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Finite Differences applied to Stochastic Problems in Pricing Derivatives

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Abstract

We firstly implement various θ-methods for the Black-Scholes model, which is a $1-d$ parabolic partial differential equation. In this project, it is used to price options dependent on a stock that follows a *stochastic process.* By assuming that the volatility is nonconstant and that it follows a *stochastic process*, we introduce an extension to the Black-Scholes equation, which is a $2-d$ partial differential equation. After carrying out a series of transformations, we solve it numerically by applying the *A.D.I. (Alternating Direction Implicit)* method and the *Explicit scheme.* By comparing the results obtained from the $1-d$ and $2-d$ equations for a fixed volatility, we find that, when the stock price is negatively correlated with the volatility, the Black-Scholes equation overestimates the out-of-the-money options and underestimates the in-the-money options. The reverse holds when the correlation is positive.

Contents

1 Introduction

Option pricing has become increasingly important in the field of finance over recent years. The trade in options has grown dramatically during the last two decades and huge volumes of options are traded globally on exchanges (such as the London International Financial Futures exchange, the American Stock Exchange and the European Options Exchange) and over the counter by banks and financial institutions. A major breakthrough in the field of option pricing has been the work of Black and Scholes in 1973 [1]. Their model is based upon the assumption that the 'instantaneous' rate of return of the underlying asset cannot be predicted, and thus they describe this rate as a random variable following a *stochastic process* known as *Brownian Motion* (processes of this kind are observed in the behaviour of many natural phenomena studied in Physics and Biology). Moreover, the *volatility* of the rate of return (i.e. the standard deviation from the average value) is taken to be constant. However, this last remark has been the subject of much attention. Empirical analysis of stock volatility has shown that it is not constant and that prices at which derivatives (especially *call* options) are traded are inconsistent with a constant volatility assumptiom.

For this reason, Garman[3] and Cox, Ingersoll and Ross[2] suggest an extension to the Black and Scholes model. It is expressed by a $2-d$ partial differential equation and the volatility obeys a stochastic process. Based on these findings, Hull and White [8] derive an expression in series form for the pricing bias caused by a stochastic volatility. This bias is the amount by which the actual option price when the volatility is stochastic exceeds the Black and Scholes price in which the volatility is taken to be constant and equal to its initial value.

The aim of this project is to apply various numerical schemes to both the $1-d$ Black-Scholes equation and the $2-d$ extension of it, which are used to price *European call* and *put* options on stocks. For the $1-d$ problem, a closed form solution exists and therefore we shall examine the results obtained from the different numerical techniques in comparison to the analytic solution and the impact of the boundary conditions on these approximations. Secondly, we shall implement the **A.D.I method (Alternating Direction Implicit)** for the $2-d$ differential equation and compare these results with the *Explicit* scheme. For these purposes, we have to transform the original equation into one without a mixed derivative. However, this brings up some difficulties which will be discussed later on.

Before moving on to explaining the numerics, it is necessary to provide some preliminaries on financial terms and definitions.

1.1 Preliminaries

In general, a **derivative product** depends for its value on the value of some other asset or assets. The simplest sort of derivative product is an **option**. Depending on whether we are dealing with **call** or **put** options, an option gives the right to buy (call) or to sell (put) an asset subject to certain conditions within a specified period of time.

- The price that is paid for the asset when the option is exercised is called the **exercise price** or **strike price**.

- The last day that this transaction takes place is called the **expiration date** or **maturity date**.

Hence, an **American option** is the one that can be exercised at any time up to the date the option expires, whilst a **European option** can only be exercised on the expiry date. In this project we will be examining the pricing of European options on stocks.

Normally, no shares are actually bought or sold (though they can be). The option acts like a straight bet between the **writer** and the **holder**. Initially, the holder pays the writer for the option. If the holder does not exercise at maturity then no further money changes hands. If the holder does exercise then the writer has to pay the holder what he owes him. The writer, however, has an obligation to settle up at expiry; the holder has no obligation to exercise.

For example, if one bought a European call option at an exercise price of £2.50 and at maturity the stock is valued at £2, then the option would not be exercised. If, on the other hand, the stock has a price of £2.80, the holder would wish to exercise the option. So, the issue here is how much to charge the holder for the option and hence how the writer can eliminate risk since he is technically exposed to possible infinite losses.

In relation to time, if the expiration date of the option is very far in the future, then the value, for example, of a call option will be approximately equal to the price of the stock. When the expiry date is very near, the option price reaches approximately its minimum value, which is the stock price minus the exercise price, or zero. Figures (2.1.1) and (2.1.2) show the behaviour of the option value and the stock price for different maturities.

Consider also the following terms:

- When stock price > exercise price, we say that the option is **in-the-money** and hence it gives the holder a positive cash flow.
- When stock price = exercise price, the option is said to be **at-the-money**.
- When stock price < exercise price, the option is said to be **out-of-the-money**.

Lastly, the **volatility** of the stock price is a measure of how uncertain we are about future stock price movements. As volatility increases, the chance that the stock will do very well or very poorly increases.

2 Black-Scholes Model

As already mentioned, an important discovery for the financial world was introduced by Black and Scholes in 1973 [1], who derived a $1-d$ backward parabolic differential equation. It is defined by

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S = rV \tag{1}$$

and it models the price of an option V on a stock S (for example), with volatility σ and riskfree interest rate r. Here, the stock price follows the stochastic process

$$dS = \mu S dt + \sigma S dX,$$

where $\mu = \mu(S,t)$ and $\sigma = \sigma(S,t)$ are the expected growth rate (drift) and volatility of S respectively and dX is a random variable drawn from a normal distribution with mean 0 and variance dt.

Let $C(S,\tau)$ be the value of a call option satisfying equation (1). The analytic solution for $C(S,t)$ given by Black and Scholes is

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{2}$$

where

$$\begin{aligned} d_1 &= \frac{log(S/E) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{log(S/E) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

and $N(.)$ is the *cumulative distribution function* for a standardised normal random variable, defined by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

(E is the exercise price and T the maturity time of the option).

If $P(S,t)$ represents the value of a put option that satisfies (1), then by the *put-call parity* relationship it holds that

$$S + P - C = Ee^{-r(T-t)}. \tag{3}$$

The final and boundary conditions for a European call option $C(S,t)$ satisfying equation (1) are given by

$$
\begin{aligned}
C(S,T) &= \max(S-E,0) && (4)\\
C(0,t) &= 0 && (5)\\
C(S,t) &\rightarrow S, \quad \text{as } S \rightarrow \infty. && (6)
\end{aligned}
$$

The corresponding conditions for a European put option $P(S,t)$ satisfying (1) are

$$
\begin{aligned}
P(S,T) &= \max(E-S,0) && (7)\\
P(0,t) &= Ee^{-r(T-t)} && (8)\\
P(S,t) &\rightarrow 0, \quad \text{as } S \rightarrow \infty. && (9)
\end{aligned}
$$

In Figures (2.1.1) and (2.1.2) we show plots of the European call and put values for several times up to expiry. Note how the curves approach the payoff functions of S, that is $\max(S-E,0)$ for a call option and $\max(E-S,0)$ for a put option, as $t \rightarrow T$.

There are also other methods available for pricing options without the need for partial differential equations such as the *Monte Carlo* simulation and *lattice* methods [6]. However, pricing through the solution of the differential equation has the advantage of greater generality and also is a faster and more accurate approach.

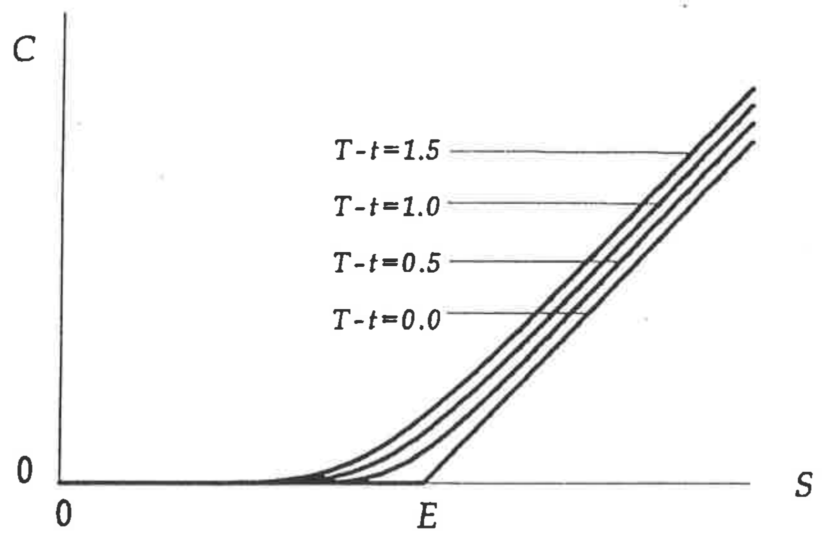

Figure 2.1.1: The European call values $C(S,t)$ as a function of S for several values of time to expiry; $r = 0.1, \sigma = 0.2, E = 1$ and $T - t = 0, 0.5, 1.0, 1.5$.

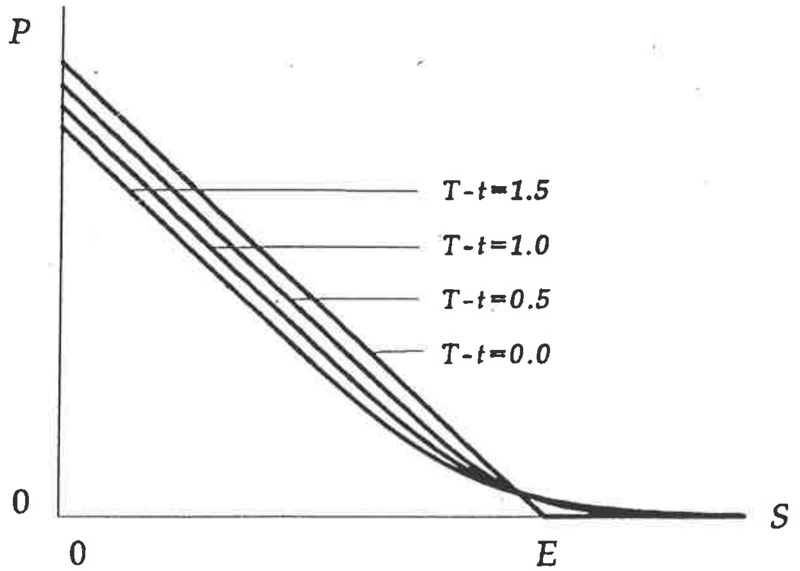

Figure 2.1.2: The European put value $P(S,t)$ as a function of S for several values of time to expiry; $r = 0.1, \sigma = 0.2, E = 1$ and $T - t = 0, 0.5, 1.0, 1.5$.

3 Finite Difference Methods

To derive an approximate solution to the $1-d$ partial differential equation (1), we shall be looking at various θ-methods. The parameter θ basically controls the implicitness of the scheme. However, we will also consider cases in which θ is chosen so that the scheme becomes explicit.

Before introducing these numerical techniques, we need to transform equation (1) by just reversing the time. Hence, by letting $\tau = T - t$ to be the time to maturity of the asset, then

$$\frac{\partial}{\partial \tau} = -\frac{\partial}{\partial t}$$

and thus (1) becomes

$$W_\tau = \frac{1}{2}\sigma^2 S^2 W_{SS} + rSW_S - rW \tag{10}$$

with $V(S,t) = W(S,\tau)$.

Clearly, we cannot solve this problem numerically for all $0 < S < \infty$ without taking an infinite number of S-steps. Instead we consider a finite, but suitably large interval such that

$$0 \leq S^- \leq S \leq S^+,$$

which we will discretise accordingly.

The initial and boundary conditions for a call option $C(S,\tau)$ satisfying (10) now become

$$C(S,0) = \max(S-E,0) \tag{11}$$

$$C(S^-,\tau) = 0 \tag{12}$$

For $S = S^+$, we choose between the following three conditions:

1. A Dirichlet condition defined by

$$C(S^+,\tau) = S^+ - Ee^{-r\tau}. \tag{13}$$

2. A Neumann condition, given by

$$C_S(S^+, \tau) = 0. \tag{14}$$

3. A different condition on the derivative, given by

$$C_S(S^+, \tau) = 1. \tag{15}$$

Similarly, the conditions for a put option $P(S, \tau)$ satisfying (10) are

$$\begin{aligned} P(S,0) &= \max(E-S,0) && (16) \\ P(S^+,\tau) &= 0 && (17) \\ P(S^-,\tau) &= Ee^{-r\tau} - S^- && (18) \end{aligned}$$

The aim here is to show the impact the boundary conditions can have on the approximations. We choose to do that only for call options as the equivalent problem for put options is not much different.

Having set all the necessary conditions, we can discretise the initial boundary value problem using finite difference methods. Let N be the number dividing the interval of S into equally spaced subintervals, such that

$$\begin{aligned} S_i &= i\delta S, \qquad i = 0, \ldots, N, \\ \delta S &= (S^+ - S^-)/N. \end{aligned}$$

Similarly, let M be the number dividing the time interval, such that

$$\begin{aligned} \tau_j &= j\delta\tau, \qquad j = 0, \ldots, M, \\ \delta\tau &= T/M. \end{aligned}$$

The grid used for the numerical schemes is shown in Figure (3.1.1).

