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Godunov Methods

by

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Abstract

This paper reviews the class of numerical schemes, known as Godunov Methods, used for the solution of hyperbolic conservation laws. Such numerical schemes can be characterised by the solution (exact or approximate) of a Riemann Problem (classical or generalised) within computational cells in order to obtain the numerical fluxes.

Since the original first order scheme, proposed by Godunov in 1959, there has been much development of the idea; for example, the MUSCL scheme of van Leer in 1979, the PPM scheme of Woodward and Colella in 1984 and the Higher Order Godunov schemes of Bell, Colella and Trangenstein (1989).

As well as considering the original scheme and its later variants, we place these developments in historical context, making links with other work in the area.

1 Introduction

The title of this paper begs the question: What is a Godunov method? In 1983, Harten, Lax & van Leer gave a technical definition of Godunov–type schemes in [17]. However, more recently, van Leer [45, 46] has given the succinct definition:

> ... we define Godunov–type methods as non–oscillatory finite–volume schemes that incorporate the solution (exact or approximate) to Riemann's initial–value problem, or a generalization of it,.....

It is this definition that we will explore.

In 1959 Godunov published his inspirational paper [10], based on the work of his Ph.D., in which he used the solution of the Riemann problem as a building block for a finite–volume scheme for compressible flow. His scheme, as originally presented, involved a Lagrangian step followed by an Eulerian remapping. However, it may be recast (see for example [44]) into a conservative Eulerian framework. Whilst Godunov used the exact solution of the Riemann problem in his work, others (Roe [28], Osher & Solomon

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[27], Harten, Lax & van Leer [17] and Einfeldt [7] to name just a few) have adopted approximations to its solution and thereby generated variants on the original method.

Godunov's method is only first order accurate but gives solutions which preserve monotonicity of the data. Indeed it was in his paper [10] that Godunov presented his now famous theorem, which states that monotonicity preserving constant coefficient schemes can be at most first order accurate. It is this theorem which has led to much research in the area of non–linear schemes for hyperbolic conservation laws, since it is through the use of non–linear schemes that both monotonicity and high order accuracy can be achieved. Pioneering work carried out in the 1970s by Boris & Book [4], van Leer [39, 40, 41, 42, 43] and Roe [30] has been built upon by others (for example [48, 34, 5, 19, 15, 1]) resulting in an abundance of high resolution non-oscillatory schemes, many of which are based on Godunov's method. We now proceed to look at these.

In Section 2 we review Godunov's original scheme and its various formulations/interpretations. We then look at extensions to the scheme, firstly by replacing the exact Riemann Solver with an approximate one in Section 3 and secondly by modifying the data representation to obtain higher order accuracy in Section 4. Finally in Section 5 we look at some other advances of the Godunov methodology.

2 Godunov's Scheme

We are considering the numerical solution of the (system of) hyperbolic conservation laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}. \tag{1}$$

Godunov's method considers the numerical values of the solution $\mathbf{u}_i^n$ to be the cell averages of the analytic solution $\mathbf{u}(x,t)$ at time level n,

$$\mathbf{u}_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{u}(x, n\Delta t)\, dx, \tag{2}$$

where for simplicity of notation we assume a regular grid. We therefore have a piecewise constant data representation (see Figure 1). At each cell boundary, the resulting Riemann problem is then solved and the union of all Riemann solutions averaged over each cell to give the updated numerical solution values (see Figure 2). Since each Riemann problem is solved in isolation, the need to avoid interaction suggests a CFL limit of $\frac{1}{2}$.

For the special case of the scalar linear advection equation the scheme is easily interpreted as advection of the cell averages (Lagrangian stage), which are then remapped back onto the Eulerian grid (see Figure 3).

Godunov's scheme can be recast into Eulerian form by integrating (1) over the cell $[x_{i-1/2}, x_{i+1/2}] \times [n\Delta t, (n+1)\Delta t]$:

$$\int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{u}_t(x,t)\, dx\, dt = -\int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{f}_x(\mathbf{u}(x,t))\, dx\, dt,$$

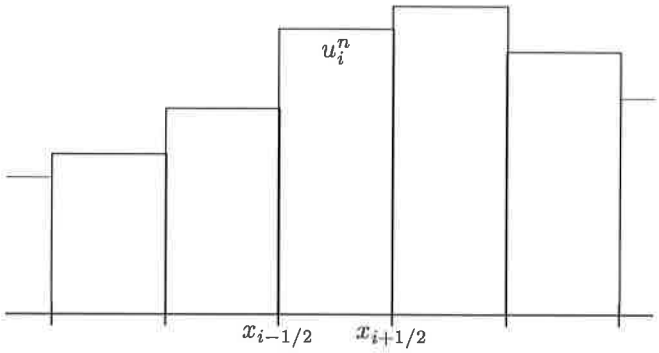

Figure 1: The piecewise constant data representation.

