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On Best Piecewise Linear L_2 Fits with Adjustable Nodes

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Numerical Analysis Report 6/90

U N I V E R S I T Y   O F   R E A D I N G

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The work reported here forms part of the research programme of the Oxford/Reading Institute for Computational Fluid Dynamics.

Abstract

In this report a simple procedure is used to determine the best continuous piecewise linear L_2 fit to a convex function of a single variable with adjustable nodes. An extension gives a very good continuous piecewise linear L_2 fit to non-convex functions, again with adjustable nodes.

§1. Theory

Let $f(x)$ be a given function of x and denote by $u_k(x)$ the best linear L_2 fit to $f(x)$ in the interval (x_{k-1}, x_k).

Then

$$\delta \int_{x_{k-1}}^{x_k} \left\{f(x)-u_k(x)\right\}^2 dx = 0 \qquad u_k \in S_k \tag{1}$$

or

$$\delta \int_{x_{k-1}}^{x_k} \left\{f(x)-u_k(x)\right\} \delta u_k(x)\, dx = 0 \qquad \delta u_k(x) \in S_k \tag{2}$$

where S_k is the family of straight lines on the interval (x_{k-1}, x_k). For an interval (x_o, x_{N+1}) which is the union of intervals (x_{k-1}, x_k), $(k=1, n+1)$, the best L_2 fit to $f(x)$ amongst piecewise linear functions discontinuous at x_k, $(k=1,n)$, is also given by (1) and (2), $(k=1, n+1)$, since the problems decouple.

Now consider the problem of determining the best L_2 fit $u(x)$ to $f(x)$ amongst all discontinous piecewise linear finctions on the fixed interval (x_o, x_{n+1}) on a variable partition $(x_1, x_2, \ldots, x_k, \ldots, x_n)$ of the interval. Then

$$\delta \int_{x_o}^{x_{n+1}} \left\{f(x)-u(x)\right\}^2 dx = \delta \sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} \left\{f(x)-u(x)\right\}^2 dx = 0 \tag{3}$$

where the x_k, $(k=1, n)$, are also varied. It is convenient to introduce here a new independent variable ξ which remains fixed, while x joins u as a dependent variable, both now depending on ξ and denoted by $\hat{x}$ and $\hat{u}$. Then (3) becomes

$$\delta \sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} \left\{f(\hat{x}(\xi))-\hat{u}(\xi)\right\}^2 \frac{d\hat{x}}{d\xi}\, dx = 0 \tag{4}$$

with $\hat{u}(\xi) = \hat{u}(x(\xi))$.

Taking the variations of the integral in (4) gives

$$\int\left\{2\left\{f(\hat{x}(\xi))-\hat{u}(\xi)\right\} \left\{f'(\hat{x}(\xi))\ \delta\hat{x}-\delta\hat{u}(\xi)\right\} \frac{d\hat{x}}{d\xi} + \left\{f(\hat{x}(\xi))-\hat{u}(\xi)\right\}^2 \frac{d}{d\xi}\,(\delta\hat{x})\right\} d\xi . \tag{5}$$

Integrating the last term by parts leads to

$$- \int 2 \left\{f(\hat{x}(\xi)) - \hat{u}(\xi)\right\} \left\{f'(\hat{x}(\xi)) \frac{d\hat{x}}{d\xi} - \frac{d\hat{u}}{d\xi}\right\} \delta\hat{x}\ d\xi$$

$$+ \sum_{k=1}^{n+1} \left\{(f(\hat{x}(\xi)) - \hat{u}(\xi))^2_{k-1}\ \delta\hat{x}_{k-1} + f(\hat{x}(\xi)) - \hat{u}(\xi))^2_k\ \delta\hat{x}_k\right\} . \tag{6}$$

Collecting terms and returning to the x,u notation, (4) yields

$$\sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} 2\left\{f(x) - u(x)\right\} \left\{\delta u - u_x \delta x\right\} dx + \sum_{k=1}^{n} \left[(f(x) - u(x))^2\right]_k \delta x_k = 0 \tag{7}$$

where the square bracket notation $[\,]_k$ denotes the jump in the quantity at the node k .

With $\delta x = 0$ this leads back to (2) and equations for the best piecewise linear discontinuous L_2 fit to $f(x)$. The full conditions are however

$$\int_{x_{k-1}}^{x_k} \left\{f(x)-u(x)\right\}\delta u \; dx = 0 \tag{8}$$

$$\int_{x_{k-1}}^{x_k} 2\left\{f(x)-u(x)\right\}(-u_x)\;\delta x_k dx + \left[(f(x_k) - u(x_k))^2\right]_k \delta x_k = 0 \quad \forall k \; . \tag{9}$$

With δu in the space of piecewise linear discontinuous functions the orthogonality condition (8) is equivalent [1] to the conditions

$$\int_{x_{k-1}}^{x_k} \left\{f(x)-u(x)\right\} \phi_{k1} \; dx = 0 \tag{10}$$

$$\int_{x_{k-1}}^{x_k} \left\{f(x)-u(x)\right\} \phi_{k2} \; dx = 0 \tag{11}$$

where ϕ_{k1}, ϕ_{k2} are the half linear basis functions in element k (see fig. 1). On the other hand, since δx lies in the space of piecewise linear continuous functions, we may set $\delta x = \alpha_k$, $\delta u = u_x \delta x$ in (7) to obtain

$$\left[\left[f(x_k)-u(x_k)\right]^2\right]_k = 0 \; . \tag{12}$$

Using L, R for left and right values at the (variable) node k, it follows from (13) that either

$$f - u_L = f - u_R \Longrightarrow u_L = u_R \tag{13}$$

and u is continuous at the new position of node k, or that

$$(f - u_L) = f - u_R \Longrightarrow u_L + u_R = 2f \tag{14}$$

there.

Now it is known [2],[3] that for convex functions $f(x)$ the best L_2 fit amongst discontinuous piecewise linear functions is continuous, which clearly corresponds to (13). The case leading to (14) cannot therefore correspond to convexity in $f(x)$ and may apply only at inflection points.

It follows that the solution of the problem (10),(11),(12) is the set of best linear fits in separate elements which have the continuity property (13) or the averaging property (14), the former in the presence of convexity of $f(x)$.

§2. The Algorithm

The algorithm used here to find the best L_2 fit with variable nodes is in two stages (carried out repeatedly until convergence), corresponding to the choices of variations referred to in §1 above.

Stage (i) $\quad \delta x = 0, \quad \delta u = \phi_{k1}$ or ϕ_{k2} $\quad (k=1,2,\ldots, n+1) \qquad (15)$

This stage of the algorithm corresponds to the best L_2 fit amongst linear functions discontinuous at prescribed nodes, as in (1),(2).

$$\text{Stage (ii)} \qquad \delta x = \alpha_k , \qquad \delta u - u_x \delta x = 0 \quad (k=1,2,\ldots, n+1) \tag{16}$$

This stage corresponds to finding x_k such that (12) holds, with variations of x,u restricted to points lying on the piecewise linear approximation (possibly linearly extrapolated) in element k.

The algorithm is analogous to minimising a quadratic function $f(x,y)$ using two search directions v1 and v2 spanning the plane. Starting from some initial guess we may alternately minimise f in the directions v1 and v2. Similarly, to find the best L_2 fit we may begin with an initial guess $\{x_k\},\{u_k\}_L,\{u_k\}_R$. Stage (i) is to find the minimum in the linear manifold specified by the variations given in (15) and so solve (10)-(11) for new $\{x_k\},\{u_k\}_L,\{u_k\}_R$ with the x_k fixed. Stage (ii) is to find the minimum in the linear manifold specified by the variations given in (16) and so solve (12) for new $\{x_k\},\{u_k\}_L,\{u_k\}_R$ by the implementation of (13),(14), more fully described below.

For regions in which $f(x)$ is convex the solution for x_k is provided by (13), i.e. the intersections of lines in adjacent elements (see fig. 2). In this case $f(x_k) - u(x_k)$ is of the same sign when approached from left or right. Where $f(x)$ has an inflection point the intersection construction may fail and need to be replaced by the averaging construction (14). This will occur when values of $f(x_k) - u(x_k)$ are of opposite sign when approached from left or right, as in fig. 3. Note that the calculation of x_k from (14) is implicit since f depends on x_k and u_L, u_R are new values, but the main iteration may be used to move towards the converged x_k by simply using the previous x_k and u values.

If $f(x)$ is convex the result of the converged iteration (stage (i) - stage (ii) - repeated) is the grid with the best <u>continuous</u> L_2 fit using piecewise linear approximation. If $f(x)$ is not convex there will in general be discontinuities in the fitted function but only at inflection points. It is simple to replace such a discontinuity locally by a continuous approximation (by say simply averaging the nodal values - in which case the result is the function value). This is of course at the expense of slightly moving away from the best fit minimisation at isolated points; the resulting approximation may however be used as an initialisation for more thorough algorithms [3].

The L_2 error of the fit described here is never worse than the error of the interpolant u_I which is well known [4] to satisfy

$$||u_I - f||_2 \leq \frac{n^{-2}}{6}||f''||_2 \tag{17}$$

on $(0,1)$. (See also Appendix).

§3 Results

We show results for five examples,

(a)	$e^{-20(1-x)}$	$0 \leq x \leq 1$	11 interior nodes
(b)	$\tanh\{20(x-0.5)\}$	$0 \leq x \leq 1$	11 interior nodes
(c)	$\sin 2\pi x$	$0 \leq x \leq 1$	11 interior nodes
(d)	$\sin 2\pi x$	$0 \leq x \leq 1$	10 interior nodes .

