the analogy goes. If we now consider a transformation $\underline{u} \mapsto \underline{\theta}$, a natural set of constraints on $\underline{\theta}$ in this case is

$$\frac{\partial\theta_i}{\partial\underline{u}} \cdot \underline{r}_i = 0\ , \qquad \text{(not summed over } i) \tag{1.38}$$

This is the opposite to that given in (1.20). If we now attempt to define the gradients $\frac{\partial\theta_i}{\partial\underline{u}}$ in terms of the left eigenvectors, we only have the equation

$$\frac{\partial\theta_i}{\partial\underline{u}} = \sum_j p_{ij}(\underline{u})\underline{l}_j^T\ . \tag{1.39}$$

for some coefficients p_{ij} . It is easy to see that $p_{ii} = 0 \ \forall i$, but this is now the only condition on the coefficients. Let P be the matrix with coefficients p_{ij} . It is possible to show that equation (1.32) is transformed to

$$\underline{\theta}_t + P \Lambda P^{-1} \underline{\theta}_x = \underline{0} . \tag{1.40}$$

But $P \Lambda P^{-1}$ is only diagonal when P is a permutation matrix multiplied by a diagonal matrix. This imposes a further $n^2 - 2n$ conditions on P leading to a badly-posed problem when $n > 2$.

1.3 Differential Invariance Theory

Consider again the system of hyperbolic equations as given in equation (1.32). Remultiplying the system by a left eigenvector of A gives

$$\underline{l}_i^T \underline{u}_t + \underline{l}_i^T A \underline{u}_x = \underline{0} . \tag{1.41}$$

Using the definition of the left eigenvectors gives

$$\underline{l}_i^T \underline{u}_t + \lambda_i \underline{l}_i^T \underline{u}_x = \underline{0} ,$$

i.e. $$\underline{l}_i^T \left[\frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial x}\right] \underline{u} = \underline{0} . \tag{1.42}$$

Let $\Gamma_i(\xi)$ be the characteristic curve satisfying the equation

$$\Gamma_i(\xi): \quad \left.\begin{aligned} \frac{dx}{dt} &= \lambda_i(\underline{u}(x,t)) \\ x\Big|_{t=0} &= \xi \end{aligned}\right\} . \tag{1.43}$$

Then we may write the differential equation

$$\underline{l}_i^T \, \underline{du} = 0 \qquad \text{on} \qquad \Gamma_i(\xi) , \tag{1.44}$$

which is a consequence of equation (1.42). This is the differential invariance theory. Note that $\underline{l}_i^T \, \underline{du}$ is generally not a perfect differential. This fact is analogous to the problems met in the previous sub-section for defining Riemann invariants for systems with $n > 2$.

An alternative way of arriving at equation (1.44) is as follows. Let us write the co-ordinate along $\Gamma_i(\xi)$ as $X_i(t;\xi)$, i.e.

$$x = X_i(t;\xi) \qquad \text{on} \qquad \Gamma_i(\xi) . \tag{1.45}$$

Now we may write

$$\underline{u} = \underline{u}(X_i(t;\xi),t) \qquad \text{on} \qquad \Gamma_i(\xi) . \tag{1.46}$$

So the time derivative of $\underline{u}$ along $\Gamma_i(\xi)$ is

$$\left.\frac{d\underline{u}}{dt}\right|_{\Gamma_i(\xi)} = \frac{d}{dt}(\underline{u}(X_i(t;\xi),t)) \tag{1.47}$$

$$= \underline{u}_t + \frac{\partial X_i}{\partial t}(t;\xi)\ \underline{u}_x$$

$$= \underline{u}_t + \lambda_i(\underline{u})\ \underline{u}_x\ , \tag{1.48}$$

from equation (1.43).

So, using equation (1.32), we obtain

$$\left.\frac{d\underline{u}}{dt}\right|_{\Gamma_i(\xi)} = (\lambda_i(\underline{u})I - A(\underline{u}))\underline{u}_x\ . \tag{1.49}$$

Premultiplying by $\underline{l}_i^T(\underline{u})$, we obtain

$$\underline{l}_i^T(\underline{u})\ \left.\frac{d\underline{u}}{dt}\right|_{\Gamma_i(\xi)} = 0\ , \tag{1.50}$$

which is equivalent to equation (1.44). Equation (1.49) gives a more general result, however.

1.4 Perturbations to Linear Transformations

The objective of this section is to attempt to preserve the canonical form of a hyperbolic system up to an appropriate order of accuracy near a constant state.

Let the constant state be $\underline{u}_o$ and let the variation be $\underline{v}$. Thus

$$\underline{u} = \underline{u}_o + \underline{v}\ . \tag{1.51}$$

Let the transformed variables again be $\underline{\theta}$, with an analogous approximation

$$\underline{\theta} = \underline{\theta}_o + \underline{\phi} \, . \tag{1.52}$$

Let the first order approximation to the transformation be given by

$$\underline{v} = M\underline{\phi} + O(|\underline{\phi}|^2) \, , \tag{1.53}$$

where M is now a general constant matrix. Substituting into equation (1.32), this gives

$$\underline{v}_t + A\underline{v}_x = \underline{0} \qquad \text{from (1.51),}$$

$$\Rightarrow \quad M\underline{\phi}_t + AM\underline{\phi}_x = O(|\underline{\phi}|^2) \qquad \text{from (1.53),}$$

$$\Rightarrow \quad \underline{\phi}_t + M^{-1}A(\underline{u}_o + \underline{v})M\underline{\phi}_x = O(|\underline{\phi}|^2) \, . \tag{1.54}$$

Now, a Taylor expansion of $A(\underline{u}_o + v)$, using (1.53) gives

$$A(\underline{u}_o + \underline{v}) = A(\underline{u}_o) + O(|\underline{\phi}|) \, . \tag{1.55}$$

So, assuming $\underline{\phi}_x$, $\underline{\phi}_t$ are both the same order as $\underline{\phi}$, we obtain

$$\underline{\phi}_t + M^{-1}A(\underline{u}_o)M\,\underline{\phi}_x = O(|\underline{\phi}|^2) \, . \tag{1.56}$$

So, if M is chosen such that

$$M^{-1}A(\underline{u}_o)M = \Lambda \, , \tag{1.57}$$

we have

$$\underline{\phi}_t + \Lambda\underline{\phi}_x = O(|\underline{\phi}|^2) \ . \tag{1.58}$$

NB, again, a sufficient condition for Λ to be real is that $A(\underline{u}_o)$ is symmetric.

This is the first order canonical form. Equation (1.57) shows that this is a tractable calculation.

Now, for the second order approximation, let the transformation be

$$v_i = \sum_j M_{ij}\phi_j + \sum_{j,k} N_{ijk}\phi_j\phi_k + O(|\underline{\phi}|^3) \ . \tag{1.59}$$

where M_{ij}, N_{ijk} are arbitrary and constant. Without loss of generality, the condition

$$N_{ijk} = N_{ikj} \qquad \forall j,k \tag{1.60}$$

may be imposed.

In suffix notation, equation (1.32) transforms to

$$\frac{\partial u_i}{\partial t} + \sum_j A_{ij} \frac{\partial u_j}{\partial x} = 0 \ . \tag{1.61}$$

Now, (1.51) and (1.59) imply

$$\frac{\partial u_i}{\partial t} = \frac{\partial v_i}{\partial t} = \sum_j M_{ij} \frac{\partial \phi_j}{\partial t} + \sum_{j,k} N_{ijk} \left\{\phi_j \frac{\partial \phi_k}{\partial t} + \phi_k \frac{\partial \phi_j}{\partial t}\right\} + O(|\underline{\phi}|^3) \ , \tag{1.62}$$

$$\Rightarrow \quad \frac{\partial u_i}{\partial t} = \sum_j M_{ij} \frac{\partial \phi_j}{\partial t} + 2 \sum_{j,k} N_{ijk} \phi_k \frac{\partial \phi_j}{\partial t} + O(|\underline{\phi}|^3) \ , \tag{1.63}$$

using (1.60).

Now let

$$P_{ij} = 2 \sum_k N_{ijk} \phi_k \ . \tag{1.64}$$

Equation (1.63) may now be converted to matrix form:

$$\frac{\partial \underline{u}}{\partial t} = M \frac{\partial \underline{\phi}}{\partial t} + P \frac{\partial \underline{\phi}}{\partial t} + O(|\underline{\phi}|^3) \ . \tag{1.65}$$

Thus, applying the same argument to the $\frac{\partial \underline{u}}{\partial x}$ term in equation (1.32), we obtain

$$(M+P) \frac{\partial \underline{\phi}}{\partial t} + A(M+P) \frac{\partial \underline{\phi}}{\partial x} = O(|\phi|^3) \tag{1.66}$$

$$\Rightarrow \quad M(I+M^{-1}P) \frac{\partial \underline{\phi}}{\partial t} + A(M+P) \frac{\partial \underline{\phi}}{\partial x} = O(|\phi|^3)$$

$$\Rightarrow \quad (I+M^{-1}P) \frac{\partial \underline{\phi}}{\partial t} + M^{-1}A(M+P) \frac{\partial \underline{\phi}}{\partial x} = O(|\phi|^3) \ . \tag{1.67}$$

Now, $M^{-1}P$ is first order in $\underline{\phi}$. Thus to leading order we have

$$(I+M^{-1}P)^{-1} = I - M^{-1}P \ . \tag{1.68}$$

So equations (1.67) and (1.68) give

$$\frac{\partial \underline{\phi}}{\partial t} + (I-M^{-1}P) \ M^{-1}A(M+P) \frac{\partial \underline{\phi}}{\partial x} = O(|\phi|^3) \ . \tag{1.69}$$

Now,

$$A = A(\underline{u}_o + v)$$
$$= A(\underline{u}_o + M\underline{\phi} + O(|\phi|^2)) \ . \qquad (1.70)$$

Also,

$$(I - M^{-1}P)M^{-1}A(M+P) = M^{-1}AM - M^{-1}PM^{-1}AM + M^{-1} AP + O(|\phi|^2)) \ . \qquad (1.71)$$

The flux matrix must be diagonal in order to have written the equations in canonical form. However, this does not mean that it is necessarily constant. The best solution seems to be to allow the eigenvalues in Λ to vary with $\underline{\phi}$. So, in general,

$$\lambda_i = \lambda_i(\underline{\theta}) = \lambda_i(\underline{\theta}_o + \underline{\phi})$$

$$= \lambda_i(\underline{\theta}_o) + \sum_j \frac{\partial \lambda_i}{\partial \theta_j} (\underline{\theta}_o) \ \phi_j + O(|\phi|^2) \ . \qquad (1.72)$$

So, if we impose

$$M^{-1}AM - M^{-1}PM^{-1}AM + M^{-1}AP = \Lambda(\underline{\theta}) + O(|\phi|^2) \ , \qquad (1.73)$$

we will have transformed (1.32) into its canonical diagonal form up to order $O(|\phi|^3)$.

Again, a necessary condition for Λ to be real is that A is symmetric.

Equation (1.73) may be multiplied by M to give:

$$AM - PM^{-1}AM + AP = M\Lambda(\underline{\theta}) + O(|\phi|^2) \ . \qquad (1.74)$$

Now, a Taylor expansion of (1.70) yields

$$A_{ij}(\underline{u}) = A_{ij}(\underline{u}_o) + \sum_{k,l} \frac{\partial A_{ij}}{\partial u_k}(\underline{u}_o)\ M_{kl}\ \phi_l + O(|\phi|^2) \ . \qquad (1.75)$$

This equating the leading order terms of (1.74) gives

$$A(\underline{u}_o)M = M\Lambda(\underline{\theta}_o) \ ,$$

$$\Rightarrow \quad M^{-1}A(\underline{u}_o)M = \Lambda(\underline{\theta}_o) \ . \qquad (1.76)$$

Substituting (1.76) in (1.74) and discounting higher order terms gives

$$\{A(\underline{u}) - A(\underline{u}_o)\}\ M - P\Lambda(\underline{\theta}_o) + A(\underline{u}_o)P = M\{\Lambda(\underline{\theta}) - \Lambda(\underline{u}_o)\} + O(|\underline{\phi}|^2) \ . \quad (1.77)$$

Writing (1.77) in suffix notation:

$$\sum_k \{A_{ik}(\underline{u}) - A_{ik}(\underline{u}_o)\}\ M_{kj} - \sum_k P_{ik}\ \Lambda_{kj}(\underline{\theta}_o)$$

$$+ \sum_k A_{ik}(\underline{u}_o)\ P_{kj} = \sum_k M_{ik}\{\Lambda_{kj}(\underline{\theta}) - \Lambda_{kj}(\underline{\theta}_o)\} + O(|\underline{\phi}|^2) \ . \qquad (1.78)$$

Recalling equation (1.64)

$$P_{ij} = 2 \sum_k N_{ijk} \phi_k \, ,$$

observing from (1.75) that

$$A_{ij}(\underline{u}) - A_{ij}(\underline{u}_o) = \sum_{k,l} \frac{\partial A_{ij}}{\partial u_k}(\underline{u}_o) \, M_{kl} \, \phi_l + O(|\phi|^2) \, ,$$

and from (1.72) that

$$\Lambda_{ij}(\underline{\theta}) - \Lambda_{ij}(\underline{\theta}_o) = \delta_{ij} \sum_k \frac{\partial \lambda_i}{\partial \theta_k} (\underline{\theta}_o) \, \phi_k + O(|\phi|^2) \, ,$$

we arrive at

$$\sum_{k,l,m} \frac{\partial A_{ik}}{\partial u_l}(\underline{u}_o) \, M_{lm} \, \phi_m \, M_{kj} - \sum_{k,l} 2N_{ikl} \, \phi_l \, \delta_{kj} \, \lambda_k(\underline{\theta}_o)$$

$$+ \sum_{k,l,m} A_{ik}(\underline{u}_o) \, 2N_{ikl} \, \phi_l = \sum_{k,l} M_{ik} \, \delta_{kj} \, \frac{\partial \lambda_k}{\partial \theta_l} (\underline{\theta}_o) \, \phi_l \, . \qquad (1.79)$$

Swapping round the suffices m and l in the first term and noting that $\underline{\phi}$ is both arbitrary and arbitrarily small, we obtain

$$\sum_{k,m} \frac{\partial A_{ik}}{\partial u_m}(\underline{u}_o) \, M_{ml} \, M_{kj} - \sum_k 2N_{ikl} \, \delta_{kj} \, \lambda_k(\underline{u}_o)$$

$$+ \sum_k A_{ik}(\underline{u}_o) \, 2N_{kjl} = \sum_k M_{ik} \, \delta_{kj} \, \frac{\partial \lambda_k}{\partial \theta_l} (\underline{\theta}_o) \, . \qquad (1.80)$$

Collapsing the kronecker deltas gives

$$\sum_{k,m} \frac{\partial A_{ik}}{\partial u_m}(\underline{u}_o)\, M_{kj}\, M_{ml} - 2\lambda_j(\underline{u}_o)\, N_{ijl} + 2\sum_k A_{ik}(\underline{u}_o)\, N_{kjl} = M_{ij}\, \frac{\partial \lambda_j}{\partial \theta_l}(\underline{\theta}_o)\ . \qquad (1.81)$$

The arbitrary constants in equation (1.81) are N_{ijk} and $\frac{\partial \lambda_j}{\partial \theta_l}(\underline{\theta}_o)$. They need to be chosen in order to satisfy this equation. This problem does not appear to be easily tractable.

Note: it is always possible to find functions $\lambda_i(\underline{\theta})$ with the correct coefficients $\frac{\partial \lambda_i}{\partial \theta_j}(\underline{\theta}_o)$. For example, a quadratic function will suffice. This problem is well-posed because we only require the derivatives at $\underline{\theta}_o$ (c.f. end of §1.2).

2. The Breakdown of Smooth Solutions

2.0 Introduction

In this section, it is attempted to formalise mathematically the concept of a characteristic solution becoming discontinuous when characteristics meet. The point where this discontinuity first forms is called the breaking point. Following the analysis of Haberman ([3]), the behaviour of the breaking point is found by backtracking along the caustic (the line on which infinitely close characteristics meet). After this, two different methods are employed to obtain the behaviour of the solution around the breaking point; namely: catastophe theory and asymptotic analysis.

2.1 Limiting Caustic Theory

The caustic is the curve along which neighbouring characteristics meet (and thus the solution curve has an infinite derivative, but is not discontinuous). It will be shown that the time when the caustic first forms is also the time when the solution first becomes discontinuous (as one would expect). Certain other properties concerning the end of the caustic (which is also the breaking point) will also be derived.

2.1.1 One Equation

The geometry for this problem is shown in figure 2. Following the notation of §1.2, let $\Gamma(\xi)$ be the curve defined by:

$$\Gamma(\xi) : \qquad x = \xi + \lambda(u_o(\xi))t \ . \tag{2.1}$$

Let us consider the intersection of neighbouring characteristics. Suppose $\Gamma(\xi-\epsilon)$ and $\Gamma(\xi+\delta)$ intersect at time t. Hence, from equation (2.1), we infer

$$\xi - \epsilon + \lambda(u_o(\xi-\epsilon))t = \xi + \delta + \lambda(u_o(\xi+\delta))t \ . \tag{2.2}$$

For convenience, let

$$\lambda_o(\xi) = \lambda(u_o(\xi)) \ . \tag{2.3}$$

Equation (2.2) then yields

$$\epsilon + \delta = \{\lambda_o(\xi-\epsilon) - \lambda_o(\xi+\delta)\}t \ . \tag{2.4}$$

Suppose λ_o is Taylor expandable near ξ and ϵ and δ are of the same order of magnitude. Then

$$\lambda_o'(\xi)t = -1 + O(|\epsilon| + |\delta|) \ . \tag{2.5}$$

Hence, taking the limit $|\epsilon| + |\delta| \to 0$ we obtain

$$\lambda_o'(\xi)t = -1 \ . \tag{2.6}$$

We may consider this equation for the caustic as giving t as a function of ξ. Thus

$$t(\xi) = -\frac{1}{\lambda_o'(\xi)} \ . \tag{2.7}$$

The minimum time satisfying equation (2.7) must also satisfy $\frac{dt}{d\xi} = 0$. Therefore, from equation (2.7), we have

$$\frac{\lambda_o''(\xi)}{\{\lambda_o'(\xi)\}^2} = 0 \tag{2.8}$$

at this value of ξ . It can be shown that the solution $\lambda_o'(\xi) = \pm\infty$ is just a geometrical anomaly. So, if we use the suffix b to denote the breaking point, we have

$$\lambda_o''(\xi_b) = 0 . \tag{2.9}$$

As this point also lies on the caustic, we can use equation (2.7) to give

$$t_b = -\frac{1}{\lambda_o'(\xi_b)} , \tag{2.10}$$

where $t_b = t(\xi_b)$.

A further point to note is that if t_b is the minimum time, we also have

$$\frac{d^2 t(\xi_b)}{d\xi^2} > 0 . \tag{2.11}$$

This, together with equation (2.7), implies

$$\frac{\lambda_o'''(\xi_b)}{\lambda_o'(\xi_b)^2} > 0 ,$$

ie $$\lambda_o'''(\xi_b) > 0 . \tag{2.12}$$

The final property to observe concerning the breaking point is to substitute into equation (2.1) to give

$$x_b = \xi_b + \lambda_o(\xi_b)t_b . \tag{2.13}$$

Now, as stated at the beginning of the section, we also need to show that, locally, no characteristics intersect before the breaking time, t_b . The following lemma is therefore proved.

Lemma 2.1

$\exists\, \delta_o > 0$ such that $\forall\, \delta_1, \delta_2 \in (-\delta_o, \delta_o)\ \delta_1, \delta_2 \neq 0$, $\Gamma(\xi_b + \delta_1)$, $\Gamma(\xi_b + \delta_2)$ intersect at a time greater than t_b .

Proof

$\Gamma(\xi_b + \delta_1)$, $\Gamma(\xi_b + \delta_2)$ intersect at

$$t = - \frac{\delta_1 - \delta_2}{\lambda_o(\xi_b + \delta_1) - \lambda_o(\xi_b + \delta_2)} \ . \tag{2.14}$$

Using the notation $\frac{d^n}{d\xi^n} \lambda_o = \lambda_o^{(n)}$ and performing a Taylor expansion with a Lagrange remainder, we have

$$\lambda_o(\xi_b + \delta_i) = \lambda_o(\xi_b) + \delta_i \lambda_o^{(1)}(\xi_b) + \frac{\delta_i^2}{2!} \lambda_o^{(2)}(\xi_b)$$

$$+ \frac{\delta_i^3}{3!} \lambda_o^{(3)}(\xi_b) + \frac{\delta_i^4}{4!} \lambda_o^{(4)}(\xi_b + \epsilon_i) \ , \tag{2.15}$$

where $i = 1,2$ and ϵ_1, ϵ_2 at least obey

$$\epsilon_1, \epsilon_2 \in (-\delta_o, \delta_o) \ . \tag{2.16}$$

Thus, using equations (2.9) and (2.10), (2.15) implies

$$\lambda_o(\xi_b + \delta_1) - \lambda_o(\xi_b + \delta_2) = (\delta_1 - \delta_2)\left[-\frac{1}{t_b}\right]$$

$$+ \frac{1}{6}(\delta_1^3 - \delta_2^3)\,\lambda_o^{(3)}(\xi_b) + \rho\ , \tag{2.17}$$

where $\rho = \frac{1}{24}\left\{\delta_1^4\,\lambda_o^{(4)}(\xi_b + \epsilon_1) - \delta_2^4\,\lambda_o^{(4)}(\xi_b + \epsilon_2)\right\}$. (2.18)

Equation (2.17) implies

$$\lambda_o(\xi_b + \delta_1) - \lambda_o(\xi_b + \delta_2) =- \frac{1}{t_b}(\delta_1 - \delta_2)$$

$$\left\{-1\ \frac{\delta_1^2 + \delta_1\delta_2 + \delta_2^2}{6}\,\lambda_o^{(3)}\ (\xi_b)t_b - \frac{\rho t_b}{\delta_1 - \delta_2}\right\} . \tag{2.19}$$

Back-substituting into equation (2.14) gives

$$t = t_b\left\{1 - \frac{\delta_1^2 + \delta_1\delta_2 + \delta_2^2}{6}\,\lambda_o^{(3)}(\xi_g)t_b - \frac{\rho t_g}{\delta_1 - \delta_2}\right\}^{-1} . \tag{2.20}$$

Now, $\delta_1^2 + \delta_1\delta_2 + \delta_2^2 > 0$ for $\delta_1\delta_2 \neq 0$, and $t_b > 0$ as we are only concerned with positive time solutions. So, with equation (2.12), we have

$$\frac{\delta_1^2 + \delta_1\delta_2 + \delta_2^2}{6}\,\lambda_o^{(3)}(\xi_b)t_b > 0\ . \tag{2.21}$$

Also, as the 'ρ' term in equation (2.20) is of higher order than the left hand side of equation (2.21), we clearly see that the lemma is true for suitably small δ_o provided all the functions defined remain bounded. A definition of δ_o may be constructed, but that is not our purpose here. ■

2.1.2 Two Equations

The two equation problem is more complicated than the single equation problem, but the basic method is unchanged.