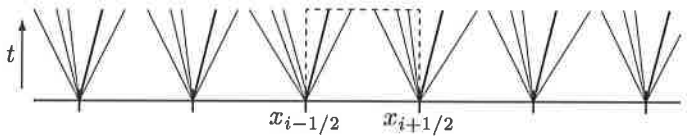

Figure 2: The resulting Riemann problems and their averaging.

whence

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{u}(x,t^{n+1}) - \mathbf{u}(x,t^n)\,dx = -\int_{t^n}^{t^{n+1}} \mathbf{f}(\mathbf{u}(x_{i+1/2},t)) - \mathbf{f}(\mathbf{u}(x_{i-1/2},t))\,dt,$$

or

$$\Delta x(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) = \\ -\Delta t\left(\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}(\mathbf{u}(x_{i+1/2},t))\,dt - \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}(\mathbf{u}(x_{i-1/2},t))\,dt\right).$$

If we now define a numerical flux as

$$\mathbf{f}_{i-1/2}^n = \mathbf{f}(\mathbf{u}_{i-1}^n, \mathbf{u}_i^n) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}(\mathbf{u}(x_{i-1/2},t))\,dt, \tag{3}$$

then we can write Godunov's method in conservation form

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{f}_{i+1/2}^n - \mathbf{f}_{i-1/2}^n). \tag{4}$$

Looking at the numerical flux (3) we see that it does not depend on the whole Riemann solution, but only the flux of the state at $x_{i-1/2}$, and because

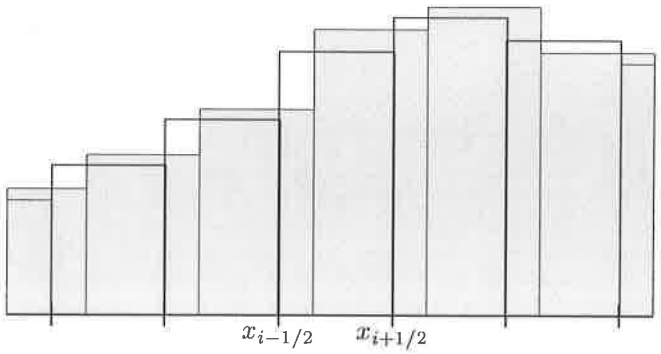

Figure 3: Lagrangian stage plus remap for linear advection.

the solution of the Riemann is self similar along rays $x/t =$ constant we can rewrite it as

$$\mathbf{f}^n_{i-1/2} = \mathbf{f}(\mathbf{u}^n_{i-1/2}). \tag{5}$$

We now see that since the detailed Riemann problem solutions within the cell are not important for calculation of the flux, they may be allowed to interact so long as the solution at $x_{i-1/2}$ does not influence the state at $x_{i+1/2}$ and vice-versa. Thus it can be seen that this form of Godunov's scheme has a CFL condition of 1.

Finally we note that using Jensen's inequality (see for example [17]) it can be shown that Godunov's scheme is entropy satisfying, i.e. all shocks are physically correct.

Godunov's method solves the Riemann problem at each cell boundary exactly. However variants of the method may be generated by utilising an approximate Riemann solver instead. We now look at some possibilities.

3 Riemann Solvers

Whilst for some scalar conservation laws the Riemann problem is easily solved, this is not the case for non-linear systems of conservation laws. Here an iterative procedure is often required which, since this must be used at every cell boundary at every time step, will make it the most computationally expensive task of the whole method. To simplify the process and reduce this overhead approximate Riemann solvers, which do not employ iteration, are often used. This can be achieved either by approximating the Riemann states and applying the physical flux, or by approximating the numerical flux directly. In this paper we look at the latter, and outline the distinguishing features of some of the approximate Riemann solvers used. A full discussion of approximate Riemann solvers can be found, for example, in [37].

3.1 Roe

Perhaps the simplest approximate Riemann solver is that due to Roe [28]. The system of conservation laws (1) may be written in quasi-linear form

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{0},$$

where $A(\mathbf{u})$ is the Jacobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}$. Roe linearises this form of the equations in each interval (x_{i-1}, x_i) by replacing the Jacobian by interval-wise constant matrices $\tilde{A}(\mathbf{u}_{i-1}, \mathbf{u}_i)$ which, for any two adjacent states $\mathbf{u}_L$, $\mathbf{u}_R$ satisfy

1. $\tilde{A}(\mathbf{u}_L, \mathbf{u}_R)$ is diagonalisable with real eigenvalues (hyperbolicity);
2. $\tilde{A}(\mathbf{u}_L, \mathbf{u}_R) \to A(\mathbf{u})$ as $\mathbf{u}_L, \mathbf{u}_R \to \mathbf{u}$ (consistency);
3. $\mathbf{f}(\mathbf{u}_L) - \mathbf{f}(\mathbf{u}_R) = \tilde{A}(\mathbf{u}_L, \mathbf{u}_R)(\mathbf{u}_L - \mathbf{u}_R)$ (conservation).