(e) $\begin{cases} e^x & 0 \leq x \leq 0.5 \\ (1.5-x)\, e^x & 0.5 \leq x \leq 1 \end{cases}$

11 interior nodes

In each case the initial grid is equally spaced. Examples (c) and (d) distinguish between the constructions (13) & (14) (see figs. 2 and 3).

In each example the trajectories of the nodes towards the final positions are shown together with the function and the fit obtained. The process is said to have converged when the ℓ_∞ norm of the nodal position updates is less than 10^{-4} . The number of iterations appears on the ordinate axis of the trajectories.

§4. References

[1] Baines, M.J. (1986) Local Adaptive Moving Finite Elements. In Numerical Methods for Fluid Dynamics II (Morton & Baines (eds)). O.U.P.

[2] Chui, C.K., Smith P.W. & Ward, J.D. (1977). On the Smoothness of Best L_2 Approximants from Nonlinear Spline Manifolds, Math. Comp., 31, 17-23.

[3] Loach, P.D. & Wathen A.J. (1988). On the Best Least Squares Approximation of Continuous Functions using Linear Splines with Free Knots. Report No. 88-14, School of Mathematics, University of Bristol, U.K.

[4] See e.g. Porter D. & Stirling D.S.G. (1990). Integral Equations; A Practical Treatment from Spectral Theory to Applications. CUP.

[5] Carey G.F. & Dinh, H.T. (1985). Grading Functions and Mesh Redistribtuion. Siam J. Numer. An. 22, 1028.

[6] Wixcey, J. (1990). Ph.D. Thesis (Reading).

Appendix A

In this appendix, following [5], we give an asymptotic equidistribution result for the convex case. From (11) and (12) it follows that $u-f$ vanishes at at least two points in each element, s_k and t_k say. Hence $u'-f'$ vanishes at at least one point in each element, r_k say. Then, since $u'' = 0$,

$$\int_{r_k}^{x} f''\, d\xi = \int_{r_k}^{x} (f'' - u'')d\xi = f'(x) - u'(x) \tag{A1}$$

and

$$\int_{s_k \text{ or } t_k}^{x} (f' - u')d\eta = f(x) - u(x) \ . \tag{A2}$$

Hence

$$\int_{x_{k-1}}^{x_k} (f-u)^2 dx = \int_{x_{k-1}}^{x_k} \left\{ \int_{s_k \text{ or } t_k}^{x} \int_{r_k}^{\eta} f''(\xi)d\xi \right\}^2 dx \tag{A3}$$

$$\leq \int_{x_{k-1}}^{x_k} \left\{ (x_k - x_{k-1})^2 \, f''_{\max,k} \right\}^2 dx \tag{A4}$$

where $f''_{\max,k}$ is the maximum norm of f'' in element k.

Now, if $E(x)$ is an equidistributing function

$$(x_k - x_{k-1})\, E(\theta_k) = \text{a constant, } C \ , \tag{A5}$$

where $x_{k-1} < \theta_k < x_k$, and we have

$$\int_{x_{k-1}}^{x_k} (f-u)^2 dx \leq C^4 \int_{x_{k-1}}^{x_k} \left\{ E(\theta_k) \right\}^{-4} \left\{ f''_{\max,k} \right\}^2 dx \tag{A6}$$

so that

$$\int_{x_o}^{x_n} (f-u)^2 dx \leq C^4 \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \left\{E(\theta_k)\right\}^{-4} \left\{f''_{max,k}\right\}^2 dx \; . \tag{A7}$$

Finally, as in [5], we approximate the right hand side of (A7) by the integral

$$C^4 \int_{x_o}^{x_n} \left\{E(x)\right\}^{-4} \left\{f''_{max,k}\right\}^2 dx \; . \tag{A.8}$$

and minimise over functions $E(x)$, yielding

$$\frac{d}{dx}\left[\left\{E(x)\right\}^{-5} \left\{f''(x)\right\}^2\right] = 0 \tag{A9}$$

or

$$E(x) = \left\{f''(x)\right\}^{2/5} \tag{A.10}$$

which may be regarded as the asymptotically equidistributed function.

Appendix B

In this appendix we extend the result in the main body of the report to general extremals.

For the problem of finding the extremal of the integral

$$\int F(x,u)dx \tag{B1}$$

over piecewise linear discontinuous functions $u(x)$ with variable nodes, we folllow the same procedure as in §1, obtaining

$$\int_{x_{k-1}}^{x_k} F_u(x,u)\delta u \; dx = 0 \tag{B2}$$

$$\int_{x_{k-1}}^{x_k} F_u(x,u)(-u_x)\delta u \; dx + \Big[F(x,u)\Big]_k \delta x_k = 0 \qquad \forall k \tag{B3}$$

in place of (8) and (9). Then (10),(11) and (12) become

$$\int_{x_{k-1}}^{x_k} F_u(x,u) \; \phi_{k,i} \; dx = 0 \qquad i = 1,2 \tag{B4}$$

$$\Big[F(x,u)\Big]_k = 0 \; . \tag{B5}$$

The corresponding algorithm is to solve (B4) for u in each element with fixed x_k (stage (i)) and then to solve (B5) for the x_k with u restricted to the stage (i) solution, possibly extrapolated (stage (ii)). Both problems are nonlinear and may or may not have unique solutions. An example in which

$$F(x,u) = Q(x).u + p(u) \tag{B6}$$

where $Q(x)$ is the given mass flow in a nozzle and $u, p(u)$ are the velocity, pressure has been treated in [6].