Recalling §1.2, it is assumed that the system is already in Riemann invariant form. Hence it is given by the equations:

$$\left.\begin{aligned} \theta_t + \lambda(\theta,\phi)\theta_x &= 0 \\ \phi_t + \mu(\theta,\phi)\phi_x &= 0 \end{aligned}\right\} . \tag{2.22}$$

We again concentrate on a single family of characteristics, $\Gamma(\xi)$, now curves not straight lines, corresponding to the first equation of (2.22). Thus, as before,

$$\Gamma(\xi): \quad \left.\begin{aligned} \frac{dx}{dt} &= \lambda(\theta_o(\xi),\phi(x(\xi,t),t)) \\ x(\xi,0) &= \xi \end{aligned}\right\} . \tag{2.23}$$

Equation (2.23) may be integrated to derive an integral equation analogous to (2.1):

$$x(\xi,t) = \xi + \int_o^t \lambda(\theta_o(\xi),\ \phi(x(\xi,\tau),\tau))d\tau . \tag{2.24}$$

Geometrically, this integration is taking place along the characteristic. Figure 3 shows the breakdown.

As in §2.1.1 we shall consider the formation of the caustic. For simplicity, the function λ will be replaced by λ_o , where

$$\lambda_o(\xi,t) = \lambda(\theta_o(\xi),\ \phi(x(\xi,t)))\ . \tag{2.25}$$

Suppose, as before, that the characteristics $\Gamma(\xi-\epsilon)$, $\Gamma(\xi+\delta)$ intersect at time t . Thus

$$x(\xi-\epsilon,t) = x(\xi+\delta,t)\ . \tag{2.26}$$

Using equations (2.24) and (2.25), we derive

$$\epsilon + \delta = \int_o^t \left\{\lambda_o(\xi-\epsilon,\tau) - \lambda_o(\xi+\delta,\tau)\right\} d\tau\ . \tag{2.27}$$

We now require the following Taylor expansion:

$$\lambda_o(\xi-\epsilon,\tau) = \lambda_o(\xi,\tau) - \epsilon\,\frac{\partial\lambda_o}{\partial\xi}(\xi,\tau) + O(\epsilon^2)\ . \tag{2.28}$$

Substituting, we obtain

$$-1 = \int_o^t \frac{\partial\lambda_o}{\partial\xi}(\xi,\tau)d\tau + O(|\epsilon| + |\delta|)\ . \tag{2.29}$$

So, again using in the limit $|\epsilon| + |\delta| \to 0$, we have

$$- 1 = \int_{o}^{t} \frac{\partial \lambda_o}{\partial \xi} (\xi, \tau) d\tau \ . \tag{2.30}$$

Now, as before, we may consider $t = t(\xi)$ which is minimised at the breaking point. So

$$\frac{dt}{d\xi} (\xi_b) = 0 \ . \tag{2.31}$$

Applying this to equation (2.30) and differentiating with respect to ξ gives

$$0 = \int_{o}^{t(\xi)} \frac{\partial^2 \lambda_o}{\partial \xi^2} (\xi, \tau) d\tau + \frac{dt}{d\xi} \frac{\partial \lambda_o}{\partial \xi} (\xi, t) \ , \tag{2.32}$$

hence

$$\int_{o}^{t_b} \frac{\partial^2 \lambda_o}{\partial \xi^2} (\xi_b, \tau) d\tau = 0 \ . \tag{2.33}$$

Again, as $t = t_b$ is the minimum of $t(\xi)$,

$$\frac{d^2 t}{d\xi^2} (\xi_b) > 0 \ . \tag{2.34}$$

Differentiating equation (2.32) with respect to ξ gives

$$0 = \int_{o}^{t(\xi)} \frac{\partial^3\lambda_o}{\partial\xi^3}(\xi,\tau)d\tau + 2\frac{dt}{d\xi}\frac{\partial^2\lambda_o}{\partial\xi^2}(\xi,t) + \frac{d^2t}{d\xi^2}\frac{\partial\lambda_o}{\partial\xi}(\xi,t) \ . \qquad (2.35)$$

Substituting $\xi = \xi_b$ gives

$$\frac{\partial^2 t}{\partial\xi^2}(\xi_b)\frac{\partial\lambda_o}{\partial\xi}(\xi_b,t_b) = -\int_{o}^{t_b}\frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b,\tau)d\tau \ .$$

Hence

$$\frac{\partial\lambda_o}{\partial\xi}(\xi_b,t_b)\int_{o}^{t_g}\frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b,\tau)d\tau \leq 0 \qquad (2.36)$$

(note the possible equality).

The last property of the breaking point, as before, is found by substituting into the integral equation (2.24) to give

$$x_b = \xi_b + \int_{o}^{t_b}\lambda_o(\xi_b,\tau)d\tau \ , \qquad (2.37)$$

using equation (2.25).

No proof that the breaking point is well-defined by this process is given here. The lemma would, however, be exactly the same as before with an analogous proof.

2.2 Catastrophe Theory Analysis

The object of this analysis is to show that the breakdown point is an example of the cusp catastrophe and then to derive any properties which naturally follow.

2.2.1 One Equation

Recalling the equations for the breaking point from §2.1.1:

$$(2.1.),\ (2.3) \Rightarrow \quad x = \xi + \lambda_o(\xi)t\ , \tag{2.38}$$

$$(2.10): \quad t_b = -\frac{1}{\lambda_o'(\xi_b)}\ ,$$

$$(2.9): \quad \lambda_o''(\xi_b) = 0\ ,$$

$$(2.12): \quad \lambda_o'''(\xi_b) > 0\ ,$$

$$(2.13): \quad x_b = \xi_b + \lambda_o(\xi_b)t_b\ .$$

We wish to perform a local rescaling of the characteristic equation (2.38) about the breaking point. Introducing rescaled variables

$$\left.\begin{aligned} x &= x_b + \tilde{x} \\ t &= t_b + \tilde{t} \\ \xi &= \xi_b + \tilde{\xi} \end{aligned}\right\}\ , \tag{2.39}$$

equation (2.38) becomes

$$x_b + \tilde{x} = \xi_b + \tilde{\xi} + \lambda_o(\xi_b + \tilde{\xi})\ \{t_b + \tilde{t}\}\ . \tag{2.40}$$

Equations (2.38) and (2.40) yield

$$\tilde{x} = \tilde{\xi} + \lambda_o(\xi_b + \tilde{\xi})\,\{t_b + \tilde{t}\} - \lambda_o(\xi_b)t_b \ . \tag{2.41}$$

rearranging gives

$$\tilde{x} = \tilde{\xi} + \{\lambda_o(\xi_b + \tilde{\xi}) - \lambda_o(\xi_b)\}t_b + \lambda_o(\xi_b + \tilde{\xi})\tilde{t} \ . \tag{2.42}$$

Performing a Taylor expansion with integral remainders we obtain

$$\tilde{x} - \lambda_o(\xi_b)\tilde{t} = \tilde{\xi} + \left\{\tilde{\xi}\lambda_o'(\xi_b) + \frac{\tilde{\xi}^2}{2!}\lambda_o''(\xi_b) + \int_0^{\tilde{\xi}} \frac{1}{2!}(\tilde{\xi} - \eta)^2 \lambda_o'''(\xi_b + \eta)d\eta\right\} t_b$$
$$+ \tilde{t}\int_0^{\tilde{\xi}} \lambda_o'(\xi_b + \eta)d\eta \ . \tag{2.43}$$

This reduces to

$$\tilde{x} - \lambda_o(\xi_b)\tilde{t} = \frac{t_b}{2}\int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 \lambda_o'''(\xi_b + \eta)d\eta + \tilde{t}\int_0^{\tilde{\xi}} \lambda_o'(\xi_b + \eta)d\eta \ . \tag{2.44}$$

Finally, introducing

$$\tilde{z} = \tilde{x} - \lambda_o(\xi_b)\tilde{t} \tag{2.45}$$

equation (2.44) becomes

$$\tilde{z} = \frac{t_b}{2}\int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 \lambda_o'''(\xi_b + \eta)d\eta + \tilde{t}\int_0^{\tilde{\xi}} \lambda_o'(\xi_b + \eta)d\eta \ . \tag{2.46}$$

This equation defines a surface on $(\tilde{z},\tilde{t},\tilde{\xi})$ space. An equivalent leading order expansion of equation (2.46) was performed by Haberman in [3], where he showed that the leading order terms (chosen in some sense) correspond to a cusp catastrophe. Our intention here is to arrive at the same results using more formal arguments.

To this end, equation (2.46) is transformed into a form relating to an unfolding. We seek a function $\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t})$ with the following properties:

$$\tilde{F}(0;\tilde{z},\tilde{t}) = 0 \qquad \forall\ \tilde{z}\ ,\ \tilde{t} \tag{2.47}$$

$$\frac{\partial\tilde{F}}{\partial\tilde{\xi}}(\tilde{\xi};\tilde{z},\tilde{t}) = 0 \qquad \forall\ \tilde{\xi}\ ;\ \tilde{z}\ ,\ \tilde{t}\ ,\ \text{satisfying (2.46).} \tag{2.48}$$

Such a function is given by

$$\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t}) = \frac{t_b}{2}\int_0^{\tilde{\xi}}\left[\int_0^{\eta}(\eta-\zeta)^2\lambda_o'''(\xi_b+\zeta)d\zeta\right]d\eta + \tilde{t}\int_0^{\tilde{\xi}}\left[\int_0^{\eta}\lambda_o'(\xi_b+\zeta)d\zeta\right]d\eta - \tilde{z}\,\tilde{\xi}\ . \tag{2.49}$$

For ease of notation, an analogous function is considered. Let

$$F(x;a,b) = \int_0^x\left[\int_0^y(y-z)^2\,g(z)dz\right]dy + a\int_0^x\left[\int_0^y h(z)dz\right]dy + bx\ . \tag{2.50}$$

This function clearly obeys

$$F(0;a,b) = 0 \ . \tag{2.51}$$

Also, the equation

$$\frac{\partial F}{\partial x}(x;a,b) = 0 \tag{2.52}$$

will analogously lead to the equation of a surface in (a,b,x) space.

Following the ideas of catastrophe theory ([4]), we attempt to show that $F(x,a,b)$ forms the first of a sequence of unfoldings which may be induced from each other, ending up with the standard form of the universal unfolding of $\frac{1}{4}x^4$ (which is the cusp catastrophe unfolding function, A_{+3}).

The first step is to show that

$$f(x) = F(x;0,0) \tag{2.53}$$

is strongly 4 - determinate (where k-determinate is defined as in [4]).

Following theorem 8.1 of [4] in the single variable case, f is strongly 4 - determinate if and only if $\exists\ a_o,\ldots, a_5 \in \mathbb{R}$ such that

$$x^5 = \overline{\left[\sum_{r=o}^{5} a_r x^r\right] j^3 \left[\frac{df}{dx}\right]}^{\,5} , \tag{2.54}$$

where $j^k\phi$ is the Taylor expansion of ϕ about the origin up to order k and $\overline{\qquad}^{\,k}$ denotes truncation at order k .

Equations (2.50) and (2.53) imply

$$f(x) = \int_0^x \left[\int_0^y (y-z)^2 \, g(z)dz \right] dy \ . \qquad (2.55)$$

Calculating successive derivatives of f gives

$$f'(x) = \int_0^x (x-y)^2 \, g(y) \, dy \ , \qquad (2.56)$$

$$f''(x) = \int_0^x 2(x-y) \, g(y) \, dy \ , \qquad (2.57)$$

$$f'''(x) = \int_0^x 2g(y) \, dy \ , \qquad (2.58)$$

$$f^{IV}(x) = 2g(x) \ . \qquad (2.59)$$

Recalling the Taylor expansion for $f'(x)$,

$$j^3 f'(x) = f'(o) + xf''(o) + \frac{x^2}{2!} f'''(o) + \frac{x^3}{3!} f^{IV}(o) \ . \qquad (2.60)$$

Therefore, we obtain

$$j^3 \, f'(x) = \frac{x^3}{3} \, g(o) \ . \qquad (2.61)$$

We may assume $g(o) > 0$ as this corresponds to equation (2.12) in our original notation. Therefore, equation (2.54) can clearly be satisfied by setting

$$a_o = 0 \ , \quad a_1 = 0 \ , \quad a_2 = \frac{3}{g(o)} \ , \quad a_3 = 0 \ , \quad a_4 = 0 \ , \quad a_5 = 0 \ . \qquad (2.62)$$

So we conclude that f is indeed strongly 4 - determinate.

Now, by definition, this implies that there exists a neighbourhood N of 0 and a function

$$\theta : N \to \mathbb{R}$$

with the property

$$\frac{d\theta}{dx}(o) = 1 \qquad (2.63)$$

and

$$\forall \ x \in N \ , \qquad f(x) = \frac{g(o)}{12}\,\theta(x)^4 \ . \qquad (2.64)$$

It is possible to determine $\theta(x)$ by a naïve polynomial expansion

$$\theta = x + \sum_{r=2}^{\infty} a_r x^r \ . \qquad (2.65)$$

In order to transform the unfolding into the standard form, we define the function ϕ by

$$\phi = \left[\frac{g(o)}{12}\right]^{1/4} \theta \ , \qquad (2.66)$$

so

$$f(x) = \frac{\phi(x)^4}{4} \, . \tag{2.67}$$

It will be assumed that this function is invertible for small x and ϕ, and that the inverse function may be approximated by a finite Taylor series. It will turn out later that we require the quadratic term in this expansion. For simplicity, let us first consider the inversion of equation (2.65):

$$x(\theta) = \theta + A\theta^2 + O(\theta)^3 \tag{2.68}$$

for some constant A.

But, differentiating equation (2.59) gives

$$f^V(x) = 2g'(x) \, . \tag{2.69}$$

Therefore,

$$j^5 f(x) = \frac{g(o)}{12} x(\theta)^4 + \frac{g'(o)}{60} x(\theta)^5 \, . \tag{2.70}$$

Substituting in equation (2.68) gives

$$f(x(\theta)) = \frac{g(o)}{12} \{\theta^4 + 4A\theta^5\} + \frac{g'(o)}{60} \theta^5 + O(\theta^6) \, . \tag{2.71}$$

Thus, equating terms of fifth order in θ gives

$$\frac{g(o)}{12} 4A + \frac{g'(o)}{60} = 0 \, , \tag{2.72}$$

implying,

$$A = -\frac{g'(o)}{20g(o)} \; . \tag{2.73}$$

We may now define a function $G(\phi;a.b)$ such that

$$G(\phi;a,b) = F(x(\phi);a,b) \; . \tag{2.74}$$

This may be written explicitly, using equation (2.50) as

$$G(\phi;a,b) = \int_o^{x(\phi)} \left[\int_o^y (y-z)^2 \, g(z)dz\right]dy + a\int_o^{x(\phi)} \left[\int_o^y h(z)dz\right]dy + bx(\phi) \; . \tag{2.75}$$

Using equations (2.53) and (2.67), this simplifies to

$$G(\phi;a,b) = \frac{\phi^4}{4} + a\int_o^{x(\phi)} \left[\int_o^y h(z)dz\right]dy + bx(\phi) \; . \tag{2.76}$$

Also, equations (2.66), (2.68) and (2.73) give

$$x(\phi) = \left[\frac{3}{g(o)}\right]^{1/4}\phi - \frac{g'(o)}{20g(o)}\left[\frac{3}{g(o)}\right]^{1/2}\phi^2 + O(\phi^3) \; . \tag{2.77}$$

The next step is to use theorem 8.7 in [4] to prove the existence of a more simple unfolding from which G may be induced. We therefore attempt to apply this theorem with the following values of the relevant constants:

$$k = 4 \; , \quad p = 5 \; , \quad q = 2 \; , \quad r = 2 \; , \quad n = 1 \; . \tag{2.78}$$

The conditions we need to meet are as follows:

i) $\phi^4/4$ is strongly 4 - determinate;

ii) $M_1^3 \subseteq \Delta_5(\phi^4/4)$ (as we are attempting to satisfy case a));

iii) G is a versal unfolding of $\phi^4/4$.

Condition i) is trivially satisfied (e.g. it is equivalent to the preceding analysis of $f(x)$ being strongly 4 - determinate with all derivatives of g zero at the origin).

Let us consider condition ii). By definition (see [4] again),

$$M_1^3 = \{A\phi^3 \text{ st } \phi \in \mathbb{R}\} \; , \tag{2.79}$$

$$\Delta_5(\phi^4/4) = \left\{ \overline{\sum_{r=o}^{5} a_r \, \phi^r \, j^5 \left[\frac{d}{d\phi} (\phi^4/4)\right]}^{\,5} \text{ st } a_o, \ldots, a_5 \in \mathbb{R} \right\} \; . \tag{2.80}$$

$$= \left\{ a_o\phi^3 + a_1 \, \phi^4 + a_2\phi^5 \text{ st } a_o \, , \, a_1 \, , \, a_2 \in \mathbb{R} \right\} \; . \tag{2.81}$$

So condition ii) is easily satisfied by setting $A = a_o$.

Finally, it is shown that condition iii) is satisfied. The following notation needs to be introduced:

$$J^k f = j^k f - j^o f \; , \tag{2.82}$$

$$v_a^k(G) = \frac{\partial}{\partial a}\left[J^k(G(\phi;a,o))\right] . \tag{2.83}$$

and similarly,

$$v_b^k(G) = \frac{\partial}{\partial b}\left[J^k(G(\phi;o,b))\right] . \tag{2.84}$$

Let $V^k(G) = \text{span}\ (v_a^k(G),\ v_b^k(G))$. (2.85)

Now, theorem 8.6 in [4] states that G is versal when $V^k(G)$ and $\Delta_k(\phi^4/4)$ are transverse subspaces of J_1^k , where $\frac{\phi^4}{4}$ is k-determinate. $\left[J_1^k = \left\{\sum_{r=1}^{k} a_r\phi^r \text{ st } a_1,\ldots,a_k \in \mathbb{R}\right\}\right]$. Clearly, here, $k = 4$. Equations (2.76) and (2.83) with $k = 4$ give

$$v_a^4(G) = J^4 \int_o^{x(\phi)} \left[\int_o^y h(z)dz\right]dy \tag{2.86}$$

$$v_b^4(G) = J^4 x(\phi) . \tag{2.87}$$

We have the general result for $u = u(\phi)$ that

$$J^4 u = \frac{du}{d\phi}(o)\phi + \frac{d^2u}{d\phi^2}(o)\ \frac{\phi^2}{2!} + \frac{d^3u}{d\phi^3}\ (o)\ \frac{\phi^3}{3!} + \frac{d^4u}{d\phi^4}\ (o)\ \frac{\phi^4}{4!} . \tag{2.88}$$

Now, similarly to equation (2.81), it can be shown that

$$\Delta_k(\phi^4/4) = \left\{a_o\phi^3 + a_1\phi^4\ \text{ st } a_o,a_1 \in \mathbb{R}\right\} . \tag{2.89}$$

Clearly,

$$\dim \; J_1^4 = 4 \; . \qquad (2.90)$$

Also, equation (2.89) simply gives

$$\dim \Delta_4(\phi^4/4) = 2 \; . \qquad (2.91)$$

Furthermore, equation (2.88) shows that the coefficients of ϕ in $v_a^4(G)$ and $V_b^4(G)$ are constants, so we must have

$$\dim \; V^4(G) = 2 \; . \qquad (2.92)$$

Therefore, $\Delta_4(\phi^4/4)$ and $V^4(G)$ are transverse subspaces of J_1^4 if and only if

$$\dim \; (\Delta_4(\phi^4/4) \cap V^4(G)) = 4 - 2 - 2 = 0 \; . \qquad (2.93)$$

It can be shown that, up to order ϕ^3,

$$v_a^4(G) = \left[\frac{dx}{d\phi}(o)\right]^2 h(o)\frac{\phi^2}{2!} + O(\phi)^3 \qquad (2.94)$$

$$v_b^4(G) = \frac{dx}{d\phi}(o)\phi + \frac{d^2x}{d\phi^2}(o)\frac{\phi^2}{2!} + O(\phi)^3 \; . \qquad (2.95)$$

Therefore, as $\Delta_4(\phi^4/4) = O(\phi)^3$, we may infer that $\Delta_4(\phi^4/4)$ and $V^4(G)$ only intersect at isolated points (assuming the higher order terms in $v_a^4(G)$ and $v_b^4(G)$ are not proportional, which verifies equations (2.93) and hence condition iii).

So we have, by theorem 8.7, that $G(\phi;a,b)$ is strongly equivalent to another unfolding $H(\psi;\alpha,\beta)$, where

$$H(\psi;\alpha,\beta) = j^5\left(\frac{\psi^4}{4}\right) + \alpha J^2\left[\frac{\partial}{\partial\alpha} G(\psi;\alpha,o)\right] + \beta J^2\left[\frac{\partial}{\partial\beta} G(\psi;o,\beta)\right] . \tag{2.96}$$

The strong equivalence condition means that G may be induced from H (so $\psi = \psi(\phi;a,b)$, $\alpha = \alpha(a,b)$, $\beta = \beta(a,b)$) with

$$\left.\frac{\partial(\psi,\alpha,\beta)}{\partial(\phi,a,b)}\right|_{\phi = o, a = o, b = o} = I_3 . \tag{2.97}$$

where I_3 is the 3×3 identity matrix.
This equation enables us to envisage another Taylor expansion (here for ψ,α and β) , but these calculations are not presented.

