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Abstract

We derive unconditionally linearly stable semi-implicit centred and upwind schemes for convection and convection-diffusion equations in 1-D.

Introduction

In a recent paper Carey & Jiang [1], using a least squares finite element approach, derived the centred implicit scheme on a regular grid

$$[1 + (\tfrac{1}{6} - \theta^2\nu^2)\delta^2]\,(U_j^{n+1} - U_j^n) = \nu\,\Delta_o U_j^n + \theta\nu^2\delta^2 U_j^n \qquad (1.1)$$

for the approximation U_j^n to the solution u of the equation

$$u_t + au_x = 0\ ,$$

where a is a constant, $\nu = a\,\Delta t/\Delta x$ is the Courant number, $\Delta_o U_j^n = \tfrac{1}{2}(U_{j+1}^n - U_{j-1}^n)$, $\delta^2 U_j^n = U_{j+1}^n - 2U_j^n + U_{j-1}^n$ and θ is an implictness parameter $(0 \le \theta \le 1)$. The scheme is conservative, second order if $\theta = \tfrac{1}{2}$ and unconditionally stable if $\theta \ge \tfrac{1}{2}$. The factor $\tfrac{1}{6}$ in (1.1) arises from the mass matrix using linear elements. Carey & Jiang note the similarity with the Taylor Galerkin form

$$[1 + \tfrac{1}{6}(1 - \nu^2)\delta^2]\,(U_j^{n+1} - U_j^n) = -\nu\,\Delta_o U_j^n + \tfrac{1}{2}\,\nu^2\delta^2 U_j^n \qquad (1.2)$$

of Donea [2].

Another similarity is with the implicit centred schemes of Lerat [3]

$$[1 + 2\alpha\nu\Delta_o + \tfrac{1}{2}(\gamma+\beta\nu^2)\delta^2](U_j^{n+1} - U_j^n) = -\nu\,\Delta_o U_j^n + (1-2\alpha)\tfrac{1}{2}\,\nu^2\delta^2 U_j^n\ . \qquad (1.3)$$

The choice $\alpha = 0$, $\beta = -\tfrac{1}{2}$, $\gamma = \tfrac{1}{3}$ regains (1.1) with $\theta = \tfrac{1}{2}$. Lerat showed that the scheme (1.3) is unconditionally stable in L_2 under the conditions

$$\alpha < \tfrac{1}{2}\ ,\quad \beta \le \alpha - \tfrac{1}{2}\ ,\quad \gamma < \tfrac{1}{2} \qquad (1.4)$$

which are satisfied by $\alpha = 0$, $\beta = -½$, $\gamma = ⅙$.

The choice $\alpha = 0$, $\beta = -⅙$, $\gamma = ⅙$ regains (1.2). These do not satisfy (1.4) and (1.2) is only conditionally stable. It is however third order accurate and, unlike (1.1), possesses the unit CFL property.

Consider schemes of the form

$$[1 + \tfrac{1}{2}(\gamma+\beta\nu^2)\delta^2](U_j^{n+1} - U_j^n) = -\nu\ \Delta_o U_j^n + \tfrac{1}{2}\phi\nu^2\delta^2 U_j^n \ . \qquad (1.5)$$

which include (1.1) with $\gamma = ⅙$, $\beta = -2\theta^2$, $\phi = 2\theta$

(1.2) with $\gamma = ⅙$, $\beta = -⅙$, $\phi = 1$.

Lerat [3] in recent work takes $\gamma = 0$, $\beta = -1$, $\phi = 1$, motivated by the most rapid convergence towards steady state.

The parameter γ arises from the mass matrix in the underlying Galerkin method : $\gamma = ⅙$ corresponds to linear elements, $\gamma = 0$ to delta functions. The parameters ϕ and β come from higher order terms in the Taylor expansion : $\beta = -⅙$ gives third order accuracy, $\phi = 1$ gives second order accuracy.

