A FULLY TWO-DIMENSIONAL FINITE DIFFERENCE METHOD ON A REGULAR GRID FOR SYSTEMS OF CONSERVATION LAWS

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ABSTRACT

We present finite difference algorithms for a scalar conservation law and for a system which are fully two-dimensional, conservative, stable, second order accurate and anti-diffusive. Shock recognition in two-dimensions is discussed together with an adaptation of a wave splitting technique for the Euler equations.

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1. Introduction

In an earlier report [1] high order algorithms were proposed for scalar conservation laws in two and three dimensions, while another report [2] dealt with shock recognition in two dimensions. In the present report we combine and extend these ideas to present a complete method for a two-dimensional system of conservation laws.

The plan of the report is as follows. We begin by restating the basis of the two dimensional algorithms for a single scalar conservation law and give a version which is exact for bilinear functions, proving at the same time that it has a local bound (LB) property.

We then summarise the procedures (detailed in [1]) for making the scheme of higher order, giving special attention to those which preserve the LB property almost everywhere. Versions of these algorithms for the scalar non-linear case are then given.

In the next section we deal with shock recognition in two dimensions (and mistaken identity!) in a manner similar to that in [2] but expanded further.

In conclusion some of the restrictions in the method are discussed.

2. First order Algorithm

For the two dimensional scalar conservation law

$$u_t + F_x + G_y \equiv u_t + a(u)u_x + b(u)u_y = 0 \tag{2.1}$$

we define the fluctuation over the quadrilateral ABCD (see Fig. 1) as

$$\phi = -\iint (F_x + G_y)\,d\Omega = -\oint_{ABCD} (F,G)\cdot d\underline{S} \;. \tag{2.2}$$

Using trapezoidal rule integration (exact for bilinear functions F,G) this gives

$$\phi = \sum_{\text{sides like CD}} -\{\tfrac{1}{2}(F_C + F_D)(y_D - y_C) + \tfrac{1}{2}(G_C + G_D)(x_C - x_D)\} \tag{2.3}$$

which, added over all cells, has the internal cancellation property of a discrete conservation law.

Specialising to the rectangle in Fig. 2 we can show (as in [1]) that

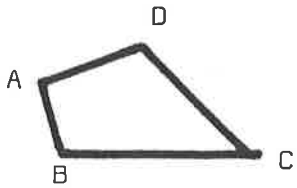

Fig. 1

$$\phi = \sum_{\substack{\text{pairs of sides}\\ \text{like DA,CB}}} -\left[\tfrac{1}{2}(G_D + G_A) - \tfrac{1}{2}(G_C + G_B)\right]\Delta x$$

$$+ \sum_{\substack{\text{pairs of sides}\\ \text{like DC,AB}}} -\left[\tfrac{1}{2}(F_D + F_C) - \tfrac{1}{2}(F_A + F_B)\right]\Delta y \tag{2.4}$$

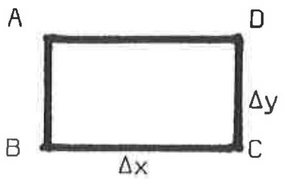

Fig. 2

$$= -\tfrac{1}{2}\sum_{\text{sides like DA}} (F_D - F_A)\Delta y - \tfrac{1}{2}\sum_{\substack{\text{sides}\\ \text{like DC}}} (G_D - G_C)\Delta x \tag{2.5}$$

Multiplying ϕ by the factor $\Delta t/(\Delta x \Delta y)$ we obtain

$$\Phi = \tfrac{1}{2}\Phi_F^{DA} + \tfrac{1}{2}\Phi_F^{CB} + \tfrac{1}{2}\Phi_G^{DC} + \tfrac{1}{2}\Phi_G^{AB} \tag{2.6}$$

where

$$\Phi_F^{DA} = -\frac{\Delta t}{\Delta x}(F_D - F_A) \qquad \Phi_G^{DC} = -\frac{\Delta t}{\Delta y}(G_D - G_C) \tag{2.7}$$

When (2.6) is summed over a network of rectangles, contributions from adjacent sides can be added so that, apart from boundaries,

$$\sum_{\substack{\text{all}\\ \text{cells}}} \Phi = \sum_{\substack{\text{all pairs}\\ \text{of sides}}} \Phi_F + \Phi_G \, , \tag{2.8}$$

dropping superfixes in (2.7).

The discharge of the fluctuations over all rectangles is therefore achieved by the discharge of these side fluctuations (2.7) over all sides (apart from boundaries). As a result, many of the techniques for one-dimensional algorithms can be used in two dimensions. The procedures are applied simultaneously in the x and y directions (parallel to the rectangle sides), as distinct from time splitting techniques in which they are applied serially.

If the signals (2.7) are used to increment values of u at the downstream end of the sides (indicated by the signs of $F'(u)$, $G'(u)$ or their approximations) in a time step Δt, we obtain a conservative first-order scheme [3]. For a,b constant, it is exact for linear functions u, and (as can easily be shown) it satisfies a local bound (see below).

Practically, for Φ_F, Φ_G arising from DA,DC (see Fig. 3), we can specify which points receive increments (2.7) in the first order algorithm once approximations to $F'(u)$ and $G'(u)$ are known. These can be taken to be (see [3])

$$F'(u) = \frac{F_D - F_A}{u_D - u_A} \, , \qquad G'(u) = \frac{G_D - G_A}{u_D - u_A} \tag{2.9}$$

$$= a_{AD}, \text{ say,} \qquad\qquad = b_{AD}, \text{ say.}$$

3. Bilinear Functions and the LB property

To make the first order scheme exact for bilinear functions u we introduce the idea of transfers, i.e. the addition (and subtraction) of some proportion of the quantity u to one point (and from another) at a fixed time: such transfers clearly preserve conservation. Transfers of a proportion of the increments (2.7) laterally, i.e. of Φ_F in the y-direction and of Φ_G in the x-direction, do not alter first-order accuracy and allow xy accuracy to be attained. In the first order method if a_{AD}, b_{AD} are both positive

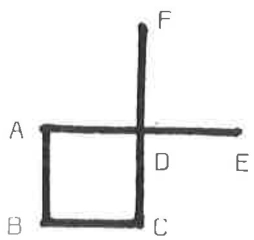

Fig. 3

the increment Φ_F of (2.7) is added to D: let a fraction α_3 of Φ_F be transferred from D to F. Likewise the increment Φ_G is added to D: let a fraction α_4 of Φ_G then be transferred from D to E. As in [1] the condition that the scheme is exact (in the linearised case) for functions xy is

$$\nu_1\alpha_3 + \nu_2\alpha_4 = \nu_1\nu_2 \tag{3.1}$$

where
$$\nu_1 = a_{AD}\frac{\Delta t}{\Delta x}, \quad \nu_2 = b_{CD}\frac{\Delta t}{\Delta y} . \tag{3.2}$$