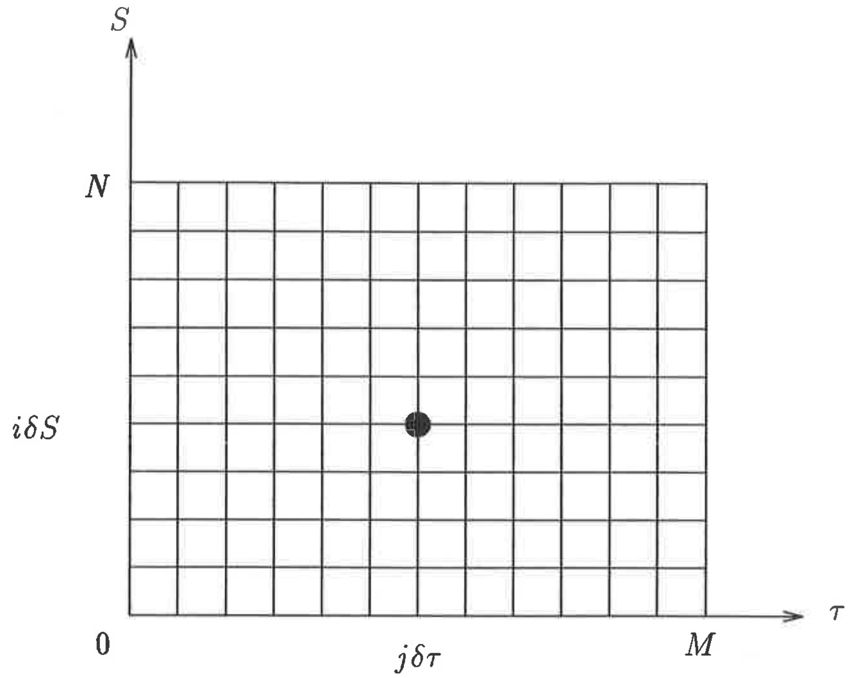

Figure 3.1.1: The mesh for a finite difference approximation

For an interior point (i,j) on the grid, $\frac{\partial W}{\partial S}$ can be approximated by a *central difference*, given by

$$\frac{\partial W}{\partial S} \approx \frac{W_{i+1}^{j} - W_{i-1}^{j}}{2\delta S}.$$

For $\frac{\partial W}{\partial \tau}$, we use the *forward difference* approximation

$$\frac{\partial W}{\partial \tau} \approx \frac{W_{i}^{j+1} - W_{i}^{j}}{\delta \tau}.$$

Lastly, $\frac{\partial^2 W}{\partial S^2}$ is approximated by

$$\frac{\partial^2 W}{\partial S^2} \approx \frac{W_{i-1}^{j} - 2W_{i}^{j} + W_{i+1}^{j}}{(\delta S)^2}.$$

Hence, the generalised θ-methods take the form

$$
\begin{aligned}
\frac{W_i^{j+1} - W_i^j}{\delta\tau} &= \frac{1}{2}\sigma^2 S_i^2\Big[\theta_1\Big(\frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{(\delta S)^2}\Big) + \theta_2\Big(\frac{W_{i-1}^{j} - 2W_i^{j} + W_{i+1}^{j}}{(\delta S)^2}\Big)\Big] \\
&\quad rS_i\Big[\theta_3\Big(\frac{W_{i+1}^{j+1} - W_{i-1}^{j+1}}{2\delta S}\Big) + \theta_4\Big(\frac{W_{i+1}^{j} - W_{i-1}^{j}}{2\delta S}\Big)\Big] \\
&\quad -r\Big[\theta_5 W_i^{j+1} + \theta_6 W_i^j\Big]. \qquad (19)
\end{aligned}
$$

For consistency, we require that $\theta_1 + \theta_2 = \theta_3 + \theta_4 = \theta_5 + \theta_6 = 1$. Rearranging (19), we obtain

$$
\begin{aligned}
W_i^{j+1} - W_i^j &= \alpha_i[\theta_1(W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}) + \theta_2(W_{i-1}^{j} - 2W_i^{j} + W_{i+1}^{j})] \\
&\quad +\beta_i[\theta_3(W_{i+1}^{j+1} - W_{i-1}^{j+1}) + \theta_4(W_{i+1}^{j} - W_{i-1}^{j})] \\
&\quad \gamma[\theta_5 W_i^{j+1} + \theta_6 W_i^j], \qquad (20)
\end{aligned}
$$

where

$$
\alpha_i = \frac{1}{2}\sigma^2 S_i^2 \frac{\delta\tau}{(\delta S)^2} > 0, \qquad \beta_i = \frac{1}{2} r S_i \frac{\delta\tau}{\delta S} > 0, \qquad \gamma = -r\delta\tau < 0. \qquad (21)
$$

Rearranging (20), we are left with

$$
c_i W_{i-1}^{j+1} + a_i W_i^{j+1} + b_i W_{i+1}^{j+1} = c_i' W_{i-1}^{j} + a_i' W_i^{j} + b_i' W_{i+1}^{j} \qquad (22)
$$

with coefficients

$$
\begin{aligned}
c_i &= -\alpha_i\theta_1 + \beta_i\theta_3 & \qquad c_i' &= \alpha_i\theta_2 - \beta_i\theta_4 \\
a_i &= 1 + 2\alpha_i\theta_1 - \gamma\theta_5 & \qquad a_i' &= 1 - 2\alpha_i\theta_2 + \gamma\theta_6 \\
b_i &= -\alpha_i\theta_1 - \beta_i\theta_3 & \qquad b_i' &= \alpha_i\theta_2 + \beta_i\theta_4.
\end{aligned}
$$

The problem is then reduced down to solving the system of equations

$$
AW^{j+1} = BW^j + \underline{d}, \qquad (23)
$$

where A and B are the following $(N-1) \times (N-1)$ *tridiagonal* matrices

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_2 & a_2 & b_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & c_{N-2} & a_{N-2} & b_{N-2} \\ 0 & \dots & 0 & c_{N-1} & a_{N-1} \end{pmatrix} \qquad B = \begin{pmatrix} a_1' & b_1' & 0 & \dots & 0 \\ c_2' & a_2' & b_2' & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & c_{N-2}' & a_{N-2}' & b_{N-2}' \\ 0 & \dots & 0 & c_{N-1}' & a_{N-1}' \end{pmatrix}$$

and the vector $\underline{d}$ adjusts the system at each timestep to allow for the Dirichlet boundary conditions (13), if we are pricing call options, and (16) if we are pricing put options;

$$\underline{d} = \begin{pmatrix} -c_1 W_0 + c_1' W_0 \\ 0 \\ \vdots \\ -b_{N-1} W_N + b_{N-1}' W_N \end{pmatrix}.$$

If we consider the Neumann condition (14), we only need to solve $AC^{j+1} = BC^j$, where A and B are now of dimension $N \times N$ ($C(S,\tau)$ is the value of a call option satisfying (10)). The only differences are the values of the coefficients c_N and c_N' of the matrices A and B respectively, which are now replaced by

$$c_N \to c_N + b_N \qquad c_N' \to c_N' + b_N'. \tag{24}$$

This is obtained by approximating the derivative $C_S(S^+, \tau) = 0$ by a central difference. Hence, we get an expression for the 'fictitious' point, say, c_{N+1}, which we then substitude into (24) for $i = N$.

Lastly, for condition (15), we approximate again the derivative by a central difference so that (24) still applies. Thus, we need to solve the system of equations (23) with A and B being $N \times N$ tridiagonal matrices. The last entry of $\underline{d}$ is now $(b_N - b_N')2\delta S$ and all the others are zero.

Solving for put options, the procedure is the same as for the call options but with different initial and boundary conditions.

For the schemes that we will be considering, the θ's take the following values:

Scheme	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
Crank-Nicolson	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Kenneth-Vetzal	1	0	1	0	0	1
Fully Implicit	1	0	1	0	1	0
Semi Implicit	1	0	0	1	1	0
Explicit 1	0	1	0	1	0	1
Explicit 2	0	1	0	1	1	0

Table 3.1.2

The first four schemes, as shown in Table (3.1.2), involve three unknown values of W on the new time level $j+1$. So, it is required to invert the tridiagonal matrix A on the l.h.s of the system of equations (23). The last two schemes considered are explicit. The *Explicit 1* scheme uses three values of W at the current time level j and *Explicit 2* scheme uses two values of W at time level j. Hence, inverting a matrix is not needed and the values of W at the new time level are found explicitly. A natural generalization of an implicit and explicit method is by taking a weighted average of the two. Crank-Nicolson is such a scheme.

We will now continue to analyse the *stability* and *convergence* of the numerical methods.

3.1 Invertibility

Before carrying out any stability analysis, it is necessary first to ensure, in the case of the implicit schemes, that the matrix A is *invertible.* So, consider the following definitions:

Definition

A tridiagonal matrix A is said to be *strictly diagonally dominant (s.d.d)* if and only if

$$| a_i | > | c_i | + | b_i |,$$

where the a_i's are the coefficients on the diagonal and the c_i's and b_i's are the coefficients on the lower and upper diagonal respectively. Then, the matrix A is *non-singular.*

Consider also the mesh *Péclet number* Pe;

$$Pe = \frac{\beta_i}{\alpha_i} = \frac{r\delta S}{\sigma^2 S_i}. \tag{25}$$

We assume that

$$r < \sigma^2 S_i \ \forall i, \tag{26}$$

and hence $Pe < 1$ ($\forall i$) provided that $\delta S < 1$. Condition (26) implies that the differential equation (10) is diffusion dominated.

If we consider, for example, the matrix A corresponding to the *Fully Implicit* scheme, its coefficients are

$$\begin{aligned} c_i &= -\alpha_i + \beta_i \\ a_i &= 1 + 2\alpha_i - \gamma \\ b_i &= -\alpha_i - \beta_i. \end{aligned}$$

Hence, from (26) it follows that c_i, $b_i < 0$, $a_i > 0$ (since $\gamma < 0$) and so

$$1 + 2\alpha_i - \gamma > (\alpha_i - \beta_i) + (\alpha_i + \beta_i)$$

$$\Rightarrow \mid a_i \mid > \mid c_i \mid + \mid b_i \mid .$$

The matrix A is therefore *s.d.d.* and thus *invertible.* However, this is a *sufficient* but not a *necessary* condition for invertibility [14].

Similarly, the matrix A corresponding to *Crank-Nicolson* and *Kenneth-Vetzal* schemes is *s.d.d*, provided that condition (26) is satisfied.

Lastly, for the *Semi-Implicit* scheme, the coefficients of A are

$$\begin{aligned} c_i &= -\alpha_i < 0 \\ a_i &= 1 + 2\alpha_i - \gamma > 0 \\ b_i &= -\alpha_i < 0. \end{aligned}$$

Clearly, in this case A is *s.d.d* without any restriction on the *Péclet number.*

3.2 Stability

We shall now introduce the problem of stability of the finite difference calculations used to solve equation (4) subject to Dirichlet condition (14). If we consider the more general boundary conditions that involve a derivative term, the analysis for stability and accuracy is concentrated on the nodes that lie on the boundary. Here, we will examine the behaviour of the schemes for the simple case.

Let Q^j and R^j be two solutions of the system of equations $AW^{j+1} = BW^j + \underline{d}$, that have the same inhomogeneous term $\underline{d}$ but with different initial data Q^0 and R^0. Then, their difference $W^j = Q^j - R^j$ satisfies the homogeneous system of equations and stability is achieved by establishing that

$$AW^{j+1} = BW^j \quad \text{and} \quad \Rightarrow \|W^j\| \leq K\|W^0\|$$

($\|.\|$ defines the norm). If the constant K is such that $\mid K \mid \leq 1$, then the scheme is said to be *stable*.

Fourier analysis is the most precise and useful tool for studying stability in the l_2 norm. It can be, however, quite restrictive since it can only be applied to linear problems with constant coefficients and periodic boundary conditions, approximated by difference schemes on uniform meshes. Noticing that the problem we are solving has variable coefficients, the method can still be applied locally as long as the stability condition is satisfied at every point in the interior of the domain [12]. Moreover, Fourier stability is, in general, both a sufficient and a necessary condition.

(i)**Fully Implicit**

Substituting $W_i^m = \lambda_m e^{ikj\delta S}$ into (22), we are left with

$$\lambda\{(-\alpha_i + \beta_i)e^{-ik\delta S} + (1 + 2\alpha_i - \gamma) + (-\alpha_i - \beta_i)e^{ik\delta S}\} = 1$$

$$\Rightarrow \{-2\alpha_i \cos(k\delta S) - 2i\beta_i \sin(k\delta S) + 1 + 2\alpha_i - \gamma\} = 1. \qquad (27)$$

Since $\cos(k\delta S) = 1 - 2\sin^2(\frac{k\delta S}{2})$, from (27) it follows that

$$\lambda = \frac{1}{4\alpha_i \sin^2(\frac{k\delta S}{2}) - 2i\beta_i \sin(k\delta S) + 1 - \gamma}$$

$$\Rightarrow |\lambda|^2 = \frac{1}{(1+4\alpha_i s^2-\gamma)^2 + 16\beta_i^2 s^2(1-s^2)}, \tag{28}$$

where $s^2 = \sin^2(\frac{k\delta S}{2}) \in [0,1]$. The quantity λ is the *amplification factor* and for stability we require that $|\lambda| \le 1$. For $s^2 = 0, |\lambda|^2$ attains its maximum value $(\forall i)$; i.e.

$$|\lambda|^2 = \frac{1}{(1-\gamma)^2}.$$

Now, $|\lambda|^2 \le 1$, since $\frac{1}{(1-\gamma)^2} = \frac{1}{(1+r\delta\tau)^2} \le 1$ and so the scheme is *unconditionally stable.*

(ii)**Semi Implicit**

Carrying out the same process as in part (i), the amplification factor in this case is found to be

$$\lambda = \frac{1+2i\beta_i^2\sin(k\delta S)}{1+4\alpha_i\sin^2(\frac{k\delta S}{2})-\gamma}$$

$$\Rightarrow |\lambda|^2 = \frac{1+16\beta_i^2 s^2(1-s^2)}{(1-\gamma+4\alpha_i s^2)^2}. \tag{29}$$

The maximum of $|\lambda|^2$ occurs at the point

$$s^2 = \frac{1-\gamma-\alpha_i/2\beta_i^2}{2(1-\gamma+2\alpha_i)}, \tag{30}$$

provided that

$$1-\gamma-\frac{\alpha_i}{2\beta_i} > 0$$

$$\Rightarrow \sigma^2 < \frac{r^2}{1+r}, \quad \text{since } \delta\tau < 1. \tag{31}$$

Hence, provided that (31) holds, substituting (30) into (29) we get

$$|\lambda|^2 = \frac{\beta_i^2(1+4\beta_i^2)}{\beta_i^2(1-\gamma)^2+4\alpha_i\beta_i^2(1-\gamma)-\alpha_i^2}. \tag{32}$$

For stability, we require that

$$1+4\beta_i^2 \le (1-\gamma)+4\alpha_i(1-\gamma)-\frac{\alpha_i^2}{\beta_i^2}. \tag{33}$$

From the assumption in (31), inequality (33) becomes

$$1+4\beta_i^2 < (1-\gamma)^2 - \frac{\alpha_i^2}{\beta_i^2}$$

and since $\alpha_i^2/\beta_i^2 > 1$ (condition (26)), then

$$4\beta_i^2 < (1-\gamma)^2$$
$$\Rightarrow \delta\tau < \frac{1}{\frac{rS_i}{\delta S} - r}$$
$$\Rightarrow \frac{\delta\tau}{\delta S} < \frac{1}{rS_i - r\delta S}. \quad (34)$$

If, on the other hand, condition (31) does not hold, then the maximum of $|\lambda|^2$ is at the point $s^2 = 0$, which substituted into (29) gives

$$|\lambda|^2 = \frac{1}{(1-\gamma)^2} = \frac{1}{(1+r\delta\tau)^2} \leq 1 \quad (35)$$

and hence the scheme's *unconditional* stability follows immediately.