Stability of (1.5) is investigated using $U_j^n = g^n e^{ijk\Delta x}$, giving

$$[1-2(\gamma+\beta\nu^2)s^2](g - 1) = -2i\nu sc - 2\phi\nu^2 s^2 \qquad (1.6)$$

where $s = \sin \tfrac{1}{2}k\Delta x$, $c = \cos \tfrac{1}{2}k\Delta x$. Thus

$$g = \frac{1-2(\gamma+\beta\nu^2)s^2 - 2\phi\nu^2 s^2 - 2i\nu sc}{1 - 2(\gamma+\beta\nu^2)s^2}$$

$$= \frac{c^2 + (1-2(\gamma+\beta\nu^2) - 2\phi\nu^2)s^2 - 2i\nu sc}{c^2 + (1-2(\gamma+\beta\nu^2))s^2} \qquad (1.7)$$

$$|g|^2 = \frac{[c^2 + (1-2(\gamma+\beta\upsilon^2)s^2-2\phi\upsilon^2s^2]^2 + 4\upsilon^2s^2c^2}{[c^2+ (1-2(\gamma+\beta\upsilon^2))s^2]^2} \tag{1.8}$$

$$= 1 + \frac{4\phi^2\upsilon^4s^4 + 4\upsilon^2s^2c^2 - 4\phi\upsilon^2s^2[c^2+(1-2(\gamma+\beta\upsilon^2))s^2]}{[c^2 + (1-2(\gamma+\beta v^2)s^2)]^2}$$

$$= 1 + \frac{4\upsilon^2s^2c^2(1-\phi) + 4\upsilon^2s^4\phi[\phi\upsilon^2 - \{1-2(\gamma+\beta\upsilon^2)\}]}{[c^2 + (1-2(\gamma+\beta\upsilon^2)s^2)]^2} \tag{1.9}$$

from which, for stability for all υ , we require

$$\phi \geq 1 \ , \quad \gamma \leq ½ \ , \quad \beta \leq -\frac{\phi}{2} \ . \tag{1.10}$$

This confirms Carey & Jiang's [1] result, that (1.1) is unconditionally stable if $\theta \geq ½$. It is still unconditionally stable if $\gamma = ½$ (but marginally).

The marginal case is of interest. Take $\phi = 1$, $\gamma = ½$, $\beta = -½$. The scheme (1.5) becomes

$$[1 + 4\ (1-\upsilon^2)\ \delta^2]\ (U_j^{n+1} - U_j^n) = -\ \upsilon\ \Delta_o U_j^n + ½\upsilon^2\delta U_j^n \tag{1.11}$$

and (1.8) is

$$g = \frac{(c^2 - \upsilon^2s^2)^2 + 4\upsilon^2s^2c^2}{(c^2 + \upsilon^2s^2)^2} = 1 \ . \tag{1.12}$$

This shows clearly the origin of the implicit operator

$$1 + ¼(1-\upsilon^2)\delta^2$$

for this purpose. Since $(c^2 - \nu^2 s^2)^2 + 4\nu^2 s^2 c^2 = (c^2 + \nu^2 s^2)^2$ then

$$\{1 - (1-\nu^2)s^2 - 2\nu^2 s^2\}^2 + 4\nu^2 s^2 c^2 = \{1-(1-\nu^2)s^2\}^2 \ . \tag{1.13}$$

The term $4\nu^2 s^2 c^2$ comes from the central differencing of u_x and the $2\nu^2 s^2$ term from that of u_{xx}. The remaining terms show the origin of the implicitness operator. When γ is reduced below ¼ we get some dissipation.