A parametric solution of (2.10) is

$$\alpha_3 = \nu_2 \cos^2\theta \qquad \alpha_4 = \nu_1 \sin^2\theta \tag{3.3}$$

(with θ a free parameter), and the allocation of the signals

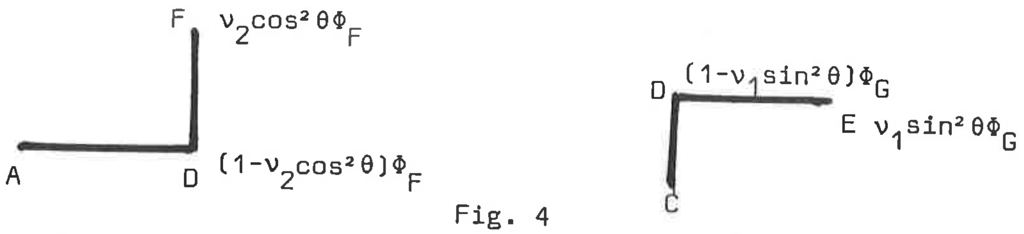

Fig. 4

(2.7) is then as shown in Fig. 4. A convenient choice of θ is $\theta = \pi/4$, for which

$$\alpha_3 = \tfrac{1}{2}\nu_2 \qquad \alpha_4 = \tfrac{1}{2}\nu_1 . \tag{3.4}$$

Turning now to the total increment received by D, the value of u_D at the next time level will be (see Fig. 3)

$$u^D = u_D + (1 - \nu_2 \cos^2\theta)\Phi_F^{DA} + (1 - \nu_1 \sin^2\theta)\Phi_G^{DC}$$
$$+ \nu_2 \cos^2\theta\ \Phi_F^{CB} + \nu_1 \sin^2\theta\ \Phi_G^{AB} \tag{3.5}$$

$$= u_D - (1 - \nu_2 \cos^2\theta)\nu_1(u_D - u_A) - (1 - \nu_1 \sin^2\theta)\nu_2(u_D - u_C)$$
$$- \nu_2 \cos^2\theta\ \nu_1(u_C - u_B) - \nu_1 \sin^2\theta\ \nu_2(u_A - u_B) \tag{3.6}$$

since, for example,

$$\Phi_f^{DA} = -\frac{\Delta t}{\Delta x}(f_D - f_A) = -\frac{\Delta t}{\Delta x} a_{AD}(u_D - u_A) = -\nu_1(u_D - u_A) \tag{3.7}$$

by (2.9) and (2.10). Thus

$$u^D = u_D\,(1-\nu_1(1-\nu_2\cos^2\theta) - \nu_2(1-\nu_1\sin^2\theta) + u_A(\nu_1(1-\nu_2\cos^2\theta) - \nu_1\nu_2\sin^2\theta)$$
$$+\, u_C(\nu_2(1-\nu_1\sin^2\theta) - \nu_1\nu_2\cos^2\theta) + u_B\nu_1\nu_2(\cos^2\theta + \sin^2\theta)$$
$$= u_D(1-\nu_1-\nu_2+\nu_1\nu_2) + u_A(\nu_1-\nu_1\nu_2) + u_C(\nu_2-\nu_1\nu_2) + u_B\nu_1\nu_2$$
$$= (1-\nu_1)(1-\nu_2)u_D + \nu_1(1-{}_2)u_A + \nu_2(1-\nu_1)u_C + \nu_1\nu_2u_B. \qquad (3.8)$$

Provided that $0 \leqq \nu_1 \leqq 1$, $0 \leqq \nu_2 \leqq 1$, all the coefficients in (3.8) are positive and it follows that u^D lies in the support of u_D, u_C, u_B, u_A (Fig. 3). We call this property the LB (locally bounded) property, viz.

$$u^D \in \text{support } [u_D, u_C, u_B, u_A] \quad . \qquad (3.9)$$

4. Second order Algorithm and B functions

We now have a conservative scheme, exact for bilinear functions u (in the linearised case) which satisfies an LB property. Since first order scheme are too diffusive in most cases we seek to make the scheme second order (i.e. also exact for x^2 and y^2) and can do this by further transfers.

Thus, by transferring $\alpha_1\Phi_F^{DA}$ from D to A and $\alpha_2\Phi_G^{DC}$ from D to C we do not alter bilinear accuracy but do obtain second order accuracy - for x^2 and y^2 - if

$$\nu_1(1 + 2\alpha_1) = \nu_1^2 \qquad \nu_2(1 + 2\alpha_2) = \nu_2^2 \qquad (4.1)$$

leading to

$$\alpha_1 = \tfrac{1}{2}(1 - \nu_1) \qquad \alpha_2 = \tfrac{1}{2}(1 - \nu_2)\;. \qquad (4.2)$$

Such a scheme is a variant of two-dimensional Lax-Wendroff type schemes. As a result, however, (as is well known) we lose the LB property.

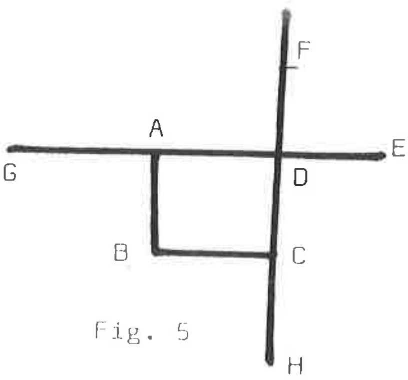

Fig. 5

An alternative is to transfer $\alpha_1\Phi_F^{AG}$ from D to A and $\alpha_2\Phi_G^{CH}$ from D to C (see Fig. 5) and second order accuracy is obtained with the same values (2.19) of α_1, α_2. This is a second order fully upwind scheme. The LB property is again lost, however, We can re-establish the LB property by using a non-linear function of the two possibilities above. As in [4] we introduce a general transfer function called the B function, which is a function of the two fluctuations Φ_F^{DA} and Φ_F^{AG} (or Φ_G^{DC} and Φ_G^{CH}).

For example, if $B(b_1,b_2) = b_1$, Φ_F^{DA} is transferred (as in the Lax-Wendroff scheme above), while if $B(b_1,b_2) = b_2$, Φ_F^{AG} is transferred (as in the second order upwind scheme). The choice $B = \frac{1}{2}(b_1+b_2)$ corresponds to a Fromm type scheme.

If $B = b_1$ the new value of u_D after a time step is

$$u^D = u_D + (1-\alpha_3)\Phi_F^{DA} + (1-\alpha_4)\Phi_G^{DC} + \alpha_3\Phi_F^{CB} + \alpha_4\Phi_G^{AB}$$

$$- \alpha_1\Phi_F^{DA} - \alpha_2\Phi_G^{DC} + \alpha_1\Phi_F^{ED} + \alpha_2\Phi_G^{FD} \qquad (4.3)$$

(see Fig. 5), but if a general B function is transferred we have

$$u^D = u_D + (1-\alpha_3)\Phi_F^{DA} + (1-\alpha_4)\Phi_G^{DC} + \alpha_3\Phi_F^{CB} + \alpha_4\Phi_G^{AB}$$

$$- \alpha_1 B\left(\Phi_F^{DA},\Phi_F^{AG}\right) - \alpha_2 B\left(\Phi_G^{DC},\Phi_G^{CH}\right) + \alpha_1 B\left(\Phi_F^{ED},\Phi_F^{DA}\right) + \alpha_2 B\left(\Phi_G^{FD},\Phi_G^{DC}\right) \qquad (4.4)$$

Setting $\beta = \dfrac{B(b_1,b_2)}{b_1}$ and $\gamma = \dfrac{B(b_1,b_2)}{b_2}$, (4.5)

we can write

$$u^D = u_D + (1-\alpha_3-\beta\alpha_1+\gamma\alpha_1)\Phi_F^{DA} + (1-\alpha_4-\beta\alpha_2+\gamma\alpha_2)\Phi_G^{DC}$$

$$+ \alpha_3\Phi_F^{CB} + \alpha_4\Phi_G^{AB} \qquad (4.6)$$

$$= u_D - (1-\alpha_3-\overline{\beta-\gamma}\alpha_1)\nu_1(u_D-u_A) - (1-\alpha_4-\overline{\beta-\gamma}\alpha_2)\nu_2(u_D-u_C)$$

$$-\alpha_3\nu_1(u_C-u_B) - \alpha_4\nu_2(u_A-u_B) \qquad (4.7)$$

$$= u_D(1-\nu_1(1-\alpha_3-\overline{\beta-\gamma}\alpha_1) - \nu_2(1-\alpha_4-\overline{\beta-\gamma}\alpha_2))$$

$$+ (\nu_1(1-\alpha_3-\overline{\beta-\gamma}\alpha_1) - \nu_2\alpha_4)u_A + (\nu_2(1-\alpha_4-\overline{\beta-\gamma}\alpha_2) + \nu_1\alpha_3)u_C$$

$$+ (\nu_1\alpha_3+\nu_2\alpha_4)u_B \qquad (4.8)$$

$$= [1-(1-\nu_1)(1-\nu_2) + \overline{\beta-\gamma}(\nu_1\alpha_1+\nu_2\alpha_2)]u_D + [\nu_1(1-\nu_2) - \overline{\beta-\gamma}\nu_1\alpha_1]u_A$$

$$+ [\nu_2(1-\nu_1) - \overline{\beta-\gamma}\nu_2\alpha_2]u_C + \nu_1\nu_2u_B , \tag{4.9}$$

using (3.1). Here β,γ vary from cell to cell.