The first two conditions are readily satisfied if $\tilde{A}$ is taken to be the Jacobian evaluated at an averaged state, i.e. $\tilde{A}(\mathbf{u}_L, \mathbf{u}_R) = A(\bar{u})$. However a straight arithmetic average will not, in general, satisfy the final condition and instead a geometric average is often used (in the form of the form of the arithmetric mean of an auxilary vector, known as the parameter vector — see [28, 31, 9]).

Once an $\tilde{A}$ has been obtained, it is diagonalised $\tilde{X}\tilde{\Lambda}\tilde{X}^{-1}$ which results in a set of decoupled linear advection equations in each interval. The flux differences $\mathbf{f}_R - \mathbf{f}_L$ in each interval are decomposed onto the local eigenvectors

$$\Delta\mathbf{f} = \mathbf{f}_R - \mathbf{f}_L = \sum_{k=1}^{n} \tilde{\alpha}^{(k)} \tilde{\lambda}^{(k)} \tilde{\mathbf{x}}^{(k)}$$

where $\tilde{\lambda}^{(k)}$, $\tilde{\mathbf{x}}^{(k)}$ and $\tilde{\alpha}^{(k)}$ are the eigenvalue, eigenvector and coefficient for $\Delta\mathbf{u}$, respectively, corresponding to the kth characteristic field of $\tilde{A}$.

Whilst Roe's original scheme updated the solution by upwinding and directly adding these flux difference components, it may be placed in the framework of intercell fluxes by integration around the half cell $(x_{i-1/2}, x_i) \times (t^n, t^{n+1})$ (see [17]) resulting in the flux

$$\mathbf{f}_{i-1/2} = \frac{1}{2}(\mathbf{f}_{i-1} + \mathbf{f}_i) - \frac{1}{2} \sum_{k=1}^{n} \tilde{\alpha}^{(k)}_{i-1/2} |\tilde{\lambda}^{(k)}_{i-1/2}| \tilde{\mathbf{x}}^{(k)}_{i-1/2}.$$

In this formulation the $\tilde{A}$ can be seen to be identified with the cell interfaces.

Because the resulting individual approximate Riemann problems are linear, their solutions contain only discontinuities and not expansion fans. For this reason Roe's original method is not entropy satisfying, although a number of *entropy fixes* have since been proposed (see for example [16, 29, 31]).

3.2 Osher

Whereas Roe's approximate Riemann solver approximates the solution of the Riemann problem using only discontinuities, Osher (Engquist & Osher [8],Osher & Solomon [27]) approximates the Riemann solution using only simple waves. The numerical flux for Osher's scheme may be written as

$$\mathbf{f}_{i-1/2} = \frac{1}{2}(\mathbf{f}_{i-1} + \mathbf{f}_i) - \frac{1}{2} \int_{\mathbf{u}_{i-1}}^{\mathbf{u}_i} |A(\mathbf{u})| \, d\mathbf{u},$$

where $|A(\mathbf{u})|$ is defined to be $X|\Lambda|X^{-1}$, (Λ is the matrix of eigenvalues of A and X the corresponding matrix of eigenvectors) and the integration path between $\mathbf{u}_{i-1}$ and $\mathbf{u}_i$ is taken in phase space over simple wave solutions. That is, for a system of n conservation laws, $n-1$ intermediate states are found, connected via n simple waves. In his original scheme Osher took the paths to be in order of decreasing eigenvalue, however the reverse order may also be selected (see [20]). Since the approximation uses only simple waves, not discontinuities, it can be shown to be entropy satisfying.

3.3 Harten, Lax & van Leer (HLL)

For a system of n conservation laws, both Roe's and Osher's approximate Riemann solvers use n intermediate states, connected in the case of Roe by discontinuities and in the case of Osher by simple waves. In 1983 Harten Lax & van Leer [17] proposed a much simpler approximation which assumes a Riemann solution consisting of just two waves separating three constant states. If $s^R_{i-1/2}$ and $s^L_{i-1/2}$ are upper and lower bounds, respectively, for the largest and smallest signal velocities resulting from the solution of the Riemann problem centred at $x_{i-1/2}$, then the approximate solution is taken to be

$$\tilde{\mathbf{u}}(x,t) = \begin{cases} \mathbf{u}_{i-1} & \text{if} \quad \frac{x}{t} \le s^L_{i-1/2} \\ \mathbf{u}^{HLL}_{i-1/2} & \text{if} \quad s^L_{i-1/2} \le \frac{x}{t} \le s^R_{i-1/2} \\ \mathbf{u}_i & \text{if} \quad s^R_{i-1/2} \le \frac{x}{t} \end{cases},$$

where the intermediate state is obtained from conservation to be

$$\mathbf{u}^{HLL}_{i-1/2} = \frac{s^R_{i-1/2}\mathbf{u}_i - s^L_{i-1/2}\mathbf{u}_{i-1}}{s^R_{i-1/2} - s^L_{i-1/2}} - \frac{\mathbf{f}_i - \mathbf{f}_{i-1}}{s^R_{i-1/2} - s^L_{i-1/2}}. \tag{6}$$