However, equation (2.96) still needs to be simplified. Clearly,

$$j^5\left(\frac{\psi^4}{4}\right) = \frac{\psi^4}{4} . \tag{2.98}$$

Let
$$u(\psi) = \int_o^{x(\psi)} \left[\int_o^y h(z)dz\right]dy . \tag{2.99}$$

Then,

$$\frac{du}{d\psi} = \frac{dx}{d\psi} \int_o^{x(\psi)} h(y)dy , \tag{2.100}$$

and

$$\frac{d^2u}{d\psi^2} = \frac{d^2x}{d\psi^2} \int_o^{x(\psi)} h(y)dy + \left[\frac{dx}{d\psi}\right]^2 h(x(\psi)) . \tag{2.101}$$

Now, $x(o) = o$, and

$$\frac{dx}{d\psi}(o) = \left[\frac{3}{g(o)}\right]^{1/4} . \qquad (2.102)$$

from equations (2.66) and (2.68). So, substituting into the above, we obtain

$$J^2 u = \left[\frac{3}{g(o)}\right]^{1/2} h(o)\frac{\psi^2}{2!} . \qquad (2.103)$$

Hence, by a similar construction to equation (2.86), we obtain

$$J^2\frac{\partial}{\partial\alpha}(G(\psi;\alpha,o)) = J^2 u = \left[\frac{3}{g(o)}\right]^{1/2} h(o)\frac{\psi^2}{2!} . \qquad (2.104)$$

It is then simple to show that

$$J^2\frac{\partial}{\partial\beta}(G(\psi;o,\beta)) = J^2 x(\psi) = \left[\frac{3}{g(o)}\right]^{1/4} \psi \; \frac{d^2x}{d\psi^2}(o) \; \frac{\psi^2}{2!} . \qquad (2.105)$$

Equation (2.78) gives

$$\frac{d^2x}{d\psi^2}(o) = - \frac{g'(o)}{20g(o)} \; 2 \left[\frac{3}{g(o)}\right]^{1/2} . \qquad (2.106)$$

Thus, combining,

$$J^2 \frac{\partial}{\partial\beta}(G(\psi;o,\beta)) = \left[\frac{3}{g(o)}\right]^{1/4} \psi - \frac{g'(o)}{10g(o)}\left[\frac{3}{g(o)}\right]^{1/2} \frac{\psi^2}{2} . \qquad (2.107)$$

Equations (2.97), (2.99), (2.105) and (2.107) now combine to give

$$H(\psi;\alpha,\beta) = \frac{\psi^4}{4} + \{\alpha h(o) - \beta \frac{g'(o)}{10g(o)}\}\left[\frac{3}{g(o)}\right]^{1/2} \frac{\psi^2}{2} + \beta\left[\frac{3}{g(o)}\right]^{1/4} \psi . \qquad (2.108)$$

The final induced transformation is the simple linear transformation of coefficents $(\alpha,\beta) \mapsto (\gamma,\delta)$ given by

$$\gamma = \left[\alpha h(o) - \beta \frac{g'(o)}{10g(o)}\right] \left[\frac{3}{g(o)}\right]^{1/2} . \tag{2.109}$$

$$\delta = \beta\left[\frac{3}{g(o)}\right]^{1/4} . \tag{2.110}$$

Giving the final unfolding function

$$I(\psi;\gamma,\delta) = H(\psi;\alpha,\beta) = \frac{\psi^4}{4} + \gamma\frac{\psi^2}{2} + \delta\psi . \tag{2.111}$$

This is the standard form for the cusp catastrophe (A_{+3}) , as already mentioned.

So, as we have been considering an analogous function to the characteristic unfolding function $\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t})$ (recall equation (2.49)), it is therefore also possible locally to transform this function into the standard form for the cusp catastrophe.

2.2.2 Two Equations

In this subsection we will only attempt to derive the characteristic manifold equation local to the breaking point and transform it into the form of an unfolding.

First of all, let us recall some of the results of §2.1.2. Equation (2.24) gave the characteristic equation:

$$x(\xi,t) = \xi + \int_o^t \lambda(\theta_o(\xi),\ \phi(x(\xi,\tau),\tau))d\tau .$$

For simplicity, we will use equation (2.25) to reduce the above to

$$x(\xi,t) = \xi + \int_0^t \lambda_o(\xi,\tau)d\tau \ . \tag{2.112}$$

Equation (2.37) was

$$x_b = \xi_b + \int_0^{t_b} \lambda_o(\xi_b,\tau)d\tau \ .$$

The local reparametrisation equation (2.39) will now be employed. Combining this with the previous two equations gives

$$\tilde{x} = \tilde{\xi} + \int_0^{t_b+\tilde{t}} \lambda_o(\xi_b+\tilde{\xi},\tau)d\tau - \int_0^{t_b} \lambda_o(\xi_b,\tau)d\tau \ . \tag{2.113}$$

implying,

$$\tilde{x} = \tilde{\xi} + \int_0^{t_b} \left\{\lambda_o(\xi_b+\tilde{\xi},\tau) - \lambda_o(\xi_b,\tau)\right\}d\tau + \int_{t_b}^{t_b+\tilde{t}} \lambda_o(\xi_b+\tilde{\xi},\tau)d\tau \ . \tag{2.114}$$

By the theory of Taylor expansions,

$$\lambda_o(\xi_b+\tilde{\xi},\tau) - \lambda_o(\xi_b,\tau) = \frac{\partial\lambda_o}{\partial\xi}(\xi_b,\tau)\tilde{\xi} + \frac{\partial^2\lambda_o}{\partial\xi^2}(\xi_b,\tau)\frac{\tilde{\xi}^2}{2!}$$

$$+ \int_0^{\tilde{\xi}} \frac{1}{2!}(\tilde{\xi}-\eta)^2 \frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b+\eta,\tau)d\eta \ . \tag{2.115}$$

Also, as the breaking point lies at the end of the caustic, equation (2.30) may be applied to give

$$\int_0^{t_b} \frac{\partial\lambda_o}{\partial\xi}(\xi_b,\tau)d\tau = -1 \ . \tag{2.116}$$

Furthermore, equation (2.33) is recalled:

$$\int_0^{t_b} \frac{\partial^2\lambda_o}{\partial\xi^2}(\xi_b,\tau)d\tau = 0 \ .$$

The previous three equations combine to give the result

$$\int_0^{t_b} \left\{\lambda_o(\xi_b+\tilde{\xi},\tau) - \lambda_o(\xi_b,\tau)\right\} d\tau = -\tilde{\xi} + \int_0^{t_b}\left[\int_0^{\tilde{\xi}} \tfrac{1}{2}(\tilde{\xi}-\eta)^2 \frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b+\eta,\tau)d\eta\right]d\tau \ . \tag{2.117}$$

This is now combined with equation (2.114) to give

$$\tilde{x} = \int_0^{t_b}\left[\int_0^{\tilde{\xi}} \tfrac{1}{2}(\tilde{\xi}-\eta)^2 \frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b+\eta,\tau)d\eta\right]d\tau + \int_0^{\tilde{t}} \lambda_o(\xi_b+\tilde{\xi},\tau_b+\tau)d\tau \ . \tag{2.118}$$

The second integral on the right hand side of equation (2.118) may be expanded as follows

$$\int_0^{\tilde{t}} \lambda_o(\xi_b + \tilde{\xi}, \tau_b + \tau)d\tau = \int_0^{\tilde{t}} \lambda_o(\xi_b, \tau_b + \tau)d\tau + \int_0^{\tilde{t}} \left[\int_0^{\tilde{\xi}} \frac{\partial \lambda_o}{\partial \xi}(\xi_b + \eta, t_b + \tau)d\eta \right] d\tau. \tag{2.119}$$

In a similar vein to the single equation case we define a new parameter as follows:

$$\tilde{z} = \tilde{x} - \int_0^{\tilde{t}} \lambda_o(\xi_b, \tau_b + \tau)d\tau . \tag{2.120}$$

Equations (2.118), (2.119) and (2.120) now imply

$$\tilde{z} = \int_0^{t_b} \left[\int_0^{\tilde{\xi}} \frac{1}{2}(\tilde{\xi}-\eta)^2 \frac{\partial^3 \lambda_o}{\partial \xi^3}(\xi_b + \eta, \tau)d\eta \right] d\tau + \int_0^{\tilde{t}} \left[\int_0^{\tilde{\xi}} \frac{\partial \lambda_o}{\partial \xi}(\xi_b + \eta, t_b + \tau)d\eta \right] d\tau . \tag{2.123}$$

Equation (2.121) defines the manifold for $\tilde{\xi} = \tilde{\xi}(\tilde{z}, \tilde{t})$. It should be noted that it has exactly analogous structure to equation (2.46).

In the final part of this subsection, equation (2.121) is converted into the form of an unfolding. As in the previous subsection, equations (2.47) and (2.48) hold for the unfolding $\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t})$:

$$\tilde{F}(0;\tilde{z},\tilde{t}) = 0 \qquad \forall\ \tilde{z},\tilde{t}$$

$$\frac{\partial\tilde{F}}{\partial\tilde{\xi}}(\tilde{\xi};\tilde{z},\tilde{t}) = 0 \qquad \forall\ \tilde{\xi},\tilde{z},\tilde{t}\ .$$

These equations are satisfied by the unfolding

$$\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t}) = \frac{1}{2}\int_0^{t_b}\left[\int_0^{\tilde{\xi}}\left\{\int_0^{\eta}(\eta-\zeta)^2\,\frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b+\zeta,\tau)d\zeta\right\}d\eta\right]d\tau$$

$$+\int_0^{\tilde{t}}\left[\int_0^{\tilde{\xi}}\left\{\int_0^{\eta}\frac{\partial\lambda_o}{\partial\xi}(\xi_b+\zeta,\tau_b+\tau)d\zeta\right\}d\eta\right]d\tau - \tilde{z}\tilde{\xi}\ . \qquad (2.122)$$

Because of the similarity of this unfolding to that given in the previous subsection, it is envisaged that it may also be transformed into the standard unfolding of $\frac{1}{4}x^4$ - the cusp catastrophe (A_{+3}).

A less constructive proof for both a single equation and two equations may well be possible by a simple application of an unfolding theorem for the cusp catastrophe given by Schaeffer ([5]). All the conditions for the proof are satisfied, but the unfolding contains an extra constant term. However, the theorem may be applicable for gaining insight into the nature of characteristic singularities for more complicated systems.

2.3 Asymptotic Analysis

The basic idea for this section is to analyse the behaviour of the solution to the single characteristic equation with a discontinuity imposed by the application of the Rankine-Hugoniot jump conditions in the region of the breaking point. Because of the intimate relationship between characteristics and the transfer of information, it will be shown that this behaviour can be analysed by considering characteristics near the breaking characteristic (e.g. $\Gamma(\xi_b+\delta)$, where δ is small).

Because of the tedious nature of these calculations, only the initial equations and the final results are presented. The results have been checked with MAPLE.

2.3.1. Initial Equations

Let the shock curve be given by

$$S : \quad x = X(t) \ . \tag{2.123}$$

As the shock starts at the breaking point, we must have

$$x_b = X(t_b) \ . \tag{2.124}$$

Recall equation (1.1)

$$u_t + f(u)_x = 0 \ .$$

It is a standard argument (e.g. see [6] §2.3.2.)that the time-dependent form of the Rankine Hugoniot jump conditions in one dimension may be written

$$\frac{dX}{dt} = \frac{[f]}{[u]} \ . \qquad (2.125)$$

where $[.]$, as usual, denotes the jump in a function over a discontinuity.

Suppose that the two characteristics $\Gamma(\xi_b - \epsilon)$ and $\Gamma(\xi_b + \delta)$ meet at the point $(X(t_b + \tilde{t}),\ t_b + \tilde{t})$. Because the characteristics meet, we have an equation analogous to equation (2.4):

$$\epsilon + \delta = \{\lambda_o(\xi_b - \epsilon) - \lambda_o(\xi_b + \delta)\}\ (t_b + \tilde{t}) \ . \qquad (2.126)$$

The jumps in f and u may be expanded in terms of their characteristic data:

$$[f] = f(u_o(\xi_b + \delta)) - f(u_o(\xi_b - \epsilon)) \qquad (2.127)$$

$$[u] = u_o(\xi_b + \delta) - u_o(\xi_b - \epsilon) \ . \qquad (2.128)$$

Furthermore, we suppose that ϵ may be expanded as a power series in δ , thus

$$\epsilon = a_1\delta + a_2\delta^2 + a_3\delta^3 + \ldots \qquad (2.129)$$

It is therefore possible to use equations (2.126) and (2.129) to find $\tilde{t}$ as a power series in δ . But, equations (2.123), (2.124) and (2.125) may be combined in the form

$$X(t_b+\tilde{t}) = x_b + \int_o^{\tilde{t}} \frac{[f]}{[u]}\bigg|_{t=t_b+\tau} d\tau \ . \tag{2.130}$$

The integral in this equation may be transformed to give

$$X(t_b+\tilde{t}(\delta)) = x_b + \int_o^{\delta} \frac{[f]}{[u]}\bigg|_{t=t_b+\tilde{t}(\tilde{\delta})} \frac{d\tilde{t}}{d\delta}(\tilde{\delta})\, d\tilde{\delta} \ , \tag{2.131}$$

whence we may obtain X as a power series in δ . The system is closed by resubstituting this result into equation (2.125) to give

$$\frac{\frac{dX}{d\delta}}{\frac{dt}{d\delta}} = \frac{[f]}{[u]} \ , \tag{2.132}$$

(noting $\frac{dt}{d\delta} = \frac{d\tilde{t}}{d\delta}$) .

This procedure enables us to calculate the coefficients $a_1, a_2 \ldots$.

2.3.2. Results

For convenience, the following notation is introduced

$$u_o^{(n)} = \frac{d^n u_o}{d\xi^n}(\xi_b) \ , \quad n \geq 0 \tag{2.133}$$

$$\lambda_o^{(n)} = \frac{d^n \lambda_o}{d\xi^n}(\xi_b) \ , \quad n \geq 0 \ . \tag{2.134}$$

It was found that a_1 obeys a cubic equation with roots $1, -1$ and -1. The second and third roots correspond to data on the same side of the breaking characteristic $\Gamma(\xi_b)$ and we therefore discounted (they actually also correspond to the caustic where $\frac{[f]}{[u]}$ is undefined).

Taking the root $a_1 = 1$, it is then possible to show that

$$a_2 = \frac{3u_o^{(1)}\lambda_o^{(4)} + 2u_o^{(2)}\lambda_o^{(3)}}{15u_o^{(1)}\lambda_o^{(3)}} , \tag{2.135}$$

and from this to show that

$$a_3 = a_2^2 . \tag{2.136}$$

This rather surprising result leads to the conjecture that ϵ may be written as an analytic function of δ thus

$$\epsilon = \frac{\delta}{1-a_2\delta} . \tag{2.137}$$

However, a first attempt at imposing this conjecture would seem to suggest that this is not the case.

The next objective is to write X as a power series in $\tilde{t}$ by eliminating δ. Thus we require b_1, b_2, ..., where

$$X(t_b + \tilde{t}) = x_b + b_1\tilde{t} + b_2\tilde{t}^2 + \,.. \tag{2.138}$$

The results are that

$$b_1 = \lambda_o^{(o)} , \tag{2.139}$$

as would have been expected (as the shock S is continuous with the limiting characteristic $\Gamma(\xi_b)$ at the breaking point (x_b, t_b)). Also

$$b_2 = \frac{\lambda_o^{(1)^3}(8u_o^{(2)}\lambda_o^{(3)} - 3u_o^{(1)}\lambda_o^{(4)})}{10u_o^{(1)}\lambda_o^{(3)^2}} . \tag{2.140}$$

This leads to the radius of curvature of the shock tip in (x,t) space to be calculated as

$$\rho_o = \frac{10\lambda_o^{(3)^2}u_o^{(1)}(1 + \lambda_o^{(o)^2})^{3/2}}{\lambda_o^{(1)^3}(3u_o^{(1)}\lambda_o^{(4)} - 8u_o^{(2)}\lambda_o^{(3)})} . \tag{2.141}$$

Finally, if we define the arc length from the shock tip,

$$\tilde{\sigma} = \sqrt{[(X(t_b + \tilde{t}) - x_b)^2 + \tilde{t}^2]} . \tag{2.142}$$

It is possible to show that

$$\tilde{\sigma} = (\lambda_o^{(o)^2} + 1)^{1/2} \frac{\lambda_o^{(3)}}{6\lambda_o^{(1)^2}} \delta^2 + O(\delta^3) . \tag{2.143}$$

From which we obtain

$$[u(\tilde{\sigma})] = 2u_o^{(1)}\lambda_o^{(1)}(1 + \lambda_o^{(o)^2})^{-1/4} \left[\frac{6}{\lambda_o^{(1)}}\right]^{1/2} \tilde{\sigma}^{1/2} + O(\tilde{\sigma}) . \tag{2.144}$$

So we see that the initial shock formation shape is the standard pitchfork suggested by catastrophe theory. Note, however, that the equal area rule for thermodynamic transitions (see [4] p. 328) does not have an analogue for the catastrophe in §2.2.1 even though the Rankine Hugoniot jump conditions may be manipulated into a similar form. This is because of the transformations necessary in changing the initial characteristic manifold equation into the standard form for the cusp catastrophe. So, for example, we cannot use an equal area argument on equation (2.111) to obtain

$$[\psi] = 2\sqrt{-\gamma} \tag{2.145}$$

and then transform back to the initial variables.

3. Burgers' Equation

3.0 Introduction

Burgers' equation is widely regarded as the simplest equation expressing the balance between convection and diffusion. The analytic solution discovered by Cole ([7]) and Hopf ([8]) is often used to provide test problems for numerical schemes. In view of this, it seems surprising that the so called Cole-Hopf transformation has not been extended to more complicated systems which would have much greater value for practical problems.

The purpose of this section, therefore, is to extend the Cole-Hopf transformation as far as is analytically possible and then to investigate what uses these generalisations might have practically.

3.1 The Cole-Hopf Transformation

In this subsection, a brief review of the Cole-Hopf transformation is given for the sake of completeness and the introduction of suitable notation. Here, and in other places, we shall follow the account of Whitham ([9], §4).

Burgers' equation ([10]) may be written

$$u_t + (\tfrac{1}{2}u^2)_x = \epsilon u_{xx} \,, \tag{3.1}$$

where $u = u(x,t)$ and ϵ is a positive constant. We introduce the substitution

$$u = \psi_x \,, \tag{3.2}$$

yielding the equation

$$\psi_{xt} + (\tfrac{1}{2}\psi_x^2)_x = \epsilon\psi_{xxx} . \tag{3.3}$$

This equation may be integrated. without loss of generality, homogeneity may be assumed to give

$$\psi_t + \tfrac{1}{2}\psi_x^2 = \epsilon\psi_{xx} . \tag{3.4}$$

Now, let us introduce a further substitution.

$$\psi = -2\epsilon \ln \phi . \tag{3.5}$$

A small amount of analysis leads us to the transformed equation

$$\phi_t = \epsilon\phi_{xx} . \tag{3.6}$$

This is just the heat equation for which there exists an exact solution when ϵ is positive. First of all, let us calculate the initial data for ϕ .

Let

$$\phi_o(x) = \phi(x,o) . \tag{3.7}$$

Suppose, as in §1 and §2, that

$$u_o(x) = u(x,o) . \tag{3.8}$$

Equations (3.2) and (3.5) can now be inverted to yield

$$\phi_o(x) = \exp\left\{-\frac{1}{2\epsilon}\int_o^x u_o(\eta)d\eta\right\} . \qquad (3.9)$$

The standard solution to the heat equation is

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}}\int_{-\infty}^{\infty}\phi_o(\eta)\ \exp\left\{-\frac{(x-\eta)^2}{4\epsilon t}\right\} d\eta . \qquad (3.10)$$

Let us introduce two new functions:

$$G(\eta;x,t) = \int_o^\eta u_o(\zeta)d\zeta + \frac{(x-\eta)^2}{2t} \qquad (3.11)$$

$$I(g(\eta;x,t)\ ,\ G(\eta;x,t)) = \int_{-\infty}^{\infty} g(\eta;x,t)\ \exp\left\{-\frac{1}{2\epsilon}G(\eta;x,t)\right\}d\eta \qquad (3.12)$$

(I is in fact a functional of g and G and implicitly a function of x and t), where $g(\eta;x,t)$ is an arbitrary function.

Then it can be shown that the general solution to Burger's equation is

$$u(x,t) = \frac{I\left[\frac{x-\eta}{t}\ ,\ G(\eta;x,t)\right]}{I(1\ ,\ G(\eta;x,t))} . \qquad (3.13)$$

The combination of the transformations given by equations (3.2) and (3.5) is collectively known as the Cole-Hopf transformation. It may be written

$$u = -2\epsilon\,\frac{\phi_x}{\phi} . \qquad (3.14)$$

3.2 Generalisations

3.2.1 Several Equations

After having played around with different attempts to generalise the Cole-Hopf transformation to systems of equations, it has become clear to the author that the generalisation is only possible if the quadratic structure of the non-linear convection term is preserved. To this end, we consider the system of equations

$$u_t^i + \sum_{j,k} A_{jk}^i (u^j u^k)_x = \sum_j E_{ij} u_{xx}^j \ , \tag{3.15}$$

where i,j,k run from 1 to n , u^i are the dependent variables and A_{jk}^i and E_{ij} are constants.

Let

$$\underline{u} = (u^1, \ldots u^n) \ . \tag{3.16}$$

We shall consider linear transforms

$$\underline{v} = M\underline{u} \ , \tag{3.17}$$

where M is a constant matrix.

The following theorem is now presented.