The LB property depends on the inequalities

$$(1 - \nu_1)(1 - \nu_2) - \overline{\beta -\gamma}(\nu_1\alpha_1 + \nu_2\alpha_2) \leqq 1 \tag{4.10}$$

$$\left.\begin{aligned} 1 - \nu_1 - \overline{\beta -\gamma}\alpha_2 \geqq 0 \\ 1 - \nu_2 - \overline{\beta -\gamma}\alpha_1 \geqq 0 \end{aligned}\right\} \tag{4.11}$$

Suppose that we take

$$B(b_1,b_2) = \begin{cases} \text{min mod}(b_1,b_2) & b_1b_2 \geqq 0 \\ 0 & b_1b_2 < 0 \end{cases} \tag{4.12}$$

(c.f. [1]). Then from (2.22) it can be seen that $0 \leqq \beta \leqq 1$ and $0 \leqq \gamma \leqq 1$. In this case sufficient conditions for the inequalities (2.27), (2.28) to hold are

$$(1 - \nu_1)(1 - \nu_2) + (\nu_1\alpha_1 + \nu_2\alpha_2) \leqq 1, \tag{4.13}$$

i.e. $$(\nu_1 - \nu_2)^2 + \nu_1 + \nu_2 \geqq 0 , \tag{4.14}$$

which is satisfied for all $\nu_1 \leqq 1$, $\nu_2 \leqq 1$; and

$$\nu_1(1-\nu_1) - \tfrac{1}{2}\nu_1(1-\nu_2) \geqq 0, \quad \nu_2(1-\nu_2) - \tfrac{1}{2}\nu_2(1-\nu_1) \geqq 0 \tag{4.15}$$

for which we require $\nu_1(\tfrac{1}{2}-\nu_1) + \tfrac{1}{2}\nu_1\nu_2 \geqq 0$, $\nu_2(\tfrac{1}{2}-\nu_2) + \tfrac{1}{2}\nu_1\nu_2 \geqq 0$. (4.16)

These conditions are satisfied if $\nu_1 \leqq \tfrac{1}{2}$, $\nu_2 \leqq \tfrac{1}{2}$.

Thus with the non-linear transfer function (4.12) it is possible to obtain a conservation scheme which is LB and second order almost everywhere. A similar analysis can be carried out for other B functions. [Note that the LB property (2.17) described here is slightly different from, and stronger than, that used in ref. [1], equation (4.17) - see below].

5. Anti-diffusive B functions

The B function (4.12) gives a scheme which is formally second order but clips extrema ($b_1 b_2 < 0$). Another approach to preserving the LB property is to specify a small range of b_1, b_2, close to extrema, for which second order accuracy is sacrificed in favour of LB. This can be done for any convenient B function, even one which on the face of it is known to violate LB when not mollified in such a way.

Thus the B function (in one dimension)

$$B(b_1,b_2) = \begin{cases} \frac{2}{\nu}\,\mathrm{minmod}(b_1,b_2) & \frac{b_1}{b_2} > \frac{2}{\nu} \\ \frac{2}{1-\nu}\,\mathrm{minmod}(b_1,b_2) & \frac{b_1}{b_2} < \frac{1}{2}(1-\nu) \\ \mathrm{maxmod}(b_1,b_2) & \text{otherwise} \end{cases} \Bigg\} \; b_1 b_2 > 0 \qquad (5.1)$$

$$\qquad 0 \qquad\qquad b_1 b_2 < 0 \qquad b_1 b_2 < 0$$

(c.f. ref. [1], equation (1.37)), for the majority of values of b_1, b_2 violates the LB property (through the part $\max(b_1,b_2)$), giving the Lax-Wendroff scheme or the second order upwind scheme in precisely those situations where such schemes produce oscillations. This is in direct opposition to the choice (4.12), which is safe in the sense that it picks out the scheme which does not produce oscillations in a given situation.

The B function (5.1), dubbed ULTRABEE in another place [4], can be constructed in an alternative way which makes clearer its relationship with the LB property and with concepts of Flux Corrected Transport [5], [6]. First, we recall that the first order upwind scheme is very diffusive. The second order correction (made through the B-function transfer) reduces the diffusion and is thus an anti-diffusive step in the FCT tradition. At the same time we seek the requirement that the LB property is preserved, for which we need positive coefficients in the expression for u^D (c.f. (3.8)). The B function of (5.1) in fact increases anti-diffusion while preserving positive coefficients.

One particular property is worth noting, throwing light on the coefficients $\frac{2}{\nu}$ and $\frac{1}{2}(1-\nu)$ in (5.1). If, for an adjacent pair of cells (say AD, DE in Fig. 5) the B function for the left hand cell is zero (at an extremum say) and an overshoot is threatened, then the coefficient $\frac{2}{\nu}$ in (5.1) is just that which leads to $u^D = u_A$. Similarly if the B function for the right hand cell is zero and an undershoot is threatened, the coefficient $\frac{1}{2}(1-\nu)$ in (5.1) is just that which makes $u^D = u_D$. Thus no new extrema are (quite) created. This description is of great value when we generalise to two dimensions, which we now do.

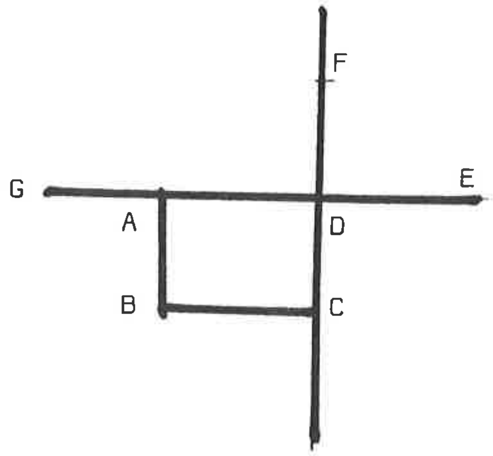

Fig. 6

To maximise anti-diffusion in two dimensions we again propose to use a B function based on

$$B(b_1,b_2) = \text{maxmod}(b_1,b_2) \qquad (5.2)$$

with limits on its range of applicability (c.f. (5.1)), to be determined. Assuming zero contribution from the fluctuations in AG and CH (close to an extremum) and a regime such that $|b_1| > |b_2|$, so that $\beta = 1$, $\gamma = b_1/b_2$, we have from (4.4)(see Fig. 6)

$$u^D = u_D - (1-\alpha_3)\nu_1(u_D-u_A) - (1-\alpha_4)\nu_2(u_D-u_C) - \alpha_3\nu_1(u_C-u_B)$$

$$- \alpha_4\nu_2(u_A-u_B) - \gamma\alpha_1\nu_1(u_D-u_A) - \gamma\alpha_2\nu_2\ (u_D-u_C) \qquad (5.3)$$

$$= [1-\nu_1(1-\alpha_3) - \nu_2(1-\alpha_4) - \gamma\alpha_1\nu_1 - \gamma\alpha_2\nu_2]u_D$$

$$+ [(1-\alpha_3)\nu_1 - \alpha_4\nu_2 + \gamma\nu_1\alpha_1]u_A + [(1-\alpha_4)\nu_2 - \alpha_3\nu_1 + \alpha_2\nu_2\ ]u_C$$

$$+ (\alpha_3\nu_1 + \alpha_4\nu_2)u_B \qquad (5.4)$$

$$= (1-\nu_1-\nu_2+\nu_1\nu_2 - \gamma\alpha_1\nu_1 - \gamma\alpha_2\nu_2)u_D + (\nu_1-\nu_1\nu_2 + \gamma\nu_1\alpha_1)u_A$$

$$+ (\nu_2-\nu_1\nu_2+\alpha_2\nu_2\gamma)u_C + \nu_1\nu_2u_B \ , \tag{5.5}$$

using (3.1). We want to choose γ in such a way that, in the worst case, the coefficient of u_D is zero, i.e.