Integrating this solution substituted into the conservation law over the half cell $(x_{i-1/2}, x_i) \times (t^n, t^{n+1})$ results in the HLL flux

$$\mathbf{f}_{i-1/2} = \begin{cases} \mathbf{f}_{i-1} & \text{if} \quad \frac{x}{t} \le s^L_{i-1/2} \\ \mathbf{f}^{HLL}_{i-1/2} & \text{if} \quad s^L_{i-1/2} \le \frac{x}{t} \le s^R_{i-1/2} \\ \mathbf{f}_i & \text{if} \quad s^R_{i-1/2} \le \frac{x}{t} \end{cases},$$

where

$$\mathbf{f}^{HLL}_{i-1/2} = \frac{s^R_{i-1/2}\mathbf{f}_{i-1} - s^L_{i-1/2}\mathbf{f}_i + s^L_{i-1/2}s^R_{i-1/2}(\mathbf{u}_i - \mathbf{u}_{i-1})}{s^R_{i-1/2} - s^L_{i-1/2}}.$$

Harten *et al.* noted that the intermediate state, as given by (6), is a mean value of the exact Riemann solution, and it therefore follows from Jensen's inequality that the scheme is entropy satisfying. They also note that if the two states $\mathbf{u}_{i-1}$ and $\mathbf{u}_i$ can be connected by a shock of the first or nth family, then the correct shock speed is obtained and the solution is exact. They then proceed to derive a scheme with two intermediate states which has the additional property that if the end states can be connected by a shock or contact discontinuity of *any* family, then the approximate Riemann solver does so. Toro, Spruce & Speares [38] also modified the original HLL approximate solver restoring missing contact or shear waves, naming it the HLLC solver.

It now remains to specify the upper and lower bounds $s^R_{i-1/2}$ and $s^L_{i-1/2}$. One possibility is to evaluate them directly, or, as suggested by Davis [6] to use the maximum eigenvalue evaluated at the right state and the minimum eigenvalue evaluated at the left state respectively, or to take the maximum of the largest eigenvalue at either state, and the minimum of the smallest. Another alternative, also suggested by Davis [6] and Einfeldt [7] is to use Roe's averaged eigenvalues as estimates.

4 Higher Order Extensions

First order methods such as Godunov's tend to be very diffusive, smearing the discontinuities that often arise in the solution of conservation laws. However, classical higher order methods (for example Lax–Wendroff [23]) whilst giving sharper features also produce spurious oscillations around discontinuities, possibly resulting in unphysical values (for example negative density) and/or violation of stability bounds thus causing a breakdown of the solution. Godunov [10] proved that this was inevitable for constant coefficient schemes, which could not be both monotonicity preserving and higher than first order accurate. This led to much work on non–linear schemes, both practical and theoretical.

One of the theoretical advances was the adoption of *total variation* as a monitor of spurious oscillations. This stems from the fact that, for the scalar conservation law, the analytic total variation,

$$\mathrm{TV}(u) = \int |u_x| \, dx,$$

does not increase (and only decreases across shocks) [22]. Harten [13] proposed that schemes should mimic this behaviour with their discrete total variation,

$$\mathrm{TV}(u^n) = \sum_i |u_i - u_{i-1}|,$$

i.e.

$$\mathrm{TV}(u^{n+1}) \leq \mathrm{TV}(u^n),$$

and christened such schemes Total Variation Diminishing (TVD). He also gave a set of algebraic criteria for the (non–linear) coefficients of a scheme to satisfy which are sufficient to show the scheme to be TVD. These algebraic conditions are more commonly used for non Godunov-type schemes, with a geometric approach being used for Godunov–type methods. Conservative schemes which are TVD can be shown to converge to a weak solution of the conservation law.

Some of the non–linear schemes developed, such as Flux Corrected Transport (FCT) [4, 48] and Flux Limiters [34], can not be considered as Godunov–type schemes except possibly for the linear case. Others however are either direct extensions of Godunov's method or at least use the same methodology in their construction.

We now look at a set of modifications to Godunov's method which result in higher order accuracy.

As already mentioned, Godunov's method assumes the data to consist of piecewise constant cell averages which are advanced in time by solving the

Riemann problems at each cell interface and then re-averaging. This results in a first order method. To achieve higher order accuracy we must change the data representation. For clarity we describe here the data representations for the scalar non–linear conservation law. The methodology is usually extended to systems via characteristic decomposition of the states.

4.1 MUSCL and variants

Van Leer[42, 43] in his MUSCL[2] scheme replaced Godunov's piecewise constant representation with a piecewise linear one (see Figure 4).

