D E P A R T M E N T   O F   M A T H E M A T I C S

THE APPLICATION OF OPTIMAL CONTROL THEORY TO TIDAL-POWER-GENERATION PROBLEMS

R. O. MOODY

Numerical Analysis Report no. 7/89

U N I V E R S I T Y   O F   R E A D I N G

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CONTENTS

1. INTRODUCTION

It has been illustrated by H.M.S.O. (1981) that the generation of electrical energy using tidal energy is economically viable. An important factor associated with assessing a tidal-power scheme is knowing that the plant is operating at maximum efficiency; Count (1980), Wilson *et al.* (1981), Jefferys (1981), and Berry (1982) have recently investigated aspects of this topic, utilising a number of mathematical models and techniques.

A particularly attractive tool for solving tidal-power-generation problems is the mathematical theory of Optimal Control. Birkett and Nichols (1983a, 1986), Berry, Birkett, Count, and Nicol (1984), Birkett, Count, and Nichols (1984), Birkett, Count, Nichols, and Nicol (1984), and Birkett (1985b, 1986) have applied optimal-control methods to the problem of maximising the average-power or revenue functional subject to the satisfaction of fluid-flow equations in an estuary. These workers then apply numerical techniques, including finite differences and an iterative Conditional Gradient algorithm, to the resulting optimal-control problem. This approach has been proven to be computationally feasible, and may be generalised to accommodate, for example, ebb or two-way schemes, nonlinear headflow relationships, and variable estuarine geometries.

In this report we describe the recent work conducted on the tidal-power-generation models of the Reading University Group, which include either ordinary differential equations or partial differential equations. Section 2 contains work on the ordinary-differential-equation models, on which almost all attention has been focussed recently. In Section 3 we describe the models governed by partial differential equations, and in Section 4 we draw conclusions and suggest future work.

2. ORDINARY-DIFFERENTIAL-EQUATION MODELS

Consider an estuary across which there is a barrier containing K_1 turbines and K_2 sluices. Let us assume that the tidal elevation is constant on either side of the barrier, and that it is a known function of time on the seaward side. Then the water-surface elevation above the datum level, η, satisfies

$$S(\eta(t))\,\dot{\eta}(t) = K_1\,\alpha_1(t)\,P(f(t)-\eta(t)) + K_2\,\alpha_2(t)\,R(f(t)-\eta(t)), \qquad 0 < t < T, \qquad (2.1)$$

and

$$\eta(0) = \eta(T), \qquad (2.2)$$

where T is the tidal period and f is the elevation above the datum level on the seaward side of the barrier (and is periodic with period T). The functions S, P, and R denote the horizontal surface-area of the water, the flow through a turbine, and the flow through a sluice respectively. The proportions of turbines and sluices in operation are respectively α_1 and α_2, which satisfy

$$\left.\begin{array}{l} 0 \leq \alpha_1(t) \leq 1 \\ 0 \leq \alpha_2(t) \leq 1 \end{array}\right\}, \qquad 0 < t < T. \qquad (2.3)$$

Let C be a tariff function, then the energy, E, obtained is given by

$$E = K_1 \int_0^T C(t)\, \alpha_1(t)\, \mathcal{F}(f(t) - \eta(t))\, dt, \qquad (2.4)$$

in which $\mathcal{F}$ is a function representative of the instantaneous power (at a particular head difference). Throughout this report, we take C to be unity, corresponding to the maximisation of energy (as opposed to revenue).

The Optimal Control Problem (Birkett, 1985a) is that of determining α_1 and α_2 so that E of (2.4) is maximised subject to (2.1)-(2.3).

The Reading Group has three ordinary-differential-equation models for optimising the generation of tidal power. A simple model, referred to as Model OD1, can simulate ebb or two-way generation for a single tide (with one constant amplitude), and maximises either power or revenue. Model OD2, on the other hand, simulates only ebb generation, but may optimise over a sequence of tides with different amplitudes; again, either power or revenue may be maximised. In addition, Model OD2 includes options for pumping water into the basin (i.e., upstream of the barrier) and expansion losses (the latter requiring testing). A hybrid, Model OD3, is essentially Model OD1, but has the capacity for optimising over a series of tides, as in Model OD2. A computer program describing the original Model OD1, written by Dr. Nick Birkett, was adjusted by Dr. Ian Johnson and the author to produce programs representing Models OD2 and OD3 respectively.

There are three sets of data for use on the three models: one set representing the Severn estuary; another set, the Mersey estuary; and a third set of test data (referred to as Test), which requires smoothing as this data is quite severe (see Section 2.4). The Mersey and Test data sets may be incorporated in all three models, whereas the Severn set is compatible only with Model OD1. It is intended to replace the three models by a sophisticated one which includes all options of the existing models, and which admits the use of all three data sets.

In the ensuing subsections we perform a variety of experiments on the ordinary-differential-equation models and discuss the outcomes. The experiments consist of analysing the variation of the generated energy with respect to (i) perturbations in the tide, (ii) ebb or two-way generation, (iii) the initial choices of controls for the turbines and sluices, and (iv) the technique used to smooth certain data. The tests corresponding to (i), (ii), (iii), and (iv) are described in Subsections 2.1, 2.2, 2.3, and 2.4 respectively. In Subsection 2.1 we vary the tide, but in 2.2, 2.3, and 2.4, it is given by

$$T_0(t) = A \cos(2\pi t/T) , \qquad 0 < t < T , \tag{2.5}$$

where A is the tidal amplitude and the tidal period, T, is taken to be 12.42 hours in all experiments.