$$(1 - \nu_1)(1 - \nu_2) - \gamma\tfrac{1}{2}\nu_1(1 - \nu_1) - \gamma\tfrac{1}{2}\nu_2(1 - \nu_2) = 0 \text{, and} \tag{5.6}$$

u_D is then a linear combination of u_A, u_B and u_C. This leads to

$$\gamma = \frac{2(1-\nu_1)(1-\nu_2)}{\nu_1(1-\nu_1)+\nu_2(1-\nu_2)} = \frac{2}{\dfrac{\nu_1}{1-\nu_2} + \dfrac{\nu_2}{1-\nu_1}} \tag{5.7}$$

The B function (5.2) is now limited as follows

$$B(b_1,b_2) = \begin{cases} \left.\begin{array}{ll} \gamma \ \mathrm{minmod}(b_1,b_2) & \dfrac{b_1}{b_2} > \gamma \\ \mathrm{maxmod}(b_1,b_2) & \text{otherwise} \end{array}\right\} & b_1b_2 \geq 0 \\ 0 & b_1b_2 < 0 \end{cases} \tag{5.8}$$

Correspondingly, assuming zero transfers across ED and FD (Fig. 6) and a regime such that $|b_1|<|b_2|$ so that $\beta = b_2/b_1$, $\gamma = 1$ we have from (4.4) (assuming different β's for the x,y directions, i.e. β_1,β_2 respectively)

$$u^D = u_D - (1-\alpha_3)\nu_1(u_D-u_A)-(1-\alpha_4)\nu_2(u_D-u_C)$$

$$- \alpha_3\nu_1(u_C-u_B)-\alpha_4\nu_2(u_A-u_B)+\alpha_1\beta_1\nu_1(u_D-u_A)$$

$$+ \alpha_2\nu_2\beta_2(u_D-u_C) \tag{5.9}$$

$$= [1-\nu_1(1-\alpha_3)-\nu_2(1-\alpha_4)+\nu_1\alpha_1\beta_1+\nu_2\alpha_2\beta_2]u_D$$

$$+ [\nu_1(1-\alpha_3)-\alpha_4\nu_2-\alpha_1\beta_1\nu_1]u_A + [\nu_2(1-\alpha_4)-\alpha_3\nu_1-\alpha_2\beta_2\nu_2]u_C$$

$$+ (\alpha_3\nu_1+\alpha_4\nu_2)u_B \tag{5.10}$$

$$= [1-\nu_1-\nu_2+\nu_1\nu_2+\nu_1\alpha_1\beta_1+\nu_2\alpha_2\beta_2]u_D + [\nu_1-\nu_1\nu_2-\nu_1\alpha_1\beta_1]u_A$$

$$+ [\nu_2-\nu_1\nu_2-\nu_2\alpha_2\beta_2]u_C + \nu_1\nu_2u_B, \tag{5.11}$$

again using (3.1).

The choice of the β's must now be made to make the coefficients of u_A, u_C positive, namely

$$\left.\begin{aligned} 1 - \nu_2 - \tfrac{1}{2}(1 - \nu_1)\beta_1 &\geq 0 \\ 1 - \nu_1 - \tfrac{1}{2}(1 - \nu_2)\beta_2 &\geq 0 \end{aligned}\right\} \tag{5.12}$$

giving

$$\beta_1 \leq \frac{2(1-\nu_2)}{1-\nu_1}, \qquad \beta_2 \leq \frac{2(1-\nu_1)}{1-\nu_2}. \tag{5.13}$$

In this case u^D is a linear combination of u_D and u_B.

We have thus devised two limiters, $B_i(b_1,b_2)$, $i = 1,2$, for the x,y directions respectively, as follows

$$B_i(b_1,b_2) = \begin{cases} \gamma \operatorname{minmod}(b_1,b_2) & \dfrac{b_1}{b_2} > \gamma \\ \beta_i \operatorname{minmod}(b_1,b_2) & \dfrac{b_2}{b_1} > \beta_i \\ \operatorname{maxmod}(b_1,b_2) & \text{otherwise} \end{cases} \Bigg\} \; b_1 b_2 > 0 \\ \quad 0 \hspace{10em} b_1 b_2 < 0 \tag{5.14}$$

where γ, β_i are given by (5.7) and (5.13). Since γ, β_i must be greater than 1 we are restricted to ν_1, $\nu_2 < \frac{1}{2}$.

6. Optimising anti-diffusion

The spirit of the anti-diffusive limiter is expressed in the idea that near extrema it is more important to obtain maximum anti-diffusion (subject to an LB principle) than it is to obtain formal accuracy. As a result discontinuities are sharper [7]. Note that conservation is built-in.

In [1] the B function idea is applied to the lateral transfers (for xy accuracy) also. As a result a sharper LB property is obtained, namely,

$$u^D \in \text{support } [u_D, u_C, u_A], \tag{6.1}$$

at the expense of local bilinear accuracy (c.f. (3.9)). In that case, using the notation of the present discussion, (4.4) becomes (see Fig. 7)

$$u^D = u_D + \Phi_F^{DA} + \Phi_G^{DC} - \alpha_3 B\left(\Phi_F^{DA},\Phi_F^{FK}\right) - \alpha_4 B\left(\Phi_G^{DC},\Phi_G^{EI}\right)$$

$$+ \alpha_3 B\left(\Phi_F^{CB},\Phi_F^{DA}\right) + \alpha_4 B\left(\Phi_G^{AB},\Phi_G^{DC}\right)$$

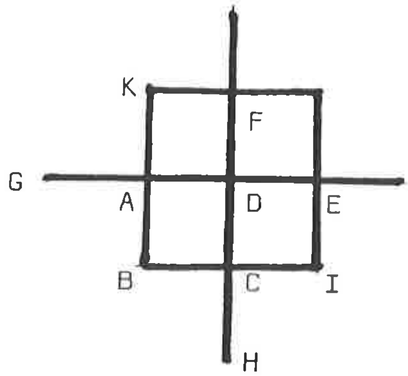

Fig. 7

$$- \alpha_1 B\left(\Phi_F^{DA},\Phi_F^{AG}\right) - \alpha_2 B\left(\Phi_G^{DC},\Phi_G^{CH}\right) + \alpha_1 B\left(\Phi_F^{ED},\Phi_F^{DA}\right) + \alpha_2 B\left(\Phi_G^{FD},\Phi_G^{DC}\right) \qquad (6.2)$$

By an argument similar to that following (5.2) and (5.8) , assuming $|b_1| < |b_2|$ and zero fluctuations for ED, FD, EI, KF (see Fig. 7) and also using longitudinal transfers with B function (5.2) and lateral transfers with B function

$$B(b_1,b_2) = \text{minmod}(b_1,b_2) \qquad (6.3)$$

(corresponding to minimum lateral diffusion - see (6.14)), we have from (6.2)

$$u^D = u_D - \nu_1(u_D-u_A) - \nu_2(u_D-u_C) - \alpha_3\nu_1(u_D-u_A) - \alpha_4\nu_2(u_D-u_C)$$

$$+ \alpha_1\beta_1\nu_1(u_D-u_A) + \alpha_2\beta_2\nu_2(u_D-u_C) + \gamma_1'\alpha_3\nu_1(u_D-u_A) + \gamma_2'\alpha_4\nu_2(u_D-u_C) \qquad (6.4)$$

$$= u_D - \nu_1(u_D-u_A)[1+(1-\gamma_1')+\alpha_3-\beta_1\alpha_1] - \nu_2(u_D-u_A)[1+(1-\gamma_2')+\alpha_4-\beta_2\alpha_2] \qquad (6.5)$$

(c.f.(5.11)),where $\beta_1\beta_2$ are the values of $\beta = \dfrac{b_1}{b_2}$ corresponding to the x,y directions, and γ_1', γ_2' lie between 0 and 1.