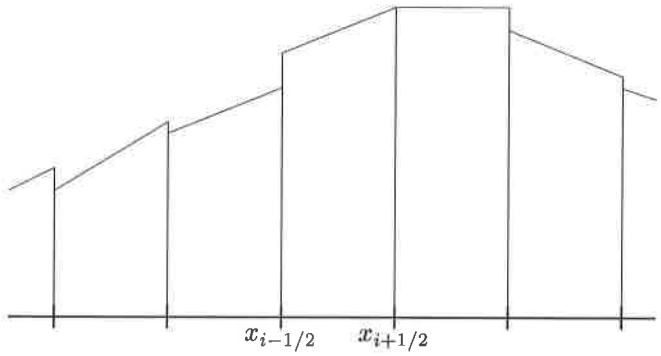

Figure 4: The piecewise linear data representation.

This piecewise linear representation was constructed to maintain conservation by defining the cell representation to be

$$u_i(x) = u_i^n + \frac{\Delta_i u}{\Delta x}(x - x_i),$$

where u_i^n is the Godunov cell average (2) and the slope $\frac{\Delta_i u}{\Delta x}$ must be defined. Van Leer [42] gave three possibilities for $\Delta_i u$,

1. centred differencing of the piecewise constant cell averages $\Delta_i u = \frac{1}{2}(u_{i+1} - u_{i-1})$;
2. differencing of the underlying continuous function $\Delta_i u = (u(x_{i+1/2}, t^n) - u(x_{i-1/2}, t^n))$, i.e. a difference of the unaveraged values – this leads to the necessity to evolve $\Delta_i u$ as well as u;
3. maintaining the first moment of the underlying analytical solution $\Delta_i u = \frac{12}{(\Delta x)^2}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n)(x - x_i)\, dx$. Again this leads to independent updates of the slope and cell average.

If we calculate the slopes in any of these fashions there is the potential of producing a data representation which has a higher maximum or lower

[2]MUSCL - Monotonic Upstream–centred Scheme for Conservation Laws - was actually the name of Paul Woodward's computer code incorporating van Leer's ideas, but the name has stuck with this type of scheme

minimum than the piecewise constant average (see Figure 5), i.e. increasing the total variation of the data representation. Viewing the Lagrangian plus remap interpretation of Godunov's scheme, it can easily be seen that this increase will be maintained, and spurious oscillations may ensue.

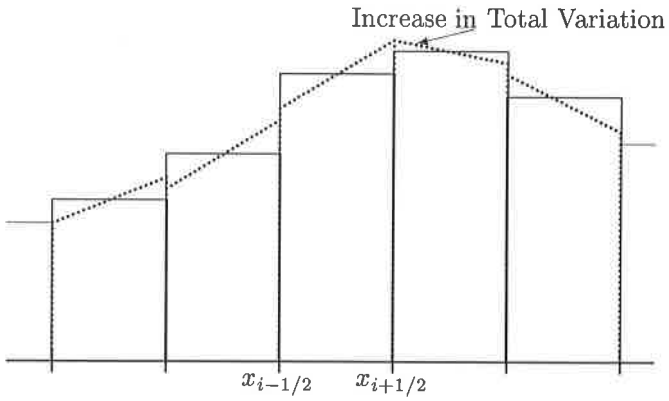

Figure 5: An increase in total variation.

To avoid this increase in total variation, van Leer limited the gradient of his slopes. He achieved this by defining a monotonised slope as

$$(\Delta_i u)_{\text{mono}} = \begin{cases} \min\{2|\Delta u_{i-1/2}|, |\Delta_i u|, 2|\Delta u_{i+1/2}|\}\text{sgn}\Delta_i u \\ \quad \text{if} \quad \text{sgn}\Delta u_{i-1/2} = \text{sgn}\Delta u_{i+1/2} = \text{sgn}\Delta_i u \\ 0 \quad \text{otherwise} \end{cases},$$

which may be applied to any definition of $\Delta_i u$ and where $\Delta u_{i-1/2} = u_i - u_{i-1}$. This prevents the linear function taking values outside of the range of the neighbouring mesh averages and reduces the slope to zero, i.e. reverting to piecewise constant, where there is an extremum of the data. For his first choice of slope, van Leer also gave an improved limiting of

$$(\Delta_i u)_{\text{mono}} = \begin{cases} \frac{2\Delta u_{i-1/2}\Delta u_{i+1/2}}{\Delta u_{i-1/2}+\Delta u_{i+1/2}} & \text{if} \quad \text{sgn}\Delta u_{i-1/2} = \text{sgn}\Delta u_{i+1/2} \\ 0 & \text{otherwise} \end{cases}, \quad (7)$$

which has the effect of taking the harmonic mean of $\Delta u_{i-1/2}$ and $\Delta u_{i+1/2}$ instead of their algebraic mean as in the non–monotonised slope.