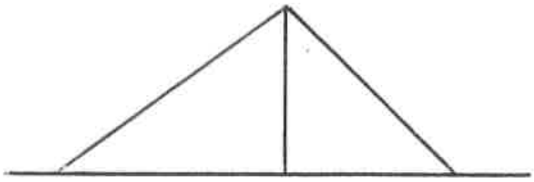


Fig. 1a α_{ik}

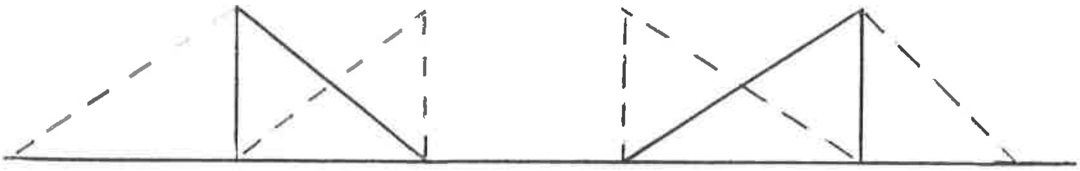


Fig. 1b ϕ_{k1} ϕ_{k2}

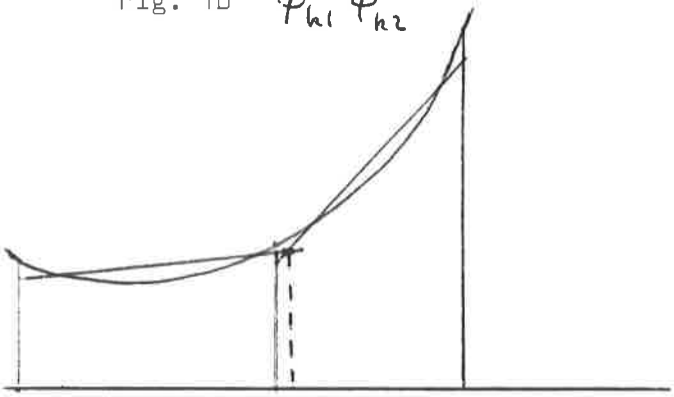


Fig. 2

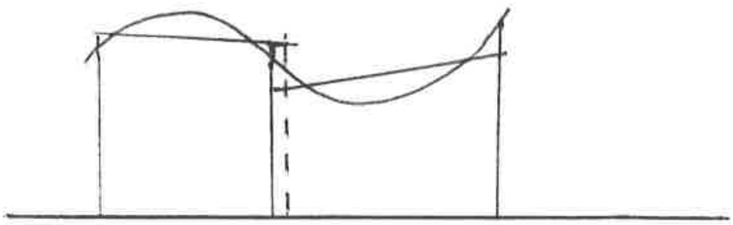
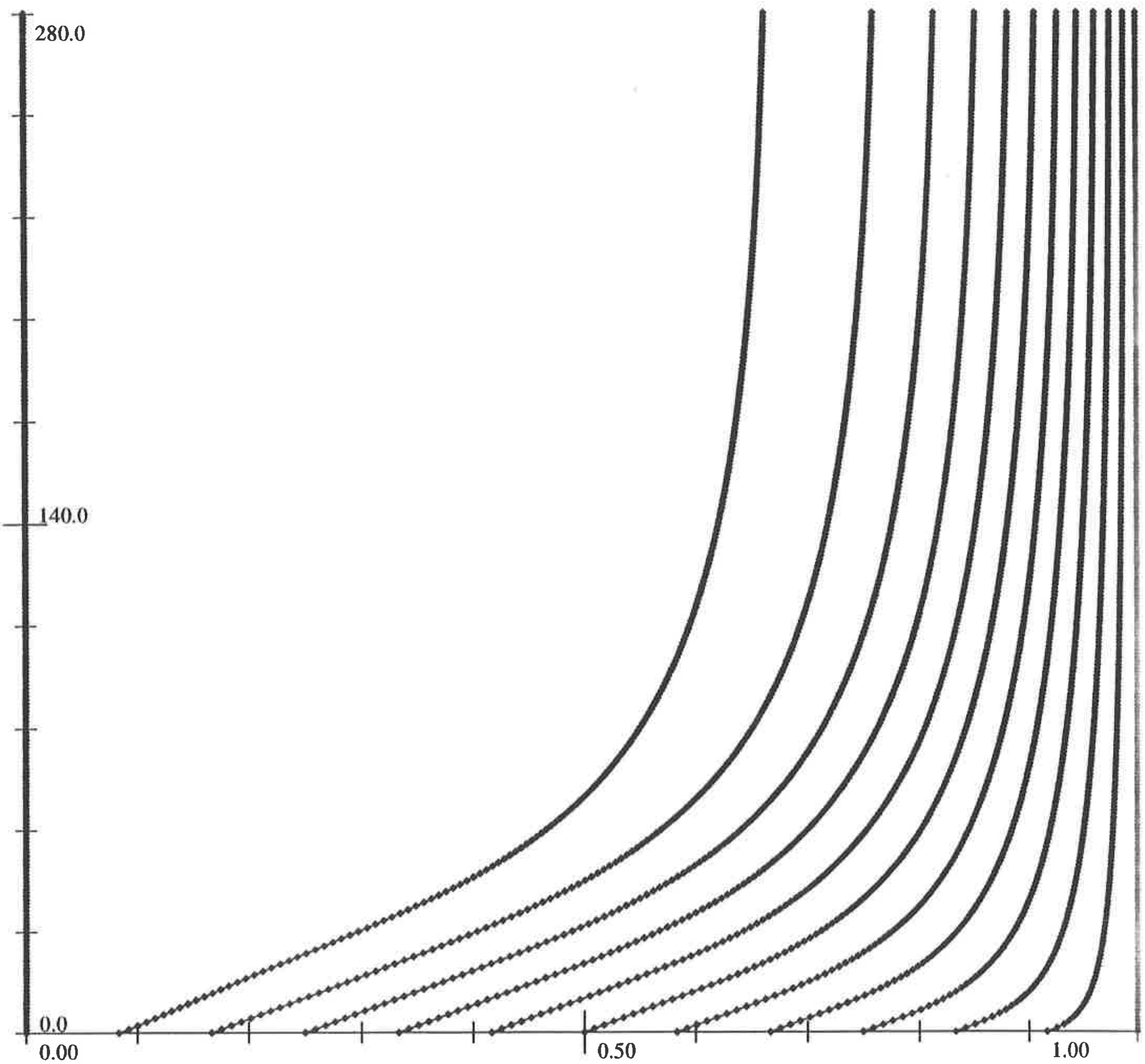


Fig. 3



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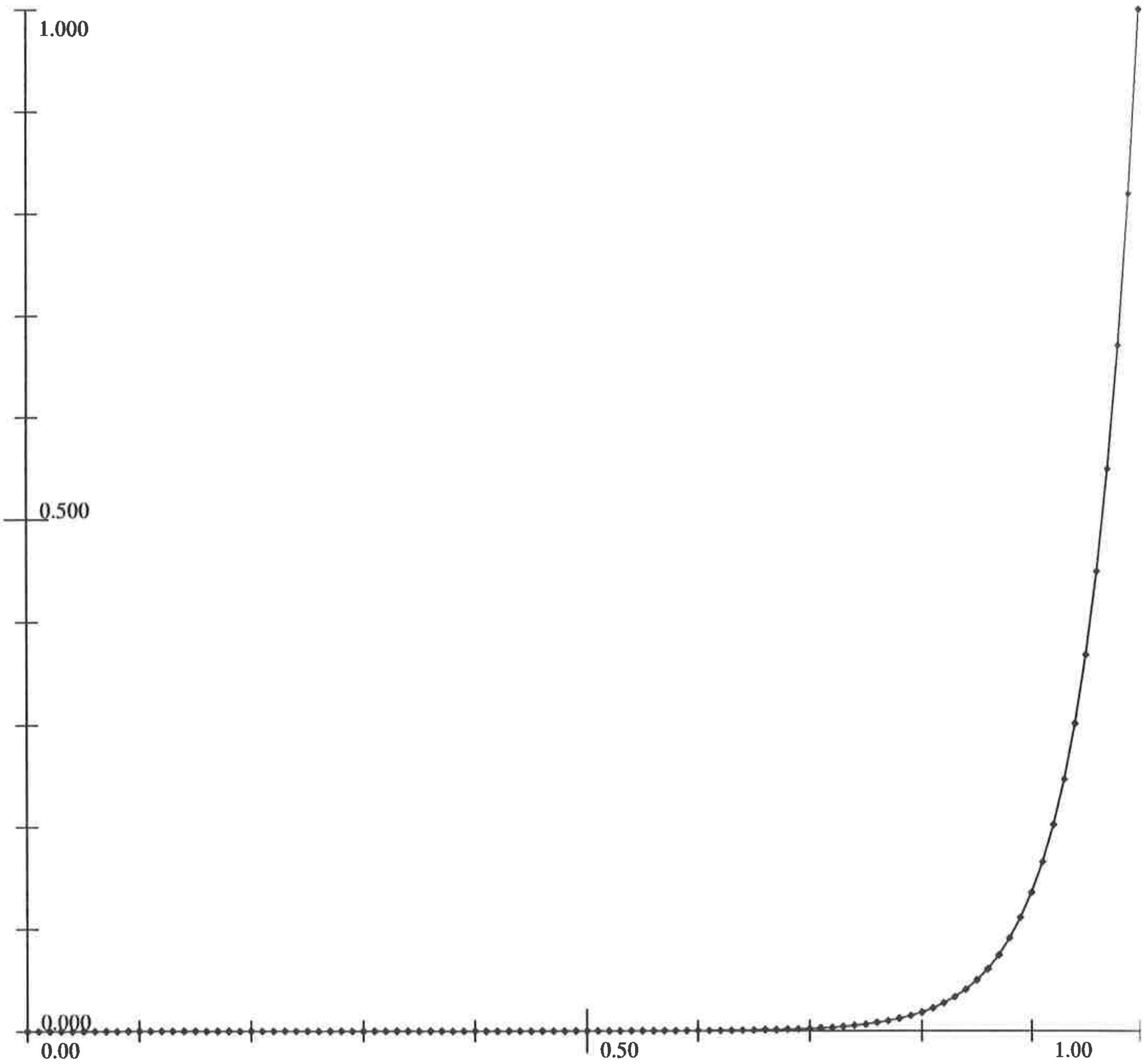

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$e^{-20(1-x)}$

function

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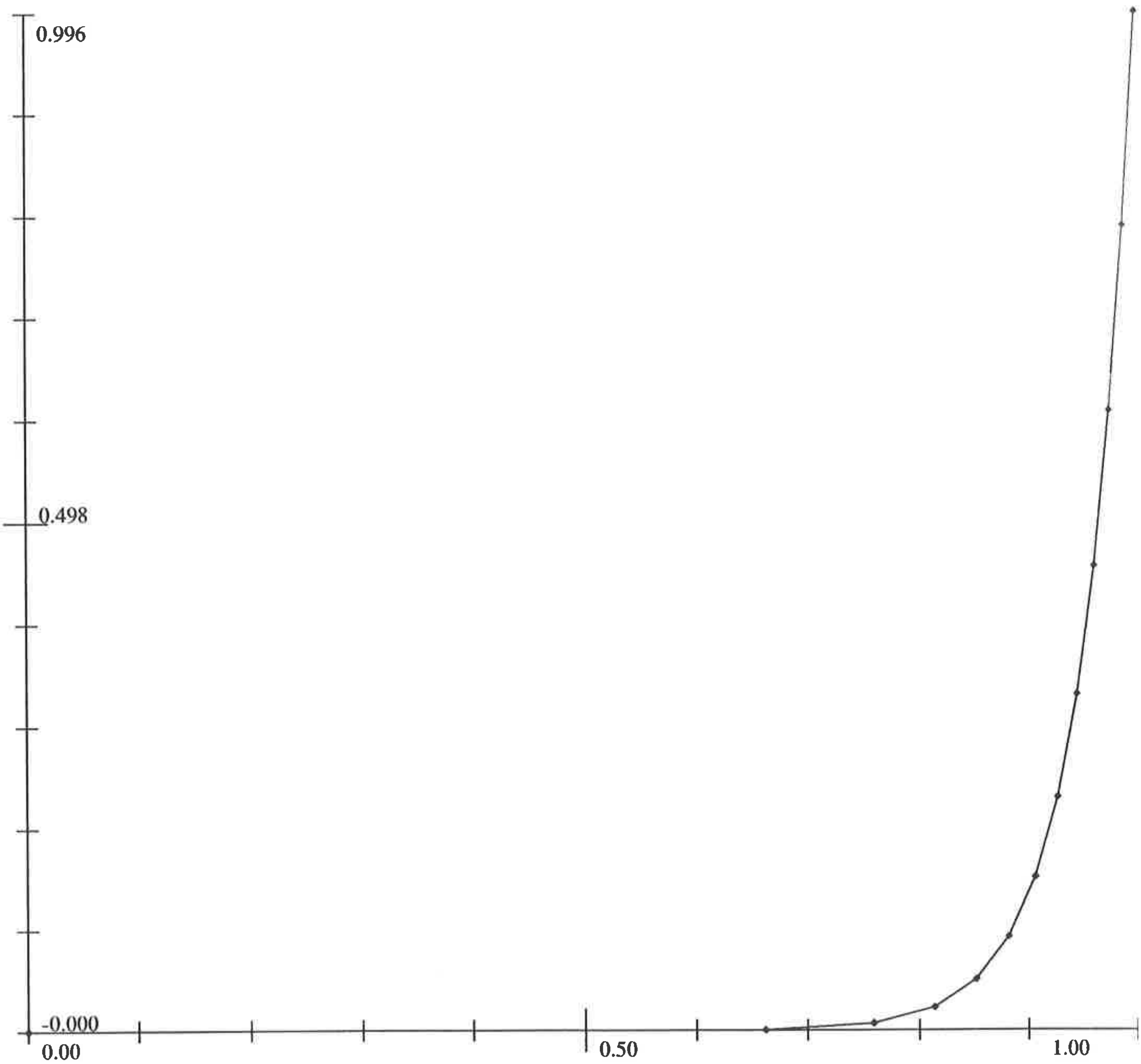