Theorem 3.1

The system of equations (3.15) may be solved exactly by the use of the Cole-Hopf transformation and the method of characteristics if there exists a constant matrix M and constant coefficients $\epsilon_1 \ldots \epsilon_n$

such that:

i) M is invertible;

ii) $\epsilon_1, \ldots, \epsilon_n \geq 0$

iii) $\forall i,j,k \; , \; A^i_{jk} = \frac{1}{2} \sum_l M^{-1}_{il} M_{lj} M_{lk}$ (3.18)

iv) $E = M^{-1} \epsilon M$, (3.19)

where

$$\epsilon = \mathrm{diag}\{\epsilon_1, \ldots, \epsilon_n\} \tag{3.20}$$

<u>**Proof**</u>

Our aim is to decouple equations (3.15) to the system

$$\forall i \; , \; v^i_t + (\tfrac{1}{2} v^{i^2})_x = \epsilon_i v^i_{xx} \; . \tag{3.21}$$

Then condition ii) allows us to solve the individual equations either by the method of characteristics (when $\epsilon_i = 0$) or by the Cole-Hopf transformation (when $\epsilon_i > 0$).

Introducing the linear transformation given by equation (3.17) gives

$$\frac{\partial}{\partial t}\left\{ \sum_j M_{ij} u^j \right\} + \frac{\partial}{\partial x}\left\{ \frac{1}{2} \left[\sum_j M_{ij} u^j \right] \left[\sum_k M_{ik} u^k \right] \right\} = \epsilon_i \frac{\partial^2}{\partial x^2} \left\{ \sum_j M_{ij} u^j \right\} . \tag{3.22}$$

As M is constant, this reduces to

$$\sum_j M_{ij} u^j_t + \frac{1}{2} \sum_{j,k} M_{ij} M_{ik} (u^j u^k)_x = \epsilon_i \sum_j M_{ij} u^j_{xx} \; . \tag{3.23}$$

We now use condition i) and pre-multiply by M^{-1}. This gives us

$$u_t^i + \frac{1}{2}\sum_{j,k,l} M_{ij}^{-1} M_{lj} M_{lk} (u^j u^k)_x = \sum_{j,l} M_{il}^{-1} \epsilon_l M_{lj} u_{xx}^j . \tag{3.24}$$

Clearly, condition iii) is satisfied by comparing equations (3.15) and (3.18) with equation (3.24). Finally, to observe condition iv), we write equation (3.19) is suffix form:

$$E_{ij} = \sum_{k,l} M_{ik}^{-1} \epsilon_{kl} M_{lj} . \tag{3.25}$$

But, by equation (3.20),

$$\epsilon_{kl} = \delta_{kl} \epsilon_l . \tag{3.26}$$

Therefore

$$E_{ij} = \sum_{k,l} M_{ik}^{-1} \delta_{kl} \epsilon_l M_{lj}$$

$$= \sum_{l} M_{il}^{-1} \epsilon_l M_{lj} , \tag{3.27}$$

whence condition iv) is satisfied. ■

Taking a look at the degrees of freedom, and observing that without loss of generality we may impose

$$A_{jk}^i = A_{kj}^i \tag{3.28}$$

(because of the quadratic term $u^j u^k$) , in general we see that theorem 3.1 is very restrictive. This is because $\{A^i_{jk}\}$ has $\frac{1}{2}n^2(n+1)$ degrees of freedom and E has $\frac{1}{2}n(n+1)$ degrees of freedom as it must be symmetric, giving a total of $\frac{1}{2}(n+1)^2$. However, the only degrees of freedom allowed in theorem 3.1 are the n coefficients $\{\epsilon_i\}$ and the n^2 coefficients of the transformation matrix M , giving a total of n(n+1) . These issues will be discussed further in §3.3.

The coefficients $\epsilon_1,\ldots,\epsilon_n$ are of course the eigenvalues of E . They may be thought of as diffusion coefficients. Condition ii) corresponds to E being a positive semi-definite matrix. This is similar to the strong diffusivity condition given in [6], §3.4.

Another property of the system with a physical interpretation is described by the following lemma.

Lemma 3.2

$\exists i_o$ such that the right hand side of equation (3.15) is zero for $i = i_o$ $\Longleftrightarrow$ E has a row of zeros $\Longleftrightarrow$ at least one eigenvector of E is zero.

No proof is given as the result is fairly obvious.

The physical application of this lemma might be to the continuity equation, which almost always has a zero right hand side.

Despite the restrictions on the degrees of freedom in equation (3.15), theorem 3.1 does seem to provide the best possible generalisation using a linear transformation. The following three theorems illustrate this point.

Theorem 3.3

Suppose the system of equations (3.15) is reduced by the linear transformation given by equation (3.17) to the system of equations

$$v^i_t + \alpha_i(\tfrac{1}{2} {v^i}^2)_x = \epsilon_i v^i_{xx} \; , \tag{3.29}$$

then there exists a linear transformation

$$\underline{w} = N\underline{u} \tag{3.30}$$

which reduces equations (3.15) to the original decoupled system of the form of equation (3.21), i.e.

$$w^i_t + (\tfrac{1}{2} {w^i}^2)_x = \epsilon_i w^i_{xx} \; . \tag{3.31}$$

Proof

We exploit the quadratic convection term by observing that the transformation

$$w^i = \alpha^i v^i \tag{3.32}$$

reduces equations (3.29) to equations (3.31) for $\alpha_i \neq o$.

We may then set

$$N = AM \; , \tag{3.33}$$

where $A = \text{diag}\,\{\alpha_1, \ldots, \alpha_n\}$, (3.34)

and equation (3.30) is satisfied. ■

This theorem shows that the most general rescaling of a single

Burgers' equation can be achieved using a linear transformation on a system of Burgers' equations.

In the next theorem we show that two successive linear transformations do not give coefficients $\{A^i_{jk}\}$ and $\{E_{ij}\}$ any more degrees of freedom. To this end, we define the following functions:

$$\hat{A}^i_{jk}(M) = \frac{1}{2}\sum_l M^{-1}_{il}M_{lj}M_{lk} \tag{3.35}$$

$$\hat{E}(M) = M^{-1}EM \ . \tag{3.36}$$

Theorem (3.1) shows that the system

$$u^i_t + \sum_{j,k} \hat{A}^i_{jk}(M)(u^j u^k)_x = \sum_j \hat{E}_{ij}(M)u^j_{xx} \tag{3.37}$$

is analytically soluble for any invertible matrix M . This leads us to the following theorem.

<u>**Theorem 3.4**</u>

The structure of equation system (3.37) is invariant under linear transformations. In other words, the two transformations

$$\left.\begin{aligned} \underline{w} &= N\underline{v} \\ \underline{v} &= M\underline{u} \end{aligned}\right\} \tag{3.38}$$

applied successively to the decoupled system of equations (3.31) only has the effect of transforming it to the system

$$u^i_t + \sum_{j,k} \hat{A}^i_{jk}(NM)(u^j u^k)_x = \sum_j \hat{E}_{ij}(NM)u^j_{xx} \tag{3.39}$$

Proof

By a change of notation in equations (3.37), it is clear that the first equation (3.38) transforms equations (3.31) to the equations

$$v_t^i + \sum_{j,k} \hat{A}^i_{jk}(N)(v^j v^k)_x = \sum_j \hat{E}_{ij}(N) v^j_{xx} . \tag{3.40}$$

Now, let us introduce the second transformation equation (3.38). Equation (3.40) will become

$$\left[\sum_j M_{jk} u^j\right]_t + \sum_{j,k} \hat{A}^i_{jk}(N) \left\{\left[\sum_l M_{jl} u^l\right] \left[\sum_m M^i_{km} u^m\right]\right\}_x$$

$$= \sum_j \hat{E}_{ij}(N) \left[\sum_k M_{jk} u^k\right]_{xx} . \tag{3.41}$$

As in the previous argument, because M is constant and invertible, we may reduce equations (3.41) to

$$u_t^i + \sum_{j,k,l,p,q} M^{-1}_{ij} \hat{A}^j_{kl}(N) M_{kp} M_{lq} (u^p u^q)_x = \sum_{j,k,l} M^{-1}_{ij} \hat{E}_{jk}(N) M_{kl} u^l_{xx} . \tag{3.42}$$

But

$$\hat{A}^i_{jk}(NM) = \frac{1}{2} \sum_l (NM)^{-1}_{il} (NM)_{lj} (NM)_{lk} . \tag{3.43}$$

from equation (3.35). This expands to

$$\hat{A}^i_{jk}(NM) = \frac{1}{2} \sum_{l,p,q,r} M^{-1}_{ip} N^{-1}_{pl} N_{lq} M_{qj} N_{lr} M_{rk}$$

$$= \sum_{p,q,r} M^{-1}{}_{ip} \hat{A}^p_{qr}(N) M_{qj} M_{rk} . \tag{3.44}$$

Changing the suffices, we obtain

$$\hat{A}^{i}_{pq}(NM) = \frac{1}{2} \sum_{j,k,l} M^{-1}_{ij} \hat{A}^{j}_{kl}(N) M_{kp} M_{lq} . \tag{3.45}$$

Also, by equation (3.36),

$$\hat{E}(NM) = (NM)^{-1} E (NM) \tag{3.46}$$

$$= M^{-1}N^{-1} E N M$$

$$= M^{-1}\hat{E}(N) M . \tag{3.47}$$

Substituting equations (3.44) and (3.47) into equation (3.42) gives us equation (3.39). Hence the result is proved. ■

The final theorem in this sequence concerns linear transformation with the Cole-Hopf transformation.

Theorem 3.5

The system of equations

$$\phi^{i}_{t} = \epsilon_{i}\phi^{i}_{xx} \tag{3.48}$$

under the transformations

$$\phi^{i} = \exp\left\{- \frac{1}{2\epsilon_{i}} \sum_{j} N_{ij}\psi^{j}\right\} . \tag{3.49}$$

and

$$\psi_x^i = \sum_j M_{ij} u^j \tag{3.50}$$

becomes the system of equations (3.39). M and N are again constant invertible matrices and the coefficients $\{\epsilon_i\}$ must be positive (rather than possibly zero).

Proof

It can be shown that

$$\phi_t^i = -\frac{1}{2\epsilon_i} \sum_j N_{ij} \psi_t^j \phi^i \tag{3.51}$$

and

$$\phi_{xx}^i = \left\{ -\frac{1}{2\epsilon_i} \sum_j N_{ij} \psi_{xx}^j + \frac{1}{4\epsilon_i^2} \sum_{j,k} N_{ij} N_{ik} \psi_x^j \psi_x^k \right\} \phi^i . \tag{3.52}$$

So, substituting in equation (3.48), we obtain

$$\sum_j N_{ij} \psi_t^j + \frac{1}{2} \sum_{j,k} N_{ij} N_{ik} \psi_x^j \psi_x^k = \epsilon_i \sum_j N_{ij} \psi_{xx}^j . \tag{3.55}$$

Using the same method as before of multiplying by N^{-1}, we obtain

$$\psi_t^i + \sum_{j,k} \hat{A}_{jk}^i(N) \psi_x^j \psi_x^k = \sum_j \hat{E}_{ij}(N) \psi_{xx}^j . \tag{3.56}$$

Differentiating with respect to x and performing the transformation given by equation (3.50) we obtain

$$\sum_j M_{ij} u_t^j + \sum_{j,k,p,q} \hat{A}^i_{jk}(N) M_{jp} M_{kq} (u^p u^q)_x = \sum_{j,k} \hat{E}_{ij}(N) M_{jk} \psi^k_{xx} \; . \tag{3.57}$$

Multiplying by M^{-1} and using equations (3.44) and (3.47), equations (3.57) transform to equations (3.39) as required. ■

<u>Corollary</u>

By setting either of the matrices M , N to be the identity matrix we are able to show that the two linear transformations within the Cole-Hopf transformation <u>individually</u> correspond to a linear transformation of the dependent variables.

3.2.2 Several Dimensions

The underlying idea of this subsection is to provide a generalisation to the Cole-Hopf transformation dependent on an arbitrary direction in space. We recall that if $\underline{e}_i$ is the unit vector in the direction of the co-ordinate x_i that

$$\underline{e}_i \cdot \nabla \equiv \frac{\partial}{\partial x_i} \; . \tag{3.58}$$

In a similar vein, let $\underline{\nu}$ be the unit vector in the direction of the co-ordinate y . Thus

$$\underline{\nu} \cdot \nabla \equiv \frac{\partial}{\partial y} \; . \tag{3.59}$$

Thus,

$$\frac{\partial x_i}{\partial y} = \sum_j \nu_j \frac{\partial x_i}{\partial x_j} = \nu_i \; . \tag{3.60}$$

Also,

$$\frac{\partial}{\partial y} (\underline{x}.\underline{\nu}) = \sum_i \frac{\partial x_i}{\partial y} \nu_i = \sum_i \nu_i^2 = 1 \; . \tag{3.61}$$

So we may consistently write

$$y = \underline{x}.\underline{\nu} \; . \tag{3.62}$$

We wish to seek solutions to a problem in several space dimensions. Let us write the dependent variable u explicitly:

$$u = u(\underline{x}, t) \; . \tag{3.63}$$

Let the initial data be given by

$$u(\underline{x}, o) = u_o(\underline{x}) \; . \tag{3.64}$$

Now, this causes a problem when our solution is only a function of y and t. It is necessary to introduce a new function $u_*(y)$ with u_o having the property,

$$\forall \underline{x} \text{ such that } y = \underline{x}.\underline{\nu} \, ,$$
$$u_o(\underline{x}) = u_*(y) \; . \tag{3.65}$$

We may now consider the Burgers' equation

$$u_t + (\tfrac{1}{2}u^2)_y = \epsilon u_{yy} \tag{3.66}$$

as this problem has an analytic solution because its initial data is well defined.

However, equation (3.66) may be transformed into its multidimensional form using equation (3.59) to obtain

$$u_t + \sum_i \nu_i \frac{\partial}{\partial x_i}(\tfrac{1}{2}u^2) = \epsilon \sum_{i,j} \nu_i \nu_j \frac{\partial^2 u}{\partial x_i \partial x_j} . \tag{3.67}$$

This technique may also be applied to the system of transformed Burgers' equations (3.37). Changing the single space variable x to the variable y gives:

$$u^i_t + \sum_{j,k} \hat{A}^i_{jk}(M)(u^j u^k)_y = \sum_i \hat{E}_{ij}(M) u^j_{yy} . \tag{3.68}$$

The initial data, by analogy to above, must obey

$$u^i(\underline{x},o) = u^i_o(\underline{x}) , \tag{3.69}$$

and $\forall i\ \forall \underline{x}$ such that $y = \underline{x}.\underline{\nu}$,

$$u^i_o(\underline{x}) = u^i_*(y) . \tag{3.70}$$

Equation (3.68) may be rewritten

$$u_t^i + \sum_{j,k} \sum_{p} \hat{A}^i_{jk}(M)\nu_p \frac{\partial}{\partial x_p}(u^j u^k) = \sum_{i} \sum_{p,q} \hat{E}_{ij}(M)\nu_p \nu_q \frac{\partial^2 u^i}{\partial x_p \partial x_q} . \qquad (3.71)$$

An alternative form for the system is to start from the following decoupled system

$$\frac{\partial v^i}{\partial t} + \frac{\partial}{\partial y^i}\left(\frac{1}{2} v^{i^2}\right) = \epsilon_i \frac{\partial^2 v^i}{\partial y^{i^2}} , \qquad (3.72)$$

where

$$\frac{\partial}{\partial y^i} \equiv \underline{\nu}^i . \nabla , \qquad (3.73)$$

where $\underline{\nu}^i$ are constant unit vectors.

In an analogous way to before we may write consistently

$$y^i = \underline{x}.\underline{\nu}^i \qquad (3.74)$$

and we have the following condition on the initial data:

$\forall i \ \forall \underline{x}$ such that $y^i = \underline{x}.\underline{\nu}^i$,

$$v_o^i(\underline{x}) = v_*^i(y^i) , \qquad (3.75)$$

where

$$v_o^i(\underline{x}) = v(\underline{x}, o) . \qquad (3.76)$$

From equations (3.72) and (3.73) we derive the system of equations

$$\frac{\partial v^i}{\partial t} + \sum_p \nu_p^i \frac{\partial}{\partial x_p} \left[\frac{1}{2} v^{i^2}\right] = \epsilon_i \sum_{p,q} \nu_p^i \nu_q^i \frac{\partial^2 v^i}{\partial x_p \partial x_q} . \qquad (3.77)$$

Applying the transformation $\underline{v} = M\underline{u}$ yields the system

$$\frac{\partial u^i}{\partial t} + \sum_{j,k,l} \sum_p \nu_p^j M_{ij}^{-1} M_{jk} \frac{\partial}{\partial x_p} \left[\frac{1}{2} u^k u^l\right] = \sum_{j,k} \sum_{p,q} \nu_p^j \nu_q^j M_{ij}^{-1} \epsilon_j M_{jk} \frac{\partial^2 u^k}{\partial x_p \partial x_q} . \qquad (3.78)$$

This may be simplified by defining

$$B_p^{ijk}(M,N) = \frac{1}{2} \sum_l \nu_p^l M_{il}^{-1} M_{lj} M_{lk} \qquad (3.79)$$

where N is the $n \times m$ rectangular matrix

$$N = (\underline{\nu}^1, \ldots, \underline{\nu}^m) , \qquad (3.80)$$

where m is the number of space dimensions. We also define

$$F_{pq}^{ij}(M,N) = \sum_l \nu_p^l \nu_q^l M_{il}^{-1} \epsilon_l M_{lj} . \qquad (3.81)$$

(note $\nu_p^i = N_{pi}$) .

Equation (3.78) simplifies to

$$\frac{\partial u^i}{\partial t} + \sum_{j,k} \sum_p B_p^{ijk}(M,N) \frac{\partial}{\partial x_p} (u^j u^k) = \sum_j \sum_{p,q} F_{pq}^{ij}(M,N) \frac{\partial^2 u^j}{\partial x_p \partial x_q} \qquad (3.82)$$

which is not much of an improvement.

Equation (3.82) represents the most complex linear generalisation of Burgers' equation known to the author.

3.2.3 Non-Linear Transformations

As already covered in §1.2, the equation

$$\frac{\partial \underline{v}}{\partial \underline{u}} = M(\underline{u}) \tag{3.83}$$

is not well-posed unless $n = 1$ or 2. Therefore we cannot consider an arbitrary matrix $M(\underline{u})$ replacing the constant matrix M in the system of equations (3.15) as this system does not in general have an analytic solution. To obtain anything apart from a constant matrix we must either solve a well-posed set of equations such as (1.38) or prescribe higher order terms in $\underline{u}$ similar to the construction of equation (1.52).

This having been done, the differential identity

$$\partial \underline{v} = \frac{\partial \underline{v}}{\partial \underline{u}} . \partial \underline{u} . \tag{3.84}$$

Shows us that the nonlinear form of equations (3.15) is

$$u^i_t + \sum_{j,k} \hat{A}^i_{jk}(M(\underline{u}))(u^j u^k)_x = \sum_j \hat{E}_{ij}(M(\underline{u}))u^j_{xx} , \tag{3.85}$$

where $M(\underline{u}) = \frac{\partial \underline{v}}{\partial \underline{u}}$ is substituted for notational convenience rather than being solved.

Similarly, the nonlinear form of equations (3.71) is

$$u_t^i + \sum_{j,k} \sum_p \hat{A}_{jk}^i(M(\underline{u}))\nu_p \frac{\partial}{\partial x_p}(u^j u^k) = \sum_j \sum_{p,q} \hat{E}_{ij}(M(\underline{u}))\nu_p \nu_q \frac{\partial^2 u^j}{\partial x_p \partial x_p} , \qquad (3.86)$$

and the nonlinear form of equations (3.82) is

$$\frac{\partial u^i}{\partial t} + \sum_{j,k} \sum_p B_p^{ijk}(M(\underline{u}),N) \frac{\partial}{\partial x_p}(u^j u^k) = \sum_j \sum_{p,q} F_{pq}^{ij}(M(\underline{u}),N) \frac{\partial^2 u^j}{\partial x_p \partial x_q} . \qquad (3.87)$$

Another possibility is to consider having variable unit vectors in the multidimensional analysis, e.g.

$$\underline{\nu} = \underline{\nu}(\underline{x}) . \qquad (3.88)$$

This corresponds to one (or many) curvilinear coordinate (or coordinates) y (or y^i).

This case is not considered here as it is not expected to be particularly applicable due to the difficulty in prescribing the initial data. The second order terms also become extremely complicated.

3.2.4 Examples

We shall limit ourselves here to linear transformations in one space dimension. We shall look at the structure of the simple cases and make an attempt to model the physical flow equations.

The case $n = 1$ is trivial as shown:

$$\left.\begin{array}{l} M = (M_{11}) \\ \{A^i_{jk}\} = \{A^1_{11}\} \\ M^{-1} = \frac{1}{M_{11}} \\ \epsilon = (\epsilon_1) \end{array}\right\} . \tag{3.89}$$

The conditions for theorem 3.1 become

$$\left.\begin{array}{ll} \text{i)} & M_{11} \neq 0 \quad ; \\ \text{ii)} & \epsilon_1 \geq 0 \quad ; \\ \text{iii)} & A^1_{11} = \frac{1}{2} M_{11} \;; \\ \text{iv)} & E = M^{-1}\epsilon M = \epsilon = (\epsilon_1) \end{array}\right\} . \tag{3.90}$$

So we can always reduce

$$u^1_t + A^1_{11}({u^1}^2)_x = \epsilon_1 u^1_{xx} \tag{3.91}$$

to

$$v^1_t + (\tfrac{1}{2}\, {v^1}^2)_x = \epsilon_1 v^1_{xx} \; . \tag{3.92}$$

and then solve analytically, provided $A^1_{11} \neq 0$ and $\epsilon_1 \geq 0$.