2.1 The Effect of Tidal Perturbations

In this subsection we consider ebb generation in Model OD1 using data describing the Severn estuary, with 216 turbines, 166 sluices, and 800 time steps. Consider the tides T_1, T_2, T_3, and T_4 (for

$t \in [0,T]$) defined by

$$T_1(t) = A\cos(2\pi t/T) + \epsilon A \ , \qquad (2.6)$$

$$T_2(t) = A\cos(2\pi t/T) + \epsilon A\cos(4\pi t/T) \ , \qquad (2.7)$$

$$T_3(t) = A\sin(2\pi t/T) + \epsilon A \ , \qquad (2.8)$$

$$T_4(t) = A\sin(2\pi t/T) + \epsilon A\sin(4\pi t/T) \ , \qquad (2.9)$$

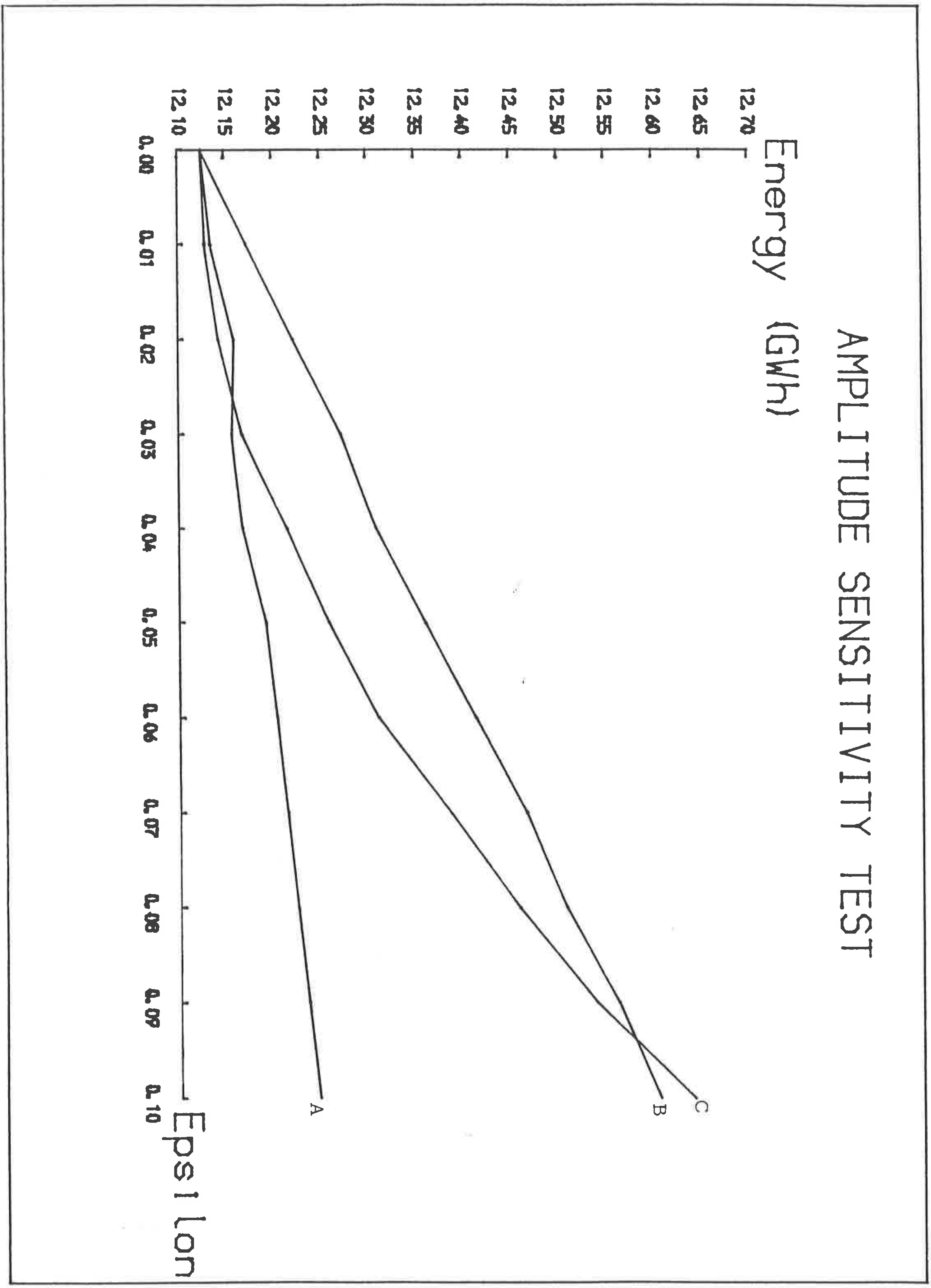
where ϵ is a perturbation parameter. Note that T_1 and T_3 are essentially the same, since

$$T_3(t + T/4) = T_1(t) \ , \qquad t \in \mathbb{R} \ ; \qquad (2.10)$$

i.e., T_1 and T_3 differ only in phase, by an amount $T/4$.

The purpose of the experiment in this subsection is to investigate the variation of the obtained energy with ϵ for different amplitudes. We select a value for A then choose ϵ as $0.00(0.01)0.10$.

Figures 1 and 2 illustrate the variation of energy with ϵ (for the four tides) with respective amplitudes of 3.25m and 4.25m . In both figures, Curve A corresponds to both tides T_1 and T_3 (verifying their equivalence), Curve B to T_2, and Curve C to T_4 . On each particular graph the greatest amount of energy obtained corresponds to $\epsilon = 0.10$ (or 10%), the largest value of the parameter. For Figure 1, the largest percentage increases for A, B, and C are respectively 1.01%, 3.97%, and 4.26% ; the corresponding values for Figure 2 are 1.12%, 6.15%, and 3.22% . These results show that the variation in obtained energy, which is not highly significant, is greater in the case of the harmonic perturbations than in that of the constant ones (i.e.,



AMPLITUDE SENSITIVITY TEST
Energy (GWh)
12.10
12.15
12.20
12.25
12.30
12.35
12.40
12.45
12.50
12.55
12.60
12.65
12.70
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
Epsilon
A
B
C