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$e^{-20(1-x)}$

fit

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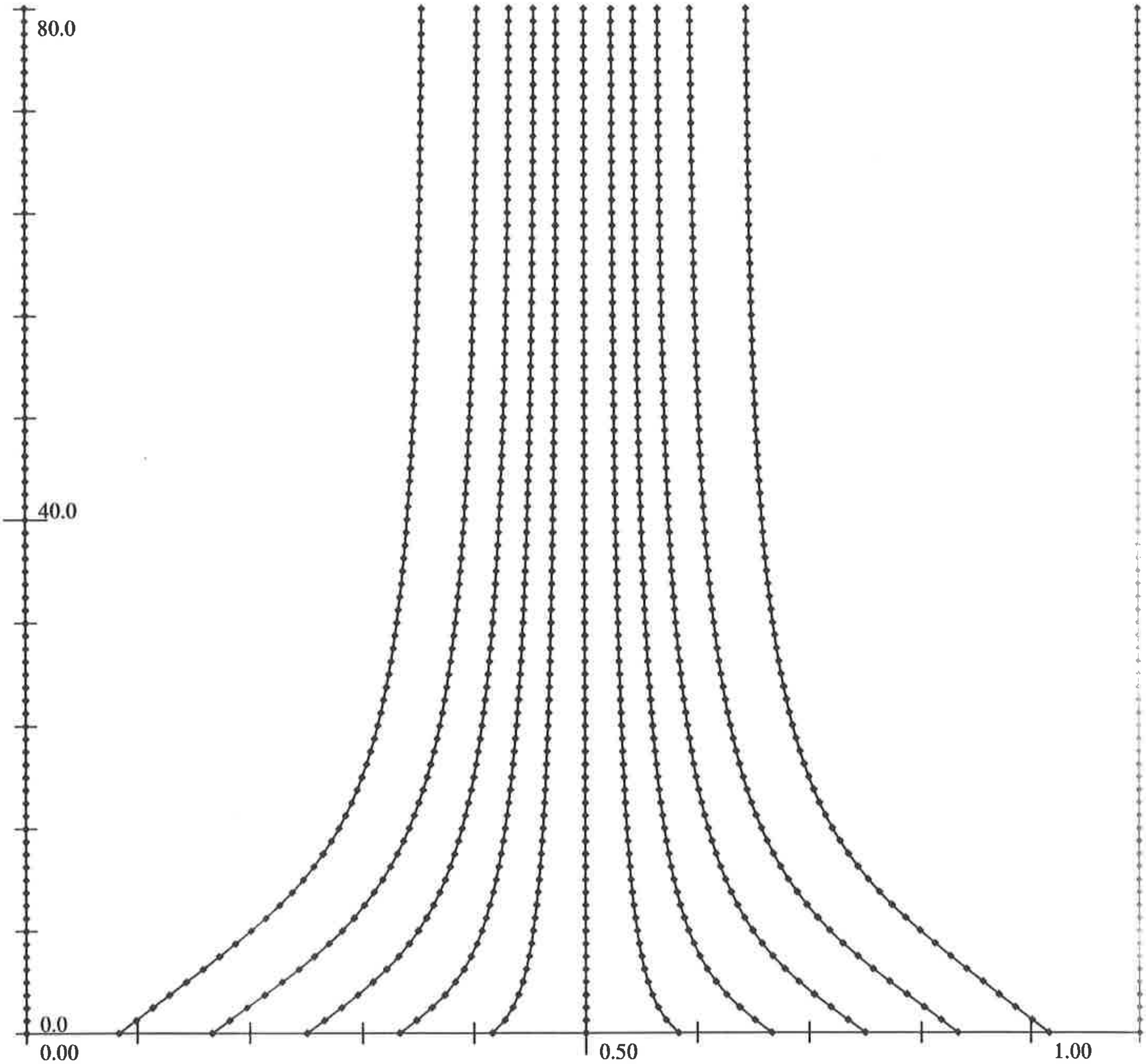

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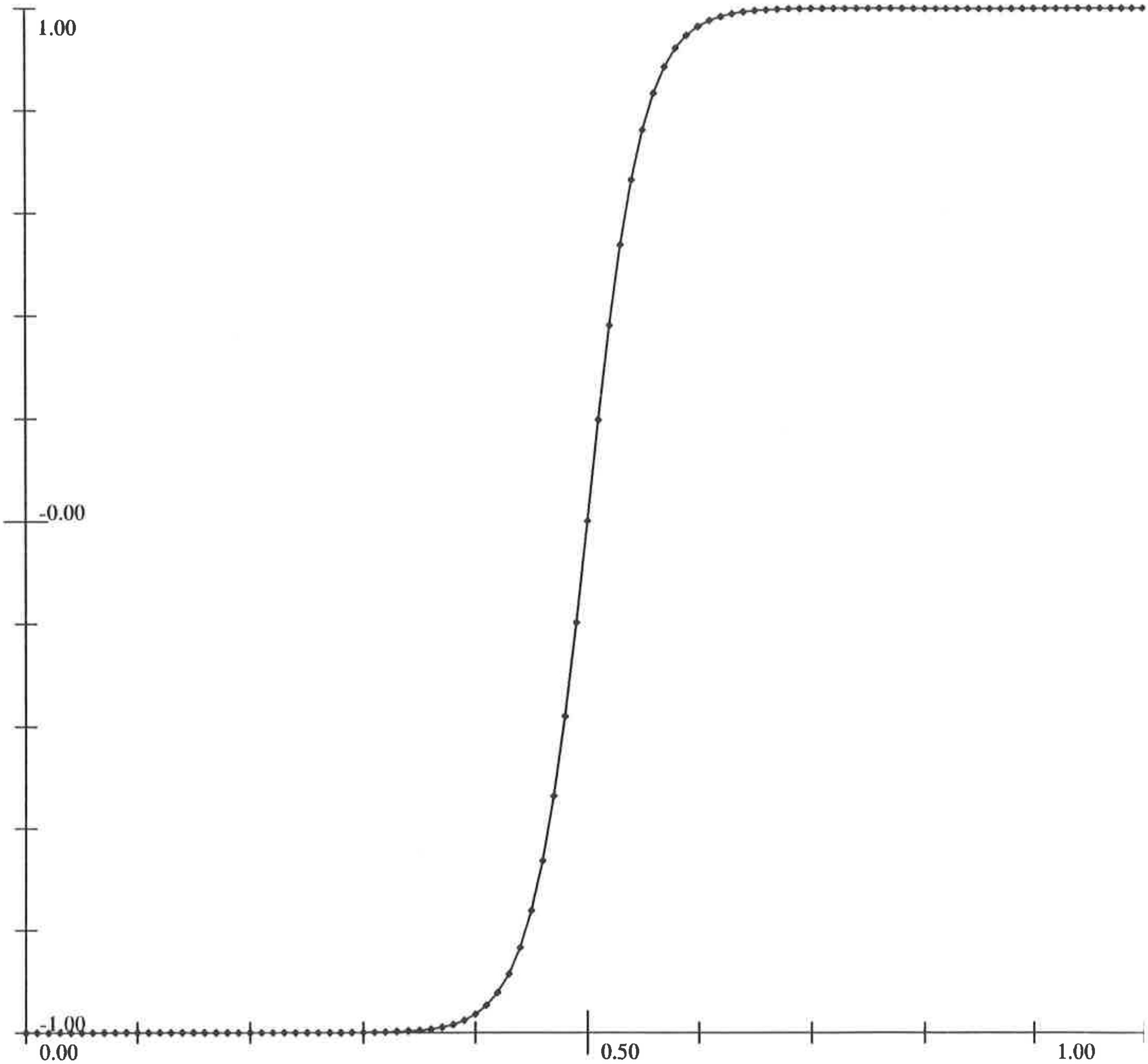