(iii)**Crank-Nicolson**

The amplification factor is now

$$\lambda = \frac{[1+\gamma/2 - 2\alpha_i \sin^2(\frac{k\delta S}{2})] + i\beta_i \sin(k\delta S)}{[1-\gamma/2 + 2\alpha_i \sin^2(\frac{k\delta S}{2})] - i\beta_i \sin(k\delta S)}.$$

If $1+\gamma/2 - 2\alpha_i \sin^2(\frac{k\delta S}{2}) \geq 0 \ \forall i$, it holds that

$$(1+\gamma/2 - 2\alpha_i \sin^2(\frac{k\delta S}{2})) \leq (1-\gamma/2 + 2\alpha_i \sin^2(\frac{k\delta S}{2}), \quad \forall i.$$

If, on the other hand, α_i and γ are such that $1+\gamma/2 - 2\alpha_i \sin^2(\frac{k\delta S}{2}) < 0$, it follows that

$$-(1+\gamma/2 - 2\alpha_i \sin^2(\frac{k\delta S}{2})) < (1-\gamma/2 + 2\alpha_i \sin^2(\frac{k\delta S}{2}), \quad \forall i.$$

In any case, plotting the numerator and denominator of λ on a real-imaginary plane, then the ratio of the two lies always in (or on) the unit circle. Hence, $|\lambda| \leq 1$ and the scheme is *unconditionally stable.*

(iv)**Kenneth-Vetzal**

The amplification factor is found to satisfy

$$|\lambda|^2 = \frac{(1+\gamma)^2}{(1+4\alpha_i s^2)^2 + 16\beta_i^2 s^2(1-s^2)}$$

For $s^2 = 0$, $|\lambda|^2$ attains its maximum value; i.e.

$$|\lambda|^2 = (1+\gamma)^2 = (1 - r\delta\tau)^2.$$

Provided that

$$1 - r\delta\tau \geq -1 \Rightarrow \delta\tau \leq \frac{2}{r}, \qquad (36)$$

the scheme is *stable*.

(v)**Explicit 1**

Here,

$$|\lambda|^2 = (1+\gamma)^2 + (16\beta_i^2 - 8(1+\gamma)\alpha_i)s^2 + 16(\alpha_i^2 - \beta_i^2)s^4. \qquad (37)$$

By considering (37) as a quadratic function of s^2, for stability we require that it lies in the interval $[0,1]$ by ensuring that the coefficient of s^2 is nonpositive and that $|\lambda|^2 \in [0,1]$ for $s^2 = 0, 1$.

Hence, we need to satisfy

$$\begin{aligned} 2\beta_i^2 - (1+\gamma)\alpha_i \leq 0 \Rightarrow \frac{1}{2}S_i^2 \frac{\delta\tau}{(\delta S)^2}\left(r^2\delta\tau - (1 - r\delta\tau)\sigma^2\right) &\leq 0 \\ \Rightarrow \delta\tau &\leq \frac{\sigma^2}{r^2 + r\sigma^2}. \end{aligned} \qquad (38)$$

Substituting $s^2 = 0$ into (37) then $|\lambda|^2 = (1+\gamma)^2$. Thus, we need

$$\begin{aligned} -1 - \gamma &\geq 1 \\ \Rightarrow \delta\tau &\leq \frac{2}{r}. \end{aligned} \qquad (39)$$

Moreover, substituting $s^2 = 1$ into (37) we get $|\lambda|^2 = (1 + \gamma - 4\alpha_i)^2$ and requiring $|\lambda|^2 \leq 1$ then

$$\begin{aligned} 4\alpha_i - \gamma - 1 &\leq 1 \\ \Rightarrow \frac{\delta\tau}{(\delta S)^2} &\leq \frac{1}{(\sigma^2 S_i^2 + \frac{r(\delta S)^2}{2})}. \end{aligned} \qquad (40)$$

From condition (26), it can be shown that the conditions (38) and (40) are equivalent.

(vi) **Explicit 2**

The amplification factor is now

$$| \lambda |^2 = \frac{(1-4\alpha_i s^2)^2 + 16\beta_i^2 s^2(1-s^2)}{(1-\gamma)^2}.$$

Since $(1-\gamma)^2 \geq 1$, it follows that

$$\frac{(1-4\alpha_i s^2)^2 + 16\beta_i^2 s^2(1-s^2)}{(1-\gamma)^2} \leq (1-4\alpha_i s^2)^2 + 16\beta_i^2 s^2(1-s^2).$$

Hence, it is sufficient to show that the quadratic $1 + (16\beta_i^2 - 8\alpha_i)s^2 + 16(\alpha_i - \beta_i)s^4$ lies in $[0,1]$. Using the same principles mentioned in part (v) it follows that

$$2\beta_i^2 - \alpha_i \leq 0 \qquad \Rightarrow \delta\tau \leq \frac{\sigma^2}{r^2}.$$

Also, require that $| \lambda |^2 \leq 1$ for $s^2 = 1$; i.e

$$(1-4\alpha_i)^2 \leq 1 \qquad \Rightarrow \frac{\delta\tau}{(\delta S)^2} \leq \frac{1}{\sigma^2 S_i^2}. \tag{41}$$

3.3 Convergence

The next issue that we need to examine is the matter of *convergence*, which consists of finding the conditions under which

$$W_{apprx}(S^*, \tau^*) - W_{exact}(S^*, \tau^*) \to 0$$

i.e the difference between the analytic and the numerical solutions of the differential and difference equations at a *fixed* point (S^*, τ^*) tend to zero uniformly as $\delta S, \delta\tau \to 0$ and $i, j \to \infty$, with $i\delta S (= S^*)$ and $j\delta\tau (= \tau^*)$ remaining fixed ((S^*, τ^*) lies in the domain $(S^-, S^+) \times (0, T)$).

We start by looking at convergence for the *Explicit 1* scheme.

(i)**Explicit 1**

Let $e_i^j = W_{apprx}(S^*, \tau^*) - W_{exact}(S^*, \tau^*)$. Now, $W_{apprx}(S^*, \tau^*)$ satisfies equation (22) exactly (for the appropriate choice of the θ's), while $W_{exact}(S^*, \tau^*)$ leaves as a remainder the *truncation error* $\rho_i^{j+1}\delta\tau$. Thus, the error is determined from the relations

$$e_i^{j+1} = (\alpha_i - \beta_i)e_{i-1}^j + (1 - 2\alpha_i + \gamma)e_i^j + (\alpha_i + \beta_i)e_{i+1}^j + \delta\tau\rho_i^{j+1} \tag{42}$$

for $i = 1, \ldots, N-1$ and $j = 0, \ldots, M-1$. We assume that the coefficients on the r.h.s of (42) are nonnegative. Thus, $\alpha_i - \beta_i \geq 0$, which is satisfied by our initial assumption (26) and also $1 - 2\alpha_i + \gamma \geq 0$. Hence,

$$\frac{\delta\tau}{(\delta S)^2} \leq \frac{1}{\sigma^2 S_i^2 - r(\delta S)^2}, \tag{43}$$

which is slightly more restrictive than the Fourier stability condition (40). We show that by assuming the coefficients on the r.h.s of (42) to be nonnegative are sufficient conditions for convergence.

Using the triangle inequality we deduce that

$$\mid e_i^{j+1} \mid \leq (\alpha_i - \beta_i) \mid e_{i-1}^j \mid + (1 + 2\alpha_i - \gamma) \mid e_i^j \mid + (\alpha_i + \beta_i) \mid e_{i+1}^j \mid + \delta\tau\rho_i^{j+1}. \tag{44}$$

From the boundary conditions, we have that $e_i^0 = e_0^j = e_N^j = 0$ and we define

$$E^j = max_{0 \leq i \leq N} \mid e_i^j \mid$$
$$\overline{\rho} = max_{1 \leq i \leq N-1} \mid \rho_i^{j+1} \mid$$

Equation (44) becomes

$$E^{j+1} \leq E^j + \delta\tau\overline{\rho} < (1-\gamma)E^j + \delta\tau\overline{\rho} \quad (\gamma < 0).$$

Because $E^0 = 0$ and $\tau \in [0, T]$, by induction, we can show that

$$E^j \leq j\delta\tau\overline{\rho} \leq T\overline{\rho}. \tag{45}$$

The bound on the truncation error is found to be

$$\rho_i^{j+1} \leq \frac{1}{2}(\delta\tau) max \mid W_{\tau\tau} \mid + \{\frac{\sigma^2 S_i^2}{24} max \mid W_{SSSS} \mid + \frac{rS_i}{12} max \mid W_{SSS} \mid\}(\delta S)^2$$
$$\leq \delta\tau\{\frac{1}{2} max \mid W_{\tau\tau} \mid + \frac{1}{12}(\frac{\sigma^2 S_i^2}{2} max \mid W_{SSSS} \mid + rS_i max \mid W_{SSS} \mid)\frac{1}{\nu}\},$$

where $\nu = \frac{\delta\tau}{(\delta S)^2}$ is a *refinement path*.

In general, to define convergence of a difference scheme which involves two meshes δS and $\delta\tau$, we need to specify what relationship we assume between them as they both tend to zero. This is defined by the refinement path ν, which is assumed to satisfy condition (43). Hence, from (45) $E^j \to 0$, as $\delta\tau \to 0$.

(ii)**Explicit 2**

The argument for convergence in this case is similar to the one in part (i). The truncation error is again of order $O(\delta\tau)+O(\delta S)^2$. This time ν must satisfy the condition

$$\nu < \frac{1}{\sigma^2 S_i^2} \quad \forall i,$$

which agrees with the Fourier stability condition (41). As before, $E^j \to 0$ as $\delta\tau \to 0$ and thus the approximate solution converges to the analytic one.

(iii)**Fully Implicit**

Here, the error e_i^j is determined by

$$(-\alpha_i + \beta_i)e_{i-1}^{j+1} + (1 + 2\alpha_i - \gamma)e_i^{j+1} + (-\alpha_i - \beta_i)e_{i+1}^{j+1} = e_i^j - \delta\tau\rho_i^{j+1}$$

$$\Rightarrow (1 + 2\alpha_i - \gamma)e_i^{j+1} = (\alpha_i - \beta_i)e_{i-1}^{j+1} + (\alpha_i + \beta_i)e_{i+1}^{j+1} + e_i^j - \delta\tau\rho_i^{j+1}, \quad (46)$$

for $i = 1, \ldots, N-1$ and $j = 0, \ldots, M-1$. As in part (i), we have that $e_i^0 = e_0^j = e_N^j = 0$. Since the coefficients of (46) are nonnegative (condition (26)), it follows that

$$E^{j+1} \leq (1-\gamma)E^{j+1} \leq E^j + \delta\tau\overline{\rho}.$$

Because $E_0 = 0$, then by induction we get

$$E^j \leq j\delta\tau\overline{\rho} \leq T\overline{\rho}.$$

The order of the truncation error is, as in the explicit case, $O(\delta\tau) + O(\delta S)^2$. In this case, there is no restriction on the refinement path ν. By fixing ν and letting $\delta\tau \to 0$ and $\delta S \to 0$ uniformly, it follows that $E^j \to 0$.

For the rest of the schemes, the process to show convergence is similar to part(iii). The difference lies essentially in the fact that the truncation error of the Crank-Nicolson scheme is of order $O(\delta\tau)^2 + O(\delta S)^2$ [11]. As a result, we can achieve good accuracy economically.

The convergence as well as the stability analysis arguments used in Sections 3.2 and 3.3 follow similar arguments to the corresponding arguments in [13] and [14].

3.4 Results

We will now implement the generalised θ-methods to give approximate solutions to the $1-d$ Black-Scholes equation (4), considering both cases of call and put options on stocks. In the computations we use the following parameters:

S^- :min. value of S	50
S^+ :max. value of S	150
E : exercise price	100
T : maturity date	0.25 (1/4 of a year)
r : interest rate	0.08
σ : volatility	0.2

Table 3.4.1

Observe that given the values in Table (3.4.1), condition (26) is satisfied and hence the problem we are solving is diffusion dominated.

Tables (3.4.2), (3.4.3), (3.4.4) and (3.4.5) show the approximations to call options and Tables (3.4.6), (3.4.7), (3.4.8) and (3.4.9) show the corresponding results for put options. The numerical solutions were obtained using the programs *theta, theta1, theta2*, which we wrote in Fortran 90 and the exact solutions were computed using Mathematica.

In all cases, it is apparent that by taking a smaller δS we obtain better accuracy, but this is done at the cost of having to take an even smaller $\delta\tau$ and hence a large number of steps. For the explicit schemes this is necessary for stability. Recalling condition (40) for *Explicit 1* scheme and taking $\delta S = 0.5$ and $\max_i\{S_i\} = 150$, then $\nu = \delta\tau/(\delta S)^2 \leq 0.001$. If N is the number of timesteps, then since $\delta\tau = T/N$, N should be at least equal to 925 to reach the final time $\tau = T$. A similar argument applies to *Explicit 2* scheme, but with a less restrictive condition on ν. Therefore, as shown in Tables (3.4.2), (3.4.4),(3.4.6) and (3.4.8), when ν does not satisfy the stability conditions (40) or (41), the *explicit* schemes become *unstable* whilst all other methods preserve stability. Also, $\sigma^2 = 0.04 > 0.005 = r^2/(1+r)$ and hence the *Semi-Implicit* scheme is expected to be unconditionally stable, which is verified from the results. In general, *Crank-Nicolson* has the fastest convergence, which is due to its truncation error of order $O(\delta\tau)^2 + O(\delta S)^2$.

Also notice that *Explicit 1* in comparison with *Explicit 2*, seems to converge faster. Taking though a relatively large timestep, we can see by comparing Table (3.4.4) with (3.4.5) and Table (3.4.8) with (3.4.9) that the two schemes produce very similar results.

Table (3.4.10) illustrates the solution for different boundary conditions. For the problem we are solving, it seems that a 'bad' choice of the boundary condition produces poor results for the nodes near the boundary region, whilst the central nodes are not affected as much. Hence, we may use 'incorrect' boundary conditions and still consider numerical solutions in the central region of the domain. For financial modelling, this is quite important since in many cases, we do not know the behaviour of the solution along the boundaries.