The coefficients of u_A, u_C are zero (and therefore $u^D = u_D$) in the worst case if

$$\beta_1 = \frac{1}{\alpha_1} \qquad\qquad \beta_2 = \frac{1}{\alpha_2} \qquad (6.6)$$

In this case we obtain

$$\beta_1 = \frac{2}{1-\nu_1} \qquad\qquad \beta_2 = \frac{2}{1-\nu_2} \quad . \tag{6.7}$$

In the $|b_1| > |b_2|$ case, taking the fluctuations in AG, CH, AB, CB to be zero (Fig. 7), with the same B functions as before, we have

$$\begin{aligned} u^D &= u_D-\nu_1(u_D-u_A)-\nu_2(u_D-u_C)+\alpha_3\nu_1(u_D-u_A)+\alpha_4\nu_2(u_D-u_C) \\ &\quad -\alpha_1\gamma\,\nu_1(u_D-u_A)-\alpha_2\gamma\,\nu_2(u_D-u_C)-\beta_1'\alpha_3\nu_1(u_D-u_A)-\beta_2'\alpha_4\nu_2(u_D-u_C) \\ &= u_D-\nu_1(u_D-u_A)(1-\alpha_3+\gamma\alpha_1)-\nu_2(u_D-u_C)(1-\alpha_4+\gamma\alpha_2) \end{aligned} \tag{6.8}$$

$$\begin{aligned} &= (1-\nu_1-\nu_2+\nu_1(1-\beta_1')\alpha_3+\nu_2(1-\beta_2')\alpha_4-\gamma\nu_1\alpha_1-\gamma\nu_2\alpha_2)u_D+(1-(1-\beta_1')\alpha_3+\gamma\alpha_1)\nu_1u_A \\ &\quad + (1-(1-\beta_2')\alpha_4+\gamma\alpha_2)\nu_2u_C \end{aligned} \tag{6.9}$$

where β_1', β_2' lie between 0 and 1.

For maximum antidiffusion we seek to make the coefficient of u_D zero, so that u^D is a linear combination of u_A and u_C. As in (5.6) we need

$$1-\nu_1-\nu_2+\nu_1(1-\beta_1')(\alpha_3-\gamma\alpha_1)+\nu_2(1-\beta_2')(\alpha_4-\gamma\alpha_2) = 0 \tag{6.10}$$

so that γ is given by

$$\gamma = \frac{2(1-\nu_1-\nu_2)}{\nu_1(1-\nu_1)+\nu_2(1-\nu_2)}$$

Since $\beta_1,\beta_2\gamma$ must be greater than 1 we require $\nu_1,\nu_2 < 0.375$.

To summarise, we have designed a two-dimensional anti-diffusive algorithm with, for longitudinal transfers (i = 1,2 refer to the x,y directions respectively), the B function

$$B_i(b_1,b_2) = \begin{cases} \gamma \min(b_1,b_2) & b_1/b_2 > \gamma \\ \beta_i \min(b_1,b_2) & b_2/b_1 > \beta_i \\ \max(b_1,b_2) & \text{otherwise} \end{cases} \quad b_1 b_2 > 0 \\ 0 \qquad\qquad b_1 b_2 < 0 \tag{6.11}$$

where (in the $\theta = \pi/4$ case)

$$\beta_1 = \frac{2}{1-\nu_1} \qquad \beta_2 = \frac{2}{1-\nu_2} \tag{6.12}$$

$$\gamma = \frac{1-\nu_1-\nu_2}{\nu_1(1-\nu_1) + \nu_2(1-\nu_2)} \tag{6.13}$$

and, for lateral transfers, the B function

$$B_i(b_1,b_2) = \begin{cases} \text{minmod}(b_1,b_2) & b_1 b_2 > 0 \\ 0 & b_1 b_2 < 0 \end{cases} \tag{6.14}$$

where the minmod function selects the argument with minimum modulus.

When all the limiters are in operation and under the assumptions made above, $u^D = u_D$ in a region where $|b_1|<|b_2|$ and $u^D = \ell u_A + m u_C$ in a region where $b_1 > b_2$. Without such assumption we can still assert that u^D is a linear combination of u_D, u_A and u_C when $|b_1|<|b_2|$ and u^D is a linear combination of u_A, u_C and u_B when $|b_1|>|b_2|$.

As in [1], [7], [8] a practical version of the one-dimensional B function (2.34) is obtained by replacing $2/\nu$ by 2 and $\frac{1}{2}(1-\nu)$ by $\frac{1}{2}$ (the so-called SUPERBEE [7]). Similarly a practical version of the two-dimensional anti-diffusive B function is obtained by replacing β_1, β_2, γ in (6.12) and (6.13) by 2 .

In full this is

$$B_i(b_1,b_2) = \left\{ \begin{array}{ll} \left.\begin{array}{ll} 2\ \min(b_1,b_2) & b_1/b_2 > 2 \\ 2\min(b_1,b_2) & b_2/b_1 > 2 \\ \max(b_1,b_2) & \text{otherwise} \\ 0 & \end{array}\right\} & \begin{array}{l} b_1b_2 > 0 \\ \\ \\ b_1b_2 < 0 \end{array} \end{array} \right\} \quad (6.15)$$

for longitudinal transfers, (where now ν_1, ν_2 have to be below 0.3 for LB), while for lateral transfers

$$B(b_1,b_2) = \left\{ \begin{array}{ll} \text{minmod}(b_1,b_2) & b_1b_2 > 0 \\ 0 & b_1b_2 < 0 \end{array} \right\} \quad (6.16$$

These could be termed 2-D Superbee.

The anti-diffusive limiters are of principal interest in the case of contact discontinuities since, for shocks, compression is automatically built in to the algorithm via accuracy and conservation and safe limiters are sufficient. Sweby [8] gives a full discussion of many of these points.

7. Scalar Shock Recognition

Much of the above analysis is written down for one-signed values of ν_1, ν_2, but there is little or no difficulty in formally extending the procedures to general ν_1, ν_2. Essentially all that is needed is careful respect of the sign of ν when assigning increments and making transfers and the replacement of ν in coefficients by $|\nu|$. Of course not all the results in previous sections automatically go through: each needs to be checked individually. Also, conditions like $b_1 b_2 < 0$ in (5.1) will be satisfied at shocks and expansions, causing further variations in conditions and results (see [9]).

A significant property of the first order scheme in one dimension (see [3]) is its capability of recognising an isolated shock by matching the shock condition exactly, in the sense described below. We now show that this property is retained in two dimensions with the present scheme.

The first order scheme may be summarised in the steps

$$\left.\begin{array}{l}\text{1. Compute } \Phi \\ \text{2. Increment } u \text{ at the downwind point by } \Phi .\end{array}\right\} \qquad (7.1)$$

In one dimension

$$\Phi = -\frac{\Delta t}{\Delta x}(F_D - F_A) \qquad (7.2)$$

(see Fig. 2 and equations (2.6), (2.7)). If the data arises from a simple shock wave between the points D and A moving with speed S_x and separating constant states u_D and u_A, then the flux per unit width swept out in time Δt is

$$-S_x \Delta t\,(u_D - u_A) \qquad (7.3)$$

which, because of the jump relation

$$(F_D - F_A) = S_x\,(u_D - u_A), \qquad (7.4)$$

reduces to the Φ of (7.2), Thus the first step of (7.1) calculates the correct flux in this case as well as in the case of smooth data. After step 2 of (7.1) has been taken we may say that the scheme moves the shock at the correct shock speed S_x.