Figure 1

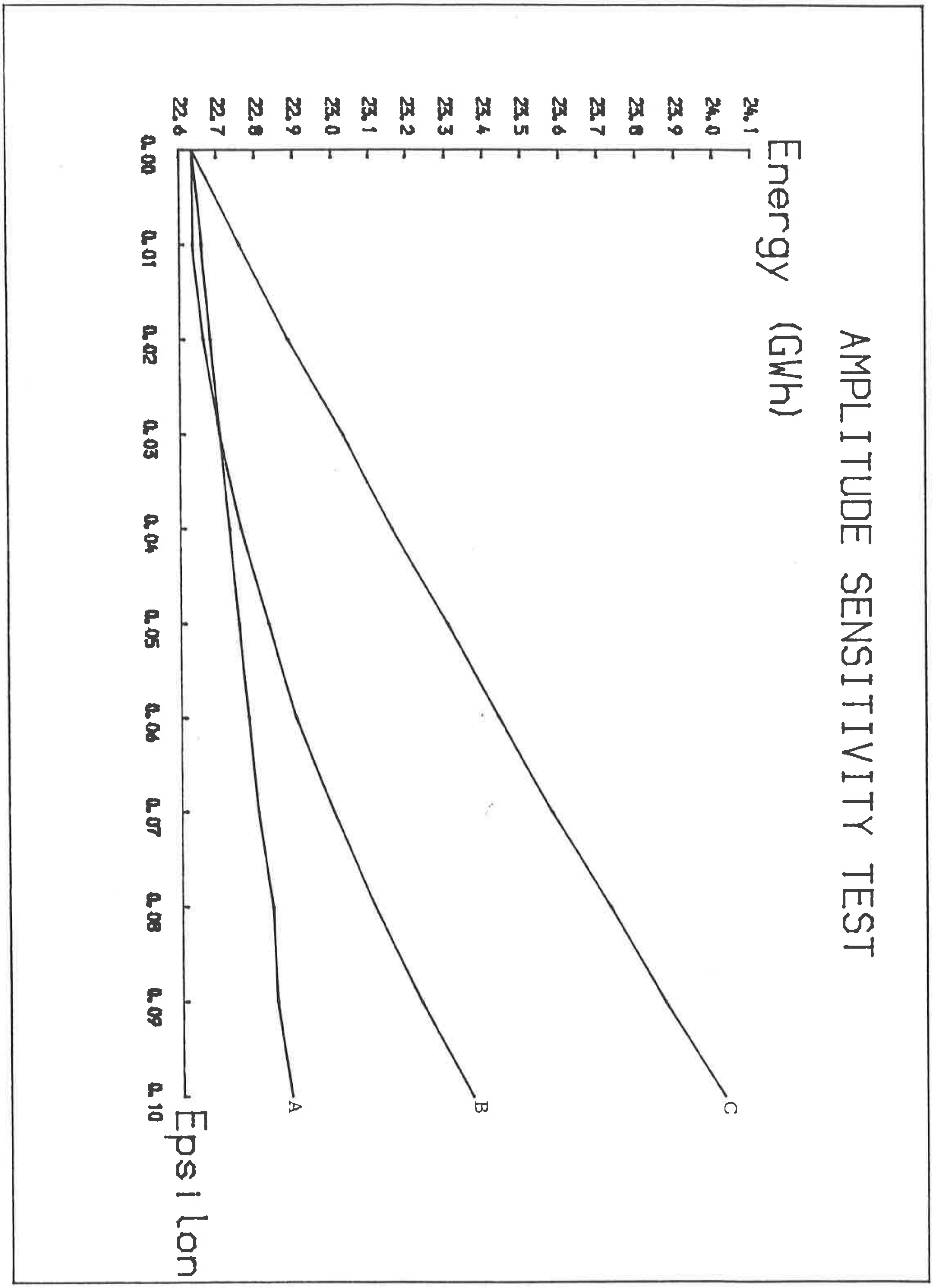

AMPLITUDE SENSITIVITY TEST
Energy (GWh)
24.1
24.0
23.9
23.8
23.7
23.6
23.5
23.4
23.3
23.2
23.1
23.0
22.9
22.8
22.7
22.6
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
Epsilon
A
B
C

Figure 2

the ϵA terms).

2.2 Comparison of Ebb and Two-way Generation

The purpose of the experiment in this subsection is to compare the energy generated from ebb and two-way schemes. On employing Model OD3 and the Mersey data with 300 time steps, we obtain the results in Table 1. Columns 1, 2, and 3 correspond to Data 1, Data 2, and Data 3 (the turbine-sluice numbers of which are respectively (13,9), (26,18), and (52,36)). The left-hand entry in each column is the ebb-generation energy; the right-hand one is the two-way-generation energy. A graphical form of the ebb-scheme results appears in Figure 3, in which A, B, and C respectively correspond to Data 1, Data 2, and Data 3. A two-way-scheme graph is not included, as its form is similar to that of the ebb one.

With reference to Table 1, we see that (as expected), for a fixed amplitude and data set, the energy generated from the two-way scheme is not less than that from the ebb scheme. Also, the amplitude at which the two-way energy first exceeds the ebb energy increases on increasing the numbers of turbines and sluices.

Curves of instantaneous power against normalised time (i.e., time divided by the tidal period) for an amplitude of 5.5m are present in Figures 4 and 5 for ebb and two-way schemes respectively. It is apparent that two-way is superior to ebb for this particular, large amplitude. This is, however, not true for the case of the Severn data with an amplitude of 5.5m: both schemes produce Figure 6.

Amp. (m)	Energy (GWh)					
	Data 1		Data 2		Data 3	
1.625	0.260	0.260	0.346	0.346	0.360	0.360
1.875	0.370	0.370	0.524	0.524	0.574	0.574
2.125	0.487	0.487	0.727	0.727	0.825	0.825
2.375	0.613	0.613	0.953	0.953	1.107	1.107
2.625	0.738	0.738	1.183	1.183	1.415	1.415
2.875	0.861	0.861	1.417	1.417	1.752	1.752
3.125	0.984	0.984	1.657	1.657	2.113	2.113
3.375	1.108	1.108	1.904	1.904	2.496	2.496
3.625	1.232	1.232	2.152	2.152	2.902	2.902
3.875	1.353	1.358	2.404	2.404	3.329	3.329
4.125	1.456	1.468	2.634	2.634	3.760	3.760
4.375	1.542	1.568	2.839	2.840	4.184	4.184
4.625	1.618	1.656	3.018	3.020	4.593	4.593
4.875	1.684	1.804	3.174	3.180	4.980	4.980
5.125	1.740	1.988	3.308	3.322	5.339	5.339