European call options

S_0	BS	CN	KV	FI	SI	E1	E2
80	0.0690	0.0691	0.0954	0.0948	0.0840	0.0	0.0
85	0.3162	0.3154	0.3536	0.3520	0.3288	0.0	0.0
90	1.0254	1.0241	1.0456	1.0423	1.0093	0.0	0.0
95	2.5253	2.5251	2.5051	2.5001	2.4718	0.0	0.0
100	5.0169	4.9567	4.9724	4.9664	4.9567	-2×10^{17}	-2×10^{17}
105	8.4585	8.4588	8.4248	8.4189	8.4267	9×10^{11}	3×10^{11}
110	12.6204	12.6202	12.6122	12.6069	12.6213	-4×10^{12}	-4×10^{12}
115	17.2281	17.2286	17.2365	17.2319	17.2445	2×10^{13}	-1×10^{13}
120	22.0666	22.0698	22.0791	22.0750	22.0831	-4×10^{13}	2×10^{13}

Table 3.4.2 $\delta S = 0.5, \delta\tau = 0.025$ (10 timesteps)

S_0	BS	CN	KV	FI	SI	E1	E2
80	0.0690	0.0691	0.0694	0.0694	0.0693	0.0688	0.0688
85	0.3162	0.3163	0.3167	0.3166	0.3164	0.3159	0.3158
90	1.0254	1.025	1.0251	1.0251	1.0247	1.0248	1.0247
95	2.5253	2.5243	2.5241	2.5239	2.5236	2.5246	2.5244
100	5.0169	5.0157	5.0153	5.0149	5.0148	5.0163	5.0159
105	8.4585	8.4576	8.4573	8.4567	8.4568	8.4580	8.4575
110	12.6204	12.6201	12.6199	12.6191	12.6193	12.6200	12.6194
115	17.2281	17.2282	17.2280	17.2270	17.2272	17.2274	17.2270
120	22.0666	22.0668	22.0666	22.0654	22.0655	22.0657	22.0654

Table 3.4.3 $\delta S = 0.5, \delta\tau = 0.0002$ (950 timesteps)

S_0	BS	CN	KV	FI	SI	E1	E2
80	0.0690	0.0690	0.0758	0.0757	0.0730	0.0	0.0
85	0.3162	0.3162	0.3259	0.3255	0.3194	0.0	0.0
90	1.0254	1.0252	1.0304	1.0295	1.0211	0.0	0.0
95	2.5253	2.5251	2.5201	2.5189	2.5119	0.0	0.0
100	5.0169	5.0134	5.0058	5.0042	5.0019	-3×10^{-17}	-3×10^{-17}
105	8.4585	8.4583	8.4499	8.4483	8.4503	8×10^{11}	3×10^{11}
110	12.6204	12.6203	12.6181	12.6166	12.6204	-6×10^{12}	-5×10^{12}
115	17.2281	17.2278	17.2302	17.2285	17.2319	3×10^{13}	-2×10^{13}
120	22.0666	22.0664	22.0699	22.0676	22.0698	-3×10^{13}	2×10^{13}

Table 3.4.4 $\delta S = 0.25, \delta\tau = 0.00625$ (40 timesteps)

S_0	BS	CN	KV	FI	SI	E1	E2
80	0.0690	0.0690	0.0691	0.0691	0.0691	0.0689	0.0689
85	0.3162	0.3163	0.3164	0.3163	0.3163	0.3162	0.3162
90	1.0254	1.0254	1.0254	1.0254	1.0253	1.0253	1.0253
95	2.5253	2.5252	2.5252	2.5251	2.5251	2.5253	2.5253
100	5.0169	5.0168	5.0168	5.0167	5.0168	5.0169	5.0171
105	8.4585	8.4586	8.4587	8.4587	8.4588	8.4584	8.4589
110	12.6204	12.6209	12.6210	12.6212	12.6213	12.6202	12.6211
115	17.2281	17.2289	17.2290	17.2292	17.2292	17.2280	17.2291
120	22.0666	22.0674	22.0677	22.0676	22.0677	22.0667	22.0680

Table 3.4.5 $\delta S = 0.25, \delta\tau = 0.000065$ (3800 timesteps)

European put options

S_0	BS	CN	KV	FI	SI	E1	E2
80	18.0889	18.0888	18.1133	18.1165	18.1057	-1×10^{10}	-1×10^{10}
85	13.3361	13.3352	13.3715	13.3737	13.3506	-5×10^{9}	1×10^{10}
90	9.0453	9.0439	9.0635	9.0641	9.0311	-5×10^{10}	1×10^{10}
95	5.5452	5.5449	5.5230	5.5220	5.4936	-9×10^{9}	-1×10^{11}
100	3.0368	2.9765	2.9903	2.9882	2.9785	-2×10^{17}	-2×10^{17}
105	1.4784	1.4786	1.4428	1.4407	1.4485	0.0	0.0
110	0.6402	0.6396	0.6301	0.6286	0.6430	0.0	0.0
115	0.2479	0.2472	0.2545	0.2536	0.2661	0.0	0.0
120	0.0863	0.0860	0.0971	0.0966	0.1047	0.0	0.0

Table 3.4.6 $\delta S = 0.5, \delta\tau = 0.025$ (10 timesteps)

S_0	BS	CN	KV	FI	SI	E1	E2
80	18.0889	18.0892	18.0894	18.0887	18.0883	18.0889	18.0879
85	13.3361	13.3363	13.3366	13.3362	13.3357	13.3359	13.3351
90	9.0453	9.0449	9.0450	9.0447	9.0443	9.0447	9.0442
95	5.5452	5.5442	5.5439	5.5437	5.5434	5.5445	5.5441
100	3.0368	3.0356	3.0351	3.0349	3.0348	3.0361	3.0359
105	1.4784	1.4774	1.4770	1.4769	1.4770	1.4778	1.4777
110	0.6402	0.6397	0.6396	0.6396	0.6397	0.6399	0.6398
115	0.2479	0.2477	0.2478	0.2477	0.2479	0.2476	0.2476
120	0.0864	0.0864	0.0865	0.0865	0.0866	0.0862	0.0862

Table 3.4.7 $\delta S = 0.5, \delta\tau = 0.0002$ (950 timesteps)

S_0	BS	CN	KV	FI	SI	E1	E2
80	18.0889	18.0891	18.0952	18.0962	18.0935	-1×10^{10}	-2×10^{10}
85	13.3361	13.3362	13.3453	13.3460	13.3400	-9×10^{8}	3×10^{8}
90	9.0453	9.0451	9.0497	9.0499	9.0416	-4×10^{9}	4×10^{11}
95	5.5452	5.5450	5.5395	5.5392	5.5323	-11×10^{9}	-2×10^{11}
100	3.0368	3.0332	3.0251	3.0246	3.0222	-3×10^{17}	-3×10^{17}
105	1.4784	1.4782	1.4693	1.4687	1.4707	0.0	0.0
110	0.6402	0.6401	0.6375	0.6371	0.6408	0.0	0.0
115	0.2479	0.2478	0.2495	0.2493	0.2526	0.0	0.0
120	0.0864	0.0864	0.0892	0.0891	0.0913	0.0	0.0

Table 3.4.8 $\delta S = 0.25, \delta\tau = 0.00625$ (40 timesteps)

S_0	BS	CN	KV	FI	SI	E1	E2
80	18.0889	18.0884	18.0872	18.0876	18.0882	18.0883	18.0883
85	13.3361	13.3356	13.3351	13.3350	13.3354	13.3355	13.3356
90	9.0453	9.0449	9.0445	9.0443	9.0444	9.0447	9.0447
95	5.5452	5.5450	5.5446	5.5446	5.5444	5.5447	5.5448
100	3.0368	3.0366	3.0363	3.0362	3.0362	3.0364	3.0365
105	1.4784	1.4782	1.478	1.4779	1.4779	1.4781	1.4782
110	0.6402	0.6402	0.6401	0.6401	0.6401	0.6402	0.6402
115	0.2479	0.2478	0.2478	0.2478	0.2478	0.2478	0.2478
120	0.0864	0.0864	0.0864	0.0864	0.0864	0.0863	0.0864

Table 3.4.9 $\delta S = 0.25, \delta\tau = 0.000065$ (3800 timesteps)

S_0	BS	CN	KV	FI	SI	E1	E2
85	0.3162	0.3163	0.3167	0.3166	0.3164	0.3159	0.3158
		0.3163	0.3167	0.3166	0.3164	0.3159	0.3158
		0.3163	0.3167	0.3166	0.3164	0.3159	0.3158
95	2.5253	2.5243	2.5241	2.5239	2.5236	2.5246	2.5244
		2.5243	2.5241	2.5239	2.5236	2.5246	2.5244
		2.5243	2.5241	2.5239	2.5236	2.5246	2.5244
100	5.0169	5.0157	5.0153	5.0149	5.0148	5.0163	5.0159
		5.0154	5.015	5.0145	5.0145	5.016	5.0156
		5.0151	5.0146	5.0142	5.0141	5.0156	5.0153
105	8.4585	8.4576	8.4573	8.4567	8.4568	8.458	8.4575
		8.4553	8.4549	8.4543	8.4544	8.455	8.4552
		8.4530	8.4525	8.4519	8.4520	8.4535	8.4529
110	12.6204	12.6201	12.6199	12.6191	12.6193	12.6202	12.6194
		12.6072	12.6068	12.6060	12.6062	12.6074	12.606
		12.5942	12.5937	12.5929	12.5932	12.5947	12.5939
120	22.0666	22.0668	22.0666	22.0654	22.0655	22.0667	22.0654
		21.8747	21.8738	21.8727	21.8731	21.8754	21.8741
		21.6827	21.6810	21.6799	21.6807	21.6841	21.6829

Table 3.4.10 European call options valued using different boundary conditions. Each cell in the table contains the numerical solution obtained using the conditions $C(S^+,\tau) = S^+ - Ee^{-r\tau}$, $C_S(S^+,\tau) = 0$ and $C_S(S^+,\tau) = 1$ ($\delta S = 0.5, \delta\tau = 0.0002$)

4 Extension of Black-Scholes Model

We will now consider the case where the stock has a *stochastic* volatility and thus the Black-Scholes $1-d$ equation cannot be applied to this problem. As a result, we shall examine an extension to it. So, define

S : stock price

V : variance of $\delta S/S$(volatility)

$\psi(t)$: drift rate of $\delta S/S$

$\eta(V)$: drift rate of δV

ξ : instantaneous standard deviation of $\delta V/\sqrt{V}$

ρ : instantaneous correlation between $\delta S/S$and $\delta V/\sqrt{V}$.

Since the volatility is now modelled as separate state variable, S and V are assumed to obey the *stochastic processes*

$$\frac{dS}{S} = \psi dt + \sqrt{V} dz \tag{47}$$

$$dV = \eta dt + \xi\sqrt{V} dw, \tag{48}$$

where dz and dw are *Wiener processes.*

Garman [3] and Cox, Ingersoll and Ross [2], showed that a security f whose price depends on stochastic variables θ_i, must satisfy the differential equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i,j} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 f}{\partial\theta_i\partial\theta_j} - rf = \sum_i \theta_i \frac{\partial f}{\partial\theta_i}[-\mu_i + \lambda_i\sigma_i], \tag{49}$$

where σ_i is the instantaneous standard deviation of the proportional change in θ_{ij}, ρ_{ij} is the instantaneous correlation between θ_i and θ_j, μ_i is the proportional drift rate of θ_i and λ_i is the market price of risk for variable θ_i. When variable i is traded, the ith element of the r.h.s of (49) is $-r\theta_i\frac{\partial f}{\partial\theta_i}$.

It follows that, for the two state variable case in question,

$$\frac{\partial f}{\partial t} + \frac{1}{2}\{VS^2\frac{\partial^2 f}{\partial S^2} + 2\rho V\xi S\frac{\partial^2 f}{\partial S\partial V} + \xi^2 V\frac{\partial^2 f}{\partial V^2}\} - rf$$
$$= -rS\frac{\partial f}{\partial S} - (\eta - \lambda_v\xi\sqrt{V})\frac{\partial f}{\partial V} \qquad (50)$$

Assuming that the market price λ_v is zero, we are left with

$$\frac{\partial f}{\partial t} + \frac{1}{2}\{VS^2\frac{\partial^2 f}{\partial S^2} + 2\rho V\xi S\frac{\partial^2 f}{\partial S\partial V} + \xi^2 V\frac{\partial^2 f}{\partial V^2}\} - rf = -rS\frac{\partial f}{\partial S} - \eta\frac{\partial f}{\partial V} \quad (51)$$

The initial and boundary conditions for (51) are:

$$f(S,V,T) = \max(S-E,0) \qquad (52)$$
$$f(0,V,t) = 0 \qquad (53)$$
$$f(S,V,t) \to S, \quad \text{as } S \to \infty \qquad (54)$$

As we did for the $1-d$ problem, we consider S in the finite interval $[S^-, S^+]$ $\subset [0,\infty]$, so that conditions (54) and (53) become

$$f(S^+,V,t) = S^+ - Ee^{-r(T-\tau)}, \qquad (55)$$

$$f(S^-,V,t) = 0. \qquad (56)$$

We also need two more boundary conditions on V. So, if $V \in [V^-, V^+]$ $\subset [0,\infty]$, then consider the *Neumann conditions*

$$f_V(S,V^-,t) = f_V(S,V^+,t) = 0, \quad \text{for } S^- < S < S^+. \qquad (57)$$

If ξ and η are zero, then (51) devolves into (1), the Black-Scholes equation. We shall assume that ξ is constant and that the drift rate of V, η, is given by

$$\eta = a + bV, \qquad (58)$$

where a, b are constants. In order to ensure that V remains nonnegative we require $a \geq 0$. From (58), we can have a constant drift ($b = 0$), a constant proportional drift ($a = 0$), or a mean reverting process ($a > 0, b < 0$). In this latter case, V tends to revert to a level $-a/b$ with a reversion rate $-b$.

A lot of research has been done on option pricing with a stochastic volatility. Hull and White [8] derive an expression for the pricing bias in series form. Also, for the special case where $\rho = 0$, i.e. the volatility is uncorrelated with the stock price, they produce analytic results for a European call option [9]. More recent work was carried out by Heston in 1993 [6], who gives a closed-form solution for other values of ρ in the interval $[0, 1]$. A.Kurpiel and T.Roncalli [10] tackled the problem numerically using *Hopschotch methods*, considering both cases of European and American options.

Here, we will deal with European call options on stocks that satisfy the $2-d$ partial differential equation (51) and derive numerical solutions by using the method called **Alternating Direction Implicit (A.D.I)**. However, in order to be able to implement this scheme, we must eliminate the mixed derivative term $\partial f / \partial S \partial V$. To achieve this, consider the following standard based transformations. The same sort of transformations were carried out in [4].