Suppose now that a shock separating constant states u_L and u_R moves across a two-dimensional region with speed S at an angle β to the direction of the grid (see Fig. 8). The shock conditions are

$$\left.\begin{aligned} \Delta_n \mathcal{F} &= S\Delta_n u \\ \Delta_n \mathcal{G} &= 0 \end{aligned}\right\} \qquad (7.5)$$

where $\mathcal{F}$ and $\mathcal{G}$ are the components of (F,G) perpendicular and parallel to the shock, and Δ_n refers to differencing in the n direction, perpendicular to the shock i.e. $\Delta_n u = u_R - u_L$

The part of the shock between P and Q sweeps out flux at the rate

$$\begin{aligned} & - S\Delta t \, PQ \, \Delta_n u \\ = & - \Delta t \, (\Delta x \sin \beta + \Delta y \cos \beta)\Delta_n \mathcal{F} \end{aligned} \qquad (7.6)$$

by the jump relation

$$\Delta_n \mathcal{F} = S \, \Delta_n u \qquad (7.7)$$

and the geometric relation

$$PQ = \Delta x \sin \beta + \Delta y \cos \beta \; ; \qquad (7.8)$$

here $\beta < \tan^{-1} (\Delta y/\Delta x)$.

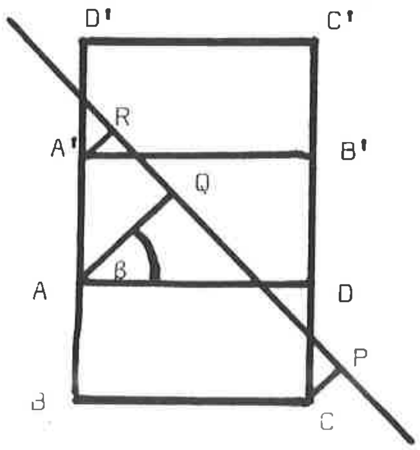

Fig. 8

To relate $\mathcal{F}$ to F and G we use

$$F = \mathcal{F}\cos \beta - \mathcal{G}\sin \beta \, , \qquad G = \mathcal{F}\sin \beta + \mathcal{G}\cos \beta \qquad (7.9)$$

so that

$$\Delta_n F = S \cos \beta \, \Delta_n u, \quad \Delta_n G = S \sin \beta \, \Delta_n u \qquad (7.10)$$

since, across the shock, $\Delta_n \mathcal{G} = 0$.

The flux rate (7.6) now becomes

$$\begin{aligned} & - \Delta t(\Delta x \Delta_n G + \Delta y \Delta_n F) \\ & = - \Delta t[\Delta x(G_D - G_C) + \Delta y(F_D - F_A)] \quad , \end{aligned} \qquad (7.12)$$

namely, the Φ of (2.6) in this particular case.

The part of the shock between Q and R sweeps out flux at a rate

$$\begin{aligned} & - S\Delta t \; QR\Delta_n u \\ & = - \Delta t \; \Delta y \cos \beta \; \Delta_n \mathcal{F} \\ & = - \Delta t \; \Delta y \; \Delta_n F \\ & = - \Delta t \; \Delta y \; (F_{D'} - F_{A'}) \quad , \end{aligned} \qquad (7.12)$$

using (7.10) and the fact that

$$QR = \Delta y \cos \beta \quad : \qquad (7.13)$$

(7.12) is again the Φ of (2.6) in this case.

In each of the cases above u receives increments at D and at D' in accordance with step 2 of (7.1) which effectively move the shock with the correct speed S.

The part of the shock between R and S is treated as for PQ and essentially PR and QR are the only types of shock crossing that can occur. The part of the shock in the rectangle A'B'C'D' of Fig. 8 is covered by QR and RS. If $\beta < \tan^{-1}(\Delta y/\Delta x)$ and the shock crosses one of the rectangles from left to right completely, the situation is as in (7.12) above with x, G replacing y, F.

In all cases, therefore, the Φ of (2.6) not only estimates the flux for smooth flows but also calculates it exactly for oblique shocks separating constant states. Step 2 of (7.1) then ensures that the shock is advanced at the correct speed S in accordance with the jump condition. With regard to the angle β, it was argued in [2] that a scalar shock should be aligned with contours of u: hence β may be obtained from a contouring routine. However, apart from a possible tuning of the algorithm in which the angle θ of (3.3) is chosen relative to β, it is not necessary to know β for the scalar shock recognition argument to hold.

For time-split schemes as the result of commuting steps 1 and 2 of (7.1), values of u in front of the shock are updated incorrectly: the shock will be distorted and may lose its linear character, although because of conservation it will on average move with the correct speed.

8. Shock Recognition for Systems

Turning to systems of conservation laws in two dimensions, of the form

$$\underline{u}_t + \underline{F}_x + \underline{G}_y = 0, \tag{8.1}$$

we aim for a decomposition of the system into a number of independent scalar equations, each of which can be solved by the procedures described above and the contributions superimposed. In one dimension Roe [3] has shown that the essential features of shock capturing and tracking can be obtained from an approximate Jacobian $\tilde{A}$, called the Roe matrix, which is an extension of one of the Jacobian matrices A,B appearing in the version

$$\underline{u}_t + A\underline{u}_x + B\underline{u}_y = 0 \tag{8.2}$$

of (8.1), where A,B are given by

$$A = \frac{d\underline{F}}{d\underline{u}}, \qquad B = \frac{d\underline{G}}{d\underline{u}} \quad . \tag{8.3}$$

In Roe's scheme $\tilde{A}$ is a function of two discrete $\underline{u}$ values, $\underline{u}_L$ and $\underline{u}_R$, with a number of crucial properties, most importantly the shock recognition property

$$\tilde{A}(u_L,u_R)(\underline{u}_L-\underline{u}_R) = \underline{F}_L - \underline{F}_R \tag{8.4}$$

which is used as follows. By the expansion

$$\underline{u}_L - \underline{u}_R = \sum_i \alpha_i \tilde{\underline{e}}_i \quad , \tag{8.5}$$

where $\tilde{\underline{e}}_i$ are the eigenvectors of $\tilde{A}(u_L,u_R)$ and α_i are coefficients, we have from (8.4)

$$\underline{F}_L - \underline{F}_R = \sum_i \tilde{\lambda}_i \alpha_i \tilde{\underline{e}}_i \tag{8.6}$$

where $\tilde{\lambda}_i$ are the eigenvalues of $\tilde{A}(u_L,u_R)$. Each component $\alpha_i \tilde{e}_i$ in (8.5) is updated as an approximate solution of the differential equation

$$\frac{\partial(\alpha_i e_i)}{\partial t} + \tilde{\lambda}_i \frac{(\partial\alpha_i e_i)}{\partial x} = 0, \qquad (8.7)$$

the resulting increments being brought together using (8.5).

In two dimensions this idea has been used via time-splitting [11], [12] to achieve very good results in difficult problems involving the Euler equations. Here we seek a more truly two-dimensional approach.

The key relationship in Roe's approach is the condition

$$\tilde{\lambda}_i(\underline{u}_L - \underline{u}_R)_i = (\underline{F}_L - \underline{F}_R)_i \qquad (8.8)$$

where the subscript i refers to the $\underline{e}_i$ component of the vector concerned. It is (8.8) which tells us that, in one dimension for a simple shock wave moving with speed λ_i, shocks are recognised exactly (c.f. §7).

To extend the idea to systems in two dimensions we may use the rotated grid idea together with a suitable Roe Matrix. In the rotated co-ordinates (c.f.(7.5)) a system like (8.2) which is invariant under co-ordinate rotations becomes

$$\underline{u}_t + \mathcal{A}\underline{u}_n = 0, \qquad (8.9)$$

where

$$\mathcal{A} = A\cos\beta + B\sin\beta \quad , \qquad (8.10)$$

so that we must evidently look for $\tilde{\mathcal{A}}$ with the property

$$\tilde{\mathcal{A}}(\underline{u}_L - \underline{u}_R) = \mathcal{F}_L - \mathcal{F}_R \quad . \qquad (8.11)$$

(c.f. (7.4), (8.5) and (8.6)).