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function

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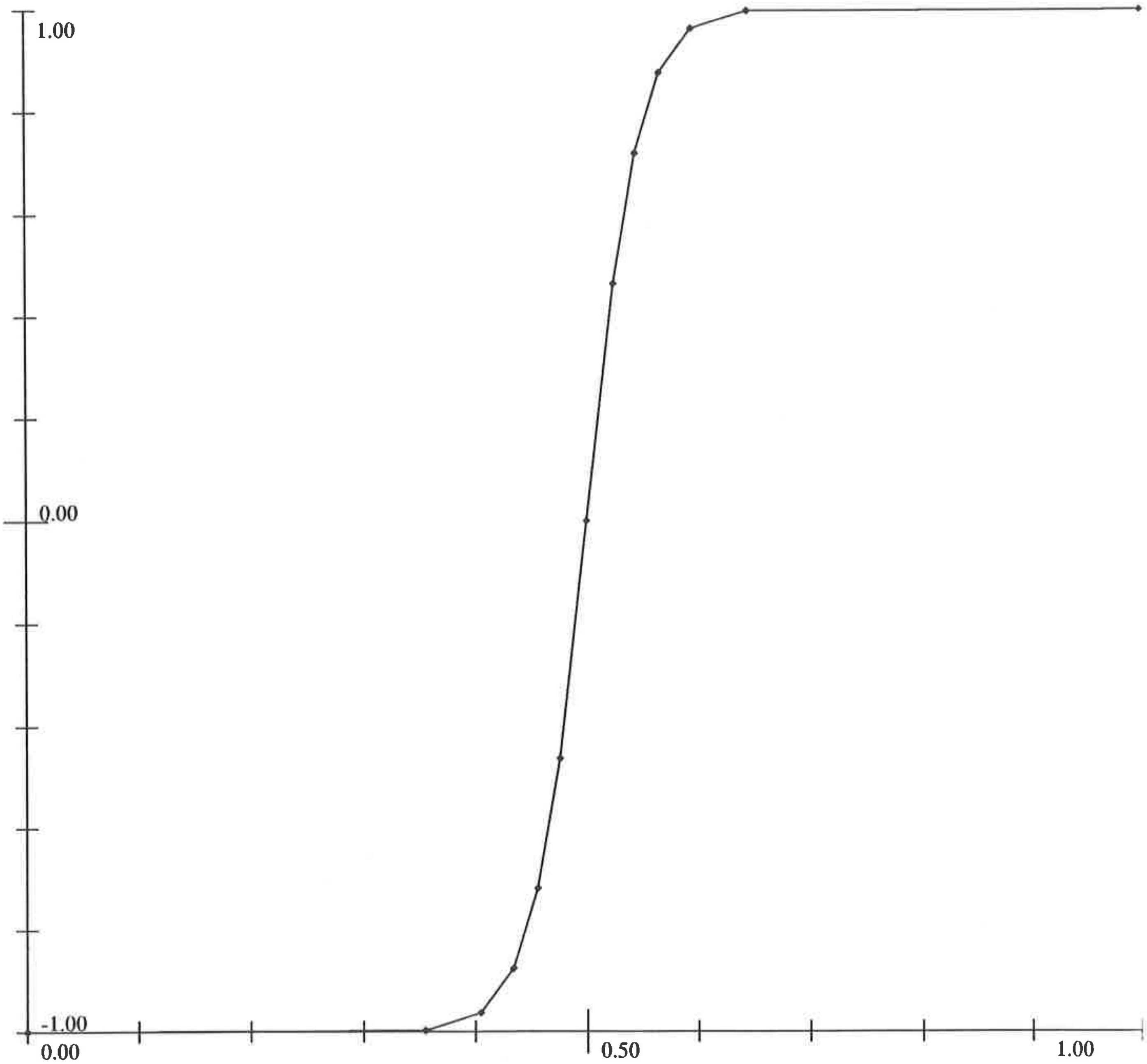

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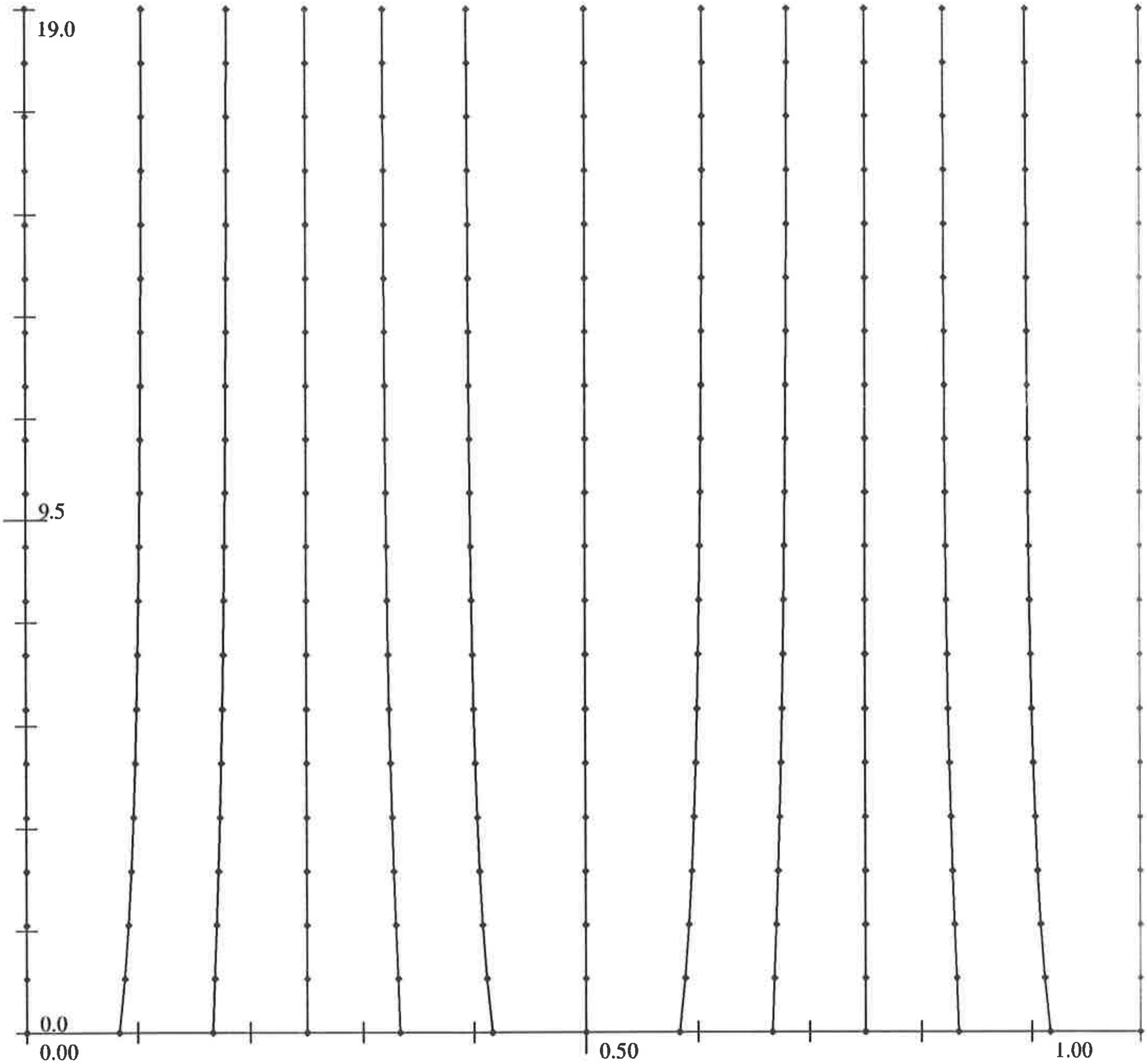
tanh 20 (x - 0.5)
fit



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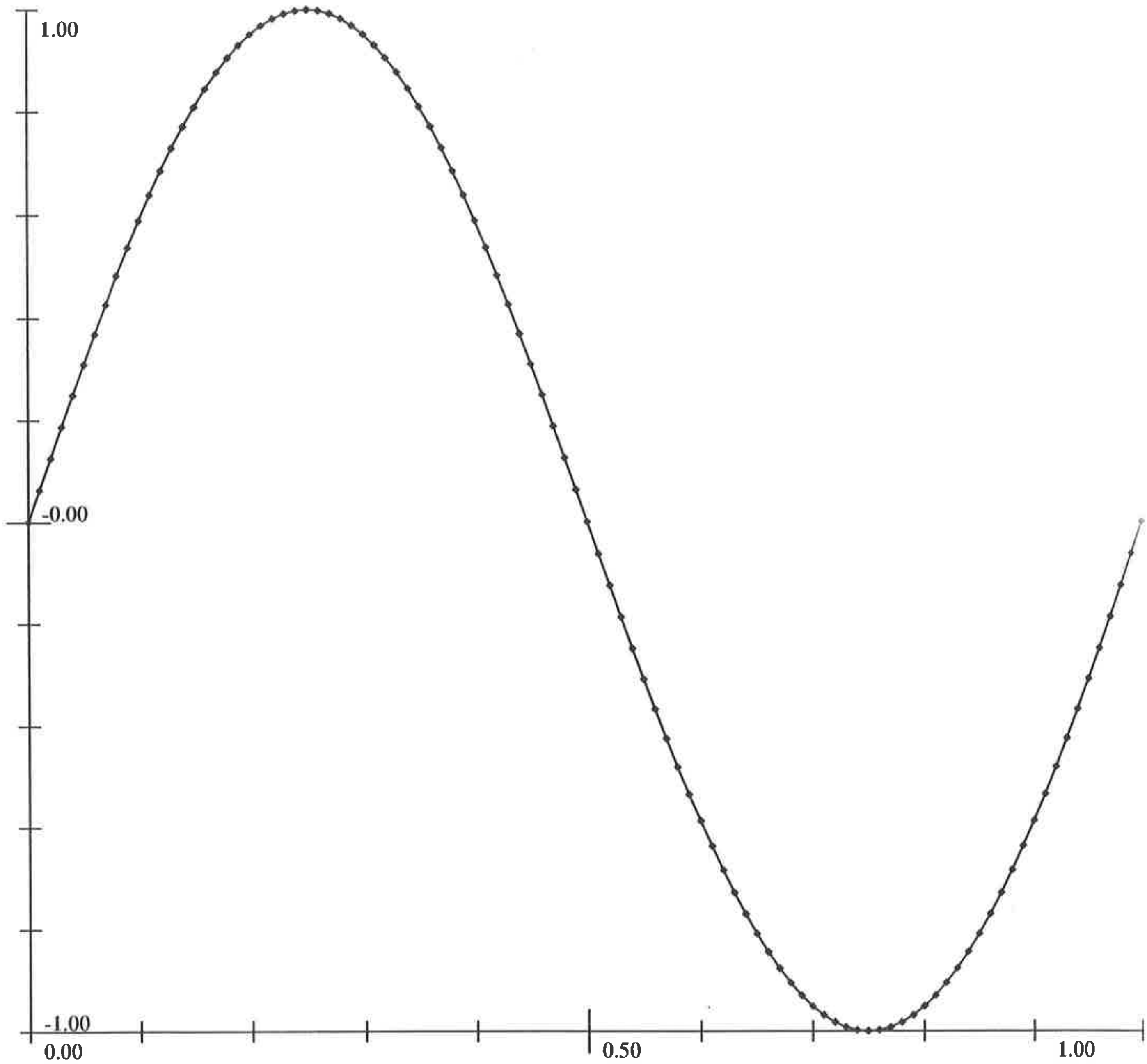