§2. Extension to Convection-Diffusion

For the convection diffusion equation

$$u_t + au_x = bu_{xx} \tag{2.1}$$

we extend the scheme to

$$[1+¼(1-\nu^2-2\mu)\delta^2]\ U_j^{n+1} = -\frac{\nu}{2}(U_{j+1}^n - U_{j-1}^n) + [1+¼(1+\nu^2+2\mu)\delta^2]\ U_j^n \tag{2.2}$$

where $\mu = \dfrac{b\Delta t}{(\Delta x)^2}$ and the bu_{xx} term has been discretised at $(n+½)\ \Delta t$. Then

$$|g|^2 = \frac{(c^2-(\nu^2+2\mu)s^2)^2 + 4\nu^2 s^2 c^2}{(c^2+(\nu^2+2\mu)s^2)^2}$$

$$= \frac{(c^2+\nu^2 s^2)^2 - 4\mu s^2(c^2-\nu^2 s^2) + 4\mu^2 s^4}{(c^2+\nu^2 s^2)^2 + 4\mu s^2(c^2+\nu^2 s^2) + 4\mu^2 s^4} \tag{2.3}$$

$$= \frac{A-B}{A+B} \tag{2.4}$$

where

$$A = (c^2 + \nu^2 s^2)^2 + 4\mu(\nu^2 + \mu)s^4$$

$$B = 4\mu s^2 c^2 \ . \qquad (2.5)$$

Both A and B are non-negative and so $|g| \leq 1$, $\forall\ \nu, \mu$. It is unconditionally stable. Dissipation is increased by reducing the factor ¼ occuring in the second term of (2.2) in each square bracket.

Re-introducing γ :

$$[1 + \tfrac{1}{2}(\gamma - \frac{\nu^2+2\mu}{2})\delta^2]\ U_j^{n+1} = - \nu\ \Delta_o U_j^n + [1 + \tfrac{1}{2}(\gamma + \frac{\nu^2+2\mu}{2})\delta^2]\ U_j^n \qquad (2.6)$$

$$|g|^2 = \frac{(1-2\gamma s^2 - (\nu^2+2\mu)s^2)^2 + 4\nu^2 s^2 c^2}{(1-2\gamma s^2 + (\nu^2+ 2\mu)s^2)^2}$$

$$= \frac{(c^2 + (1-2\gamma)s^2 - (\nu^2+2\mu)s^2)^2 + 4\nu^2 s^2 c^2}{(c^2 + (1-2\gamma)s^2 + (v^2+2\mu)s^2)^2}$$

$$= 1 + \frac{4v^2 s^2 c^2 - 4[c^2 + (1-2\gamma)s^2](\nu^2+2\mu)s^2}{(c^2 + (1-2\gamma)s^2 + (v^2+2\mu)s^2)^2}$$

$$= 1 - \frac{\{4(1-2\gamma)s^4(\nu^2+2\mu) + 4c^2 2\mu s^2\}}{(c^2 + (1-2\gamma)s^2 + (\nu^2+2\mu)s^2)^2} \qquad (2.7)$$

so that it is clear that $\gamma < \tfrac{1}{2}$ introduces dissipation.

If $\gamma = \tfrac{1}{3}$

$$|g|^2 = 1 - \frac{(\tfrac{4}{3}s^4(\nu^2+2\mu) + 4c^2 2\mu s^2)}{(c^2 + \frac{1}{3}s^2 + (\nu^2+2\mu)s^2)^2} \ . \qquad (2.8)$$

§3. Two-dimensional Convection

We attempt the same procedure in the 2-D case. Starting with

$$u_j^{n+1} = u^n + \Delta t u_t^n + \frac{\Delta t^2}{2!} u_{tt}^n \tag{3.1}$$

where

$$u_t + au_x + bu_y = 0 \text{ , we obtain} \tag{3.2}$$

$$u^{n+1} - u^n = - a\Delta t\, u_x - b\Delta t\, u_y + ½(\Delta t)^2(a^2 u_{xx} + 2ab u_{xy} + b^2 u_{yy})$$

which discretises to

$$U_{jk}^{n+1} - U_{jk}^n = - \nu_1 \Delta_{ox} U_{jk}^n - \nu_2 \Delta_{oy} U_{jk}^n + ½(\nu_1^2 \delta_x^2 + 2\nu_1\nu_2 \Delta_{ox}\Delta_{oy} + \nu_2^2 \delta_y^2) U_{jk}^n \tag{3.3}$$