4.1 The transformed equation

As in the $1-d$ case, we need to solve (51) *backwards* in time, so define

$$\tau = T - t.$$

With $c(S, V, \tau) = f(S, V, t)$, equation (51) becomes

$$L[c] \equiv \frac{\partial c}{\partial \tau} - \{\frac{VS^2}{2}\frac{\partial^2 c}{\partial S^2} + \rho V \xi S \frac{\partial^2 c}{\partial S \partial V} + \frac{\xi^2 V}{2}\frac{\partial^2 c}{\partial V^2} + rS\frac{\partial c}{\partial S} + \eta \frac{\partial c}{\partial V}\} + rc \quad (59)$$

The operator $L[.]$ is *parabolic* if and only if the differential operator

$$l[c] \equiv \frac{VS^2}{2}\frac{\partial^2 c}{\partial S^2} + \rho V \xi S \frac{\partial^2 c}{\partial S \partial V} + \frac{\xi^2 V}{2}\frac{\partial^2 c}{\partial V^2} + rS\frac{\partial c}{\partial S} + \eta \frac{\partial c}{\partial V} \quad (60)$$

is *elliptic*. This is achieved by letting

$$(\rho \xi V S)^2 - 4(\frac{VS^2}{2})\frac{\xi^2 V}{2} = V^2 S^2 \xi^2 (\rho^2 - 1) < 0$$

Hence, we require that $\rho \in (-1, 1)$, which agrees with Hull and White [8].

Having established that the operator $l[.]$ is elliptic, then at each point, there are coordinates (x, y) such that the second derivative terms in (60) can be written in the form $\beta(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2})$, where $\beta = \beta(x, y)$. Hence, consider the following series of transformations.

Define the new variables x and y by

$$x = \Phi(S, V)$$
$$y = \Psi(S, V).$$

A necessary condition on the functions Φ and Ψ to ensure a coordinate transformation for every point (S, V), there corresponds a unique point (x, y) and vice versa, such that the Jacobian

$$J(S, V) = \Phi_S \Psi_V - \Psi_S \Phi_V,$$

does not vanish in the region of interest.

For simplicity, rewrite the coefficients in (60) as

$$a = \frac{vS^2}{2}, \quad b = \rho \xi V S, \quad d = \frac{\xi^2 V}{2}, \quad g = rS, \quad h = \eta \tag{61}$$

so that

$$l \equiv a \frac{\partial^2}{\partial S^2} + b \frac{\partial^2}{\partial S \partial V} + d \frac{\partial^2}{\partial V^2} + g \frac{\partial}{\partial S} + h \frac{\partial}{\partial V} \tag{62}$$

Now, let $c(S, V) = u(x, y)$. Using the chain rule, the operator $l[.]$ with respect to the 'new' variables x, y becomes

$$M \equiv A(x,y)\frac{\partial^2}{\partial x^2} + B(x,y)\frac{\partial^2}{\partial x \partial y} + D(x,y)\frac{\partial^2}{\partial y^2} + G(x,y)\frac{\partial}{\partial x} + H(x,y)\frac{\partial}{\partial y} \tag{63}$$

where

$$A(x, y) = a\Phi_S^2 + b\Phi_S\Phi_V + d\Phi_V^2$$
$$B(x, y) = 2a\Phi_S\Psi_S + b(\Phi_S\Psi_V + \Psi_S\Phi_V) + 2d\Phi_V\Psi_V$$
$$D(x, y) = a\Psi_S^2 + b\Psi_S\Psi_V + d\Psi_V^2$$
$$G(x, y) = g\Phi_S + h\Phi_V$$
$$H(x, y) = g\Psi_S + h\Psi_V$$

So, equation (59) is transformed into

$$M[u] - ru = \frac{\partial u}{\partial \tau}.$$

Setting $B = 0$ and $A = D$, then

$$a(\Phi_S^2 - \Psi_S^2) + b(\Phi_S\Phi_V - \Psi_S\Psi_V)d(\Phi_V^2 - \Psi_V^2) = 0 \tag{64}$$

$$2a\Phi_S\Psi_S + b(\Phi_S\Psi_V + \Psi_S\Phi_V) + 2d\Phi_V\Psi_V = 0 \tag{65}$$

and so we need to solve the above set of equations for Φ and Ψ. Hence, multipling (65) by i and adding it to (64) we end up with the following quadratic equation of the complex variable $\zeta = x + iy$;

$$a\zeta_S^2 + b\zeta_S\zeta_V + d\zeta_V^2 = 0. \tag{66}$$

A root of (66) is found to be

$$\frac{\zeta_S}{\zeta_V} = \frac{x_S + iy_S}{x_V + iy_V} = -\frac{b + i\sqrt{4ad - b^2}}{2a}.$$

Separating real and imaginary parts, we get

$$x_S = \frac{dy_V - by_S}{\sqrt{4ad - b^2}}$$
$$x_V = -\frac{by_V + 2ay_S}{\sqrt{4ad - b^2}}.$$

By arbitrarily fixing $y = S$, the transformation of the 'old' variables to the 'new' variables x and y is given by

$$\begin{cases} y = S \\ x = \frac{S(\xi\rho - V)}{\xi\sqrt{1-\rho^2}} \end{cases} \tag{67}$$

Finally, the original partial differential equation (51) is transformed into the general form of a $2-d$ *parabolic* equation;

$$A(x,y)\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\} + G(x,y)\frac{\partial u}{\partial x} + H(x,y)\frac{\partial u}{\partial y} - ru = \frac{\partial u}{\partial \tau}, \tag{68}$$

where the coefficients of (68) are found to be

$$A(x,y) = \frac{y\xi(y\rho - x\sqrt{1-\rho^2})}{2} \tag{69}$$

$$G(x,y) = \frac{y(r\rho\xi - \eta)}{\xi\sqrt{1-\rho^2}} \tag{70}$$

$$H(x,y) = ry. \tag{71}$$

Notice that $A(x,y) > 0$, since substituting the expression in (67) for x, we get

$$A = \frac{y\xi(y\rho - y\rho + yV/\xi)}{2} = \frac{y^2V}{2} > 0, \text{ since } V > 0.$$

Also, the drift rate is now a function of x and y and from (67) it follows that

$$\eta = a + bV = a + b\left(\xi\rho - \frac{x\xi\sqrt{1-\rho^2}}{y}\right).$$

The conditions (52), (55) and (56) w.r.t. the 'new' variables x and y become

$$u(x,y,\tau) = \max(y - E, 0) \tag{72}$$

$$u(x,y^-,\tau) = 0 \tag{73}$$

$$u(x,y^+,\tau) = y^+ - Ee^{-r(T-\tau)}. \tag{74}$$

Our domain is no longer a square (see Figure (5.1.1)), since x is a function of both S and V. Therefore, by fixing $V = V^-$, $V = V^+$ in turn and letting S vary in the interval (S^-, S^+) , we obtain a set of boundary values for x, say, $x^{(1)}$ and $x^{(2)}$ for each case respectively. The boundary conditions (57) that before applied to the extreme values of V, V^- and V^+, now become

$$u_x(x^{(1)},y,\tau) = u_y(x^{(1)},y,\tau) = 0, \quad y^- < y < y^+ \tag{75}$$

$$u_x(x^{(2)},y,\tau) = u_y(x^{(2)},y,\tau) = 0, \quad y^- < y < y^+. \tag{76}$$

5 Numerical Schemes

As already mentioned, our aim is to solve the $2-d$ partial differential equation (68) using the **A.D.I** method. We will also apply to the problem the *Explicit* scheme and then compare the results.

In general, however, because of poor stability properties, explicit difference methods are rarely used to solve initial boundary value problems in two or more space dimensions. *Implicit* methods are used more, but they require a set of equations to be solved at the advanced time level, which is not always easy to accomplish directly. Accordingly, A.D.I is introduced, which is a *two step* method involving the solution of tridiagonal sets of equations along lines parallel to the x- and y- axes at the first and second steps respectively. Such a method was first proposed by Peaceman and Rachford in 1955 [15] who used it in oil reservoir modelling.

5.1 Discretization of the problem

Having transformed the original $2-d$ equation (51) into equation (68), we now need to specify the region in which to solve the problem and thus define our numerical grid.

As already specified, we cannot solve the problem for all $0 < S < \infty$, but instead we consider a suitably large finite interval;

$$0 \leq S^- \leq S \leq S^+$$

Also, the volatility V varies so that

$$0 \leq V^- \leq V \leq V^+.$$

Recalling from the transformations (67), the extreme values for y are

$$(S^- =)y^- \leq y \leq y^+(= S^+). \tag{77}$$

Now, x is a function of both variables S and V and most importantly, its sign depends on the sign of the correlation ρ. To define the interval in which x varies, consider the following values:

$S^+ = 9,$	$S^- = 1$
$V^+ =0.100$	$V^- = 0.001$
$\xi = 0.15$	
$\rho =$ -0.5 ,0.5	

Table 5.1.1

- If $\rho = 0.5$, then $V^- < \xi\rho < V^+$ and hence x takes up both negative and positive values;

$$\frac{S^+(\xi\rho - V^+)}{\xi\sqrt{1-\rho^2}} = x^- \leq x \leq \frac{S^+(\xi\rho - V^-)}{\xi\sqrt{1-\rho^2}} = x^+ \tag{78}$$

  The region of the tranformed problem is shown in Figure (5.1.2).

- If $\rho = -0.5$, then x is always negative and so

$$\frac{S^+(\xi\rho - V^+)}{\xi\sqrt{1-\rho^2}} = x^- \leq x \leq \frac{S^-(\xi\rho - V^-)}{\xi\sqrt{1-\rho^2}} = x^+ < 0 \tag{79}$$

  In this case the region of interest changes its shape and is shown in Figure (5.1.4).

The difficulty now arises in the discretization of the domains. For computational ease, we divide up the domain so that the nodes lie exactly on the boundaries AB, CD, AG, BJ (see Figures (5.1.2) and (5.1.4)). However, the slopes of AB and CD and the ones of AG and BJ are not *equal* and so for the first case ($\rho > 0$) we have *unequal* spacing along the x-axis and for the second one ($\rho < 0$) both x and y are unequally divided.

We now show the way we discretized the two types of domains.

- $\rho = 0.5$

  Firstly, we divide the y-axis into, say, M equally spaced subintervals;

$$y_j = j\delta y, \qquad \text{for } j = 0, \ldots, M,$$

  where $\delta y = (y^+ - y^-)/M = (S^+ - S^-)/M$.

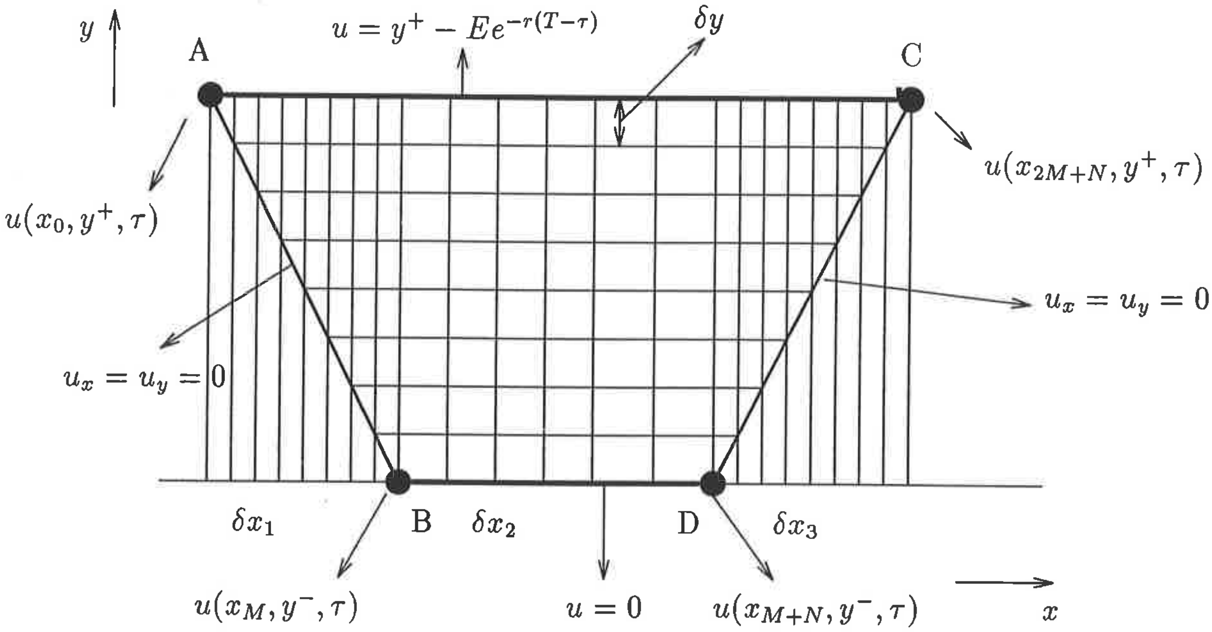

Figure 5.1.2: The domain of the transformed equation when $\rho = 0.5$

We then calculate the meshes δx_1 and δx_3 along the x-axis. As shown in Figure (5.1.3) it holds that

$$\delta x_1 = \sqrt{(\mathrm{AB}/M)^2 - (\delta y)^2} \tag{80}$$

$$\delta x_3 = \sqrt{(\mathrm{CD}/M)^2 - (\delta y)^2} \tag{81}$$

The remaining interval corresponding to BD in Figure (5.1.2) is divided into, say, N equally spaced subintervals; i.e.

$$\delta x_2 = \mathrm{BD}/N. \tag{82}$$

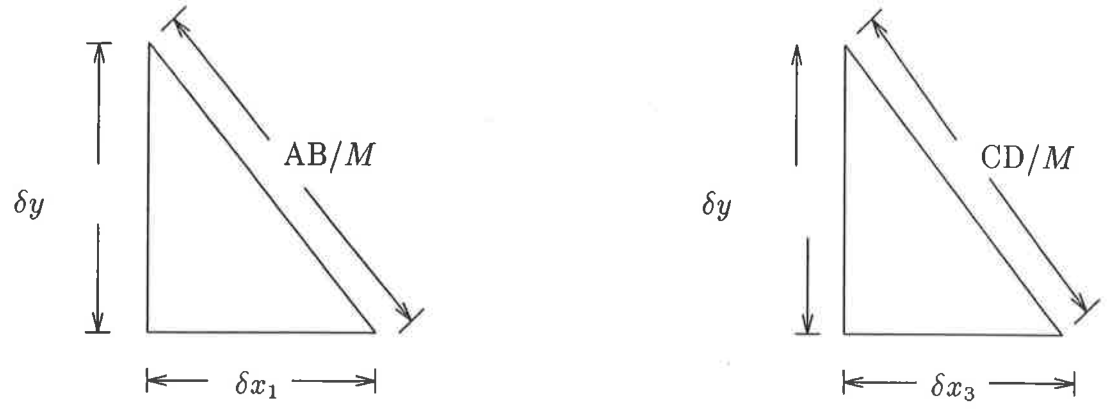

Figure 5.1.3: Calculating the different meshes along the x-axis

- $\rho = -0.5$

  Here, we start by dividing BC into, say, M subintervals;

$$\delta y_1 = BC/M \tag{83}$$

$$\Rightarrow \delta x_1 = \sqrt{(AC/M)^2 - (\delta y_1)^2} \tag{84}$$

This discretization allows the edge CD of triangle BCD to be divided into M subintervals of length δx_2;

$$\delta x_2 = \sqrt{(BD/M)^2 - (\delta y_1)^2} \tag{85}$$

In effect, we introduce a new spacing, δy_2, along the edge DE of triangle CDE;

$$\delta y_2 = \sqrt{(CE/M)^2 - (\delta x_2)^2} \tag{86}$$

Doing that, we then divide the edge EF of triangle DEF into M subintervals of length δx_3;

$$\delta x_3 = \sqrt{(DF/M)^2 - (\delta y_2)^2} \tag{87}$$

Lastly, we automatically introduce another spacing , δy_3, in y;

$$\delta y_3 = \sqrt{(EG/K)^2 - (\delta x_3)^2} \tag{88}$$

Clearly, $K \neq M$ and it can be found compuationally by taking the integer part when dividing FH by δx_3.