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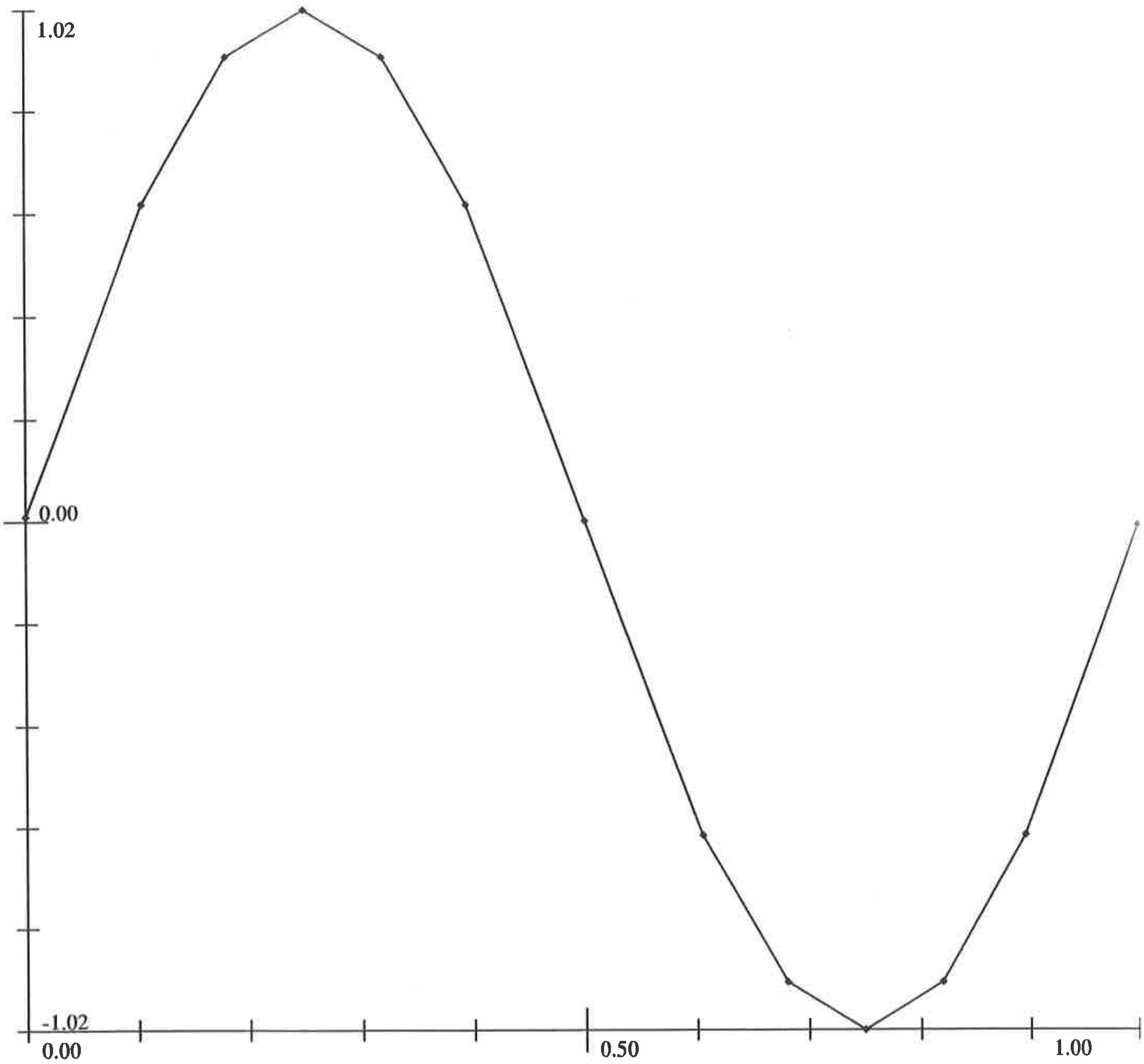

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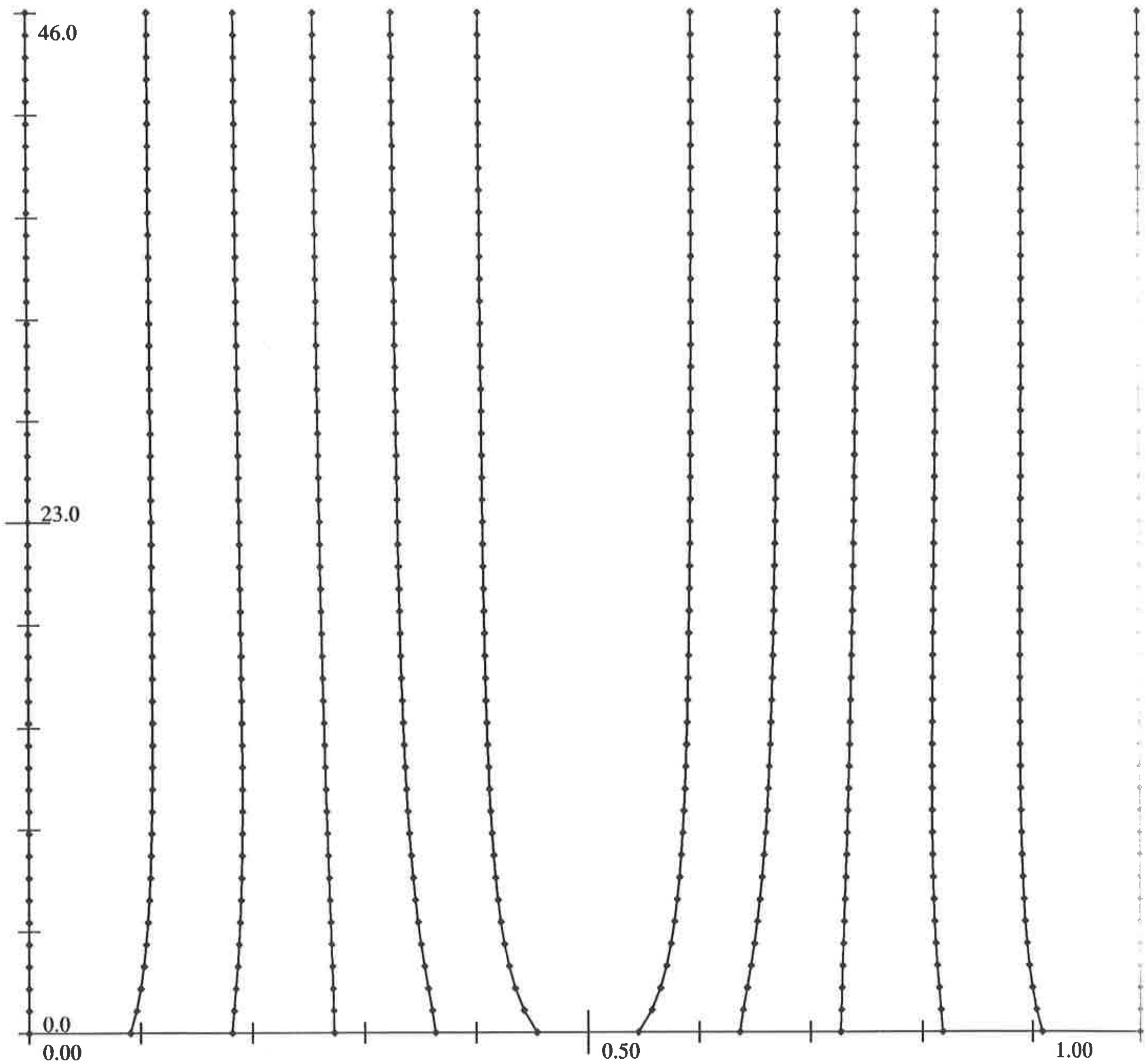