For the case $n = 2$, by the argument in §3.2.1, we have 10 degrees of freedom in our initial equation and 6 degrees of freedom in our final equation. Let us introduce the following notation

$$\left.\begin{aligned} &\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix} \\ &A^1_{ij} = A_{ij} \\ &A^2_{ij} = B_{ij} \\ &M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &\epsilon_1 = \epsilon \\ &\epsilon_2 = \delta \\ &\det M = ad - bc = \Delta \end{aligned}\right\} \tag{3.93}$$

Then it can be shown that

$$A = -\frac{1}{\Delta}\begin{bmatrix} da^2 - bc^2 & bd(a - c) \\ bd(a-c) & bd(b-d) \end{bmatrix} \tag{3.94}$$

$$B = -\frac{1}{\Delta}\begin{bmatrix} ac(c-a) & ac(d-b) \\ ac(d-b) & ad^2 - cb^2 \end{bmatrix} \tag{3.95}$$

$$E = -\frac{1}{\Delta}\begin{bmatrix} \epsilon ad - \delta bc & (\epsilon-\delta)bd \\ (\delta-\epsilon)ac & \delta ad - \epsilon bc \end{bmatrix} . \tag{3.96}$$

For the case $n = 3$, we have 27 degrees of freedom being reduced to 12. The equations are obviously much more complicated.

We now turn our attention to the one-dimensional unsteady Euler equations and attempt to fit them into this structure. The equations are (e.g. see [11] §7 & §8):

$$\begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ p+\rho u^2 \\ \rho u H \end{bmatrix}_x = 0 \tag{3.97}$$

where

$$H = E + \frac{p}{\rho} , \tag{3.98}$$

and, for a perfect gas,

$$E = \frac{1}{2} u^2 + \frac{1}{\gamma-1} \frac{p}{\rho} . \tag{3.99}$$

Now, we need to convert equations (3.97) into the form of equations (3.15) and then try to see whether the conditions of theorem 3.1 are satisfied. In order not to confuse the notation, we shall write the transformed dependent variables as U^1 , U^2 , ...

The simplest of the three equations in (3.97) is the first which suggests the transformation

$$\left.\begin{aligned} U^1 &= \rho \\ U^2 &= u \end{aligned}\right\} . \tag{3.100}$$

It then becomes

$$U^1_t + (U^1U^2)_x = 0 . \tag{3.101}$$

We now require an equation in U^2_t , i.e. u_t . The first two equations in (3.97) imply

$$u_t + uu_x + \frac{1}{\rho} p_x = 0 \qquad (3.102)$$

Clearly $uu_x = (\frac{1}{2} u^2)_x = (\frac{1}{2} \{U^2\}^2)_x$, (3.103)

which is in the correct form. We shall, however, need to convert the third term so that

$$\frac{1}{\rho} p_x = \Big[\sum_{i,j} c_{ij} U^i U^j \Big]_x \qquad (3.104)$$

for some constants c_{ij} where i and j run from 1 to either 2 or 3.

The only dependence on p (if p is an independent variable from ρ and u) will come from U^3 . We write

$$U^3 = U^3(p, \rho, u) \qquad (3.105)$$

and look at the terms in the right hand side of equation (3.104) containing U^3 . By symmetry, these are

$$2(c_{13} U^1 U^3)_x + 2(c_{23} U^2 U^3)_x$$

so, the terms containing p_x will be

$$2c_{13} U^1 \frac{\partial U^3}{\partial p} p_x + 2c_{23} U^2 \frac{\partial U^3}{\partial p} p_x$$

we may therefore equate coefficients in p_x on the left and right hand side of equation (3.104) to yield

$$\frac{1}{\rho} = 2(c_{13}\, U^1 + c_{23}\, U^2)\, \frac{\partial U^3}{\partial p}$$

i.e.

$$\frac{1}{\rho} = 2(c_{13}\, \rho + c_{23}\, u)\, \frac{\partial U^3}{\partial p} \qquad (3.16)$$

This may be written

$$\frac{\partial U^3}{\partial p} = \frac{1}{2\rho(c_{13}\rho+c_{23}u)} \; . \qquad (3.107)$$

This can be integrated to give

$$U^3 = \frac{p}{2\rho(c_{13}\rho+c_{23}u)} + \phi(\rho,u) \; , \qquad (3.108)$$

for some arbitrary function $\phi(\rho,u)$. From this we obtain

$$p = 2U^3\rho(c_{13}\rho + c_{23}u) - \phi(\rho,u) \; . \qquad (3.109)$$

Finally, we may differentiate with respect to x to give

$$\frac{1}{\rho}\, p_x = (c_{13}U^1U^3 + c_{31}\, U^3U^1 + c_{23}U^2U^3 + c_{32}U^3U^2)_x$$

$$- 2(c_{13}\rho_x + c_{23}U_x)U^3 + 4c_{13}\rho_x + 2c_{23}\,\frac{p_x}{\rho}\, u + 2\, c_{23}U_x - \phi_x \; . \qquad (3.110)$$

The terms outside the first bracket on the right hand side cannot be converted into the correct form as the U^3 term cannot be eliminated (because ϕ is not a function of p).

Thus it is impossible to have p independent of ρ and u. For this case, equation (3.104) may be written in the form

$$\frac{1}{\rho} p_x = (a\rho^2 + b\rho u + cu^2)_x . \tag{3.111}$$

Thus,

$$p_x = \rho(2a\rho\rho_x + b\rho u_x + bu\rho_x + 2cuu_x) . \tag{3.112}$$

The right hand side is homogeneous third order in ρ and u. We therefore write

$$p = A\rho^3 + B\rho^2 u + C\rho u^2 + Du^3 . \tag{3.113}$$

We shall differentiate equation (3.113) with respect to x and equate coefficients with equation (3.112). Equation (3.113) gives

$$p_x = 3A\rho^2\rho_x + 2B\rho u\rho_x + B\rho^2 u_x + Cu^2\rho_x + 2C\rho uu_x + 3Du^2 u_x . \tag{3.114}$$

Equating coefficients gives

$$\rho^2\rho_x] \qquad 2a = 3A \tag{3.115}$$

$$\rho^2 u_x] \qquad b = B \tag{3.116}$$

$$\rho u \rho_x] \qquad b = 2B \tag{3.117}$$

$$\rho u u_x] \qquad 2C = 2c \tag{3.118}$$

$$u^2 \rho_x] \qquad 0 = C \tag{3.119}$$

$$u^2 u_x] \qquad 0 = 3D \ . \tag{3.120}$$

Thus,

$$p = A\rho^3 \ . \tag{3.121}$$

This corresponds to a polytropic gas with $\gamma = 3$, which is theoretically impossible. It can be shown that the third equation of (3.97) is degenerate in this case. Putting this together and using the standard letter κ for the constant in equation (3.121), we arrive at the system

$$\begin{bmatrix} U^1 \\ U^2 \end{bmatrix} + \begin{bmatrix} U^1 U^2 \\ \frac{3\kappa}{2}\{U^1\}^2 + \frac{1}{2}\{U^2\}^2 \end{bmatrix}_x = 0 \ . \tag{3.122}$$

Using the previous notation, this corresponds to matrices

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \tag{3.123}$$

$$B = \begin{bmatrix} 3\kappa/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \ . \tag{3.124}$$

Using equations (3.94) and (3.95) , we obtain the equations

$$-\begin{bmatrix} da^2 - bc^2 & bd(a-c) \\ bd(a-c) & bd(b-d) \end{bmatrix} = (ad-bc)\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \tag{3.125}$$

$$-\begin{bmatrix} ac(c-a) & ac(d-b) \\ ac(d-b) & ad^2 - cb^2 \end{bmatrix} = (ad-bc)\begin{bmatrix} 3\kappa/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} . \tag{3.126}$$

These yield the relations

$$\left.\begin{array}{l} c = -a \\ d = b \end{array}\right\} \tag{3.127}$$

which imply

$$\left.\begin{array}{r} 2b^2a = -\frac{1}{2} \\ a^3 = -\frac{3\kappa}{2} \end{array}\right\} . \tag{3.128}$$

So, if we introduce

$$\mu = \left[\frac{3\kappa}{2}\right]^{1/6} \tag{3.129}$$

the transformation matrix is

$$M = \begin{bmatrix} -\mu^2 & \pm\frac{1}{2\mu} \\ \mu^2 & \pm\frac{1}{2\mu} \end{bmatrix} . \tag{3.130}$$

So, for this example, we can satisfy the conditions of theorem 3.1.

Therefore an exact solution exists for this problem.

Finally, we shall consider adding diffusion to the system of equations (3.122), to transform it to the system

$$\begin{bmatrix} U^1 \\ U^2 \end{bmatrix}_t + \begin{bmatrix} \underline{U}^T & A\ \underline{U} \\ \underline{U}^T & B\ \underline{U} \end{bmatrix}_x = E \begin{bmatrix} U^1 \\ U^2 \end{bmatrix}_{xx} . \qquad (3.131)$$

From the above argument and equation (3.96) it can be inferred that

$$E = (\epsilon + \delta) \begin{bmatrix} 1 & \pm \frac{1}{2\alpha} \\ \pm 2\alpha & 1 \end{bmatrix} , \qquad (3.132)$$

where

$$\alpha = \frac{\epsilon-\delta}{\epsilon+\delta}\ 2\mu^3$$

$$= \sqrt{6\kappa}\ \frac{\epsilon-\delta}{\epsilon+\delta} . \qquad (3.133)$$

Unfortunately, the two degrees of freedom present in E are insufficient to use it to model physical diffusion. The one-dimensional steady Navier-Stokes equation for an ideal polytropic gas with constant kinematic viscosity ν correspond to equations (3.131) with

$$E = \begin{bmatrix} 0 & 0 \\ 0 & \frac{4}{3}\nu \end{bmatrix} \qquad (3.134)$$

so, equations (3.132) and (3.134) are clearly inconsistent.

Other exact models of physical systems may be possible (e.g. we were not forced to choose $U^1 = \rho$ - another possibility might be

$U^1 = \rho^{1/2}$ - see [12]). The theory may still be useful to solving physical problems, as described in the next subsection.

3.3 Implications for Numerical Schemes

The basic idea of this subsection is to attempt to extend the concept of locally solving exact equations to the more complicated systems of equations decribed in §3.2. First of all, we review the major techniques of this type employed in existing numerical schemes.

The classical work in this field is Godunov's method for solving the one dimensional unsteady Euler equations in the form of a Riemann problem (see [13], §12.15). His method was excellent for solving the Riemann problem, but suffers from unbalanced inaccuracy when it is used locally and then reprojected onto constant states for the solution of more general initial-value problems.

To counter this, Roe ([12]) suggested using only approximate local Riemann solvers in order to increase efficiency without affecting accuracy. His method has been widely used.

An alternative related method is the 'random choice' method of Glimm ([14]). This has been improved and extended by Harten and Luc ([15]).

Finally, Roe ([16]) has considered a way of incorporating diffusion terms into his scheme. This appears to be the only attempt within the class of 'local exact solvers' so far.

It is anticipated that the exact solutions obtained in this section, in the same vein, will be useful for constructing local exact solutions. The global solution, however, will now be for a system of diffusive (rather than non-diffusive) conservation laws. A general procedure for accomplishing this objective is described.

We start off with a system of diffusive conservation laws (see [6], § 2.2.2).

$$\frac{\partial u^i}{\partial t} + \sum_j \sum_{p,q} \sum_m \frac{\partial}{\partial x_p} \left\{ V^{ij}_{pqm}(\underline{u}) \frac{\partial u^j}{\partial x_q} d_m \right\} = S^i \ . \tag{3.135}$$

We shall assume $S^i = 0$. The standard combination

$$|\underline{d}| W^{ij}_{pq}(\underline{u}) = \sum_m V^{ij}_{pqm}(\underline{u}) \, d_m \tag{3.136}$$

is employed. The domain, Δ , is subdivided thus

$$\Delta = \bigoplus_r \Delta_r \ . \tag{3.137}$$

On each subdomain Δ_r we use the initial solution $\underline{u}_o$ to obtain a Taylor expansion of equation (3.135)

$$\frac{\partial u^i}{\partial t} + \sum_j \sum_p A^{ij}_p \frac{\partial u^j}{\partial x_p} + \sum_{j,k} \sum_p B^{ijk}_p \frac{\partial}{\partial x_p}(u^j u^k)$$

$$= \sum_j \sum_{p,q} C^{ij}_{pq} \frac{\partial^2 u^j}{\partial x_p \partial x_q} \tag{3.138}$$

C^{ij}_{pq} is proportional to $|\underline{d}|$. The linear term on the left hand side can hopefully be removed by a change of variables $\underline{u} \mapsto \underline{v}$, giving

$$\frac{\partial v^i}{\partial t} + \sum_{j,k} \sum_p D^{ijk}_p \frac{\partial}{\partial x_p}(v^i v^k) = \sum_j \sum_{p,q} C^{ij}_{pq} \frac{\partial^2 v^j}{\partial x_p \partial x_q} \ . \tag{3.139}$$

This is only possible when there exists a matrix M such that

$$\forall p,\ M^{-1} A_p M = 0 \tag{3.140}$$

where A_p is the matrix of coefficients A_p^{ij}.
This is now in a form compatible with equation (3.82). Our objective now is to choose $\epsilon_1, \dots \epsilon_n$, M and N such that

$$\left.\begin{aligned} B_p^{ijk}\ (M,N) &\simeq D_p^{ijk} \\ B_{pq}^{ij}\ (M,N) &\simeq C_{pq}^{ij} \end{aligned}\right\} \tag{3.141}$$

in some optimized sense (the initial conditions will also need to be approximated). These equations may now be solved exactly using the Cole-Hopf transformation (or approximately if this improves efficiency without the expense of inaccuracy). The exact (or approximate) solution is then found for the next time step and the process is repeated (perhaps with a new partition of Δ, similar to Glimm in [14]).

The concept of dividing the domain up into subsystems is similar to Orlov's approach to approximating irreversible thermodynamics equations (see [17]). The incorporation of the thermodynamical issues to the system derived here will be discussed in the author's next report.

Naturally, if any theory or test cases are going to be produced, they will need to be much simpler than this general structure. However, it is anticipated that the principles will remain unchanged.

4. Limiting Diffusion Theory

4.0 Introduction

The aim of this section is to consider models with limitingly small diffusion, relating them to some of the work covered in the previous sections. Once this has been accomplished, implications of this investigation to numerical schemes is discussed.

4.1 Asymptotic Limit of the Cole-Hopf Transformation

This subsection contains an extension to the asymptotic analysis of Burgers' equation in the limit $\epsilon \to 0$ given by Whitham ([9], §4.2). Our purposes here require the next term in the expansion. The procedure used may, however, be extended to higher terms to give an apparently regular perturbation expansion.

First of all, we recall some of the equations of §3.1. Burgers' equation was given as (3.1)

$$u_t + (\tfrac{1}{2} u^2)_x = \epsilon \, u_{xx} \, . \tag{4.1}$$

We had the two definition equations (3.11) and (3.12):

$$G(\eta;x,t) = \int_0^\eta u_o(\zeta) \, d\zeta + \frac{(x-\eta)^2}{2t} \tag{4.2}$$

$$I(x,t \; ; g,G) = \int_{-\infty}^{\infty} g(\eta; \, x,t) \, e^{-G(\eta; \, x,t)/2\epsilon} \, d\eta \, . \tag{4.3}$$

This enabled us to write the solution to Burgers' equation as

$$u(x,t) = \frac{I(x,t;\ \left[\frac{x-n}{t}\right],G)}{I(x,t;\ 1,G)} \quad . \tag{4.4}$$

Now, let us consider the behaviour of $I(x,t;\ g,G)$ as $\epsilon \to 0$. Because of its dependence on the integral term $e^{-G/2\epsilon}$, this will be dominated by the contribution near the value of η where G is minimised. We shall introduce the variable $\xi(x,t)$ such that

$$G(\eta;x,t) \text{ is minimised when } \eta = \xi(x,t) \ . \tag{4.5}$$

Performing elementary calculus on equation (4.2) gives us

$$x = \xi(x,t) + u_o(\xi(x,t))t \ , \tag{4.6}$$

and

$$u_o'(\xi(x,t)) > -\frac{1}{t} \ . \tag{4.7}$$

Note the similarity between equations (4.6) and (1.7).

We shall limit ourselves to the cases where G has at most two minima with approximately equal values. These are as follows:

i) G has a single dominating minimum;

ii) G Has two minima at $\eta = \xi_1$, and $\eta = \xi_2$, with

$$|G(\xi_1;x,t) - G(\xi_2;x,t)| = O(\epsilon) \ , \tag{4.8}$$

but

$$|\xi_1 - \xi_2| = O(1) \ ; \tag{4.9}$$

iii) G has two minima as in ii) but with

$$|\xi_1 - \xi_2| = O(\epsilon) \ . \tag{4.10}$$

These three cases are represented in figure 5. We shall consider each of these cases in turn.

<u>Case i)</u>: G has a single dominating minimum. As the major contribution to I will come from near $\eta = \xi$, we introduce a change of variable

$$\eta = \xi + \epsilon z \ . \tag{4.11}$$

We shall also use the notation

$$G^{(n)} = \frac{\partial^n G}{\partial \eta^n}(\xi(x,t);x,t) \tag{4.12}$$

($G^{(n)}$ is in fact a function of x and t). We can use the change of variable to obtain the Taylor expansion

$$G(\eta;x,t) = G^{(o)} + \frac{\epsilon^2 z^2}{2!} G^{(2)} + \frac{\epsilon^3 z^3}{3!} G^{(3)} + \frac{\epsilon^4 z^4}{4!} G^{(4)} + O(\epsilon^5) \tag{4.13}$$

(we clearly have $G^{(1)} = 0$) .

From equation (4.13), we derive

$$e^{-G/2\epsilon} = e^{-G^{(0)}/2\epsilon} \exp\left\{-\frac{\epsilon z^2}{4} G^{(2)} - \frac{\epsilon^2 z^3}{12} G^{(3)} - \frac{\epsilon^3 z^4}{48} G^{(4)} + O(\epsilon^4)\right\} . \tag{4.14}$$

We need only consider functions $g(\eta;x,t)$ of the form

$$g(\eta) = a\eta + b \ , \tag{4.15}$$

where a and b are functions of x and t . Using

$$\epsilon dz = d\eta \ , \tag{4.16}$$

and equations (4.13) to (4.15), we obtain

$$I = e^{-G^{(0)}/2\epsilon}\,(a\xi+b)\int_{-\infty}^{\infty}\left[1 + \frac{az}{a\xi+b}\,\epsilon\right]\exp\left\{-\frac{\epsilon z^2}{4}\,G^{(2)} - \frac{\epsilon^2 z^3}{12}\,G^{(3)}\right.$$

$$\left. - \frac{\epsilon^3 z^4}{48}\,G^{(4)} + O(\epsilon^4)\right\}\epsilon dz \ . \tag{4.17}$$

We recall the two standard integral formulae

$$\int_{-\infty}^{\infty} z^{2n}\,e^{-\alpha z^2}\,dz = \frac{(2n-1)(2n-3)\,..(1)}{(2\alpha)^n}\left[\frac{\pi}{\alpha}\right]^{1/2} \tag{4.18}$$

$$\int_{-\infty}^{\infty} e^{-\alpha z^2}\,dz = \left[\frac{\pi}{\alpha}\right]^{1/2} . \tag{4.19}$$

Also, by symmetry,

$$\int_{-\infty}^{\infty} z^{2n+1} e^{-\alpha z^2}\,dz = 0 \ . \tag{4.20}$$

So, expanding the z^3 and z^4 terms in the expotential in the right hand side of equation (4.17) gives

$$I = \left[\frac{\pi\epsilon}{G^{(2)}}\right]^{\frac{1}{2}} (a\xi+b)\ e^{-G^{(0)}/2\epsilon} \left\{1 - \frac{\epsilon}{G^{(2)^2}} \left[\frac{aG^{(3)}}{a\xi+b} + \frac{G^{(4)}}{4}\right] + O(\epsilon^2)\right\} \qquad (4.21)$$

Thus, substituting for g , and hence for a and b , we obtain

$$u = \left[\frac{x-\xi}{t}\right]\left\{1 - \frac{\epsilon}{G^{(2)^2}} \left[- \frac{G^{(3)}}{x-\xi} + \frac{G^{(4)}}{4}\right] + O(\epsilon^2)\right\} \left\{1 - \frac{\epsilon}{G^{(2)^2}} \left[\frac{G^{(4)}}{4}\right] + O(\epsilon^2)\right\}^{-1} \qquad (4.22)$$

by cancellation. This reduces further to

$$u = \left[\frac{x-\xi}{t}\right]\left\{1 + \frac{\epsilon G^{(3)}}{G^{(2)^2}(x-\xi)} + O(\epsilon^2)\right\} . \qquad (4.23)$$

We also have the following

$$\frac{x-\xi}{t} = u_o(\xi) \qquad (4.24)$$

$$G^{(2)} = u'_o(\xi) + \frac{1}{t} \qquad (4.25)$$

$$G^{(3)} = u''_o(\xi) . \qquad (4.26)$$

Substituting these into equation (4.23) yields

$$u(x,t) = u_o(\xi) + \frac{u''_o(\xi)t}{[1+u'_o(\xi)t]^2}\,\epsilon + O(\epsilon^2) . \qquad (4.27)$$

As already mentioned, this method could be used to obtain further terms in the expansion of u. The expansion appears to be regular, condition (4.7) ensuring the order ϵ term does not become singular.

Case ii): G has two separated nearly equal minima. In this case, we introduce two changes of variable:

$$\left.\begin{aligned} \eta &= \xi_1 + \epsilon z_1 \\ \eta &= \xi_2 + \epsilon z_2 \end{aligned}\right\} \tag{4.28}$$

we also introduce $\alpha(x,t)$ such that

$$G(\xi_2) = G(\xi_1) + \alpha\epsilon \; . \tag{4.29}$$

Using these substitutions, the expansion procedure used on case i) may also be employed here. To leading order this gives

$$u = \frac{u_o(\xi_1) + Ke^{-\alpha/2} u_o(\xi_2)}{1 + Ke^{-\alpha/2}} \; , \tag{4.30}$$

where

$$K = \left[\frac{\frac{\partial^2 G}{\partial \eta^2}(\xi_1;x,t)}{\frac{\partial^2 G}{\partial \eta^2}(\xi_2;x,t)} \right]^{1/2} \tag{4.31}$$

we again have relations such as

$$\frac{\partial^2 G}{\partial \eta^2}(\xi_1;x,t) = u_o'(\xi_1) + \frac{1}{t} \; . \tag{4.32}$$

The analysis above may easily be extended to higher order expansions.