This slope limiting has much in common with flux limiters [34], except that here it is the slope which is being limited. If we define $r_{i-1/2} = \Delta u_{i-1/2}/\Delta u_{i+1/2}$ it can be seen that (7) can be written as

$$(\Delta_i u)_{\text{mono}} = \frac{r_{i-1/2} + |r_{i-1/2}|}{1 + |r_{i-1/2}|} = \phi_{VL}(r_{i-1/2}),$$

where $\phi_{VL}(r)$ is van Leer's flux limiter [40, 34]. Indeed other flux limiters can be used as slope limiters, for example Goodman & Le Veque [11] use Roe's minmod flux limiter [30, 34]. It is important to note however that,

even though slope limiter and flux limiter schemes are equivalent for linear scalar equations, they are two distinct types of method, with only the former fitting into the Godunov-type framework.

At the cell interfaces of the piecewise linear data representation we now have a set of so–called generalised Riemann problems, i.e. a discontinuity separating two linear states. These are not as easily solved as the basic Riemann problem and the wave paths are now curves rather than straight lines in $x-t$ space. This means that the Lagrangian advection step is not so readily achieved except for linear problems, and in conservation form (4) the numerical flux (3) no longer reduces to (5). Whilst some methods are based on the generalised Riemann problem [2, 3], more often an approximate Riemann solution is used, for example Goodman & Le Veque [11] approximate the flux by a linear function near the cell interfaces.

An alternative approach, due to Hancock [12] was noted by van Leer [44] whereby a second order accurate scheme is obtained by evolving the (extrapolated) cell boundary values

$$u_i^{n,L} = u_i - \frac{1}{2}(\Delta_i u)_{\text{mono}}, \qquad \text{and} \qquad u_i^{n,R} = u_i + \frac{1}{2}(\Delta_i u)_{\text{mono}},$$

obtained by the piecewise linear data representation, by a time $\frac{1}{2}\Delta t$. This is achieved via a Taylor series expansion

$$\begin{aligned} u_i^{n+1/2,L} &= u_i^{n,L} + \frac{\Delta t}{2} u_t, i \\ &= u_i^{n,L} - \frac{\Delta t}{2} f_x, i \qquad \text{using (1)} \\ &\approx u_i^{n,L} - \frac{1}{2}\frac{\Delta t}{\Delta x}(f(u_i^{n,R}) - f(u_i^{n,L})), \end{aligned}$$

and similarly

$$u_i^{n+1/2,R} \approx u_i^{n,R} - \frac{1}{2}\frac{\Delta t}{\Delta x}(f(u_i^{n,R}) - f(u_i^{n,L})).$$

These advanced states are then used as piecewise constant data for a conventional Riemann problem at the intercell boundary,

$$u_t + f(u)_x = 0 \qquad u(x,0) = \begin{cases} u_{i-1}^{n+1/2,R}, & x < 0, \\ u_i^{n+1/2,L}, & x > 0, \end{cases}$$

to obtain the similarity solution $u_{i-1/2}(x/t)$ which is in turn used to obtain the intercell numerical flux via (5) to be used in (4).

In their Higher Order Godunov method, Bell, Collela & Trangenstein [1] extended this technique to general systems of hyperbolic conservation laws, using Osher's approximate Riemann solver, with additional modifications needed to treat loss of strict hyperbolicity which arise due to eigenvector deficiencies that may arise, for example in the black–oil model of petroleum engineering.

4.2 PPM and ENO

Woodward & Colella [47, 5] extended the idea of MUSCL further by constructing a piecewise parabolic data representation. This representation is

limited so as to avoid overshoots and undershoots and incorporates a discontinuity detection mechanism so as to sharpen any discontinuities of the data. This piecewise parabolic representation is then advanced using either a Lagrangian step followed by a remap, or in conservation form, resulting in the third order Piecewise Parabolic Method (PPM). (For simple problems the method is third order in space and time; however for more complicated situations approximations in the Riemann solution can degrade the time accuracy to second order.)

Harten, Osher, Chrakavarthy and Engquist [19, 18, 15] and later Osher & Shu [32, 33] extended the idea of polynomial data representation even further. The technique used is similar to that employed in the MUSCL and PPM schemes, except that the data representation constructed from the cell averages $\{u_i^n\}$ does not damp the values of local extrema, as in the aforementioned schemes, and is even allowed occasionally to accentuate these local features.

The Essentially Non-Oscillatory (ENO) scheme, as it is called, starts from the cell averages $\bar{u}^n = \{u_i^n\}$ and constructs the approximate function $u_{\Delta x}(x;t^n) = R(x;\bar{u}^n)$, where $R(x;\bar{u}^n)$ is a piecewise polynomial in x of degree $p-1$ satisfying:

1. $R(x;\bar{u}^n) = u(x,t^n) + O(\Delta x^p)$ where the functions are smooth;
2. $R(x;\bar{u}^n)$ is conservative, i.e. $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} R(x;\bar{u}^n)\,dx = u_i^n$;
3. $R(x;\bar{u}^n)$ is essentially non-oscillatory, i.e. $\mathrm{TV}(R(\cdot;\bar{u}^n)) \leq \mathrm{TV}(u(\cdot,t^n)) + O(\Delta x^p)$.