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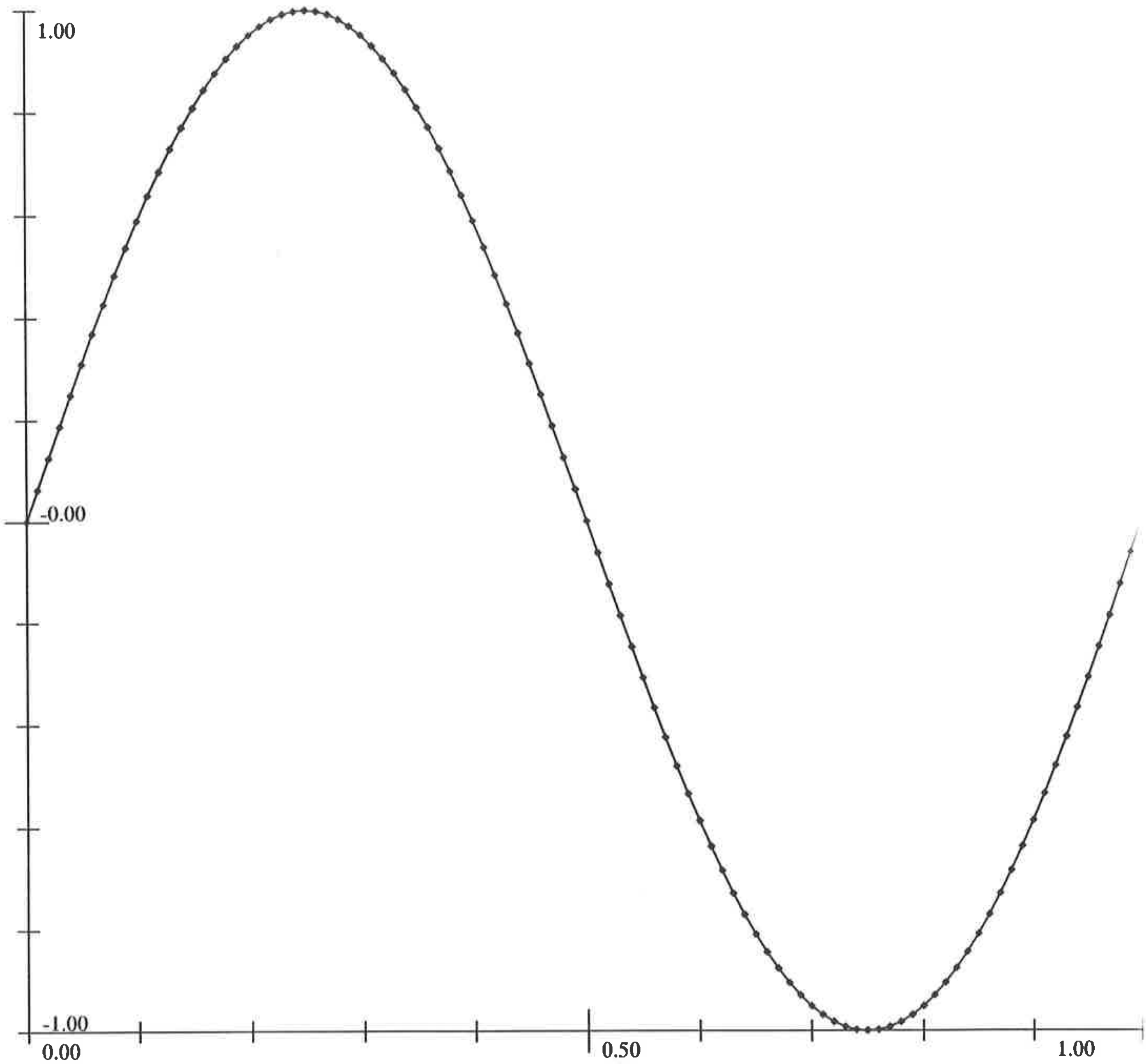

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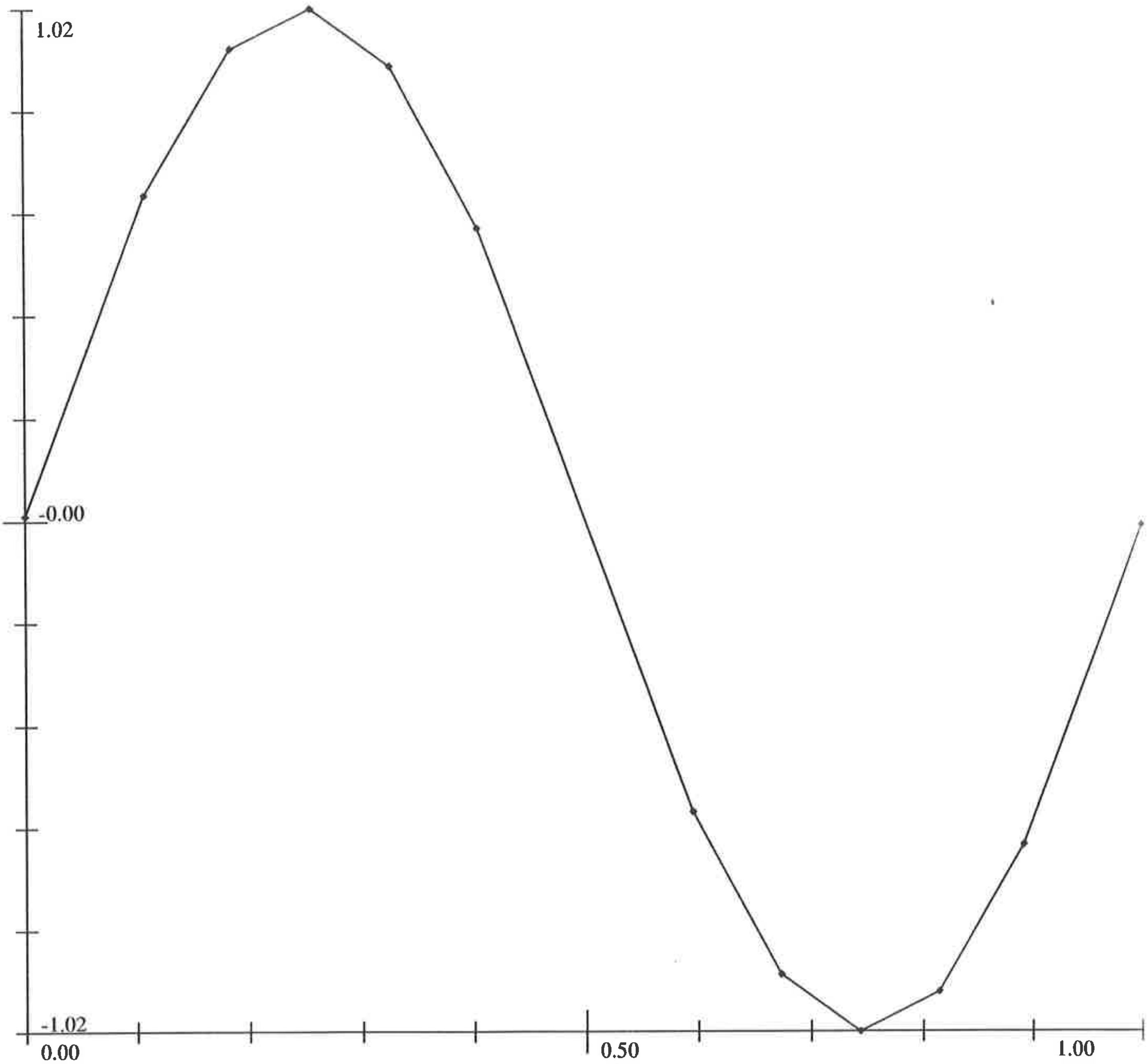
sin 2πx
function



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sin $2\pi x$

fit

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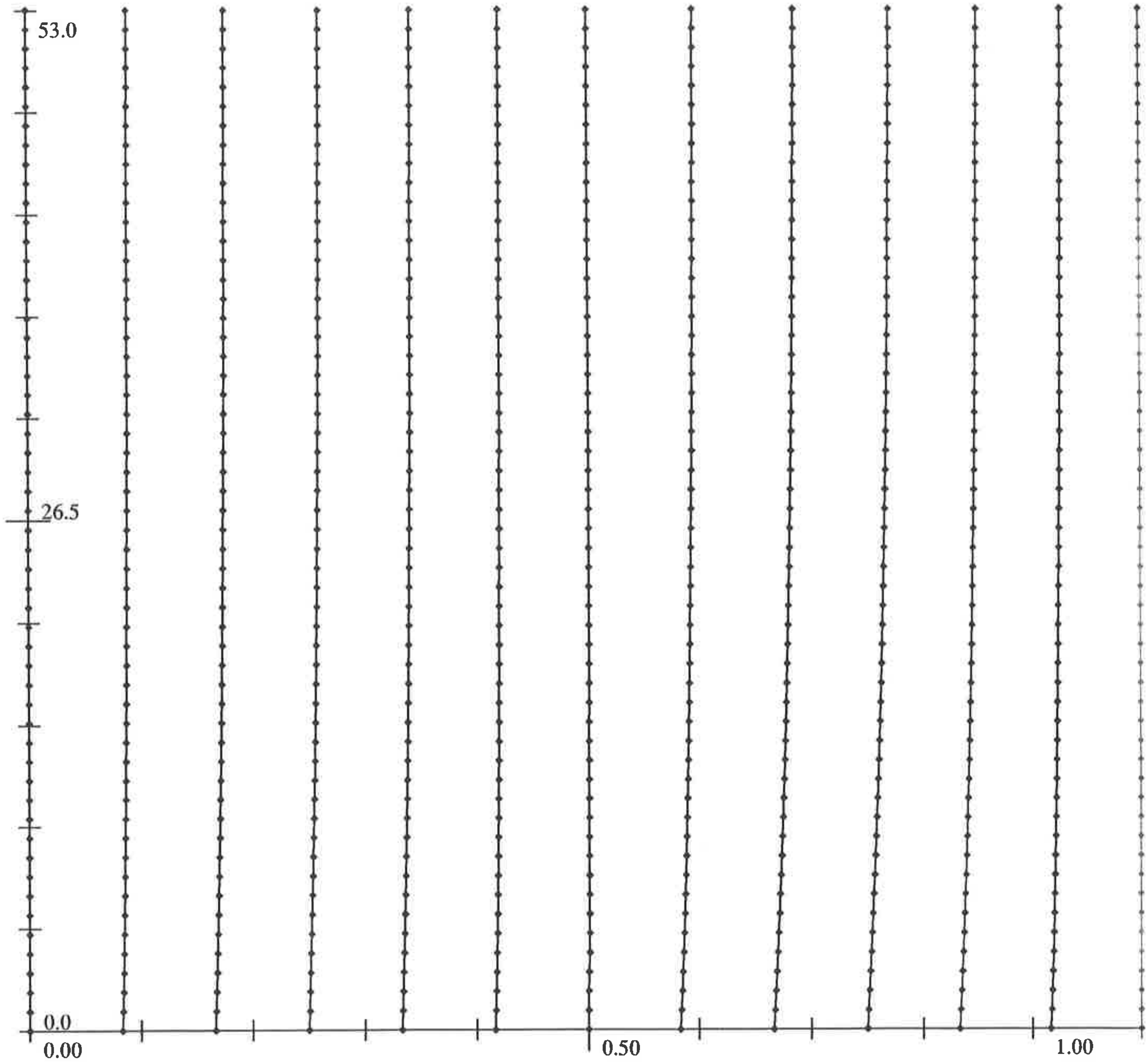