Table 1

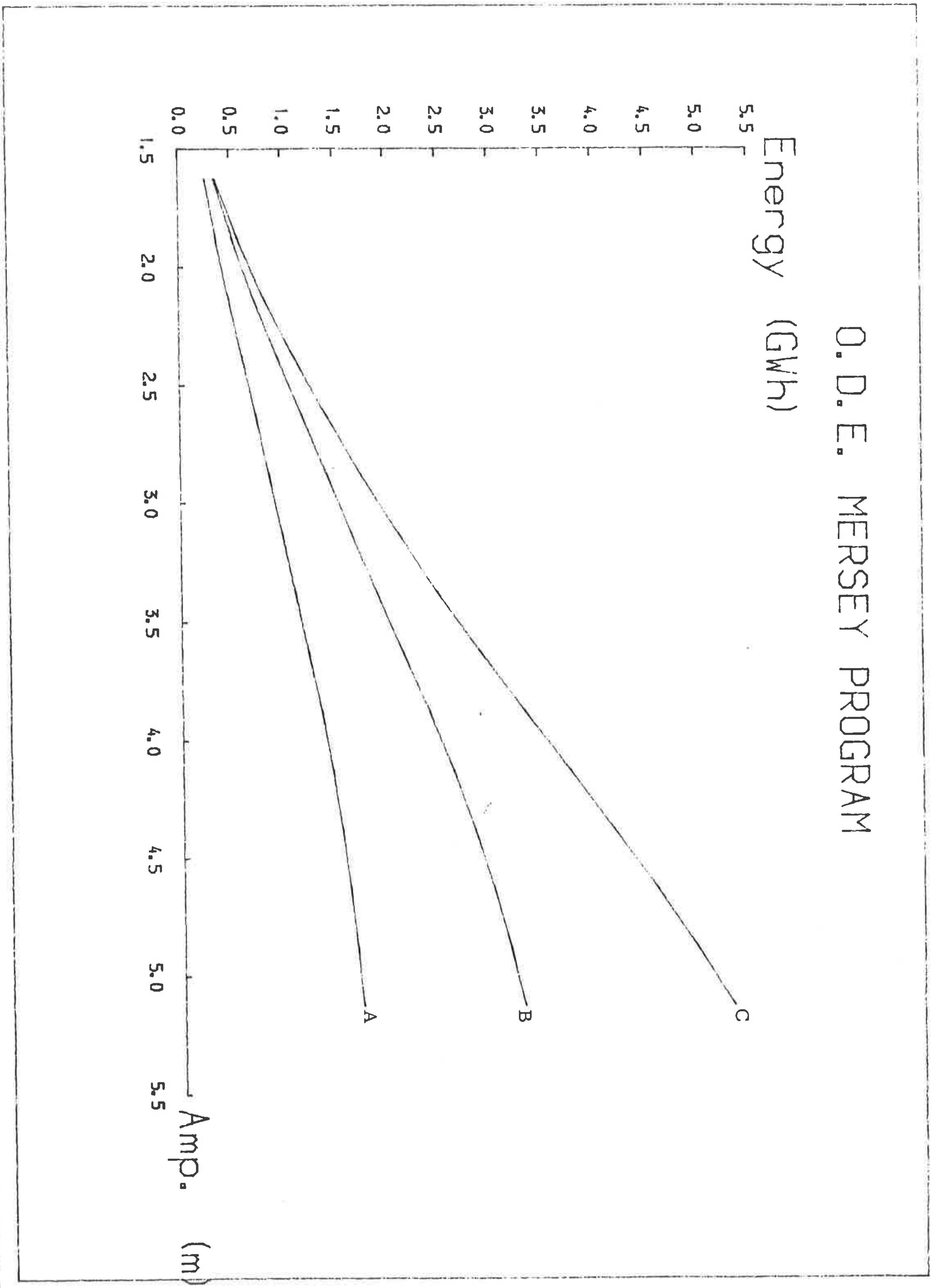

O.D.E. MERSEY PROGRAM
Energy (GWh)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
Amp. (m)
A
B
C