Here $\Delta_{ox} U_{jk} = ½(U_{j+1k} - U_{j-1k})$, $\delta_x^2 = U_{j+1k} - 2U_{jk} + U_{j-1k}$ etc. Then the stability analysis gives

$$g-1 = -2i\nu_1 s_1 c_1 - 2i\nu_2 s_2 c_2 - 2\nu_1^2 s_1^2 - 4\nu_1\nu_2 s_1 c_1 s_2 c_2 - 2\nu_2^2 s_2^2 \tag{3.4}$$

where $s_1 = \sin \frac{1}{2} k_1 \Delta x$, $c_1 = \cos \frac{1}{2} k_1 \Delta x$, $s_2 = \sin \frac{1}{2} k_2 \Delta y$, $c_2 = \cos \frac{1}{2} k_2 \Delta y$.

Introducing an implicitness operator with real and imaginary parts N_1, N_2 this becomes

$$|g|^2 = \frac{(N_1 - 2\nu_1^2 s_1^2 - 4\nu_1\nu_2 s_1 c_1 s_2 c_2 - 2\nu_2^2 s_2^2)^2 + (N_2 - 2\nu_1 s_1 c_1 - 2\nu_2 s_2 c_2)^2}{N_1^2 + N_2^2} . \tag{3.5}$$

The object is to choose N_1, N_2 simply so that $|g|^2 \leq 1$. In fact N_2

must be zero if one wishes to avoid having to invert an unsymmetric matrix. Also N_1 should be symmetric.

§4. Unconditional stability for First Order Upwind Schemes.

The first order upwind scheme for

$$u_t + au_x = 0 \qquad\qquad a > 0 \tag{4.1}$$

is

$$U_j^{n+1} - U_j^n = -\nu(U_j^n - U_{j-1}^n) . \tag{4.2}$$

Consider

$$[1 + \tfrac{1}{4}(1-\nu-\nu^2)\delta^2](U_j^{n+1} - U_j^n) = -\nu(U_j^n - U_{j-1}^n) + \tfrac{1}{2}\nu\delta^2 U_j^n \tag{4.3}$$

Then

$$\begin{aligned}[1 - (1-\nu-\nu^2)s^2](g-1) &= -\nu(1-e^{-ik\Delta x}) - 2\nu^2 s^2 \\ &= -2\nu(s^2 + isc) - 2\nu^2 s^2 .\end{aligned} \tag{4.4}$$

So

$$g = \frac{1 - (1+\nu+\nu^2)s^2 - 2i\nu sc}{1 - (1-\nu-\nu^2)s^2}$$

$$\begin{aligned}|g|^2 &= \frac{(c^2 - (\nu+\nu^2)s^2)^2 + 4\nu^2 s^2 c^2}{(c^2 + (\nu+\nu^2)s^2)^2} \\ &= 1 - \frac{4\nu s^2 c^2}{(c^2 + (\nu+\nu^2)s^2)^2} \le 1 \qquad \nu \ge 0 .\end{aligned} \tag{4.5}$$

If $a < 0$, the corresponding scheme is

$$[1 + \tfrac{1}{4}(1+\nu-\nu^2)\delta^2](U_j^{n+1} - U_j^n) = -\nu(U_j^n - U_{j-1}^n) + \tfrac{1}{2}\nu^2\delta^2 U_j^n \tag{4.6}$$

so a general unconditionally stable scheme (only 1st order) is

$$[1 + \tfrac{1}{4}(1-|\nu|-\nu^2)\delta^2](U_j^{n+1} - U_j^n) = - \nu(U_j^n - U_{j\pm1}^n) + \tfrac{1}{2}\nu^2\delta^2 U_j^n \, . \qquad (4.7)$$

Note that this method is quite distinct from (1.1) with $\theta \neq \frac{1}{2}$, for which (1.1) is first order, and from (1.2) which has no linear term in ν multipying δ^2 .

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