We illustrate with the Euler equations for a perfect gas (c.f. [3]), for which, in the usual notation, $\underline{u} = (\rho,\rho u,\rho v,e)^T$, $\underline{F} = (\rho u,\ p+\rho u^2,\ \rho uv,\ u(p+e))^T$, $\underline{G} = (\rho v,\ \rho uv,\ p+\rho v^2, v(p+e))$, and $e = \frac{p}{\gamma-1} + \frac{1}{2}\rho q^2$, $(q^2 = u^2+v^2)$. Using the enthalpy $H = \frac{a^2}{\gamma-1} + \frac{1}{2}q^2$. the eigenvectors of $\mathcal{A}$ are

$$\underline{e}_{1,2} = \begin{bmatrix} 1 \\ u \pm a\cos\beta \\ v \pm a\sin\beta \\ H \pm a(u\cos\beta + v\sin\beta) \end{bmatrix} \quad \underline{e}_3 = \begin{bmatrix} 0 \\ a\sin\beta \\ -a\cos\beta \\ a(u\sin\beta - v\cos\beta) \end{bmatrix} \quad \underline{e}_4 = \begin{bmatrix} 1 \\ u \\ v \\ \frac{1}{2}q^2 \end{bmatrix} \qquad (8.12)$$

(where a is the speed of sound) and the corresponding eigenvalues are

$$\lambda_{1,2} = u\cos\beta + v\sin\beta \pm a, \qquad \lambda_3 = \lambda_4 = u\cos\beta + v\sin\beta \quad . \qquad (8.13)$$

These correspond respectively to two Mach waves, a slip line and a contact.

Replacing ρ, u, v, H in $\mathcal{A}$ by $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{H}$, where

$$f = \frac{\rho_L^{\frac{1}{2}} f_L + \rho_R^{\frac{1}{2}} f_R}{\rho_L^{\frac{1}{2}} + \rho_R^{\frac{1}{2}}} , \tag{8.14}$$

as in [3], leads to $\tilde{\mathcal{A}}$ with the property (8.11): the eigenvectors and eigenvalues of $\tilde{\mathcal{A}}$ are (8.12), (8.13) with the same replacements. For further details, see the Appendix.

The procedure we propose is as follows. For each pair of points P,Q at the extremes of a side of one of the rectangles of Fig. 2, we evaluate

$$\Delta \underline{u} = \underline{u}_Q - \underline{u}_P . \tag{8.15}$$

This is then projected onto the eigenvectors (8.12) with the quantities u, v, H taking their average values given by (8.14): (a^2 is given by (A44) of the Appendix). The coefficients are $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ where

$$\left.\begin{aligned} \sigma_1 &= \tfrac{1}{2}\left[\frac{\Delta p}{a^2} - \frac{\rho\Delta}{a}(u\cos\beta + v\sin\beta)\right] \\ \sigma_2 &= \tfrac{1}{2}\left[\frac{\Delta p}{a^2} + \frac{\rho\Delta}{a}(u\cos\beta + v\sin\beta)\right] \\ \sigma_3 &= \frac{\rho}{a}\Delta(u\sin\beta - v\cos\beta) \qquad \sigma_4 = \Delta\rho - \frac{\Delta p}{a^2} \end{aligned}\right\} \tag{8.16}$$

all values being average values and p being given by

$$\frac{\gamma p}{\gamma - 1} = \rho H - \tfrac{1}{2}\rho q^2 . \tag{8.17}$$

The projections $\sigma_i \underline{e}_i$ are then treated as follows. For each individual component of one of these projections, say, $\sigma_i e_{ij}$, we take the appropriate Φ_{PQ} occurring in (2.6) to be

$$\left(\frac{\Delta t}{\Delta x}\right) \lambda_i \sigma_i e_{ij} \tag{8.18}$$

and increment u_j accordingly. If the angle β is ignored, as in time-split schemes, then apart from possible errors in the shock speeds, as discussed in the previous section, incorrect physical situations are described. For example, an oblique plane Mach wave, with $\underline{u}_L - \underline{u}_R$ proportional to $\underline{e}_1$, if decomposed into components along the eigenvectors corresponding to $\beta = 0$, gives

$$\tfrac{1}{2}(1-\cos\beta)\begin{bmatrix}1\\u-a\\v\\H-ua\end{bmatrix} + \sin\beta\begin{bmatrix}0\\0\\a\\av\end{bmatrix} + \tfrac{1}{2}(1-\cos\beta)\begin{bmatrix}1\\u+a\\v\\H+ua\end{bmatrix} \qquad (8.19)$$

and appears to be composed of two Mach waves and a slip line. I am indebted to P.L. Roe for this example.

There remains the question of the angle β which, in the case of systems, is not uniquely defined. In ref. [9] Davis carries the suggestion that, in the case of a shock, the angle β should be chosen such that the tangential component of the velocity is continuous. This is easy to implement within the details of the present scheme since, for any side AD, an angle β is obtained from

$$\tan\beta = \frac{v_D - v_A}{u_D - u_A} \qquad (8.20)$$

and decomposition of the local fluctuation onto the eigenvectors of $\tilde{\mathcal{A}}$ follows at once.

If the discontinuity is a slip line the decomposition is redundant but the β orientation (or its 90^0 rotation) is still appropriate.

9. To summarise, the scheme proposed here for a system of equations consists of:-

(i) the evaluations of fluctuations $F_D - F_A$, $G_D - G_C$ etc. (see Fig. 5)

(ii) calculation of their projections onto the eigenvectors of $\tilde{\mathcal{A}}$, (c.f. (8.12)).

(iii) a first order directed scheme for each projection (see (7.1)).

(iv) higher order or anti-diffusive transfers for each projection.

(v) recombination if increments.

10. Numerical Results

11. Conclusion

In this report we have considered a fully two-dimensional scheme for conservation laws which is either second order accurate everywhere or anti-diffusive and second order accurate except close to extrema. This scheme has an LB property from which it is easy to deduce a maximum principle: approximations therefore remain bounded.

The capturing of two-dimensional shocks with such a scheme has been analysed and a suitable wave-splitting technique for systems of two-dimensional conservation laws has been described.

Finally, we consider some of the advantages and disadvantages of the scheme. The difficulties of shock capture in two dimensions have already been discussed. The limitation of the scheme to regular grids is a strong disadvantage at present but it is hoped that the scheme may be adapted for orthogonal curvilinear grids. Also there are hopes that a multigrid method may readily be attached to the method in the case of steady problems.

The case for a fully two-dimensional scheme rests on rotational accuracy and a better treatment of oblique shocks. A time-splitting technique, although offering slightly larger time steps, may get oblique shock speeds wrong, and a wave-splitting version of such a scheme will, as we have seen, process components in an inappropriate way physically.

Acknowledgement

I am indebted to Professor P.L. Roe for useful discussions.