Files used:
exp(1-15)

$$\begin{cases} e^{x} & 0 \leq x \leq 0.5 \\ (1.5-x)e^{x} & 0.5 \leq x \leq 1 \end{cases}$$

trajectories

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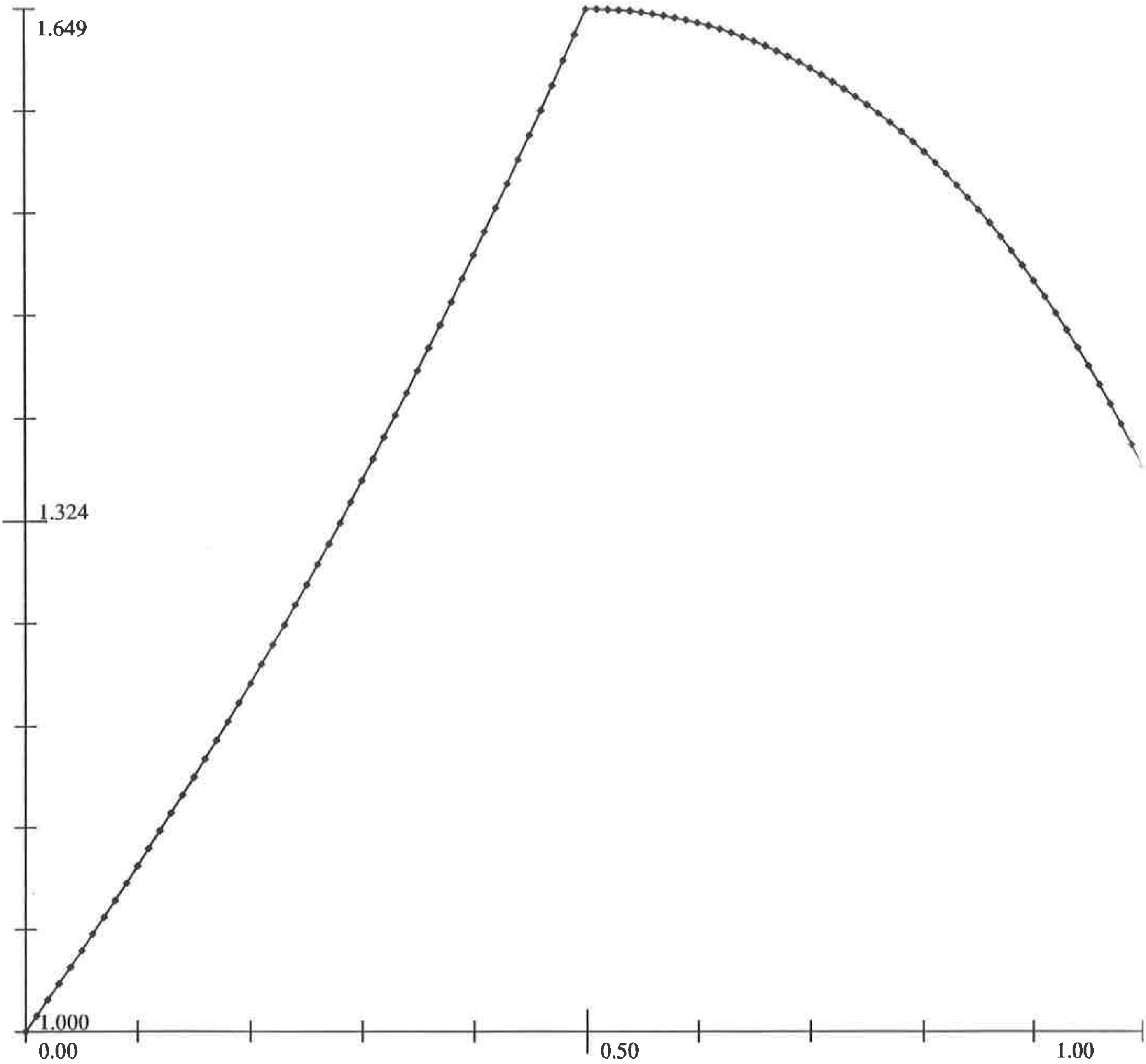

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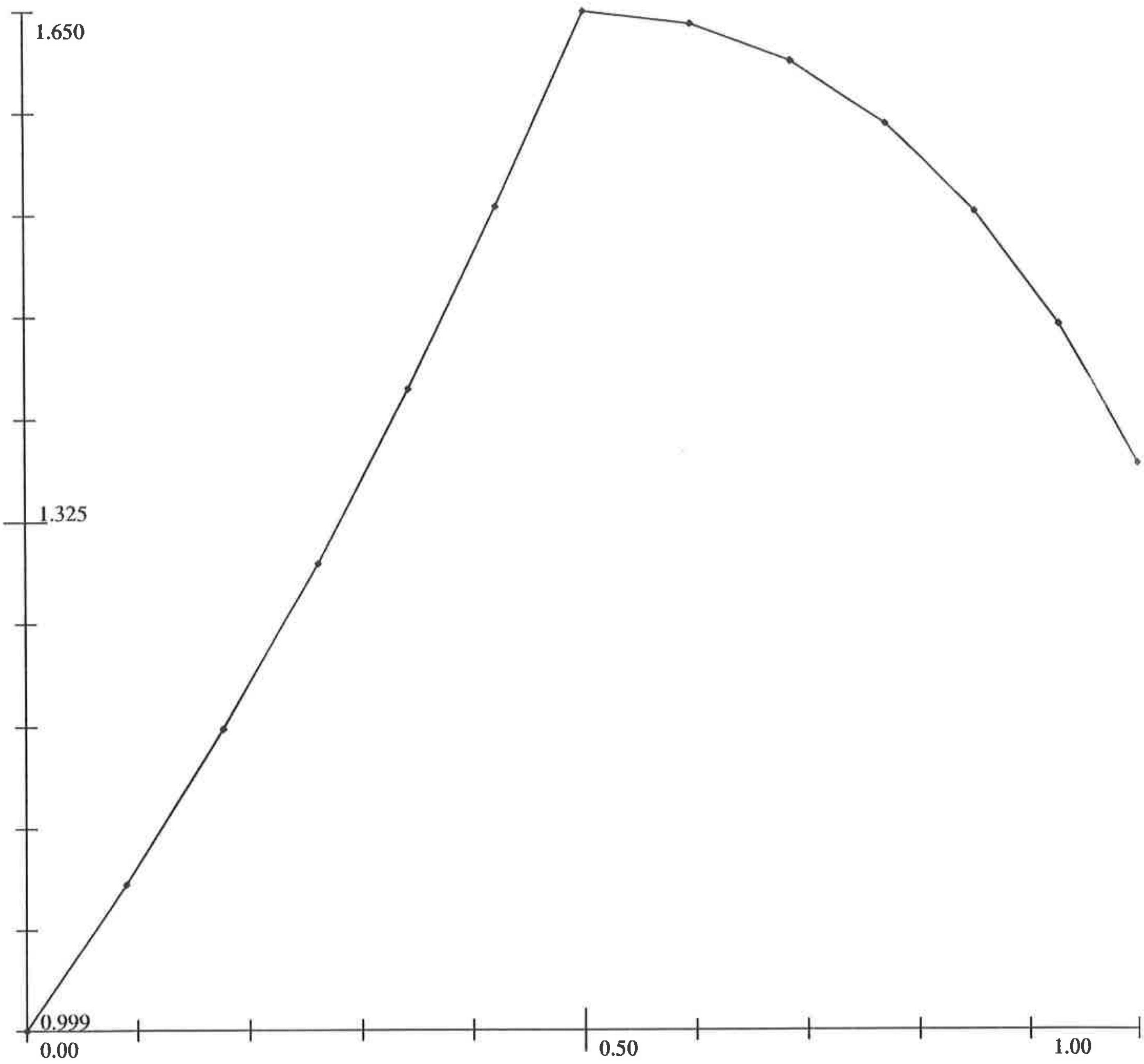

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$$\begin{cases} e^x & 0 \leqslant x \leqslant 0.5 \\ (1.5-x)e^x & 0.5 \leqslant x \leqslant 1 \end{cases}$$

fit