Now, from triangle FHJ, δy_3 introduces another spacing in x, δx_4 defined by

$$\delta x_4 = \sqrt{(FJ/K)^2 - (\delta y_3)^2}. \tag{89}$$

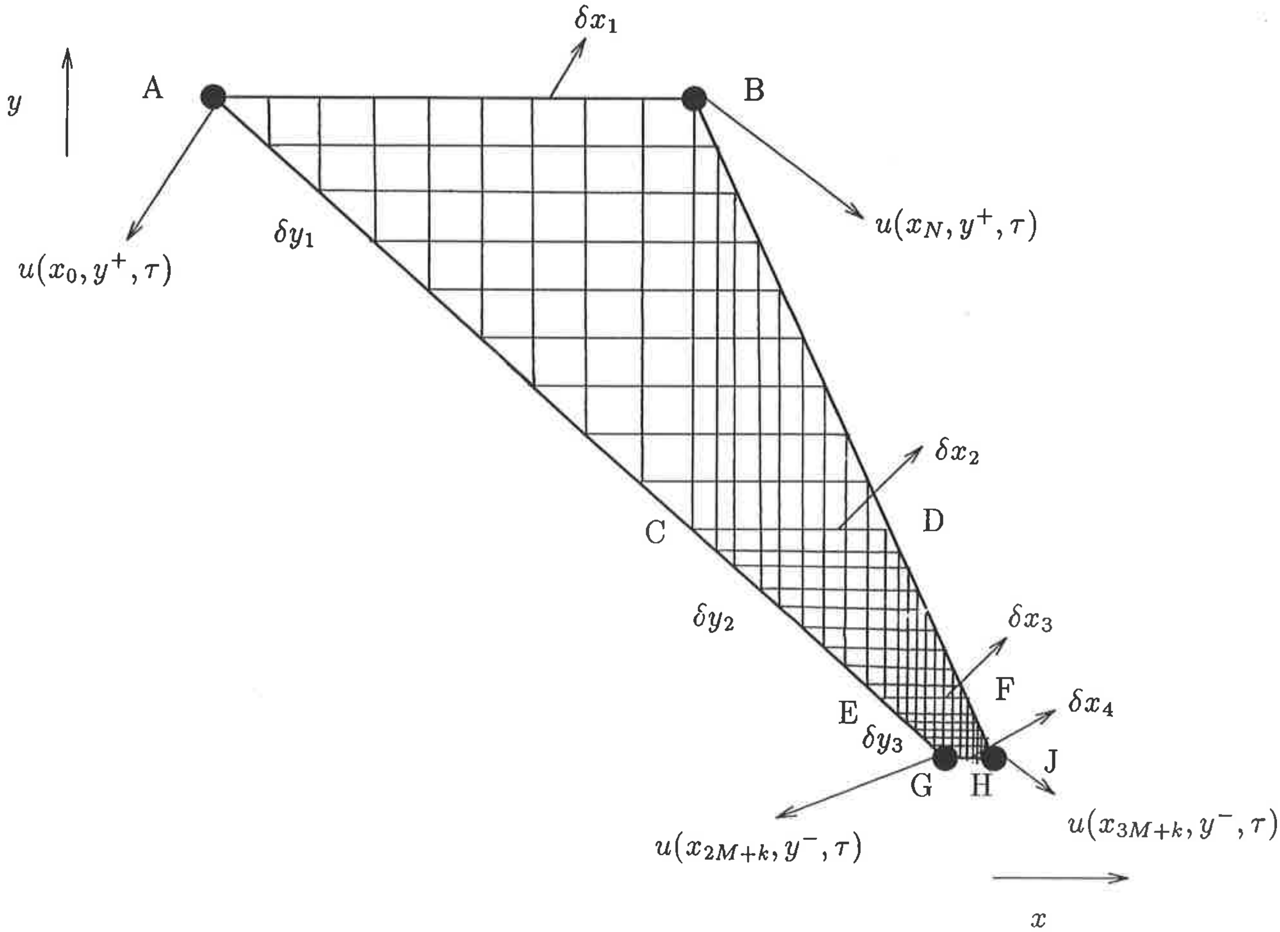

Figure 5.1.4: The domain of the transformed equation when $\rho = -0.5$

The time interval $[0, T]$ is discretized such that if R is the number of timesteps, then

$$\tau_l = l\delta\tau, \quad \text{for } l = 0, \ldots, R,$$

where $\delta\tau = T/R$.

5.2 The Alternating Direction Implicit Method (A.D.I)

The idea behind the A.D.I. method is to apply implicit finite differences in the one space dimension and explicit finite differences in the other and vice versa, each at an intermediate time level $k + \frac{1}{2}$. This implies having to solve tridiagonal sets of equations at each time step.

Since we are dealing with *unequal* spacing in x direction if $\rho > 0$ and in both x and y directions if $\rho < 0$, we approximate the derivatives as follows: let h_i, h_{i-1} be the two unequal spacings in x and g_j, g_{j-1} the corresponding spacings in y. The derivative term can be thought of as the weighted average of the two gradients; i.e.

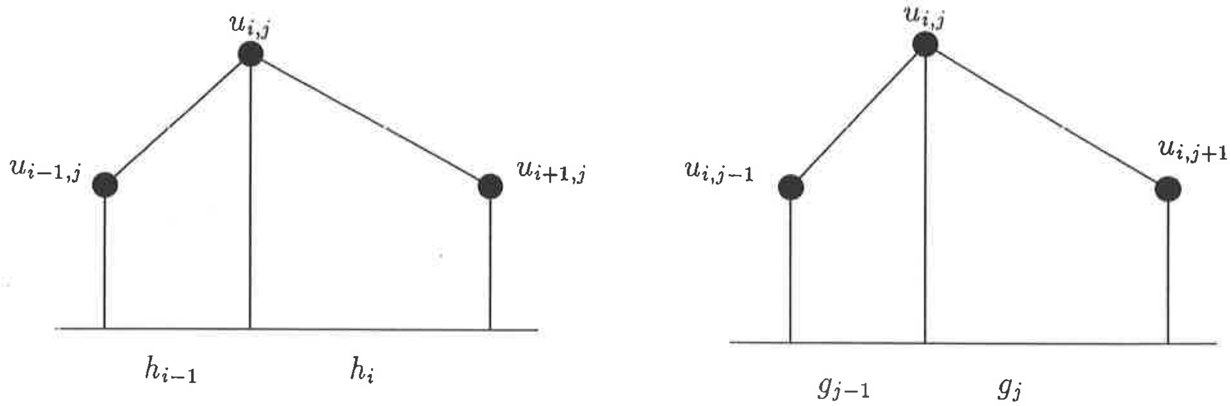

Figure 5.2.1: Unequal mesh spacing along the x and y axes.

$$\begin{aligned}\frac{\partial u(x_i, y_j)}{\partial x} &\approx \frac{h_i}{h_{i-1} + h_i}\left(\frac{u_{i,j} - u_{i-1,j}}{h_{i-1}}\right) + \frac{h_{i-1}}{h_{i-1} + h_i}\left(\frac{u_{i+1,j} - u_{i,j}}{h_i}\right) \\ &= \frac{h_{i-1}^2 u_{i+1,j} - h_i^2 u_{i-1,j} + (h_i^2 - h_{i-1}^2)u_{i,j}}{h_i h_{i-1}(h_i + h_{i+1})} \end{aligned} \quad (90)$$

$$\frac{\partial^2 u(x_i,y_j)}{\partial x^2} \approx \frac{1}{\frac{1}{2}h_{i-1}+\frac{1}{2}h_i}(u'_{i+\frac{1}{2},j}-u'_{i-\frac{1}{2},j})$$
$$= 2[\frac{h_{i-1}u_{i+1,j}+h_i u_{i-1,j}-(h_{i-1}+h_i)u_{i,j}}{h_i h_{i-1}(h_i+h_{i-1})}] \qquad (91)$$

Similarly, the approximations w.r.t. y are given by

$$\frac{\partial u(x_i,y_j)}{\partial y} \approx \frac{g_j}{g_{j-1}+g_j}\left(\frac{u_{i,j}-u_{i,j-1}}{g_{j-1}}\right)+\frac{g_{j-1}}{g_{j-1}+g_j}\left(\frac{u_{i,j+1}-u_{i,j}}{g_j}\right)$$
$$= \frac{h_{j-1}^2 u_{i,j+1}-g_j^2 u_{i,j-1}+(g_j^2-g_{j-1}^2)u_{i,j}}{g_j g_{j-1}(g_j+g_{j+1})} \qquad (92)$$

$$\frac{\partial^2 u(x_i,y_j)}{\partial y^2} \approx \frac{1}{\frac{1}{2}g_{j-1}+\frac{1}{2}g_j}(u'_{i,j+\frac{1}{2}}-u'_{i,j-\frac{1}{2}})$$
$$= 2[\frac{g_{j-1}u_{i,j+1}+g_j u_{i,j-1}-(g_{j-1}+g_j)u_{i,j}}{g_j g_{j-1}(g_j+g_{j-1})}] \qquad (93)$$

In the case where $g_j = g_{j-1}$ (90) and (93) reduce down to the usual central differences.

By treating the x direction implicitly at a half timestep, the difference equations approximating (68) are

$$\frac{u_{i,j}^{k+\frac{1}{2}}-u_{i,j}^k}{\frac{\delta\tau}{2}} = A_{i,j}\left\{2(\frac{h_{i-1}u_{i+1,j}^{k+\frac{1}{2}}+h_i u_{i-1,j}^{k+\frac{1}{2}}-(h_{i-1}+h_i)u_{i,j}^{k+\frac{1}{2}}}{h_i h_{i-1}(h_i+h_{i-1})})\right\}$$
$$+A_{i,j}\left\{2(\frac{g_{j-1}u_{i,j+1}^k+g_j u_{i,j-1}^k-(g_{j-1}+g_j)u_{i,j}^k}{g_j g_{j-1}(g_j+g_{j-1})})\right\}$$
$$+G_{i,j}\left\{\frac{h_{i-1}^2 u_{i+1,j}^{k+\frac{1}{2}}-h_i^2 u_{i-1,j}^{k+\frac{1}{2}}+(h_i^2-h_{i-1}^2)u_{i,j}^{k+\frac{1}{2}}}{h_i h_{i-1}(h_i+h_{i+1})}\right\}$$
$$+H_{i,j}\left\{\frac{g_{j-1}^2 u_{i,j+1}^k-g_j^2 u_{i,j-1}^k+(g_j^2-g_{j-1}^2)u_{i,j}^k}{g_j g_{j-1}(g_j+g_{j+1})}\right\}$$
$$-ru_{i,j}^{k+\frac{1}{2}}. \qquad (94)$$

Rearranging equation(94), we get

$$\Big(-\frac{A_{i,j}\delta\tau}{h_{i-1}(h_i+h_{i-1})}+\frac{G_{i,j}h_i\delta\tau}{2h_{i-1}(h_i+h_{i-1})}\Big)u_{i-1,j}^{k+\frac{1}{2}}+\Big(1+\frac{A_{i,j}\delta\tau}{h_ih_{i-1}}-\frac{G_{i,j}(h_i-h_{i-1})\delta\tau}{2h_ih_{i-1}}+\frac{r\delta\tau}{2}\Big)u_{i,j}^{k+\frac{1}{2}}$$

$$+\Big(-\frac{A_{i,j}\delta\tau}{h_i(h_i+h_{i-1})}-\frac{G_{i,j}h_{i-1}\delta\tau}{2h_i(h_i+h_{i-1})}\Big)u_{i+1,j}^{k+\frac{1}{2}}$$

$$=\Big(\frac{A_{i,j}\delta\tau}{g_{j-1}(g_j+g_{j-1})}-\frac{H_{i,j}g_j\delta\tau}{2g_{j-1}(g_j+g_{j-1})}\Big)u_{i,j-1}^{k}+\Big(1-\frac{A_{i,j}\delta\tau}{g_jg_{j-1}}+\frac{H_{i,j}(g_j-g_{j-1})\delta\tau}{2g_jg_{j-1}}\Big)u_{i,j}^{k}$$

$$+\Big(\frac{A_{i,j}\delta\tau}{g_j(g_j+g_{j-1})}+\frac{H_{i,j}g_{j-1}\delta\tau}{2g_j(g_j+g_{j-1})}\Big)u_{i,j+1}^{k} \qquad (95)$$