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APPENDIX

In this Appendix we construct the two-dimensional wave decomposition for the Euler equations. First we seek $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ such that

$$\begin{bmatrix} \delta\rho \\ \delta(\rho u) \\ \delta(\rho v) \\ \delta e \end{bmatrix} = \sigma_1 \begin{bmatrix} 1 \\ u-ac \\ v-as \\ H-ua \end{bmatrix} + \sigma_2 \begin{bmatrix} 1 \\ u+ac \\ v+as \\ H+ua \end{bmatrix} + \sigma_3 \begin{bmatrix} 0 \\ as \\ -ac \\ -aV \end{bmatrix} + \sigma_4 \begin{bmatrix} 1 \\ u \\ v \\ \frac{1}{2}q^2 \end{bmatrix} \tag{A1}$$

(see §8) where $c = \cos\beta$, $s = \sin\beta$, $U = u\cos\beta + v\sin\beta$, $V=-u\sin\beta+v\cos\beta$ (A2)

i.e. $$\delta\rho = \sigma_1 + \sigma_2 + \sigma_4 \tag{A3}$$

$$\delta(\rho u) = \sigma_1(u-ac) + \sigma_2(u+ac) + \sigma_3 as + \sigma_4 u \tag{A4}$$

$$\delta(\rho v) = \sigma_1(v-as) + \sigma_2(v+as) - \sigma_3 ac + \sigma_4 v \tag{A5}$$

$$\delta e = \sigma_1(H-Ua) + \sigma_2(H+Ua) - \sigma_3 aV + \sigma_4 \tfrac{1}{2}q^2. \tag{A6}$$

From (A 3) and (A 4) $$\delta(\rho u) = u\delta\rho - (\sigma_1-\sigma_2)ac + \sigma_3 as \tag{A7}$$
and, similarly, $$\delta(\rho v) = v\delta\rho - (\sigma_1-\sigma_2)as - \sigma_3 ac \tag{A8}$$

which leads to

$$-(\sigma_1-\sigma_2)ac + \sigma_3 as = \rho\delta u + O(\delta^2) \tag{A9}$$

$$-(\sigma_1-\sigma_2)as - \sigma_3 ac = \rho\delta v + O(\delta^2). \tag{A10}$$

Solving for $-(\sigma_1-\sigma_2)$ and σ_3,

$$-(\sigma_1-\sigma_2) = \rho\delta U/a + O(\delta^2) \tag{A11}$$

$$\sigma_3 = -\rho(\delta V)/a + O(\delta^2) \quad . \tag{A12}$$

Now consider (A6). Using

$$H = \frac{a^2}{\gamma-1} + \tfrac{1}{2}q^2, \quad e = \frac{p}{\gamma-1} + \tfrac{1}{2}\rho q^2 \tag{A13}$$

we have

$$\frac{1}{\gamma-1}\,\delta p + \tfrac{1}{2}\delta(\rho q^2) = (\sigma_1+\sigma_2)\left(\frac{a^2}{\gamma-1} + \tfrac{1}{2}q^2\right) - (\sigma_1-\sigma_2)Ua - \sigma_3 aV + \sigma_4\tfrac{1}{2}q^2. \quad \text{(A14)}$$

which, using (A11) and (A12) reduces to

$$(\sigma_1+\sigma_2)\left(\frac{a^2}{\gamma-1} + \tfrac{1}{2}q^2\right) + \sigma_4\tfrac{1}{2}q^2 = \frac{1}{\gamma-1}\,\delta p + \tfrac{1}{2}q^2\delta\rho + O(\delta^2), \quad \text{(A15)}$$

(A15) and (A3) may then be solved for $\sigma_1 + \sigma_2$ and σ_4, yielding

$$\sigma_1 + \sigma_2 = \frac{\delta p}{a^2} + O(\delta^2) \quad \text{(A16)}$$

$$\sigma_4 = \delta\rho - \frac{\delta p}{a^2} + O(\delta^2). \quad \text{(A17)}$$

From (A11) and (A17) we find

$$\sigma_1 = \tfrac{1}{2}\left(\frac{\delta p}{a^2} - \frac{\rho\delta U}{a}\right) + O(\delta^2) \quad \text{(A18)}$$

$$\sigma_2 = \tfrac{1}{2}\left(\frac{\delta p}{a^2} + \frac{\rho\delta U}{a}\right) + O(\delta^2) \quad . \quad \text{(A19)}$$

We now seek average values of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, of the form

$$\tilde{\sigma}_1 = \tfrac{1}{2}\left(\frac{\Delta p}{\tilde{a}^2} - \frac{\tilde{\rho}\Delta U}{\tilde{a}}\right), \quad \tilde{\sigma}_2 = \tfrac{1}{2}\left(\frac{\Delta p}{\tilde{a}^2} + \frac{\tilde{\rho}\Delta U}{\tilde{a}}\right),$$

$$\tilde{\sigma}_3 = \frac{\tilde{\rho}}{\tilde{a}}\,V, \quad \tilde{\sigma}_4 = \Delta\rho - \frac{\Delta p}{\tilde{a}^2} \quad . \quad \text{(A20)}$$

such that

$$\Delta \underline{u} = \sum \tilde{\sigma}_i\, \underline{\tilde{e}}_i$$
$$\Delta \underline{\mathcal{F}} = \sum \tilde{\lambda}_i\, \tilde{\sigma}_i\, \underline{\tilde{e}}_i \quad \text{(A21)}$$

(c.f. 8.12), where

$$\tilde{\lambda}_1 = \tilde{U} - \tilde{a}, \quad \lambda_2 = \tilde{U} + \tilde{a}, \quad \tilde{\lambda}_3 = \tilde{\lambda}_4 = \tilde{U} \quad \text{(A22)}$$

In what follows all quantities except those involving Δ will be assumed to take these average values.

Now

$$\underline{\mathcal{F}} = \underline{F}\cos\beta + \underline{G}\sin\beta = \begin{bmatrix} \rho U \\ pc + \rho u U \\ ps + \rho v U \\ (p+e)U \end{bmatrix} \tag{A 23}$$

and to satisfy (A21) we need both

$$\Delta\rho = \sigma_1 + \sigma_2 + \sigma_4 \tag{A24}$$

$$\Delta(\rho u) = (u-ac)\sigma_1 + (u+ac)\sigma_2 + as\sigma_3 + u\sigma_4 \tag{A25}$$

$$\Delta(\rho v) = (v-as)\sigma_1 + (v+as)\sigma_2 - ac\sigma_3 + v\sigma_4 \tag{A26}$$

$$\Delta e = (H-aU)\sigma_1 + (H+aU)\sigma_2 - aV\sigma_3 + \tfrac{1}{2}q^2\sigma_4 \tag{A27}$$

and

$$\Delta(\rho U) = (U-a)\sigma_1 + (U+a)\sigma_2 + U\sigma_4 \tag{A28}$$

$$\Delta(pc+\rho uU) = (U-a)\sigma_1(u-ac) + (U+a)\sigma_2(u+ac) + U(\sigma_3 as+\sigma_4 u) \tag{A29}$$

$$\Delta(ps+\rho vU) = (U-a)\sigma_1(v-as)+(U+a)\sigma_2(v+as) + U(-\sigma_3 ac+\sigma_4 v) \tag{A30}$$

$$\Delta((p+e)U) = (U-a)\sigma_1(H-aU) + (U+a)\sigma_2(H+aU) + U\sigma_3 aV + U\sigma_4\tfrac{1}{2}q^2 \tag{A31}$$

(A24) is automatically satisfied and (A25) x cosβ + (A26) x sinβ gives (A28), which is

$$\Delta(\rho U) = \rho\Delta U + U\Delta\rho \tag{A32}$$

Similarly (A25) x sinβ + (A26) x cosβ gives

$$\Delta(\rho V) = \rho\Delta V + V\Delta\rho \ . \tag{A33}$$

(A27) can be re-arranged to give

$$\Delta(\rho V^2) = 2\rho V\Delta V + V^2\Delta\rho \ . \tag{A34}$$

We thus have to solve (A29) to (A34). Now (A29) x cosβ + (A30) x sinβ gives

$$\Delta(\rho U^2) = 2\rho U\Delta U + U^2\Delta\rho \tag{A35}$$

and (A29) x sinβ - (A30) x cosβ gives

$$\Delta(\rho UV) = \rho U\Delta V + \rho V\Delta U + UV\Delta\rho \ . \tag{A36}$$

Finally (A31) can be rearranged to give

$$\Delta(\rho UH) = \rho H\Delta U + U\Delta(\rho H). \tag{A37}$$

We now have to solve (A32) to (A37).

(A32) and (A35) taken together yield

$$U = \frac{\sqrt{\rho_L}U_L + \sqrt{\rho_R}U_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{A38}$$