<u>Case iii)</u> : G has two minima near in position and value. As in case ii), let us use ξ_1 and ξ_2 for the two minima. Clearly, from figure 5, we also have a maximum between the two minima. Let us call it ξ_3 . We introduce an arbitrary value ξ_o such that

$$\left.\begin{aligned} \xi_1 &= \xi_o + \alpha\epsilon \\ \xi_2 &= \xi_o + \beta\epsilon \\ \xi_3 &= \xi_o + \gamma\epsilon \end{aligned}\right\} \tag{4.33}$$

and α,β,γ are not all of the same sign (so, in effect, $\xi_o \in (\xi_1, \xi_3)$). This leads to the change of variable

$$\eta = \xi_o + z\epsilon \ . \tag{4.34}$$

It is claimed that there exists constants λ,μ (functions of x and t) such that

$$G = \lambda + \mu \int_o^z (\zeta-\alpha)(\zeta-\beta)(\zeta-\gamma)d\zeta \tag{4.35}$$

to leading order for η near to ξ_o . Now

$$\frac{d^2G}{dz^2} = \mu \frac{d}{dz} \{(z-\alpha)(z-\beta)(z-\gamma)\} = \epsilon^2\{u_o' + \frac{1}{t}\} \ . \tag{4.36}$$

Whence we obtain the three relationships

$$\left.\begin{aligned} \mu(\alpha-\beta)(\alpha-\gamma) &= \epsilon^2\{u_o'(\xi_1) + \tfrac{1}{t}\} \\ \mu(\beta-\gamma)(\beta-\alpha) &= \epsilon^2\{u_o'(\xi_3) + \tfrac{1}{t}\} \\ \mu(\gamma-\beta)(\gamma-\alpha) &= \epsilon^2\{u_o'(\xi_2) + \tfrac{1}{t}\} \end{aligned}\right\} . \tag{4.37}$$

The next derivative gives

$$\frac{d^3G}{dz^3} = \mu \frac{d^2}{dz^2}\{(z-\alpha)(z-\beta)(z-\gamma)\} = \epsilon^3 u_o'' . \tag{4.38}$$

Whence we obtain the further three relationships

$$\left.\begin{aligned} \mu(2\alpha-\beta-\gamma) &= \epsilon^3 u_o''(\xi_1) \\ \mu(2\beta-\gamma-\alpha) &= \epsilon^3 u_o''(\xi_3) \\ \mu(2\gamma-\alpha-\beta) &= \epsilon^3 u_o'''(\xi_2) \end{aligned}\right\} . \tag{4.39}$$

It is conjectured that μ is order ϵ^4 as G is a quartic local to ξ_o . The error in equation (4.35) is therefore order ϵ^5 .

In principle we may obtain α,β,γ and μ from equations (4.37) and (4.39), given u_o' and u_o'' at ξ_1,ξ_2 and ξ_3 .

However, the quartic function for G in equation (4.35) means that we cannot find I analytically.

To conclude this subsection some striking similarities between the analysis of G and the method of characteristics are noted.

Firstly, the minimization of G led to the characteristic equation for Burger's equation (as $\lambda_o = u_o$ here). Secondly, to first order, the solution for u given by equation (4.27) is just the characteristic equation. Thirdly, equation (4.30) yields a tanh curve for u as a function of x, which will stiffen up into a discontinuity in the limit $\epsilon \to 0$. Fourthly, the conditions on u_o and its derivatives in the third case tend to the conditions on the breaking point as $\epsilon \to 0$.

These similarities are investigated in further subsections.

4.2 The Diffusive Method of Characteristics

This section basically picks up on the second similarity noted above. We use the methods and results of the previous section to devise a new method for obtaining asymptotic solutions in ϵ. It is shown that this method is not just applicable to Burgers' equation, but also to other more complicated model diffusion equations. The name 'the diffusive method of characteristics' is appropriate because information passes along the same characteristic curves, but is modified according to the gradients of the data and the value of the diffusion coefficient. The three cases distinguished in §4.1 are shown to also be present here.

4.2.1 Burgers' Equation

We start off again with Burgers' equation

$$u_t + uu_x = \epsilon u_{xx} \tag{4.40}$$

we wish to impose a solution of the form

$$u = u_o(\xi) + \epsilon\theta(\xi,t) + O(\epsilon^2) \tag{4.41}$$

where $x = \xi + u_o(\xi)t$ (4.42)

Using the differential transformation equations

$$\left.\frac{\partial u}{\partial t}\right|_x = \left.\frac{\partial u}{\partial \xi}\right|_t \left.\frac{\partial \xi}{\partial t}\right|_x + \left.\frac{\partial u}{\partial t}\right|_\xi \quad . \tag{4.43}$$

$$\left.\frac{\partial u}{\partial x}\right|_t = \left.\frac{\partial u}{\partial \xi}\right|_t \left.\frac{\partial \xi}{\partial x}\right|_t \quad , \tag{4.44}$$

and differentiating equation (4.42) with respect to x and t :

$$0 = \left.\frac{\partial \xi}{\partial t}\right|_x + u_o(\xi) + u_o'(\xi)t \left.\frac{\partial \xi}{\partial t}\right|_x \quad , \tag{4.45}$$

$$1 = \left.\frac{\partial \xi}{\partial x}\right|_t + u_o'(\xi)t \left.\frac{\partial \xi}{\partial x}\right|_t \quad , \tag{4.46}$$

we derive

$$\left.\frac{\partial u}{\partial t}\right|_x = - \frac{u_o(\xi)(u_o'(\xi) + \epsilon\frac{\partial \theta}{\partial \xi})}{1 + u_o'(\xi)t} + \epsilon \frac{\partial \theta}{\partial t} + O(\epsilon^2) \tag{4.47}$$

$$\left.\frac{\partial u}{\partial x}\right|_t = - \frac{(u_o'(\xi) + \epsilon\frac{\partial \theta}{\partial \xi})}{1 + u_o'(\xi)t} + O(\epsilon^2) \quad . \tag{4.48}$$

By repeating the differentiation, it may also be shown that

$$\epsilon \left.\frac{\partial^2 u}{\partial x^2}\right|_t = - \frac{\epsilon u_o''(\xi)}{(1 + u_o'(\xi)t)^3} + O(\epsilon^2) \quad . \tag{4.49}$$

substituting equations (4.47) to (4.49) in equation (4.40) and equating coefficients in ϵ gives

$$\epsilon^0] \qquad 0 = 0$$

$$\epsilon^1] \quad -\frac{u_o(\xi)\,\frac{\partial\theta}{\partial\xi}}{1 + u_o'(\xi)t} + \frac{\partial\theta}{\partial t} + \frac{u_o'(\xi)\,\frac{\partial\theta}{\partial\xi} + u_o'(\xi)\theta}{1 + u_o'(\xi)t} = \frac{u_o''(\xi)}{(1 + u_o'(\xi)t)^2} \tag{4.50}$$

This leads to the equation

$$\left.\frac{\partial}{\partial t}\right|_{\xi}\{1 + u_o'(\xi)t)\theta\} = \frac{u_o''(\xi)}{(1 + u_o'(\xi)t)^2} \tag{4.51}$$

which is easily integrated to give

$$\theta = -\frac{u_o''(\xi)t}{(1 + u_o'(\xi)t)^2} + \frac{f(\xi)}{(1 + u_o'(\xi)t)} \tag{4.52}$$

for some function $f(\xi)$. We obviously have the initial condition $u = u_o$ when $t = 0$, so we therefore have

$$\theta(\xi,0) = 0 \qquad \forall\xi . \tag{4.53}$$

This transforms equation (4.52) into

$$\theta = \frac{u_o''(\xi)t}{(1 + u_o'(\xi)t)^2} . \tag{4.54}$$

This is exactly the same term as we obtained in §4.1 .

The expansion is clearly invalid when

$$1 + u'(\xi)t = 0 \ . \qquad (4.55)$$

This represents an infringement of equation (4.7), thus identifying this analysis with the case i) of §4.1.

Also, when $u_o''(\xi) = 0$, we have the right hand side of equation (4.51) zero except when equation (4.55) holds. In this latter case it is indeterminate and a different analysis is required. This case corresponds to case iii) in §4.1.

4.2.2 A Generalised Model

The generalised model is

$$u_t + \lambda(u)u_x = \epsilon u_{xx} \ . \qquad (4.56)$$

We again impose equation (4.41) and equations (4.43) and (4.44) obviously still hold. However, in place of equation (4.42) we impose

$$x = \xi + \lambda_o(\xi)t \ , \qquad (4.57)$$

where $\lambda_o(\xi) = \lambda(u_o(\xi))$ as before. Performing the analysis and equating of coefficients as before proves to be more laborious but of the same structure, giving consistency for ϵ^o and a perfect differential for θ from ϵ^1 . Applying the boundary conditions as before gives the result:

$$\Big\{\theta(\xi,t) = \{(\lambda_o''(\xi)u_o'(\xi) - \lambda_o'(\xi)u_o''(\xi))(1+\lambda_o'(\xi)t) \ \ln(1 + \lambda_o'(\xi)t)$$

$$+ \lambda_o'(\xi)\lambda_o''(\xi)u_o'(\xi)t(\lambda_o'(\xi)^2u_o''(\xi) \ (1+\lambda_o'(\xi)t)^2\Big\}^{-1} \ . \qquad (4.58)$$

We observe that in the special case,

$$\lambda_o(\xi) = au_o(\xi) + b \ , \tag{4.59}$$

equation (4.58) reduces to equation (4.54). The comments at the end of §4.2.1 are also applicable here.

4.2.3 A Two-Dimensional Steady Flow Model

Here we shall use the model obtained in [6] §3.3, which could be written in the notation of this section as

$$\lambda(u)u_x + \mu(u)u_y = \epsilon(au_{xx} + 2bu_{xy} + cu_{yy}) \ . \tag{4.60}$$

We again impose equation (4.41) with y replacing t. The other condition is now

$$x = \xi + \frac{\lambda_o(\xi)y}{\mu_o(\xi)} \ , \tag{4.61}$$

as this is the characteristic equation corresponding to the non-diffusive form of equation (4.60).

The method of §4.2.1 may again be followed exactly here. We again obtain consistency when equating ϵ^o coefficients and a perfect differential equation for θ when equating ϵ^1 coefficients. The latter may again be integrated and boundary conditions imposed. The function θ obtained is, however, rather complicated so it is not presented here. It again displays the three cases of §4.1. In a way this result is quite surprising as it shows two-dimensional steady diffusion can be treated in the same way as one dimensional unsteady diffusion (in the sense described) despite the fact that the structure of their diffusion terms is different.

4.3 Relationship to Catastrophe Theory

In this subsection, we shall attempt to relate the function $G(\eta;x,t)$ of §4.1 to the unfolding function $\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t})$ of §2.2.1.

To this end, we require an expansion of G about the breaking point. We need to make a distinction between η and ξ as ξ obeys the characteristic equation (4.6), whereas η is an independent variable. However, as the major contribution to G will come from about the breaking point, we can rescale η about ξ_b, thus

$$\eta = \xi_b + \tilde{\eta} , \tag{4.62}$$

where $\tilde{\eta}$ is now an independent variable. The rescalings for x and t are those of equation (2.39). We introduce

$$G_b = G(\xi_b;x_b,t_b) , \tag{4.63}$$

and

$$G(\eta;x,t) = G_b + \tilde{G}(\tilde{\eta};\tilde{x},\tilde{t}) . \tag{4.64}$$

This leads us to

$$\tilde{G}(\tilde{\eta};\tilde{x},\tilde{t}) = \int_o^{\tilde{\eta}} u_o(\xi_b+\zeta)d\zeta + \frac{(x_b+\tilde{x}-\xi_b-\tilde{\eta})^2}{2(t_b+\tilde{t})} - \frac{(x_b-\xi_b)^2}{2t_b} . \tag{4.65}$$

The relations for the breaking point from §2.1.1 hold here, but with $\lambda(u) = u$ (as we are dealing with Burgers' equation and not the general characteristic equation). These are

$$x_b = \xi_b + u_o(\xi_b)t_b \ ; \tag{4.66}$$

$$u_o'(\xi_b) = -\frac{1}{t_b} \ ; \tag{4.67}$$

$$u_o''(\xi_b) = 0 \ ; \tag{4.68}$$

and

$$u_o'''(\xi_b) > 0 \ . \tag{4.69}$$

We use these relations to obtain

$$\int_o^{\tilde{\eta}} u_o(\xi_b + \zeta)d\zeta = \tilde{\eta}\, u_o(\xi_b) - \frac{\tilde{\eta}^2}{2t_b} + \frac{\tilde{\eta}^4}{24} u_o'''(\xi_b) + O(\tilde{\eta}^5) \tag{4.70}$$

We also have

$$\frac{(x_b+\tilde{x}-\xi_b-\tilde{\eta})^2}{2(t_b+\tilde{t})} - \frac{(x_b-\xi_b)^2}{2t_b} = \frac{(x_b-\xi_b)^2}{2t_b}\left\{\left[1 + \frac{\tilde{x}-\tilde{\eta}}{x_b-\xi_b}\right]^2\left[1 + \frac{\tilde{t}}{t_b}\right]^{-1} - 1\right\} \tag{4.71}$$

$$= \frac{u_o(\xi_b)^2 t_b}{2}\left\{\left[1 + \frac{2(\tilde{x}-\tilde{\eta})}{u_o(\xi_b)t_b} + \frac{2(\tilde{x}-\tilde{\eta})^2}{u_o(\xi_b)^2 t_b^2}\right]\left[1 - \frac{\tilde{t}}{t_b} + \frac{\tilde{t}^2}{2t_b^2} + O(\tilde{t}^3)\right]^{-1} - 1\right\} \tag{4.72}$$

We now have the problem of knowing what order the rescaled variables are. For simplicity, we shall assume the rescaling orders given by Haberman ([3]) are valid. It is hoped that in some sense they will be proved to be consistent. The rescaling orders are

$$\tilde{\eta} = O(\epsilon^{1/4}) \; ; \tag{4.73}$$

$$\tilde{t} = O(\xi^{1/2}) \; ; \tag{4.74}$$

$$\tilde{x} = O(\xi^{1/2}) \; ; \tag{4.75}$$

and

$$\tilde{z} = O(\epsilon^{3/4}) \; . \tag{4.76}$$

(Note: we assume $\tilde{\eta} = O(\tilde{\xi})$).

Following equation (4.70), we are concerned with the leading terms up to order $\epsilon^{3/4}$.

The right hand side of equation (4.72) now becomes

$$u_o(\xi_b)(\tilde{x}-\tilde{\eta}) + u_o(\xi_b)\frac{\tilde{\eta}\tilde{t}}{t_b} + \frac{(\tilde{x}-\tilde{\eta})^2}{2t_b} - \frac{\tilde{\eta}^2\tilde{t}}{2t_b^2} + \frac{\tilde{t}^2 u_o(\xi_b)^2}{4t_b} + O(\epsilon^{3/4}) \tag{4.77}$$

Substituting equations (4.70) and (4.77) into equation (4.65) gives

$$\tilde{G}(\tilde{\eta};\tilde{x},\tilde{t}) = \tilde{\eta}u_o(\xi_b) - \frac{\tilde{\eta}^2}{2t_b} + \frac{\tilde{\eta}^4}{24}u_o'''(\xi_b) + u_o(\xi_b)(\tilde{x}-\tilde{\eta})$$

$$+ u_o(\xi_b)\frac{\tilde{\eta}\tilde{t}}{t_b} + \frac{(\tilde{x}-\tilde{\eta})^2}{2t_b} - \frac{\tilde{\eta}^2\tilde{t}}{2t_b^2} + \frac{\tilde{t}^2 u_o(\xi_b)^2}{4t_b} + O(\epsilon^{3/4}) \tag{4.78}$$

This cancels down to give

$$\tilde{G}(\tilde{\eta};\tilde{x},\tilde{t}) = \left[\tilde{x}\, u_o(\xi_b) + \frac{\tilde{x}^2}{2t_b} + \frac{\tilde{t}^2 u_o(\xi_b)^2}{4t_b}\right] + \left[\frac{u_o(\xi_b)\tilde{t}}{t_b} - \frac{\tilde{x}}{t_b}\right]\tilde{\eta}$$

$$- \left[\frac{\tilde{t}}{2t_b^2}\right]\tilde{\eta}^2 + \frac{\tilde{\eta}^4}{24}\, u_o'''(\xi_b) + O(\epsilon^{5/4}) \tag{4.79}$$

Let $$\phi(\tilde{x},\tilde{t}) = t_b\tilde{x}\, u_o(\xi_b) + \frac{\tilde{x}^2}{2} + \frac{\tilde{t}^2 u_o(\xi_b)^2}{4} \tag{4.80}$$

We then obtain

$$\tilde{G}(\tilde{\eta};\tilde{x},\tilde{t}) = \frac{1}{t_b}\left\{\frac{t_b}{2}\,\frac{\tilde{\eta}^4}{12}u_o'''(\xi_b) - \frac{\tilde{t}}{t_b}\tilde{\eta}^2 - \tilde{z}\tilde{\eta} + \phi(\tilde{x},\tilde{t})\right\} + O(\epsilon^{5/4}) \tag{4.81}$$

This leading order quartic in $\tilde{\eta}$ is identical (up to the factor $\frac{1}{t_b}$) to the leading order expansion of $\tilde{F}(\tilde{\xi},\tilde{x},\tilde{t})$ in $\tilde{\xi}$.

When calculating u near the breaking point, the exponential in G_b and ϕ cancel and the terms of order $\epsilon^{5/4}$ become of order $\epsilon^{1/4}$ so can be ignored. So, if we introduce the function

$$\tilde{H}(\tilde{\eta};\tilde{z},\tilde{t}) = \frac{1}{t_b}\left\{\frac{t_b}{2}\,\frac{\tilde{\eta}^4}{12}u_o'''(\xi_b) - \frac{\tilde{t}}{t_b}\tilde{\eta}^2 - \tilde{z}\tilde{\eta}\right\} \tag{4.82}$$

equation (3.13) implies

$$u(x_b+\tilde{x}, t_b+\tilde{t}) \sim \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x_b+x-\xi_b-\tilde{\eta}}{t_b+\tilde{t}}\, e^{-\tilde{H}(\tilde{\eta};\tilde{x},\tilde{t})2\epsilon}\, d\tilde{\eta}}{\displaystyle\int_{-\infty}^{\infty} e^{-\tilde{H}(\tilde{\eta};\tilde{x},\tilde{t})2\epsilon}\, d\tilde{\eta}} \tag{4.83}$$

$$\sim \frac{x_b - \xi_b}{t_b} = u_o(\xi_b) \tag{4.84}$$

To obtain the higher order terms for u, we shall need to explicitly write the terms of order $\epsilon^{5/4}$ and above in the preceeding argument. This is beyond the scope of this report. We only note that the behaviour of u near the breaking point is governed by the highest order terms of the unfolding of the characteristic catastrophe function, $\tilde{F}$. An interesting diagramatic representation of this relationship is shown in figure 6.

Finally, we note that Haberman ([3]) has shown that the deviation to u about the breaking point satisfies Burgers' equation in the rescaled variables. His method is to use leading order expansions similar to §4.2.

4.4 Relationship to Shock Fitting

In this section we pick up on the similarities between the analysis of G and the method of characteristics that we noted at the end of §4.1.

We recall that in shock fitting, a discontinuity is imposed within the flow field in order to model a shock wave. When the discontinuity ends, it will either meet the boundary or another discontinuity or its strength will diminish to zero. In the last case, this end point will have to be tracked along with the front itself.

The separate treatment of these three cases in shock fitting corresponds very closely with the separate treatment of the three cases in §4.1. In fact, we could say that the analysis in §4.1 to §4.3 represents an extension of non-diffusive systems with shocks to the case of having a small amount of diffusion. Correspondingly, this small amount of diffusion means that, instead of treating shocks and shock tips separately, we just treat the regions near shocks and the regions near shock tips separately from the rest of the flow.

The outworking of these ideas is discussed on the final subsection below.

4.5 Implications for Numerical Schemes

We have successfully shown in this section that it is in principle possible to extend analytic solutions of non-diffusive equations with shocks to analytic solutions of corresponding equations with small amounts of diffusion and thick shock layers.

It could be argued that this analysis is useless to numerical applications because a discretization process (or a scheme itself) will introduce effects similar to diffusion and hence smear out discontinuities at least slightly. However, this approach (of shock capturing) contains many difficulties so it is asserted that fitting a shock layer with a solution could be a useful process.

One way in which this 'viscous shock fitting' could be implemented is as follows:

i) Introduce a shock tip region when the characteristics of the corresponding non-diffusive equations overturn.

ii) Introduce a shock region when the shock tip region becomes too large.

iii) Solve the equations in these three separate regions using the methods introduced in this section.

iv) Match the solution between the regions somehow.

v) Propagate the boundaries of the shock and shock tip regions somehow (generalising the Rankine-Hugoniot jump conditions).

If this process is possible, it will potentially yield better solutions than both shock capturing and shock fitting.

5. Front Tracking Thoery

5.0 Introduction

Front tracking theory concerns the rules for the propagation of discontinuity curves (in two dimensions) or surfaces (in three dimensions). The standard result is that the normal speed is given by the Rankine-Hugoniot jump conditions in an appropriate form. However, when a surface is propagating tangentially as well as normally (for expample a rotating wedge shock - see figure 7), the surface normal speed is insufficient for predicting the position of points within the discontinuity surface a finite time ahead.

In order to overcome this, a 'shock velocity' will need to be prescribed in some sense. This is achieved here by first defining a second Lagrangian type frame to give a velocity to each point in the flow which is then constrained for shock waves.

After this, implications for numerical schemes are discussed.

5.1 The Second Lagrangian Frame Formulation

5.1.1 Initial Equations

Suppose we have a system of flux equations given by

$$\frac{\partial u^i}{\partial t} + \nabla \cdot \underline{f}^i(u) = 0 \ , \quad i = 1 , \ldots , n \tag{5.1}$$

Now, let us consider the propagation of surfaces on which $\phi(\underline{u})$ is constant. Suppose that the velocity of propagation of these surfaces is $\underline{Q}$. We wish to find an analytic expression for $\underline{Q}$. It will be shown in §5.1.2 and §5.1.3 that this may be achieved in two ways.