Figure 3

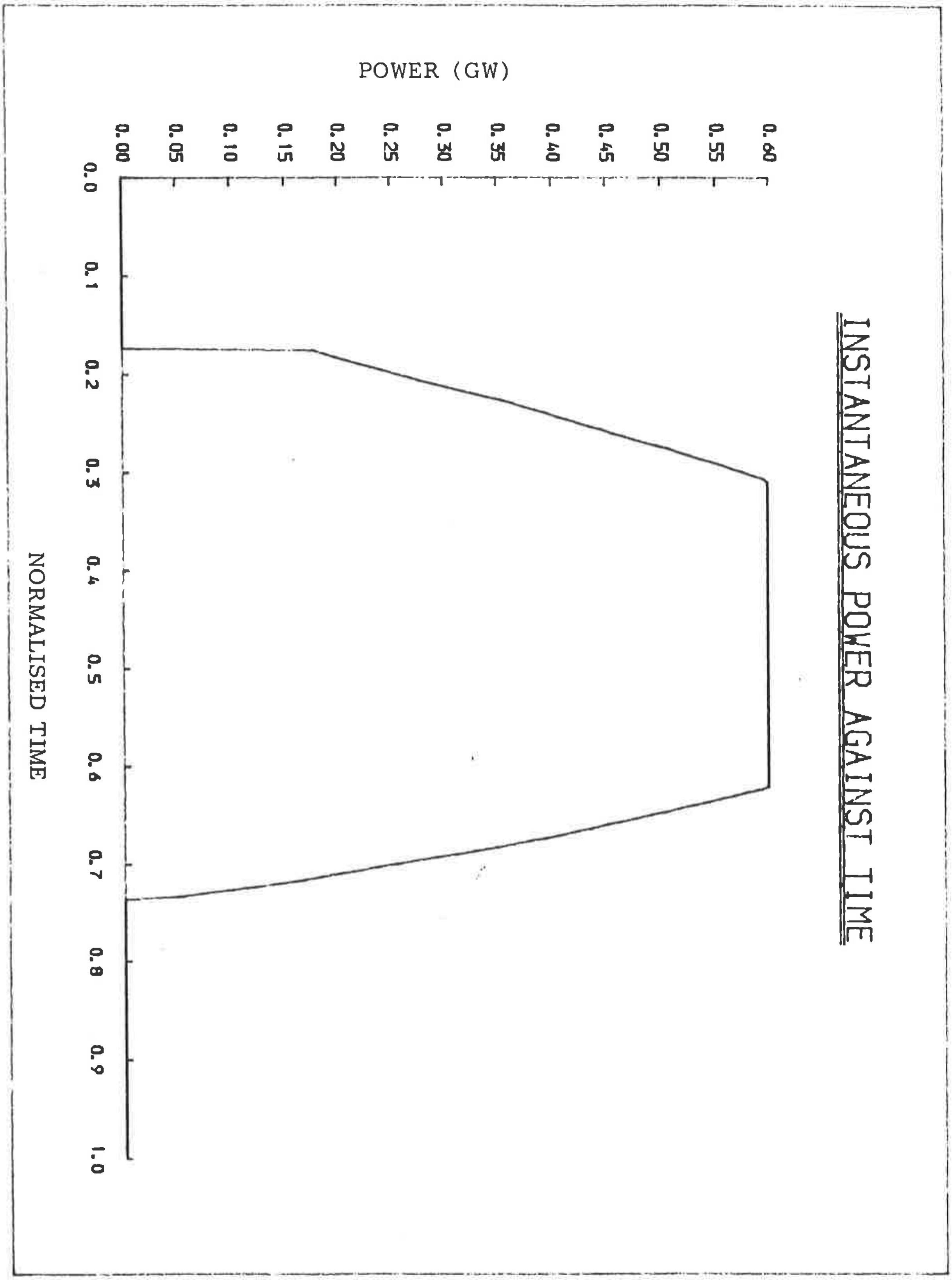

INSTANTANEOUS POWER AGAINST TIME
POWER (GW)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
NORMALISED TIME

Figure 4

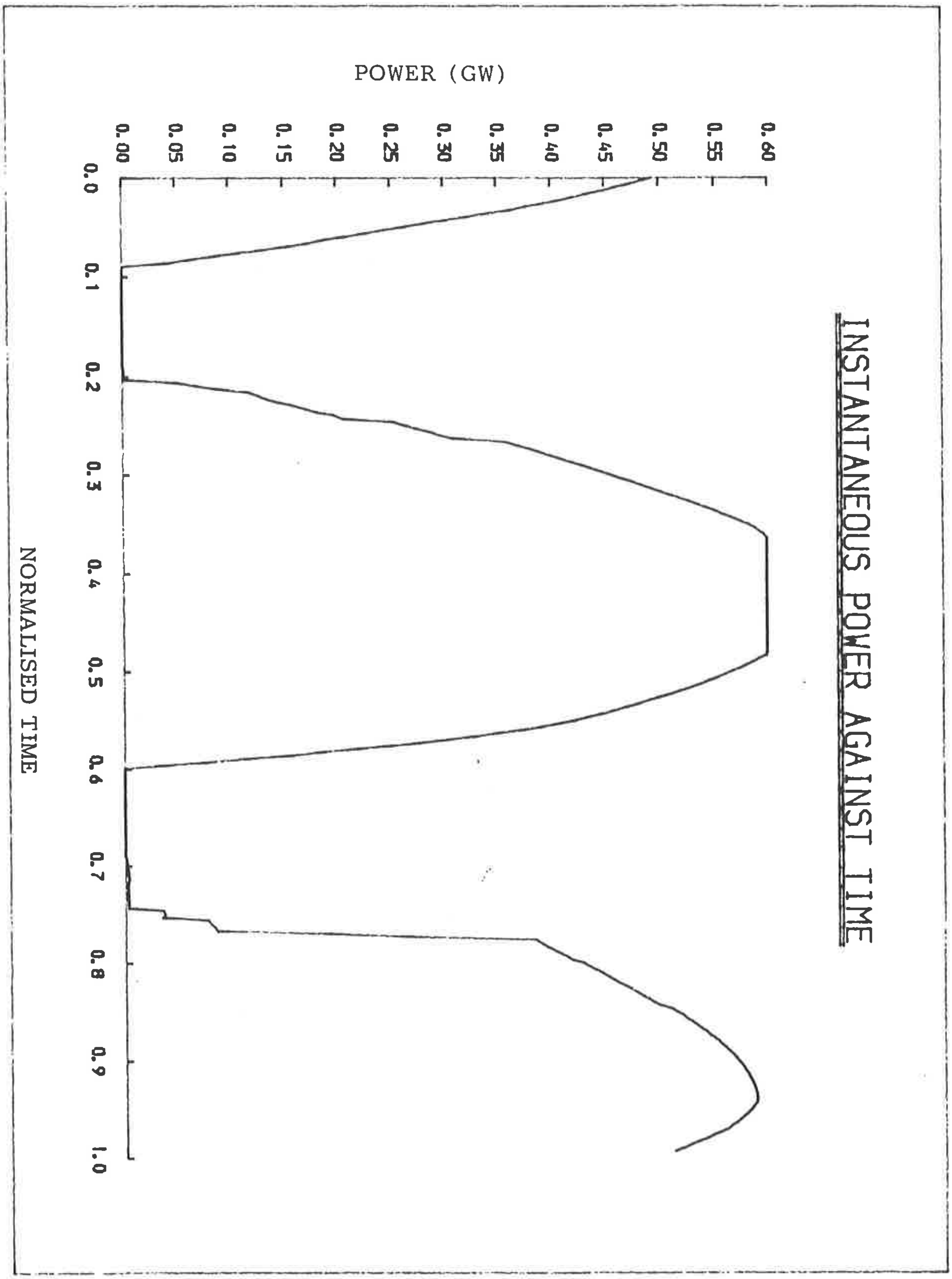
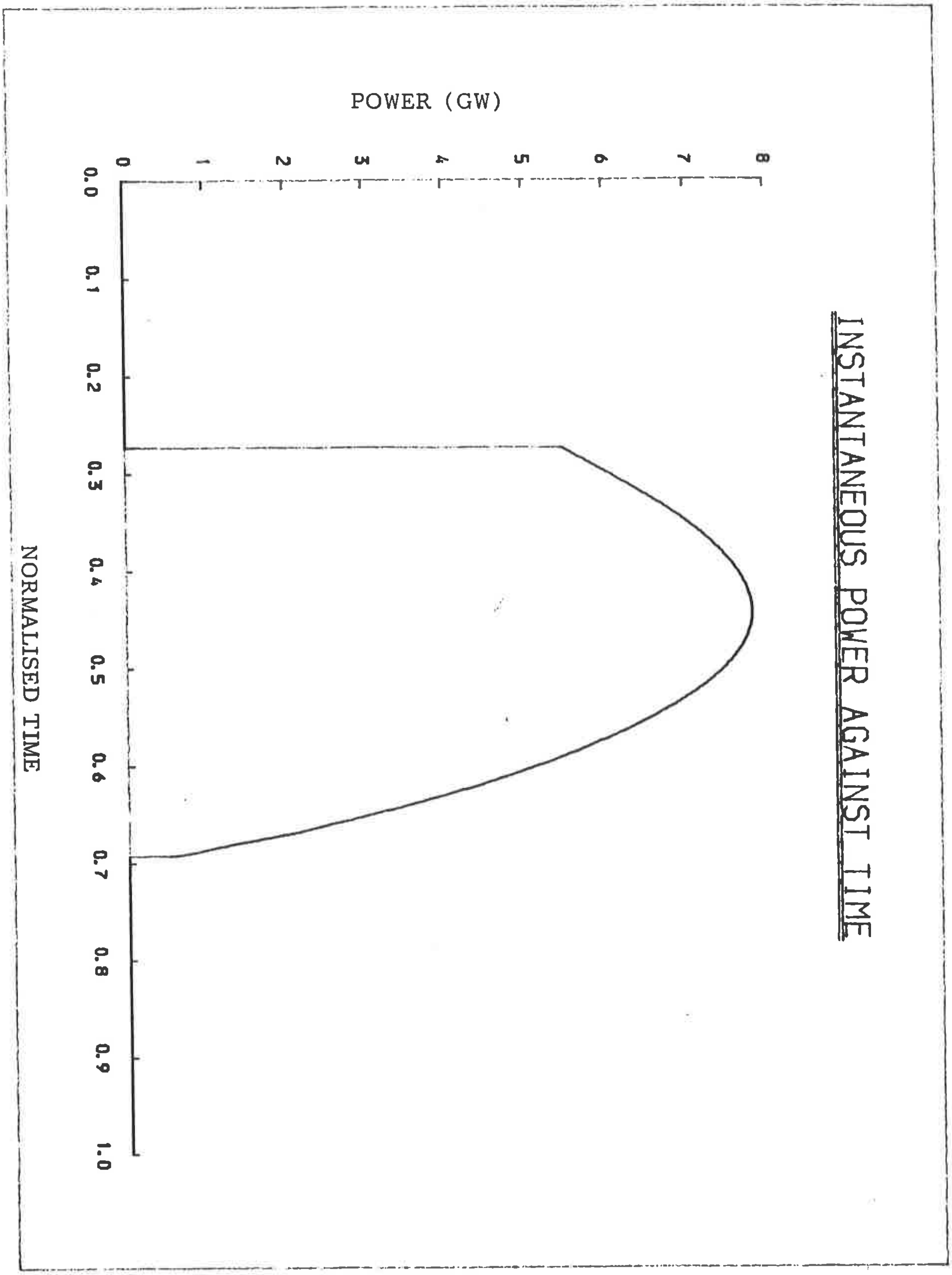