Once equations (95) are solved for each point (x_i, y_j), the solutions $u^{k+\frac{1}{2}}$ are then used in the next set of equations to determine u over the whole timestep $k+1$;

$$\frac{u_{i,j}^{k+1}-u_{i,j}^{k+\frac{1}{2}}}{\frac{\delta\tau}{2}} = A_{i,j}\Big\{2\Big(\frac{h_{i-1}u_{i+1,j}^{k+\frac{1}{2}}+h_iu_{i-1,j}^{k+\frac{1}{2}}-(h_{i-1}+h_i)u_{i,j}^{k+\frac{1}{2}}}{h_ih_{i-1}(h_i+h_{i-1})}\Big)\Big\}$$

$$+A_{i,j}\Big\{2\Big(\frac{g_{j-1}u_{i,j+1}^{k+1}+g_ju_{i,j-1}^{k+1}-(g_{j-1}+g_j)u_{i,j}^{k+1}}{g_jg_{j-1}(g_j+g_{j-1})}\Big)\Big\}$$

$$+G_{i,j}\Big\{\frac{h_{i-1}^2u_{i+1,j}^{k+\frac{1}{2}}-h_i^2u_{i-1,j}^{k+\frac{1}{2}}+(h_i^2-h_{i-1}^2)u_{i,j}^{k+\frac{1}{2}}}{h_ih_{i-1}(h_i+h_{i+1})}\Big\}$$

$$+H_{i,j}\Big\{\frac{g_{j-1}^2u_{i,j+1}^{k+1}-g_j^2u_{i,j-1}^{k+1}+(g_j^2-g_{j-1}^2)u_{i,j}^{k+1}}{g_jg_{j-1}(g_j+g_{j+1})}\Big\}$$

$$-ru_{i,j}^{k+1} \qquad (96)$$

Rearranging equation (96), we get

$$\Big(\frac{H_{i,j}g_j\delta\tau}{2g_{j-1}(g_j+g_{j-1})}-\frac{A_{i,j}\delta\tau}{g_{j-1}(g_j+g_{j-1})}\Big)u_{i,j-1}^{k+1}+\Big(1+\frac{A_{i,j}\delta\tau}{g_jg_{j-1}}-\frac{H_{i,j}(g_j-g_{j-1})\delta\tau}{2g_jg_{j-1}}+\frac{r\delta\tau}{2}\Big)u_{i,j}^{k+1}.$$

$$+\Big(-\frac{H_{i,j}g_{j-1}\delta\tau}{2g_j(g_j+g_{j-1})}-\frac{A_{i,j}\delta\tau}{g_j(g_j+g_{j-1})}\Big)u_{i,j+1}^{k+1}$$

$$=\Big(\frac{A_{i,j}\delta\tau}{h_{i-1}(h_i+h_{i-1})}-\frac{G_{i,j}h_i\delta\tau}{2h_{i-1}(h_i+h_{i-1})}\Big)u_{i-1,j}^{k+\frac{1}{2}}+\Big(1-\frac{A_{i,j}\delta\tau}{h_ih_{i-1}}+\frac{G_{i,j}(h_i-h_{i-1})\delta\tau}{2h_ih_{i-1}}\Big)u_{i,j}^{k+\frac{1}{2}}$$

$$+\Big(\frac{A_{i,j}\delta\tau}{h_i(h_i+h_{i-1})}+\frac{G_{i,j}h_{i-1}\delta\tau}{2h_i(h_i+h_{i-1})}\Big)u_{i+1,j}^{k+\frac{1}{2}} \qquad (97)$$

5.3 The Explicit Scheme

An alternative way to solve equation (68) numerically is to apply an explicit scheme. The value of $u_{i,j}$ at the 'new' time level is expressed as a linear combination of $u_{i,j}$ and its neighbouring values $u_{i-1,j}$, $u_{i+1,j}$, $u_{i,j-1}$ and $u_{i,j+1}$ at the 'old' time step. Hence, the difference equations are

$$
\begin{aligned}
u_{i,j}^{k+1} = & \left(1 - \frac{2A_{i,j}\delta\tau}{h_{i-1}h_i} - \frac{2A_{i,j}\delta\tau}{g_j g_{j-1}} + \frac{G_{i,j}(h_i - h_{i-1})\delta\tau}{h_i h_{i-1}} + \frac{H_{i,j}\delta\tau(g_j - g_{j-1})}{g_j g_{j-1}} - r\delta\tau\right)u_{i,j}^k \\
& + \left(\frac{2A_{i,j}\delta\tau}{h_{i-1}(h_i + h_{i+1})} - \frac{G_{i,j}h_i\delta\tau}{h_{i-1}(h_i + h_{i+1})}\right)u_{i-1,j}^k \\
& + \left(\frac{2A_{i,j}\delta\tau}{h_i(h_i + h_{i+1})} + \frac{G_{i,j}h_{i-1}\delta\tau}{h_i(h_i + h_{i+1})}\right)u_{i+1,j}^k \\
& + \left(\frac{2A_{i,j}\delta\tau}{g_{j-1}(g_j + g_{j-1})} - \frac{H_{i,j}g_j\delta\tau}{g_{j-1}(g_j + g_{j-1})}\right)u_{i,j-1}^k \\
& + \left(\frac{2A_{i,j}\delta\tau}{g_j(g_j + g_{j-1})} + \frac{H_{i,j}g_{j-1}\delta\tau}{g_j(g_j + g_{j-1})}\right)u_{i,j+1}^k. \qquad (98)
\end{aligned}
$$

5.4 Discretization along the boundaries

As we did for the $1-d$ equation, we approximate the derivative condition on the boundaries AB and CD (as shown in Figure (5.4.1)) by finite differences.

$$\frac{\partial u_O}{\partial x} = \frac{u_E - u_W}{2\delta x} = 0 \;\Rightarrow u_E = u_W \qquad (99)$$

$$\frac{\partial u_O}{\partial x} = \frac{u_S - u_N}{2\delta y} = 0 \;\Rightarrow u_S = u_N \qquad (100)$$

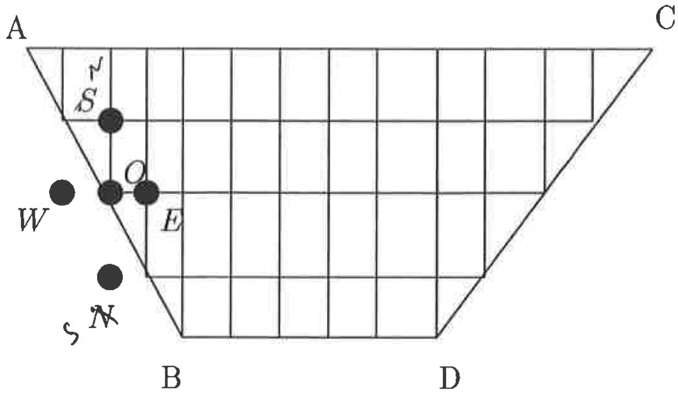

Figure 5.4.1: Discretization of the nodes along AB and CD

Hence, we substitute the above expressions for the 'fictitious' points u_N and u_W into the difference equations that correspond to the nodes along the boundary AB. We treat the difference equations for the nodes lying on CD in the same way.

5.5 Stability

(i) **Alternating Direction Implicit (A.D.I.)**

To examine the stability of this scheme, we shall apply *Fourier* analysis and in order to do that we assume equal spacing in both x and y directions; i.e. let $h_i = h_{i-1} = \delta x$ and $g_j = g_{j-1} = \delta y$. Then substituting

$$u_{i,j}^k = \lambda_k e^{z(i\delta x)b_1} e^{z(j\delta y)b_2}, \tag{101}$$

where $z = \sqrt{-1}$, into the difference equations (95) we obtain

$$\lambda_{\frac{1}{2}} = \frac{1 - 2\gamma_{i,j}\sin^2(\frac{\delta y b_2}{2}) + i\epsilon_{i,j}\sin(\delta y b_2)}{1 + 2\alpha_{i,j} sin^2(\frac{\delta x b_1}{2}) - i\beta_{i,j}\sin(\delta x b_1) + \frac{r\delta\tau}{2}}. \tag{102}$$

The coefficients $\alpha_{i,j}$, $\beta_{i,j}$, $\gamma_{i,j}$ and $\epsilon_{i,j}$ are such that

$$\alpha_{i,j} = \frac{A_{i,j}\delta\tau}{(\delta x)^2}$$

$$\beta_{i,j} = \frac{G_{i,j}\delta\tau}{2\delta x}$$

$$\gamma_{i,j} = \frac{A_{i,j}\delta\tau}{(\delta y)^2}$$

$$\epsilon_{i,j} = \frac{H_{i,j}\delta\tau}{2\delta y}.$$

Substituting now (101) into the difference equations (97), we get

$$\lambda = \lambda_{\frac{1}{2}}\left[\frac{1 - 2\alpha_{i,j}\sin^2(\frac{\delta x b_1}{2}) + i\beta_{i,j}\sin(\delta x b_1)}{1 + 2\gamma_{i,j}\sin^2(\frac{\delta y b_2}{2}) - i\epsilon_{i,j}\sin(\delta y b_2) + \frac{r\delta\tau}{2}}\right]. \tag{103}$$

Hence, substituting the expression (102) for $\lambda_{\frac{1}{2}}$, we find that the *amplification factor* satisfies

$$\lambda = \left[\frac{1 - 2\alpha_{i,j}\sin^2(\frac{\delta x b_1}{2}) + i\beta_{i,j}\sin(\delta x b_1)}{1 + \frac{r\delta\tau}{2} + 2\alpha_{i,j}\sin^2(\frac{\delta x b_1}{2}) - i\beta_{i,j}\sin(\delta x b_1)}\right]$$

$$\times\left[\frac{1 - 2\gamma_{i,j}\sin^2(\frac{\delta y b_2}{2}) + i\epsilon_{i,j}\sin(\delta y b_2)}{1 + \frac{r\delta\tau}{2} + 2\gamma_{i,j}\sin^2(\frac{\delta y b_2}{2}) - i\epsilon_{i,j}\sin(\delta y b_2)}\right]. \tag{104}$$

By arguing in the same manner as we did for the stability of the *Crank-Nicolson* scheme, consider the ratio

$$\frac{(1 - 2\alpha_{i,j}\sin^2(\frac{\delta x b_1}{2})) + i\beta_{i,j}\sin(\delta x b_1)}{(1 + \frac{r\delta\tau}{2} + 2\alpha_{i,j}\sin^2(\frac{\delta x b_1}{2})) - i\beta_{i,j}\sin(\delta x b_1)}. \tag{105}$$

Since $A_{i,j} > 0$, then $\alpha_{i,j} > 0$ and hence

$$-(1 - 2\alpha_{i,j}\sin^2(\frac{\delta x b_1}{2})) \leq (1 + \frac{r\delta\tau}{2} + 2\alpha_{i,j}\sin^2(\frac{\delta x b_1}{2})).$$

It follows that the ratio (105) is ≤ 1, $\forall i, j$.

Similarly, since $\gamma_{i,j} > 0$, it holds that

$$\frac{(1 - 2\gamma_{i,j}\sin^2(\frac{\delta y b_2}{2})) + i\epsilon_{i,j}\sin(\delta y b_2)}{(1 + \frac{r\delta\tau}{2} + 2\gamma_{i,j}\sin^2(\frac{\delta y b_2}{2})) - i\epsilon_{i,j}\sin(\delta y b_2)} \leq 1.$$

Hence, $|\lambda| \leq 1$ and thus the scheme is *unconditionally stable.*

The question of stability along the nodes where we have unequal spacings still remains. The results in Section 5.6 show that in practice the scheme is stable everywhere in the domain.

Considering the *consistency* of the scheme, it can be shown that the truncation error $\rho^{k+\frac{1}{2}}$ is of order $O((\delta\tau)^2 + (\delta x)^2 + (\delta y)^2)$ [14].

(ii)**Explicit scheme**

In this case, we analyse stability by using *Maximum Principles.* Considering the difference equations (98), $u_{i,j}^{k+1}$ is expressed as a linear combination of its neighbouring values at the previous time level; i.e. $u_{i,j}^k$, $u_{i-1,j}^k$, $u_{i+1,j}^k$, $u_{i,j-1}^k$, $u_{i,j+1}^k$. Thus, assuming that the coefficients on the l.h.s of (98) are all nonnegative and by using the triangle inequality, we have

$$\begin{aligned}
\mid u_{i,j}^{k+1} \mid \leq & \left(1 - \frac{2A_{i,j}\delta\tau}{h_{i-1}h_i} - \frac{2A_{i,j}\delta\tau}{g_j g_{j-1}} + \frac{G_{i,j}(h_i - h_{i-1})\delta\tau}{h_i h_{i-1}} + \frac{H_{i,j}\delta\tau(g_j - g_{j-1})}{g_j g_{j-1}} - r\delta\tau\right) \mid u_{i,j}^k \mid \\
& + \left(\frac{2A_{i,j}\delta\tau}{h_{i-1}(h_i + h_{i+1})} - \frac{G_{i,j}h_i\delta\tau}{h_{i-1}(h_i + h_{i+1})}\right) \mid u_{i-1,j}^k \mid \\
& + \left(\frac{2A_{i,j}\delta\tau}{h_i(h_i + h_{i+1})} + \frac{G_{i,j}h_{i-1}\delta\tau}{h_i(h_i + h_{i+1})}\right) \mid u_{i+1,j}^k \mid \\
& + \left(\frac{2A_{i,j}\delta\tau}{g_{j-1}(g_j + g_{j-1})} - \frac{H_{i,j}g_j\delta\tau}{g_{j-1}(g_j + g_{j-1})}\right) \mid u_{i,j-1}^k \mid \\
& + \left(\frac{2A_{i,j}\delta\tau}{g_j(g_j + g_{j-1})} + \frac{H_{i,j}g_{j-1}\delta\tau}{g_j(g_j + g_{j-1})}\right) \mid u_{i,j+1}^k \mid
\end{aligned} \tag{106}$$

By defining

$$U^k = \max_{i,j} \mid u_{i,j}^k \mid,$$

it follows that

$$U^{k+1} \leq (1 - r\delta\tau)U^k < U^k, \quad \text{since } r\delta\tau < 1. \tag{107}$$

So the *Maximum Principle* shows that the numerical values are bounded by the maximum and minimum values on the boundaries, provided that the following hold:

$$h_{i-1}, h_i \leq \frac{2A_{i,j}}{G_{i,j}} \tag{108}$$

$$g_{j-1}, g_j \leq \frac{2A_{i,j}}{H_{i,j}} \tag{109}$$

$$\delta\tau \leq \frac{1}{\frac{2A_{i,j} - G_{i,j}(h_i - h_{i-1})}{h_i h_{i-1}} + \frac{2A_{i,j} - H_{i,j}(g_j - g_{j-1})}{g_j g_{j-1}} + r}, \quad \forall i, j. \tag{110}$$

However, conditions (108)-(110) are only sufficient but not necessary to achieve stability.

Also, the truncation error is now first order accurate in time (and second order accurate in x and y). So, convergence is best achieved with the *A.D.I.* method.

5.6 Results

To approximate European call options when both the stock price and the volatility follow the stochastic processes (47) and (48) we use the parameters in Table (5.1.1). Also, we choose the coefficients a and b that define the drift rate η (58) to be

$$a = 0.003, \qquad b = -0.3.$$

Hence, the volatility reverts to the level $-a/b = 0.01$.