5.1.2 The Taylor Expansion Method

Consider a small timestep Δt and a small displacement $\underline{\Delta x}$. Suppose $\phi(\underline{u})$ remains constant from $(\underline{x},t)$ to $(\underline{x}+\underline{\Delta x},t+\Delta t)$. Hence

$$\phi(\underline{u}(\underline{x}+\underline{\Delta x},t+\Delta t)) = \phi(\underline{u}(\underline{x},t)) \ . \qquad (5.2)$$

Now,

$$\underline{u}(\underline{x}+\underline{\Delta x},t+\Delta t) = \underline{u}(\underline{x},t) + \sum_i \Delta x_i \frac{\partial \underline{u}}{\partial x_i}(\underline{x},t)$$

$$+ \Delta t \frac{\partial \underline{u}}{\partial t}(\underline{x},t) + O(|\underline{\Delta x}|^2 + |\Delta t|^2) \qquad (5.3)$$

Thus,

$$\phi(\underline{u}(\underline{x}+\underline{\Delta x},t+\Delta t) + \sum_i \Delta x_i \frac{\partial \underline{u}}{\partial x_i}(\underline{x},t) + \Delta t \frac{\partial \underline{u}}{\partial t}(\underline{x},t) + O(|\underline{\Delta x}|^2 + |\Delta t|^2))$$

$$= \phi(\underline{u}(\underline{x},t)) \ . \qquad (5.4)$$

This implies

$$\sum_j \frac{\partial \phi}{\partial u^j} \left\{ \sum_i \Delta x_i \frac{\partial u^j}{\partial x_i}(\underline{x},t) + \Delta t \frac{\partial u^j}{\partial t}(\underline{x},t) \right\} = O(|\underline{\Delta x}|^2 + |\Delta t|^2) \qquad (5.5)$$

We now divide through by Δt and introduce

$$\underline{Q} = \lim_{\Delta t \to o} \frac{\Delta \underline{x}}{\Delta t} . \tag{5.6}$$

Equation (5.5) then gives

$$\sum_j \frac{\partial \phi}{\partial u^j} \left\{ \sum_i Q_j \frac{\partial u^j}{\partial x_i} + \frac{\partial u^j}{\partial t} \right\} = O(|\underline{Q}.\underline{\Delta x}| + |\Delta t|) . \tag{5.7}$$

So, assuming $\underline{Q}$ remains bounded, we infer

$$\frac{\partial \phi}{\partial \underline{u}} \cdot \left\{ \frac{\partial \underline{u}}{\partial t} + \underline{Q}.\nabla \underline{u} \right\} = 0 . \tag{5.8}$$

Equation (5.6) shows that $\underline{Q}$ is clearly well-defined as the velocity of the surface.

5.1.3 The Second Lagrangian Frame Method

Let $\underline{\xi}(\underline{a},s;t)$ be the position of a particle of fluid a time t which is at position $\underline{a}$ at time s. Let $\underline{\Xi}(\underline{a},s;\tau)$ be the position of the particle of fluid at time τ which has the same value of $\phi(\underline{u})$ as the particle of fluid at position $\underline{a}$ at time s. We shall assume these maps are continuous. We introduce

$$\underline{x} = \underline{\xi}(\underline{a},s;t) \tag{5.9}$$

$$\underline{X} = \underline{\Xi}(\underline{a},s;t) \tag{5.10}$$

The fluid and surface velocities are defined as follows:

$$\underline{q}(\underline{\xi}(\underline{a},s;t),t) = \frac{\partial\underline{\xi}}{\partial t}(\underline{a},s;t) \tag{5.11}$$

$$\underline{Q}(\underline{\Xi}(\underline{a},s;\tau),\tau) = \frac{\partial\underline{\Xi}}{\partial\tau}(\underline{a},s;\tau) \ . \tag{5.12}$$

We also have the identities,

$$\forall\underline{x},t, \qquad \underline{\xi}(\underline{x},t;t) = \underline{x} \tag{5.13}$$

$$\forall\underline{X},\tau, \qquad \underline{\Xi}(\underline{X},\tau;\tau) = \underline{X} \tag{5.14}$$

and the inverse relations

$$\underline{a} = \underline{\xi}(\underline{x},t;s) \tag{5.15}$$

$$\underline{a} = \underline{\Xi}(\underline{X},\tau;s) \tag{5.16}$$

The equation for the conservation of $\phi(\underline{u})$ is

$$\forall\tau, \qquad \underline{\phi}(\underline{u}(\underline{\Xi}(a,s;\tau),\tau)) = \text{const} \tag{5.17}$$

i.e.,

$$\left.\frac{\partial}{\partial\tau}\right|_{\underline{a},s} \left\{\phi(\underline{u}(\underline{\Xi}(a,s;\tau),\tau))\right\} = 0 \ . \tag{5.18}$$

Let us treat $\underline{x},t,s$ and τ as the independent variables. Instead of equation (5.18), equations (5.15) and (5.17) give

$$\frac{\partial}{\partial\tau}\bigg|_{\underline{x},t,s}\left\{\phi(\underline{u}(\underline{\Xi}(\underline{\xi}(\underline{x},t;s),s;\tau),\tau))\right\} = 0 \ . \tag{5.19}$$

The chain rule gives

$$\sum_i \frac{\partial\phi}{\partial u^i} \frac{\partial}{\partial\tau}\bigg|_{\underline{x},t,s}\left\{u^i(\underline{\Xi}(\underline{\xi}(\underline{x},t;s),s;\tau),\tau))\right\} = 0 \ . \tag{5.20}$$

Now,

$$\frac{\partial}{\partial\tau}\bigg|_{\underline{x},t,s}\left\{u^i(\underline{\Xi}(\underline{\xi}(\underline{x},t;s),s;\tau),\tau))\right\} =$$

$$\sum_j \frac{\partial u^i}{\partial \Xi_j} (\underline{\Xi}(\underline{\xi}(\underline{x},t;s),s;\tau),\tau)) \frac{\partial \Xi_j}{\partial\tau}(\underline{\xi}(\underline{x},t;s),s;\tau)\bigg|_{\underline{x},t,s}$$

$$+ \frac{\partial u^i}{\partial\tau}(\underline{\Xi}(\underline{\xi}(\underline{x},t;s),s;\tau),\tau))\bigg|_{\underline{\Xi}} \ . \tag{5.21}$$

And

$$\frac{\partial \Xi_j}{\partial\tau}(\underline{\xi}(\underline{x},t;s),s;\tau)\bigg|_{\underline{x},t,s} = Q_j(\underline{\Xi}(\underline{\xi}(\underline{x},t;s),s;\tau),\tau) \ . \tag{5.22}$$

Equations (5.20) to (5.22) combine to give

$$\sum_i \frac{\partial\phi}{\partial u^i}\left\{\sum_j \frac{\partial u^i}{\partial \Xi_j} (\underline{\Xi}(\underline{\xi}(x,t;s),s;\tau),\tau))Q_j(\underline{\Xi}(\underline{\xi}(x,t;s),s;\tau),\tau)\right.$$

$$\left. + \frac{\partial u^i}{\partial\tau}(\underline{\Xi}(\underline{\xi}(x,t;s),s;\tau),\tau))\bigg|_{\underline{\Xi}}\right\} = 0 \ . \tag{5.23}$$

Now it is possible to consider $\underline{\Xi}$ and τ as the independent variables and to rewrite equation (5.23) as

$$\sum_i \frac{\partial\phi}{\partial u^i}\left\{\sum_j \frac{\partial u^i}{\partial \Xi_j}(\underline{\Xi},\tau)Q_j(\underline{\Xi},\tau) + \frac{\partial u^i}{\partial \tau}(\underline{\Xi},\tau)\Big|_{\underline{\Xi}}\right\} = 0 \ . \tag{5.24}$$

Rewriting $\underline{x}$ for $\underline{\Xi}$ and t for τ we finally obtain

$$\sum_i \frac{\partial\phi}{\partial u^i}\left\{\frac{\partial u^i}{\partial t}(x,t) + \sum_j \frac{\partial u^i}{\partial x_j}(x,t)Q_j(\underline{x},t)\right\} = 0 \ , \tag{5.25}$$

which is identical to equation (5.8) in suffix form.

5.1.4 Relevant Form of the Frame Condition

Let us generalise the function $\phi(\underline{u})$ to several functions $\phi^i(\underline{u})$ with associated velocity $\underline{Q}^i$. Let

$$\phi^i(\underline{u}) = u^i \ . \tag{5.26}$$

Substituting into equation (5.8) gives

$$\frac{\partial u^i}{\partial t} + \underline{Q}^i.\nabla u^i = 0 \tag{5.27}$$

We shall only consider the case of two space dimensions for the moment. Let $\underline{N}^i$ and $\underline{T}^i$ be the u^i = const. Thus we may write

$$\underline{Q}^i = V^i\underline{N}^i + W^i\underline{T}^i \tag{5.28}$$

for some functions V^i, W^i . We also have

$$\underline{N}^i = \frac{\nabla u^i}{|\nabla u^i|} \tag{5.29}$$

Hence, equations (5.27) and (5.29) imply

$$\frac{\partial u^i}{\partial t} + \left\{\frac{V^i \nabla u^i}{|\nabla u^i|} + W^i \underline{T}^i\right\}.\nabla u^i = 0 \tag{5.30}$$

Clearly, by definition,

$$\underline{T}^i.\nabla u^i = 0 \ . \tag{5.31}$$

Hence

$$\frac{\partial u^i}{\partial t} + V^i |\nabla u^i| = 0 \ . \tag{5.32}$$

Combining this with equation (5.1) gives

$$V^i = \frac{\nabla . \underline{f}^i}{|\nabla u^i|} \ . \tag{5.33}$$

5.2 Incorporation of Shock Waves

5.2.1 Additional Constraints

It is clearly suitable for a point lying within a shock surface $S(t)$ at time t , identified u^i = const, to be moved with velocity $\underline{Q}^i$ to a new point in $S(t')$ with u^i equal to the same constant value.

We know, however, that $S(t)$ has normal speed s , where

$$\forall i \quad s = \frac{[\underline{f}^i].\underline{\nu}}{[u^i]} \,, \tag{5.34}$$

where $\underline{\nu}$ is the normal to $S(t)$. Hence we must have

$$\underline{Q}^i = s\underline{\nu} + \alpha^i\underline{\tau} \,, \tag{5.35}$$

for some function α^i , where $\underline{\tau}$ is the tangent to $S(t)$.

5.2.2 Derivation of Shock Velocity

We assume that $\underline{\nu}$ and $\underline{\tau}$ are known functions. Clearly, equation (5.28) implies

$$\underline{Q}^i.\underline{N}^i = V^i \,. \tag{5.36}$$

Combining this with equation (5.35) gives

$$V^i = s\underline{\nu}.\underline{N}^i + \alpha^i\underline{\tau}.\underline{N}^i \,. \tag{5.37}$$

Thus,

$$\alpha^i = \frac{V^i - \underline{\nu}.\underline{N}^i}{\underline{\tau}.\underline{N}^i} \,. \tag{5.38}$$

In this way, the shock velocity $\underline{Q}^i$ is defined in terms of quantities already known.

It turns out that the above argument is only possible in two dimensions. If we attempt it for a higher number of dimensions, we will only be able to find $\underline{\alpha}^i.\underline{N}^i = 0$, where $\underline{Q}^i = s\underline{\nu} + \alpha^i$ and $\underline{\alpha}^i.\underline{\nu} = 0$.

5.3 Implications for Numerical Schemes

For two-dimensional unsteady systems of conservation laws, the above argument shows how to calculate shock velocities $\underline{Q}^i$ on which u^i is constant. In fact we obtain 2n values of $\underline{Q}^i$ (where $i = 1,\ldots,n$) as u^i (and hence $V^i, \underline{N}^i$ etc.) can be prescribed on both sides of the shock $S(t)$.

Suppose $u^i = u^i_o$ at $\underline{x}_o \in S(t_o)$ at time t_o . Then, provided there exists a point $\underline{x}_o + \underline{\Delta x}$ in $S(t_o+\Delta t)$ where $u^i = u^i_o$, Q^i will give the velocity separating the two space-time points. For normal length timesteps this will usually be possible for most points of $S(t)$. The problems occur at maxima and minima and at the boundaries of $S(t)$.

Hence we should look for new points of discontinuity with approximate position

$$\underline{x}_o + \underline{\Delta x} = \underline{x}_o + s\underline{\nu}\Delta t + \alpha^i\underline{\tau}\Delta t \quad (5.39)$$

These are likely to be better estimates than the points $\underline{x}_o + s\underline{\nu}\Delta t$ currently used in front tracking algorithms.

It should also be noted that, at least highest order α^i can only take one or possibly two values due to purely geometrical restrictions - see figure 8.

6. Conclusions

In this report, it has been shown how the breaking of smooth solutions is related to the cusp catastrophe. The analysis is carried out for a single equation and should be possible for two equations. It is shown how the same 'breaking point' in systems with diffusion is also related to the same unfolding of the cusp catastrophe, at least to highest order for Burgers' equation.

Along the way, new generalisations of the Cole-Hopf transformation are provided for systems of equations and various asymptotic analysis methods are devised and instigated.

In the last section, a potential improvement is found to the current theory of front tracking.

These theoretical results will hopefully lead to improved numerical schemes and a better understanding of the formation of shock waves on more complicated systems, such as the Euler of Navier-Stokes equations.

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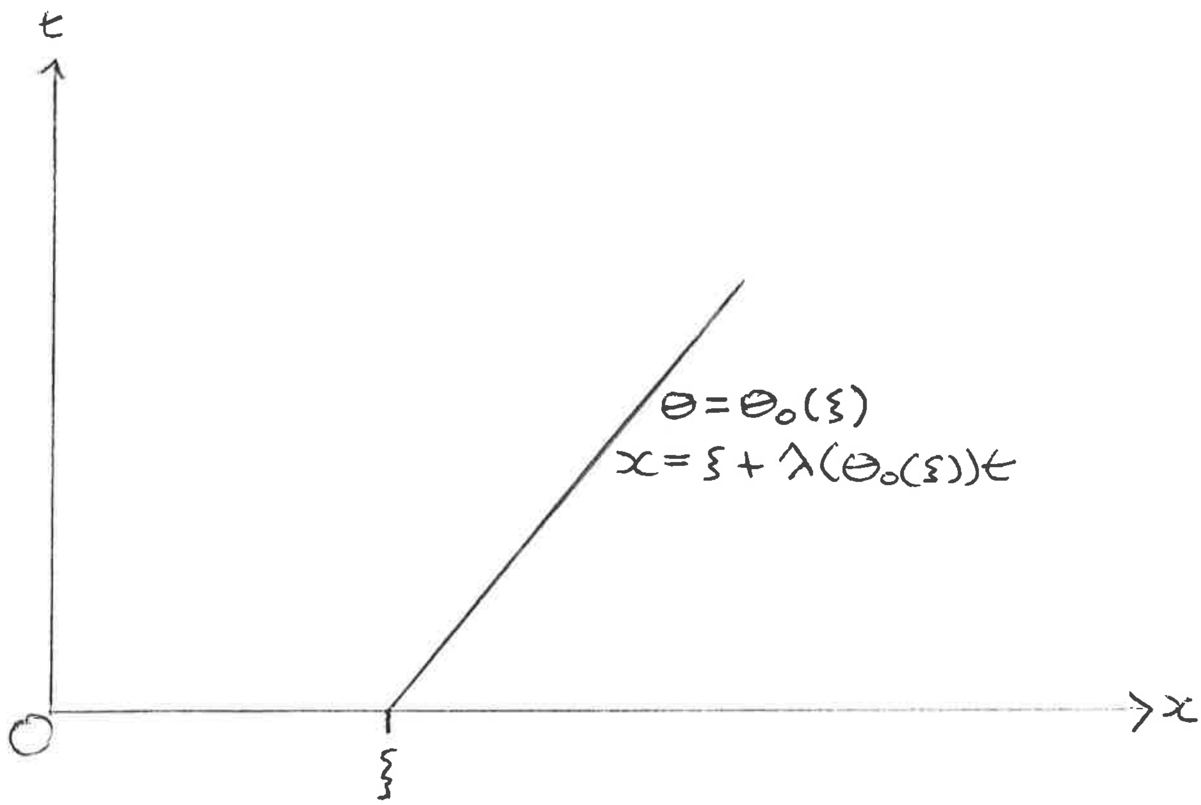


Figure 1

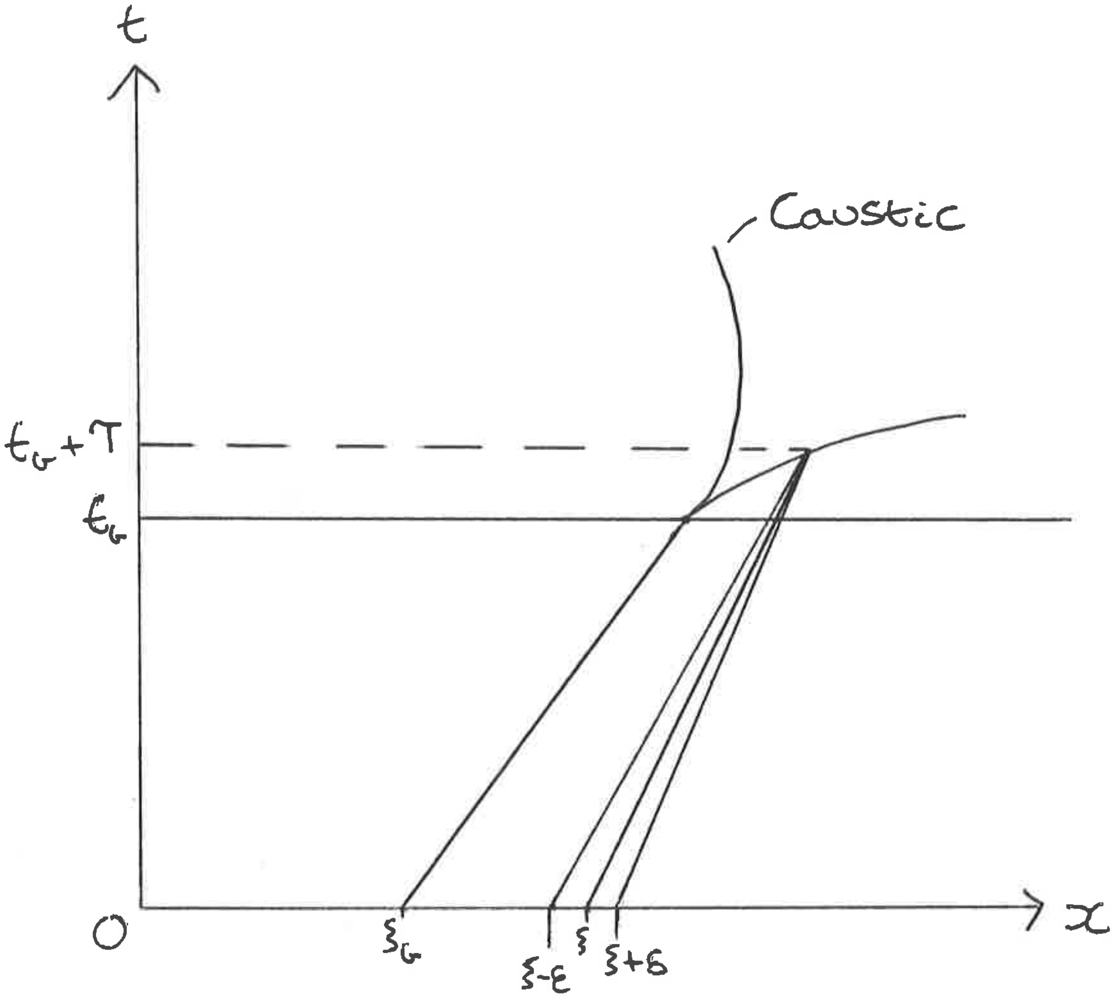


Figure 2.