INSTANTANEOUS POWER AGAINST TIME
POWER (GW)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
NORMALISED TIME
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Figure 5



INSTANTANEOUS POWER AGAINST TIME
POWER (GW)
0
1
2
3
4
5
6
7
8
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
NORMALISED TIME

Figure 6

2.3 Control-sensitivity Tests

We employ the two-way Mersey scheme and Model OD3 to investigate how sensitive the generated energy is to the choice of the initial sluice and turbine controls. Let f_i , $i = 1(1)7$, be defined by

$$f_1(t) = \begin{cases} 0 , & 0 \leq t < 0.25 \\ 0.1 , & 0.25 \leq t \leq 0.75 \\ 0 , & 0.75 < t \leq 1 \end{cases} , \tag{2.11}$$

$$f_2(t) = \begin{cases} 0 , & 0 \leq t < 0.75 \\ 0.1 , & 0.75 \leq t \leq 1 \end{cases} , \tag{2.12}$$

$$f_3(t) = \begin{cases} 0.1 , & 0 \leq t \leq 0.25 \\ 0 , & 0.25 < t < 0.75 \\ 0.1 , & 0.75 \leq t \leq 1 \end{cases} , \tag{2.13}$$

$$f_4(t) = \begin{cases} 0.1 , & 0 \leq t \leq 0.125 \\ 0 , & 0.125 < t < 0.375 \\ 0.1 , & 0.375 \leq t \leq 0.625 \\ 0 , & 0.625 < t < 0.875 \\ 0.1 , & 0.875 \leq t \leq 1 \end{cases} , \tag{2.14}$$

$$f_5(t) = \begin{cases} 0\ , & 0 \leq t < 0.125 \\ 0.1\ , & 0.125 \leq t \leq 0.375 \\ 0\ , & 0.375 < t < 0.625 \\ 0.1\ , & 0.625 \leq t \leq 0.875 \\ 0\ , & 0.875 < t \leq 1 \end{cases}\ , \tag{2.15}$$

$$f_6(t) = 0.1|\cos(2\pi t)|\ , \qquad 0 \leq t \leq 1\ , \tag{2.16}$$

$$f_7(t) = 0.1[1 - \cos(2\pi t)]\ , \qquad 0 \leq t \leq 1\ , \tag{2.17}$$

in which t is the normalised time variable. Then with 18 sluices, 27 turbines, 300 time steps, and a tidal amplitude of 3.625m, we obtain the results in Table 2.