Both MUSCL ($p=2$) and PPM ($p=3$) fit into this framework, except that they have the more restrictive condition of $\mathrm{TV}(R(\cdot;\bar{u}^n)) \leq \mathrm{TV}(u(\cdot,t^n))$, i.e. TVD.

Once the data has been reconstructed, the solution of the conservation law (1) with initial data $u_{\Delta x}(\cdot,t^n)$ is calculated and the solution re-averaged to obtain updated cell averages u_i^{n+1}.

The key step of ENO is in the reconstruction, the essence of which is as follows. The interpolant $R(x;\bar{u}^n)$ is built up in stages using Newton interpolation. Initially we may construct a local linear interpolant in the cell $(x_{i-1/2}, x_{i+1/2})$ either using u_{i+1} and u_i or u_i and u_{i-1}. The pair with smallest difference is chosen; this process being repeated for each cell. Next a quadratic interpolant for each cell is constructed by adding an additional interpolation point — this can be either the value to the left or right of the previous stencil. For example, if u_{i+1} and u_i had been chosen to form the linear interpolation for our cell, then we can add in either u_{i+2} or u_{i-1}. The one which gives the smoothest interpolant (as monitored by comparison of divided differences) is chosen. This is done for each cell and the method applied recursively until the desired degree of interpolation is reached. A variant of ENO is Weighted ENO (WENO) [25] where a linear combination of the candidate stencils for interpolation is taken. Subcell resolution has also been used [14] in order to sharpen contact discontinuities.

ENO can be shown to be Total Variation Bounded (TVB), i.e. $\mathrm{TV}(u^n) \leq C\mathrm{TV}(u^0)$ which means theoretically that solutions still converge as for TVD schemes, and practically that small oscillations on the scale of truncation error may appear but usually vanish if the solution is adequately resolved.

4.3 WAF

Toro [35, 36, 37] achieves second order accuracy in a different way. Instead of enhancing the data representation, as in the methods previously described, he exploits more of the information provided by the full solution of the conventional Riemann problem. Whilst Godunov's method evaluates the intercell flux only along $x = x_{i-1/2}$ (see (5)), Toro takes a weighted average of the flux vector across the whole wave structure of the Riemann solution at time $t^n + \frac{\Delta t}{2}$. His intercell flux is defined to be

$$\mathbf{f}_{i-1/2} = \frac{1}{\Delta x} \int_{x_{i-1}}^{x_i} \mathbf{f}(\mathbf{u}_{i-1/2}(x, t^n + \frac{\Delta t}{2}))\, dx, \tag{8}$$

where the integration is depicted in Figure 6.

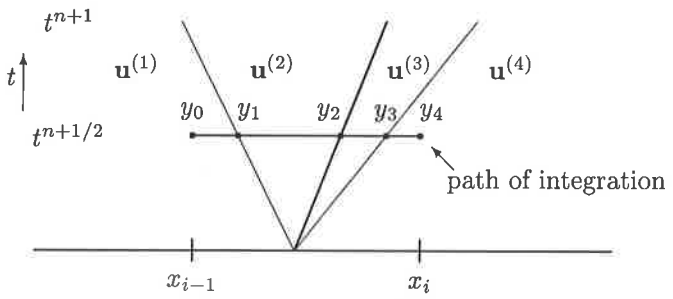

Figure 6: Calculation of the WAF flux.

If we consider first the case where there is no expansion wave (as shown in Figure 6) we see that the integral in (8) may be evaluated to give

$$\mathbf{f}_{i-1/2} = \sum_{k=1}^{n+1} \beta_k \mathbf{f}_{i-1/2}^{(k)}, \tag{9}$$

where the fluxes $\mathbf{f}_{i-1/2}^{(k)}$ are evaluated at the constant states, i.e. $\mathbf{f}_{i-1/2}^{(k)} = \mathbf{f}(\mathbf{u}^{(k)})$ and the weights β_k are the normalised lengths

$$\beta_k = \frac{y_k - y_{k-1}}{\Delta x}.$$

The weighted sum (9) leads to the name of the Weighted Average Flux (WAF) method. If there is an expansion wave present in the Riemann solution then Toro [37] suggests that the integral across it could be evaluated exactly (giving an extra weight) or the expansion could be combined with the closest constant state (taking the value of the state to be $\mathbf{u}_{i-1/2}(0)$ for a sonic rarefaction). In order to solve the Riemann problem to obtain the solution at $t^{n+1/2}$ either the exact solution can be used or an approximate Riemann solver employed.