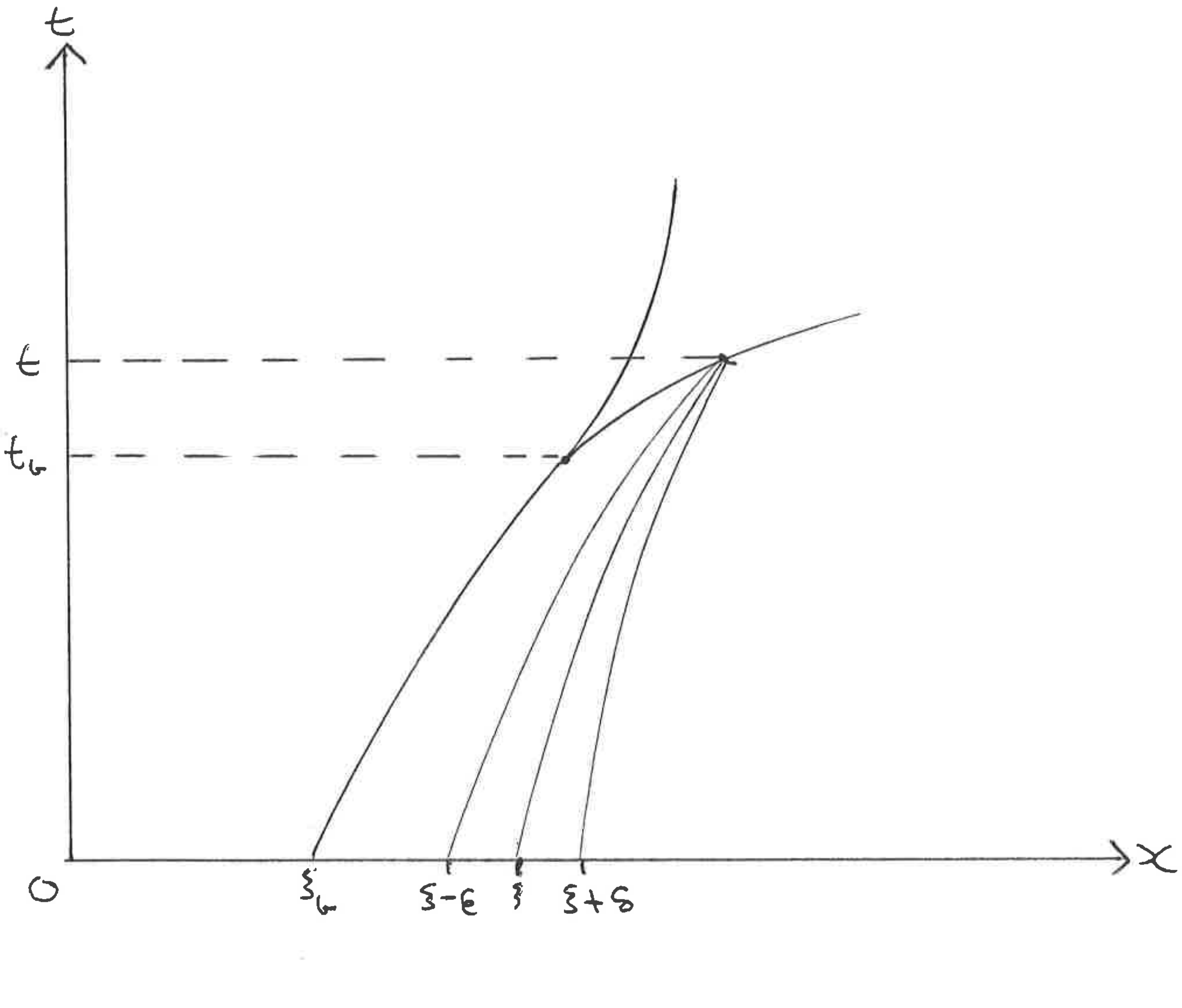


Figure 3.

$$f(\xi; a, b) = 0$$

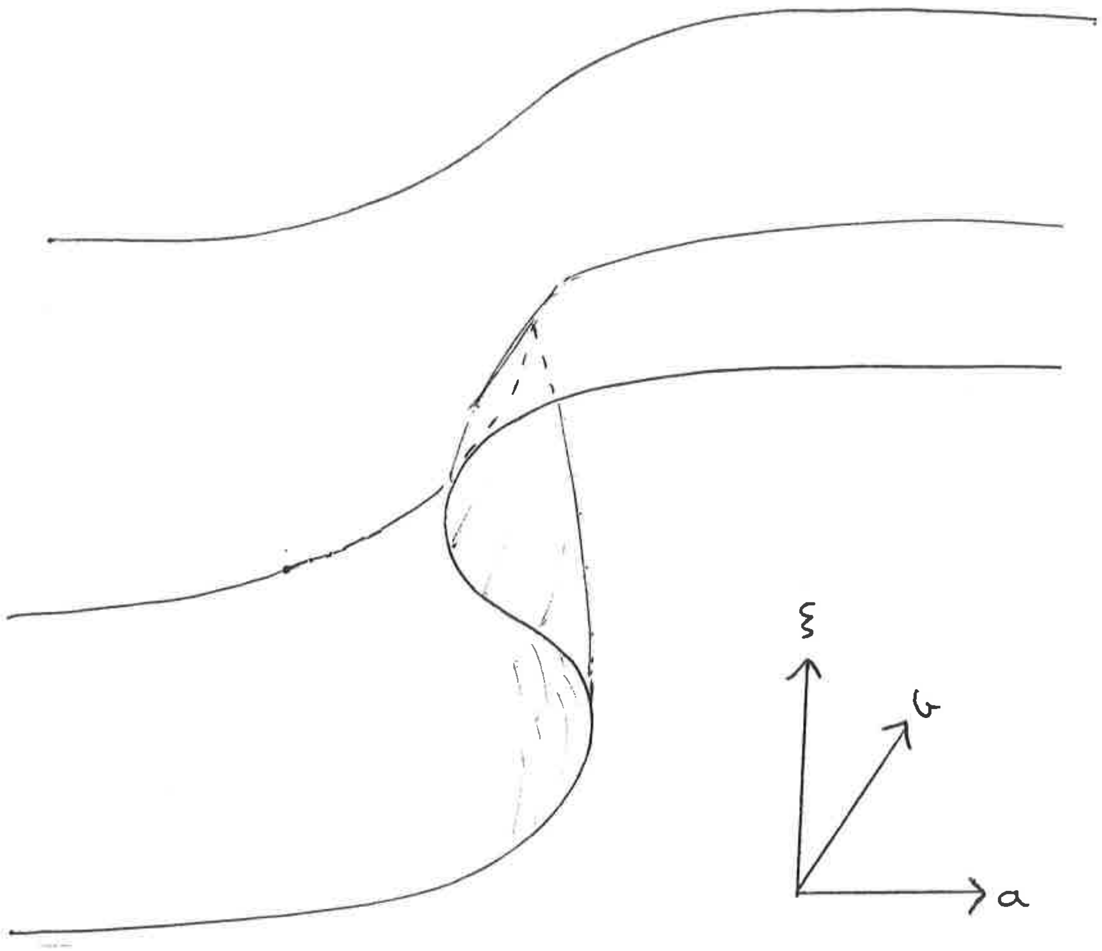
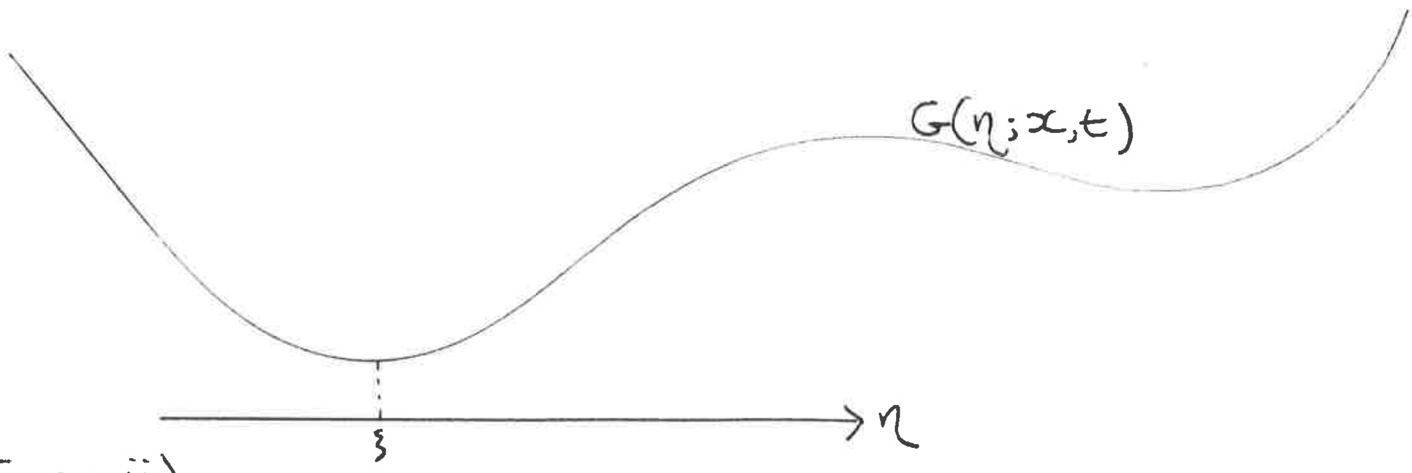
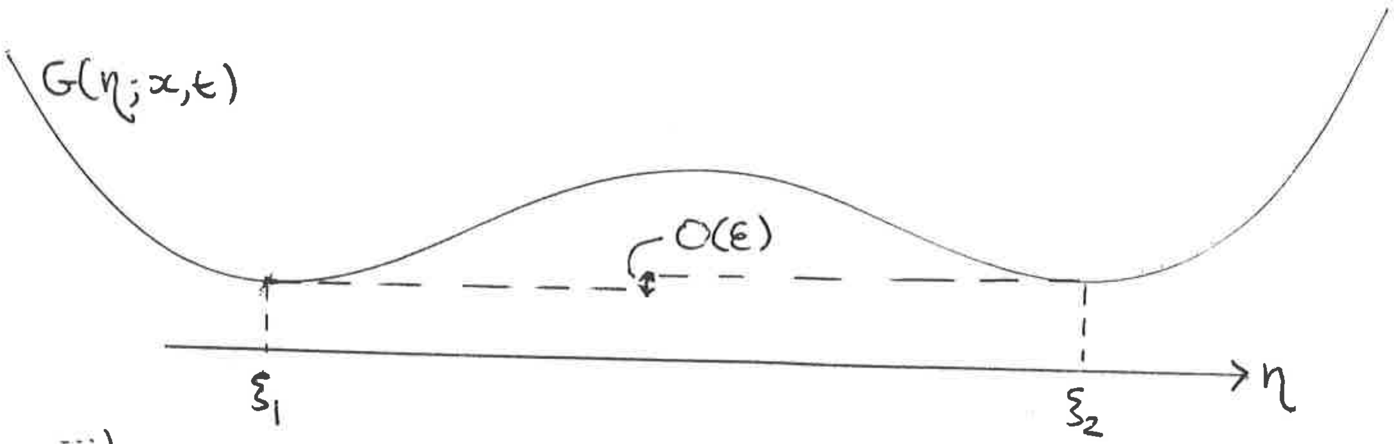


Figure 4.

Case i)



Case ii)



Case iii)

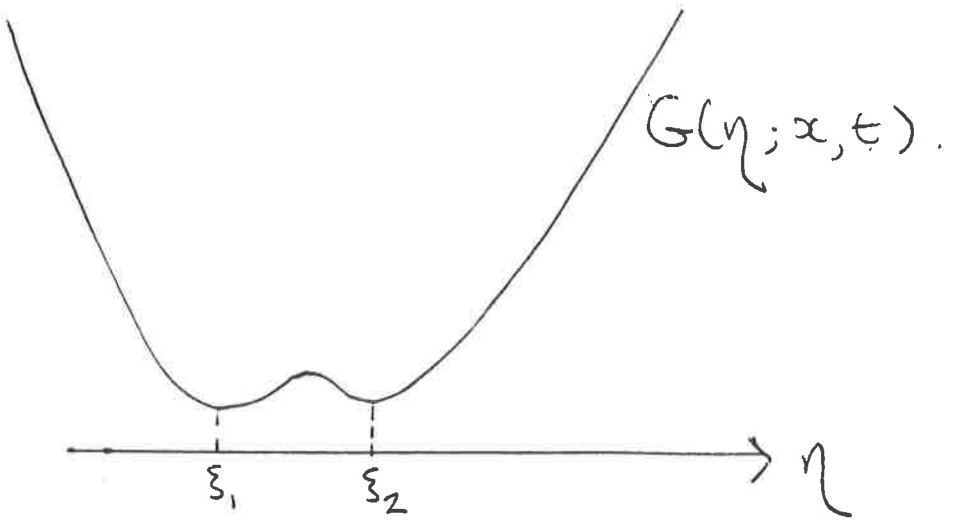


Figure 5.

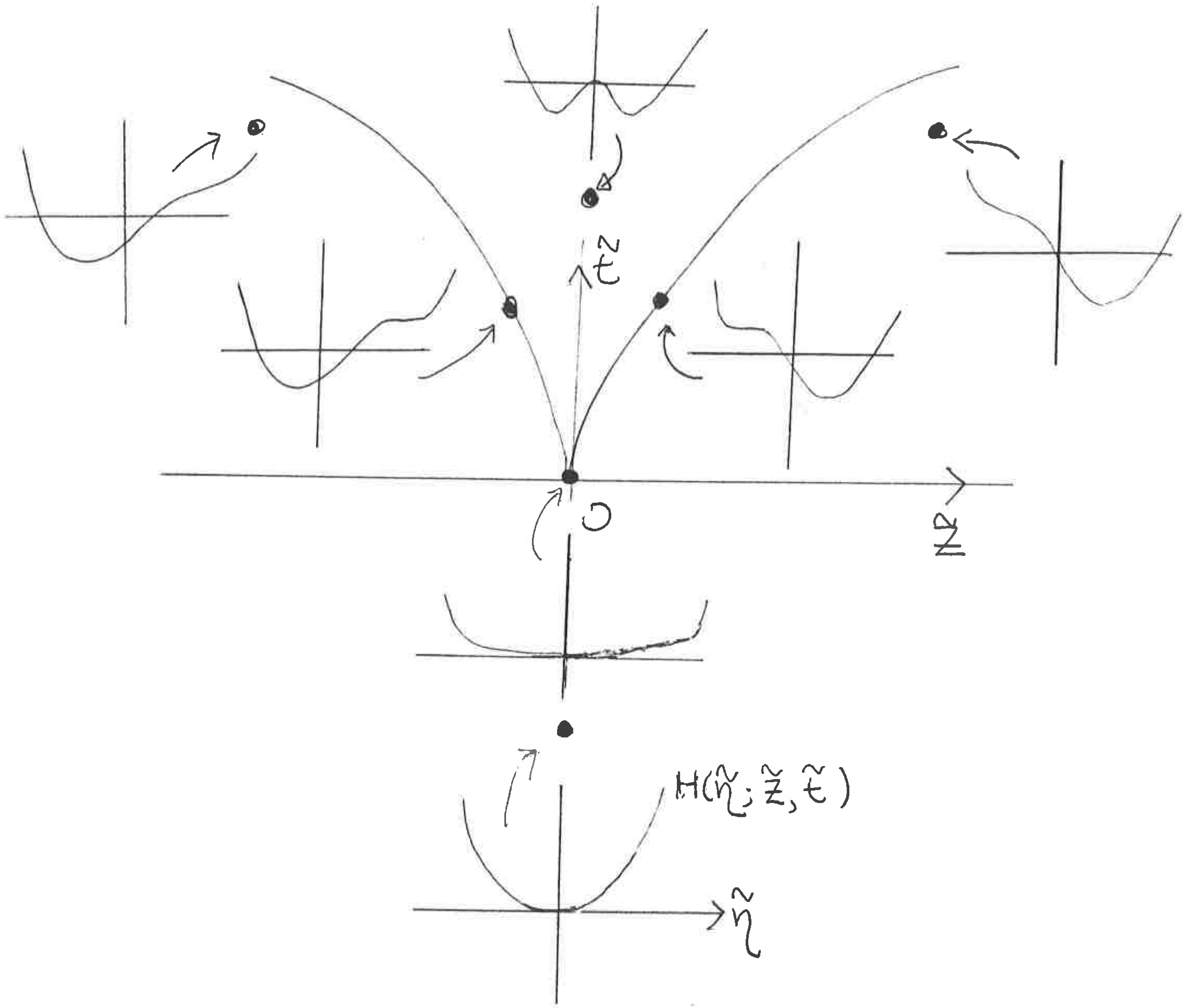


Figure 6.

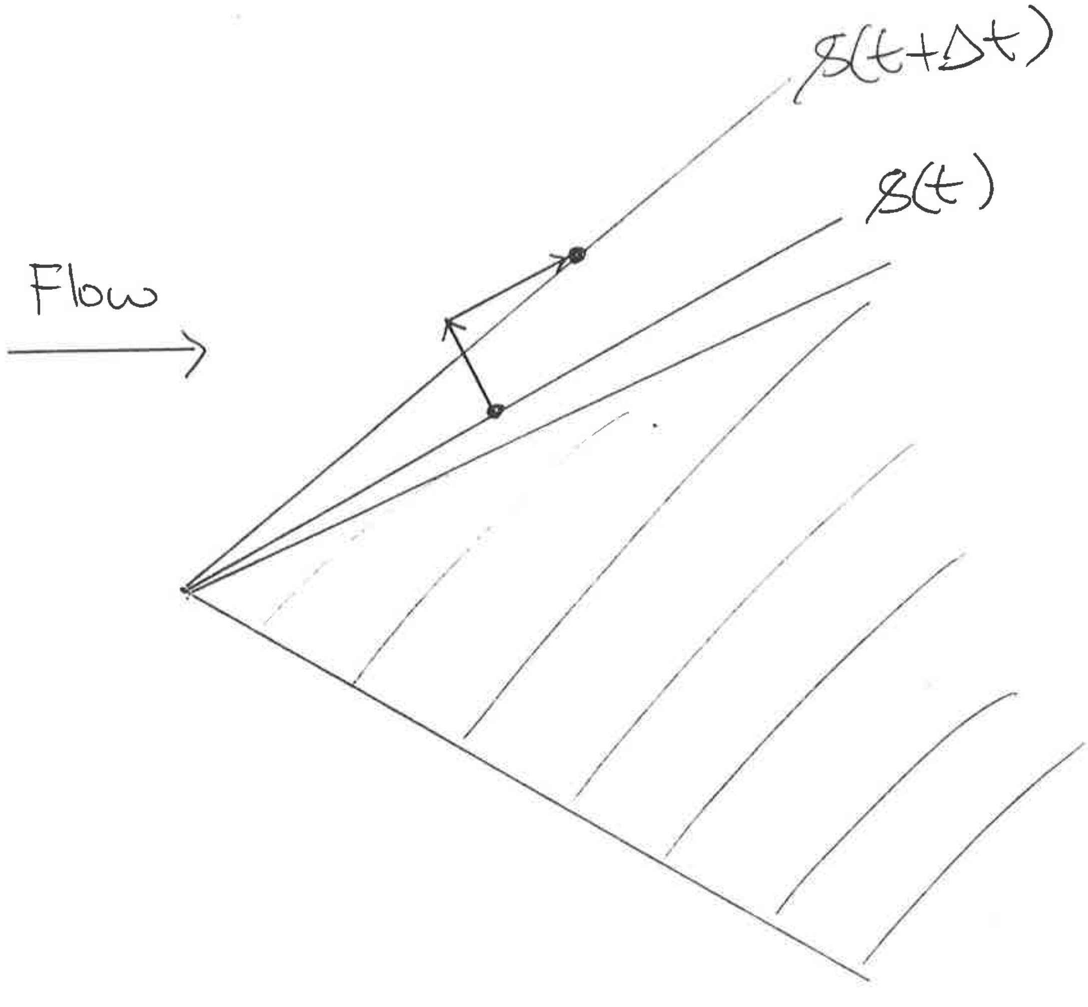
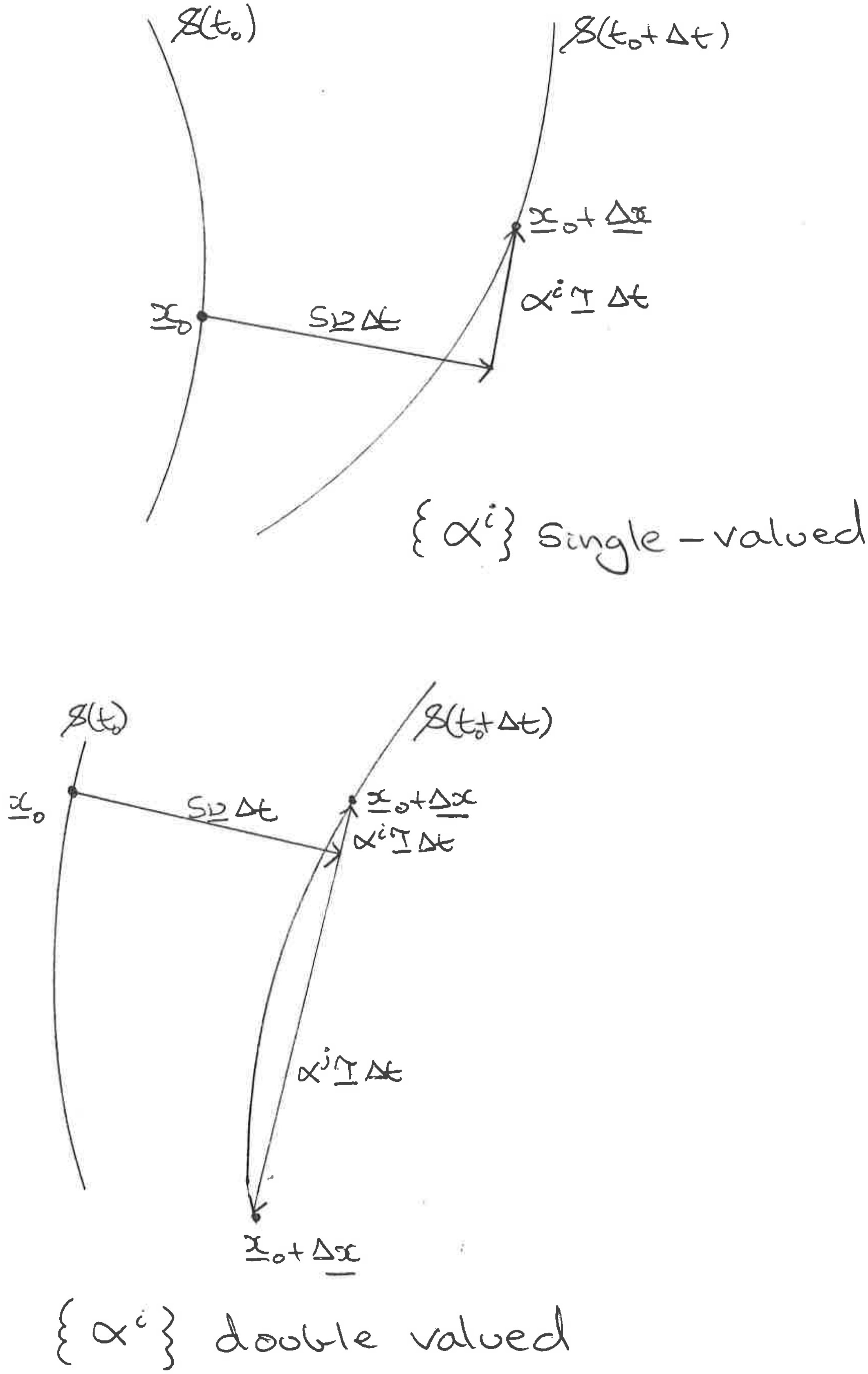


Figure 7.



S(t₀)
S(t₀+Δt)
x₀+Δx
α^i T Δt
x₀
SνΔt
{α^i} single-valued
S(t₀)
S(t₀+Δt)
x₀
SνΔt
x₀+Δx
α^i T Δt
α^j T Δt
x₀+Δx
{α^i} double valued

Figure 8.