Initial Sluice Control	Initial Turbine Control	Number of Iterations	Energy (GWh)
f_1	f_2	10	2.193
f_3	f_1	9	2.190
f_4	f_5	8	2.191
f_6	f_7	7	2.190
f_7	f_6	5	2.194

<u>Table 2</u>

Table 2 illustrates that, even though the number of iterations required for convergence of the control may vary considerably, the variation in obtained energy is insignificant. This conclusion holds also for initial-turbine-sluice-control combinations of functions not defined in (2.11)-(2.17). (Note that, in every experiment conducted, the initial turbine (or sluice) control did converge to the same turbine (or sluice) control.)

2.4 Smoothing Discontinuous Functions

Let $P(y)$ denote the flow through a turbine when the head difference (i.e., $f(t) - \eta(t)$, at time t, in Equation (2.1)) is y. Then P is sufficiently smooth for the Severn and Mersey cases, but not for the Test case. The function P which represents the Test case is depicted in Figure 7.

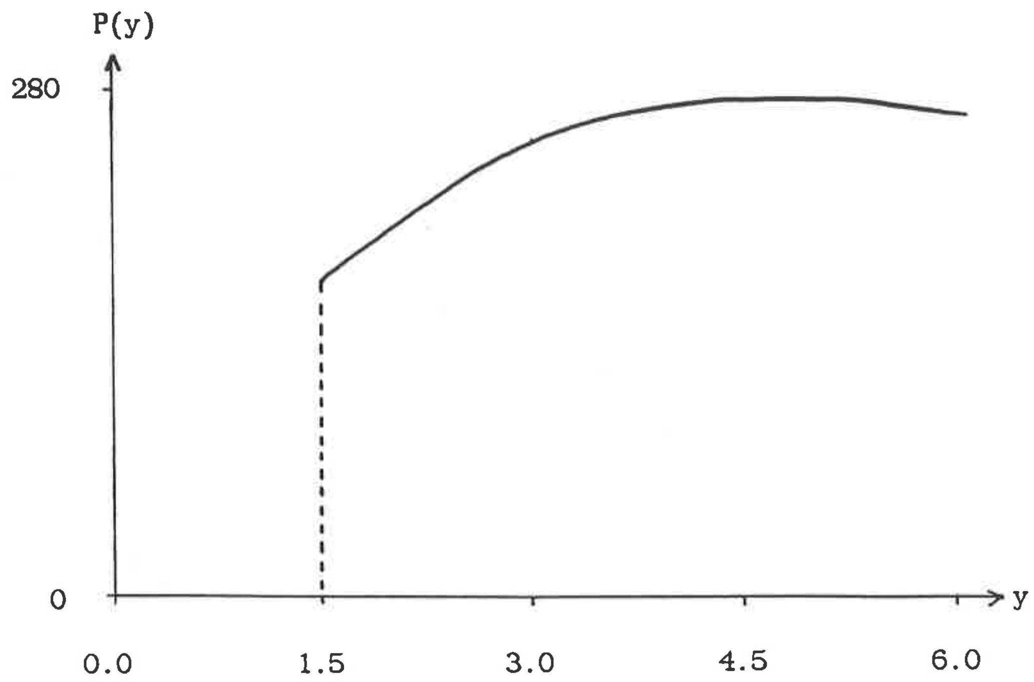


Figure 7

We now focus our attention on previous and present methods for smoothing P (and functions of similar character) in the Test case. The data for P is supplied in the form of point values at several selected head differences. The approach of Johnson (1988) is to approximate P using (a) the value of zero in $[0,y_0]$, where $0 < y_0 < 1.5$, (b) piecewise-quadratic functions through three consecutive points in $[1.5,6]$, and (c) a hyperbolic tangent function to ensure the continuity of P in $(y_0,1.5)$. The disadvantage with this technique, however, is that P' is discontinuous at some nodes - a severe drawback since the evaluation of P' is necessary in the Conditional Gradient algorithm (see Section 1). Consequently, the code of Johnson (1988) is capable of simulating ebb generation but not two-way generation.

As an alternative approach to that of Johnson (1988), the idea of a cubic spline springs readily to mind, since P' would then be continuous. The singularity at 1.5, however, prevents sensible approximations arising from this treatment. We therefore consider (a) taking P to be zero on $[0,1.5-\epsilon]$, for some small number ϵ (like Johnson, (1988)), (b) implementing a natural cubic spline on $[1.5,6]$, and (c) introducing a linking function on $(1.5-\epsilon,1.5)$. The obvious choice for the linking function is a cubic satisfying both function and derivative values at $1.5-\epsilon$ and 1.5, thereby ensuring the continuity of P'. This approximation to P is illustrated in Figure 8.