If $c(S, V, \tau)$ is the call price satisfying the $2-d$ partial differential equation (59) and $C(S, \tau)$ is the corresponding analytic solution of the $1-d$ Black-Scholes equation (10), then we compare $c(S, V, \tau)$ with $C(S, \tau)$ at the points $S = S_0$ and $V = V_0$. For the results obtained in this section, we use $V_0 = 0.01$.

From the transformations defined in (67), it follows that

$$x_0 = \frac{S_0(\xi\rho - V_0)}{\xi\sqrt{1-\rho^2}}, \qquad y_o = S_0.$$

However, x is not a linear function of V and hence for a fixed $V = V_0$ and $S = S_0$, x_0 may not be equal to any of the discretized values x_i. To overcome this we argue as follows; if $x_i \leq x_0 \leq x_{i+1}$, then

$$c(x_0, y_0, \tau) \approx \frac{x_{i+1} - x_0}{x_{i+1} - x_i} c(x_i, y_0, \tau) + \frac{x_0 - x_i}{x_{i+1} - x_i} c(x_{i+1}, y_0, \tau)$$

i.e. we take a weighted average of the neighbouring call values $c(x_i, y_0, \tau)$ and $c(x_{i+1}, y_0, \tau)$.

As shown in Tables (5.6.1)-(5.6.8), when $\rho < 0$, Black-Scholes model tends to overestimate the price of out-of-the money options and underestimate the in-the-money options. This is because, when the stock price increases, volatility tends to decrease, making it less likely that really high stock prices will be achieved. On the other hand, when the stock price decreases, volatility tends to increase and so it is more likely that really low prices will be obtained. When $\rho > 0$, the reverse happens; i.e. the Black-Scholes price is too high for in-the-money options and too low for out-of-the money options. These remarks agree with the ones made in [8] and [11].

From the results inluded in Tables (5.6.1)-(5.6.8) it is clear that the *A.D.I.* method is unconditionally stable whilst the *Explicit* scheme is only stable for sufficiently small timesteps.

For the case when $\rho < 0$, consider the maximum value of $A_{i,j}$ which, from the expression in (69), occurs when $\max_j\{y_j\} = 9$ for which $\min_i\{x_i\} = -1.7321$. Thus, $\max_{i,j}\{A_{i,j}\} = 4.05$. If we let $g_j = g_{j-1} = \delta y = 0.1$ and $h_i = h_{i-1} = \min\{\delta x\} = \delta x_1 = 0.019$, then via condition (110) we get that

$$\delta\tau = 0.000043.$$

Hence, from the results in Tables (5.6.1) and (5.6.2) it follows that the *Explicit* scheme is unstable for $\delta\tau = 0.00125,\ 0.000080$ (the results obtained for any S_0 are equal to Nan) , whilst for $\delta\tau = 0.000038$ it becomes stable.

By reducing the mesh size in the y direction and keeping the ratio $\delta\tau/(\delta y)^2$ fixed, as we did in Tables (5.6.3) and (5.6.4), again the *Explicit* scheme is unstable if not small enough step sizes are taken.

Similar arguments apply to the case when $\rho < 0$. We again observe the instability of the *Explicit* scheme over large timesteps whereas the *A.D.I* method preserves stability throughout.

To obtain the results included in this section, we wrote the programs *dim2, dim2e, dim2b, dim2eb* written in *Fortran 90* and for the Black-Scholes exact solution we used *Mathematica*.

	ρ	S_0	BS	ADI	EXPL
$\delta y = 0.100$ ($M = 80$)	0.5	4.7	0.0000	0.0312	-
$\delta x_1 = 0.019$		4.8	0.0000	0.0553	-
$\delta x_2 = 0.025$ ($N = 30$)		4.9	0.0092	0.0929	-
$\delta x_3 = 0.056$		5.0	0.0990	0.1477	-
		5.5	0.5990	0.5986	-
$\delta\tau = 0.00125$		6.0	1.0990	1.0985	-
(50 timesteps)		6.5	1.5990	1.5985	-
		7.0	2.0990	2.0987	-
$\delta y = 0.100$ ($M = 80$)	0.5	4.7	0.0000	0.0312	-
$\delta x_1 = 0.019$		4.8	0.0000	0.0553	-
$\delta x_2 = 0.025$ ($N = 30$)		4.9	0.0092	0.0929	-
$\delta x_3 = 0.056$		5.0	0.0990	0.1477	-
		5.5	0.5990	0.5986	-
$\delta\tau = 0.00008$		6.0	1.0990	1.0983	-
(2800 timesteps)		6.5	1.5990	1.5985	-
		7.0	2.0990	2.0986	-

Table 5.6.1

	ρ	S_0	BS	ADI	EXPL
$\delta y = 0.100$ ($M = 80$)	0.5	4.7	0.0000	0.0312	0.0312
$\delta x_1 = 0.019$		4.8	0.0000	0.0553	0.0553
$\delta x_2 = 0.025$ ($N = 30$)		4.9	0.0092	0.0929	0.0929
$\delta x_3 = 0.056$		5.0	0.0990	0.1476	0.1476
		5.5	0.5990	0.5984	0.5985
		6.0	1.0990	1.0981	1.0982
		6.5	1.5990	1.5985	1.5983
		7.0	2.0990	2.0982	2.0983

Table 5.6.2: $\delta\tau = 0.000038$ (6500 timesteps)

	ρ	S_0	BS	ADI	EXPL
$\delta y = 0.050$ ($M = 160$)	0.5	4.7	4×10^{-16}	0.0315	-
$\delta x_1 = 0.009$		4.8	8×10^{-8}	0.0562	-
$\delta x_2 = 0.012$ ($N = 60$)		4.9	0.0092	0.0947	-
$\delta x_3 = 0.028$		5.0	0.0990	0.1500	-
		5.5	0.5990	0.5999	-
$\delta\tau = 0.00125$		6.0	1.0990	1.0997	-
(200 timesteps)		6.5	1.5990	1.5998	-
		7.0	2.0990	2.1000	-
$\delta y = 0.050$ ($M = 160$)	0.5	4.7	4×10^{-16}	0.0326	-
$\delta x_1 = 0.009$		4.8	8×10^{-8}	0.0573	-
$\delta x_2 = 0.012$ ($N = 60$)		4.9	0.0092	0.0948	-
$\delta x_3 = 0.028$		5.0	0.0990	0.1500	-
		5.5	0.5990	0.5985	-
$\delta\tau = 0.00002$		6.0	1.0990	1.0982	-
(11200 timesteps)		6.5	1.5990	1.5983	-
		7.0	2.0990	2.0984	-

Table 5.6.3

	ρ	S_0	BS	ADI	EXPL
$\delta y = 0.050$ ($M = 160$)	0.5	4.7	4×10^{-16}	0.0319	0.0327
$\delta x_1 = 0.009$		4.8	8×10^{-8}	0.0568	0.0567
$\delta x_2 = 0.012$ ($N = 60$)		4.9	0.0092	0.0940	0.0946
$\delta x_3 = 0.028$		5.0	0.0990	0.1490	0.1492
		5.5	0.5990	0.5983	0.5987
		6.0	1.0990	1.0982	1.0983
		6.5	1.5990	1.5980	1.5982
		7.0	2.0990	2.0979	2.0982

Table 5.6.4: $\delta\tau = 0.000009$ (26000 timesteps)

	ρ	S_0	BS	ADI	EXPL
$\delta y_1 = 0.072$ $(M = 70)$	-0.5	2.01	2.5×10^{-16}	3.0×10^{-42}	-
$\delta y_2 = 0.031$		3.02	1.6×10^{-16}	1.1×10^{-19}	-
$\delta y_3 = 0.018$		4.05	7.5×10^{-17}	1.4×10^{-6}	-
$\delta x_1 = 0.097$		4.99	0.0890	0.1536	-
$\delta x_2 = 0.042$		5.50	0.5990	0.6077	-
$\delta x_3 = 0.018$		6.01	1.1090	1.1135	-
$\delta x_4 = 0.011$		6.52	1.6190	1.6215	-
$\delta\tau = 0.00125$		7.03	2.1290	2.1308	-
(200 timesteps)					
$\delta y_1 = 0.072$ $(M = 70)$	-0.5	2.01	2.5×10^{-16}	2.4×10^{-42}	-
$\delta y_2 = 0.031$		3.02	1.6×10^{-16}	1.1×10^{-19}	-
$\delta y_3 = 0.018$		4.05	7.5×10^{-17}	1.2×10^{-6}	-
$\delta x_1 = 0.097$		4.99	0.0890	0.1536	-
$\delta x_2 = 0.042$		5.50	0.5990	0.6077	-
$\delta x_3 = 0.018$		6.01	1.1090	1.1130	-
$\delta x_4 = 0.011$		6.52	1.6190	1.6216	-
$\delta\tau = 0.00010$		7.03	2.1290	2.1308	-
(1500 timesteps)					

Table 5.6.5

	ρ	S_0	BS	ADI	EXPL
$\delta y_1 = 0.072$ $(M = 70)$	-0.5	2.01	2.5×10^{-16}	2.0×10^{-42}	8.8×10^{-42}
$\delta y_2 = 0.0315$		3.02	1.6×10^{-16}	4.3×10^{-20}	4.2×10^{-20}
$\delta y_3 = 0.018$		4.05	7.5×10^{-17}	6.7×10^{-7}	7.0×10^{-7}
$\delta x_1 = 0.097$		4.99	0.0890	0.1519	0.1518
$\delta x_2 = 0.042$		5.50	0.5990	0.6080	0.6074
$\delta x_3 = 0.018$		6.01	1.1090	1.1132	1.1131
$\delta x_4 = 0.011$		6.52	1.6190	1.6243	1.6222
		7.03	2.1290	2.1343	2.1318

Table 5.6.6: $\delta\tau = 0.00001$ (25000 timesteps)

	ρ	S_0	BS	ADI	EXPL
$\delta y_1 = 0.036$ ($M = 140$)	-0.5	2.01	2.5×10^{-16}	9.8×10^{-45}	-
$\delta y_2 = 0.015$		3.02	1.6×10^{-16}	8.8×10^{-21}	-
$\delta y_3 = 0.009$		4.05	7.5×10^{-17}	6.0×10^{-7}	-
$\delta x_1 = 0.048$		4.99	0.0890	0.1552	-
$\delta x_2 = 0.021$		5.50	0.5990	0.6084	-
$\delta x_3 = 0.009$		6.01	1.1090	1.1138	-
$\delta x_4 = 0.005$		6.52	1.6190	1.6226	-
$\delta \tau = 0.00031$		7.03	2.1290	2.1317	-
(800 timesteps)					
$\delta y_1 = 0.036$ ($M = 140$)	-0.5	2.01	2.5×10^{-16}	5.8×10^{-44}	-
$\delta y_2 = 0.015$		3.02	1.6×10^{-16}	8.8×10^{-21}	-
$\delta y_3 = 0.009$		4.05	7.5×10^{-17}	6.2×10^{-7}	-
$\delta x_1 = 0.048$		4.99	0.0890	0.1552	-
$\delta x_2 = 0.021$		5.50	0.5990	0.6084	-
$\delta x_3 = 0.009$		6.01	1.1090	1.1135	-
$\delta x_4 = 0.005$		6.52	1.6190	1.6221	-
$\delta \tau = 0.00004$		7.03	2.1290	2.1312	-
(6000 timesteps)					

Table 5.6.7

	ρ	S_0	BS	ADI	EXPL
$\delta y_1 = 0.036$ ($M = 140$)	-0.5	2.01	2.5×10^{-16}	1.1×10^{-45}	2.3×10^{-45}
$\delta y_2 = 0.015$		3.02	1.6×10^{-16}	2.1×10^{-22}	7.1×10^{-22}
$\delta y_3 = 0.009$		4.05	7.5×10^{-17}	5.9×10^{-8}	6.0×10^{-8}
$\delta x_1 = 0.048$		4.99	0.0890	0.1533	0.1531
$\delta x_2 = 0.021$		5.50	0.5990	0.6087	0.6079
$\delta x_3 = 0.009$		6.01	1.1090	1.1139	1.11233
$\delta x_4 = 0.005$		6.52	1.6190	1.6249	1.6240
		7.03	2.1290	2.1350	2.1341

Table 5.6.8: $\delta \tau = 0.00004$ (100000 timesteps)

6 Conclusions

When we implemented the generalised θ-methods for the $1-d$ Black-Scholes equation, we found that the *explicit* schemes are unstable if $\delta\tau/(\delta S)^2$ does not satisfy the Fourier stability condition. All other schemes are unconditionally stable. They also produce similar results with the exception that the *Crank-Nicolson* scheme converges faster to the analytic solution.

Different transformations can be made to the Black-Scholes equation to turn it into one with constant coefficients. In [5], the transformation $Z = lnS$ is introduced so that (1) becomes

$$\frac{\partial f}{\partial t} + (r - \frac{\sigma^2}{2})\frac{\partial f}{\partial Z} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial Z^2} = rf.$$

The generalised θ-methods mentioned in this project (apart from the *Explicit* schemes) were applied to the transformed equation along with the *Semi-Lagrangian* scheme. The *Kenneth-Vetzal* scheme gave more accurate results than any other method, whilst the *Semi-Lagrangian* scheme started to diverge for relatively small timesteps.

Moreover, in [17], the Black-Scholes equation is transformed into the diffusion equation;

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial x^2},$$

where $t = T - \tau/\frac{1}{2}\sigma^2$, $x = logS/(k+1)\frac{1}{2}$ and $k = r/\frac{1}{2}\sigma^2$.

In this paper the *Fully Implicit*, *Crank-Nicolson* and *Explicit 1* schemes for European put options were tested. In agreement with the results obtained here, the *Crank-Nicolson* scheme was found to be more accurate than the *Fully Implicit* scheme. Also, the *Explicit 1* scheme was unstable when big timesteps were taken.

To approximate the extension of the Black-Scholes model, we applied the *A.D.I* method and the *Explicit* scheme. We compared the Black-Scholes price for call options with the one obtained when the volatility is stochastic. In result, we found that the correlation ρ has a great impact on the out-of-the-money and in-the-money options. When $\rho < 0$, the Black-Scholes overprices the out-of-the-money options and underprices the in-the-money options. When $\rho > 0$, the reverse holds.

The *A.D.I* scheme is shown to be unconditionally stable, whilst the *Explicit* one becomes unstable if relatively large timesteps are taken.

In this project, due to time restrictions, we considered the cases where the correlation is either negative or positive. Thus, further study can be made on zero correlation. Also, numerical approximations can be derived when the $2-d$ partial differential equation (68) is used to price European put options and American options.

Lastly, different numerical methods can be tested on the extension of the Black-Scholes model without transforming it and thus keeping the mixed derivative $\partial/\partial V \partial S$.

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