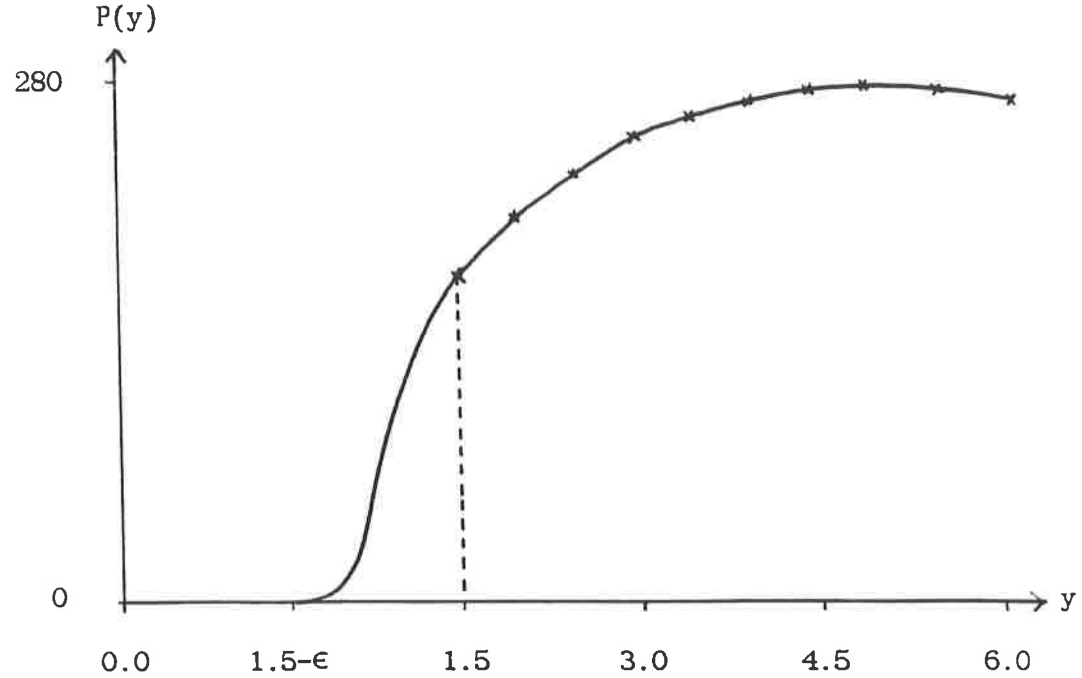


Figure 8

For comparison purposes only, we consider modifying the previous method to: (a) retain both the zero function on $[0,1.5-\epsilon]$ and the cubic spline on $[1.5,6]$, (b) replace the cubic on $(1.5-\epsilon,1.5)$ by the straight line which admits the continuity of P, and (c) force P' to be continuous by appropriately defining it to be linear on $(1.5-\epsilon,1.5)$.

The smoothing techniques of the previous two paragraphs are incorporated in the Test data (comprising 160 turbines and 166 sluices) to produce simulations for a tide of which the amplitudes vary from 1.95m to 3.75m. In both cases, the smallest value of ϵ that does not cause the simulation to fail is 0.5 - an unacceptably high value! Table 3 contains a comparison of ebb and two-way results using this optimal value of ϵ .

Approximation for P	Ebb Energy (GWh)	Two-way Energy (GWh)
Spline + cubic	63.815	76.372
Spline + line	63.900	76.461

Table 3

The results in Table 3 illustrate that the two approximation methods described above give rise to similar results. This conclusion is also present when other approximation techniques (e.g., a cubic spline and a quadratic for P , and the derivative of the spline and a linear for P') are employed. The use of a centred interval (i.e., $(1.5-\frac{1}{2}\epsilon, 1.5+\frac{1}{2}\epsilon)$) also produces results which are very similar to those in Table 3; the same trend is followed when different numbers of turbines and sluices are used. The problem appears to be a consequence of the stability of the finite-difference scheme (the Trapezoidal Rule), which requires that the time increment be less than $2/||P'||_\infty$. This stability restriction is consistent with the conjecture that the value of $||P'||_\infty$ increases as ϵ is decreased.

3. PARTIAL-DIFFERENTIAL-EQUATION MODELS

The Optimal Control problems of this section are of the same form as the one described in Section 2, but are more sophisticated.

Suppose that a one-dimensional estuary begins at $x = -\ell_1$, ends at $x = \ell_2$, and has a barrier located at $x = 0$ (ℓ_1 and ℓ_2 are positive constants). Then for $x \in [-\ell_1, \ell_2]\backslash\{0\}$, the analogy to Equation (2.1) is the pair of one-dimensional shallow-water equations: two coupled partial differential equations in terms of η, the water elevation above a datum level, and u, the velocity. In addition, there are periodic conditions for η and u (see Equation (2.2)), together with a boundary condition for η at one end of the estuary and one for u at the other. The flow across the barrier (i.e., at $x = 0$) is continuous and is prescribed, thereby supplying the two required interface conditions. The turbine and sluice controls, α_1 and α_2, are constrained according to (2.3), as they are in the previous models.

As is the case in Section 2, the Optimal Control problem is to determine α_1 and α_2 to maximise the energy derived from the scheme (Birkett and Nichols, 1983b; Birkett, 1985a, 1986).

The Reading Group has three one-dimensional partial-differential-equation models for the generation of tidal power. All models (which were translated into computer programs by Dr. Nick Birkett) are capable of simulating ebb or two-way generation, and can maximise either power or revenue. The simplest model, named LPD, contains linearised shallow-water equations and optimises the energy over a single tide.

Model NPD is a nonlinear shallow-water model which allows for the possibility of optimising over a sequence of tides; it also has options for the inclusion of pumping and losses. The most sophisticated model, which is still under development, is CPD. Like NPD, CPD is a nonlinear model which can optimise over many tides and allows pumping; unlike NPD, CPD attempts to optimise the route through the turbine hill chart (i.e., the function P is not prescribed, but is solved for in an optimal way).

Each of the three models can simulate only with data describing the Severn estuary. At present, work is being performed on the NPD model to facilitate compatibility with Mersey data. More recent work on this model includes interpolating sets of point values for low-water breadths, high-water breadths, and mean cross-sectional areas onto equispaced meshes.

4. CONCLUSIONS

In this report we have described the generation of tidal power, and have presented the Optimal Control approach of the Reading University Group. This approach has been applied to both the three ordinary-differential-equation and three partial-differential-equation models, with several data sets (not all data sets being compatible with each model). We have outlined recent work, and experiments, all of which were conducted on the simpler models. These experiments, reported in Section 2, resulted in the following conclusions: (i) small perturbations in the tide do not produce significant effects in the energy obtained from the scheme, (ii) in certain circumstances, more energy may be extracted from a two-way scheme than from an ebb one, (iii) the models are virtually insensitive to the initial choices for the turbine and sluice controls, and (iv) it is very difficult to adequately smooth the severe Test data.

Future work should include replacing the three ordinary-differential-equation models with one model which incorporates all their features, and can run with either data set. The Test data set will, however, certainly cause problems in this setup. It is therefore desirable to investigate the possibility of utilising a derivative-free optimal-control algorithm (i.e., one which does not require the evaluation of P').

Regarding the partial-differential-equation models, we need to carry out more work on NPD and CPD, including an amalgamation to produce one model which can simulate generation with any of the three data sets.

(This latter task may require the previously mentioned derivative-free control loop.) Finally, a two-dimensional model would be a useful contribution to the Reading work.

ACKNOWLEDGEMENTS

I would like to thank Dr. Nick Birkett for his invaluable assistance since my setting out on this project. Also, I am grateful to Dr. Nancy Nichols for her enthusiastic supervision.

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