As it stands the method is not TVD, however Toro applies a flux limiter type device to the flux differences across waves of the Riemann solution - see [37] for details.

5 And there's more...

In the previous sections we have only been able to give a brief description of just some of the Godunov methods in the literature. Any literature search will reveal a wealth of material on the subject, some of it extensions and new applications of established techniques and some of it novel in its own right. We conclude by looking at a couple of these developments.

5.1 Source Terms

In the exposition of the various Godunov methods, we have restricted our attention to homogeneous conservation laws. However, many models of physical situations involve source terms, giving the inhomogeneous equation

$$\mathbf{u}_t + \mathbf{f}_x(\mathbf{u}) = \mathbf{S}(x, \mathbf{u}). \tag{10}$$

Although much work has been carried out in the investigation of how to incorporate source terms into numerical methods for conservation laws, this is still an open issue. The simplest approach is to use fractional steps, splitting the non–homogeneous equation (10) into the homogeneous equation (1) supplemented by the ordinary differential equation

$$\mathbf{u}_t = \mathbf{S}(x, \mathbf{u}),$$

solving them alternatively for each timestep. This avoids incorporating the source term directly into the numerical solver for the conservation law. However this approach can perform badly when solving near-steady state flows, since the contributions from the two separate stages can be large but non-cancelling. The most obvious inclusion of a source term into a Godunov method would add extra complication to the Riemann solution and calculation of the intercell fluxes since the Riemann solution is no longer constant on rays $x/t =$ constant.

Le Veque [24], however, takes a novel approach. He introduces a new discontinuity in the centre of each cell by decomposing the piecewise constant data representation into two different states $\mathbf{u}_i^-$ and $\mathbf{u}_i^+$, (see Figure 7). This is done in such a manner so as to be conservative, and if possible so that

$$\mathbf{f}(\mathbf{u}_i^+) - \mathbf{f}(\mathbf{u}_i^-) = \mathbf{S}(x_i, \mathbf{u}_i)\Delta x. \tag{11}$$

This should then ensure that the effect of the source term in the cell is exactly cancelled by the waves resulting from this new Riemann problem. This has the implication that this new Riemann problem need not be solved, nor the source term explicitly included in the scheme, except for choosing the new states $\mathbf{u}_i^-$ and $\mathbf{u}_i^+$, via (11), which are then used with the standard Godunov approach or its high order extensions. If it is not possible to choose the new states to satisfy (11) then an additional term accounting for the discrepancy is added to the update for the cell (see [24]).

5.2 Central Schemes

In [26] Nessyahu & Tadmor introduced a central scheme based on the Godunov philosophy, although it avoids explicit solution of Riemann problems. The

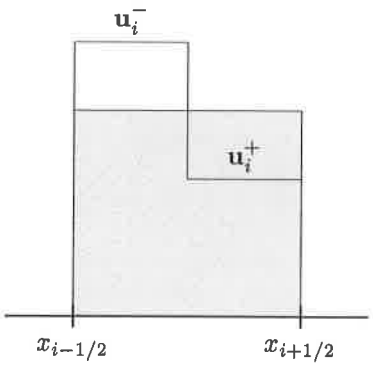

Figure 7: The additional Riemann problem.

first order case is a staggered grid version of the Lax–Friedrichs scheme, and is constructed in the following manner. Like Godunov's scheme the data is taken to be the set of cell averages, resulting in a set of Riemann problems at their interfaces (Figures 1 and 2). However, instead of re-averaging the solution of adjacent Riemann problems over the cell $(x_{i-1/2}, x_{i+1/2})$, the solution of just a single Riemann problem is averaged over the staggered cell (x_{i-1}, x_i), see Figure 8.

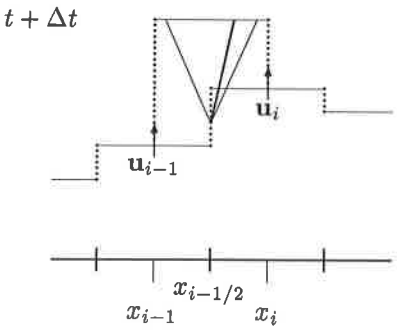

Figure 8: Central differencing by Godunov type scheme.

Equations (4) and (3) now become

$$\mathbf{u}_{i-1/2}^{n+1} = \mathbf{u}_{i-1/2}^{n} - \frac{\Delta t}{\Delta x}(\mathbf{f}_i^n - \mathbf{f}_{i-1}^n),$$

and

$$\mathbf{f}_i^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{f}(\mathbf{u}(x_i, t))\, dt \tag{12}$$

where

$$\mathbf{u}^n_{i-1/2} = \frac{1}{\Delta x}\left[\int_{x_{i-1}}^{x_{i-1/2}} \mathbf{u}_{i-1}(x)\,dx + \int_{x_{i-1/2}}^{x_i} \mathbf{u}_i(x)\,dx\right]. \qquad (13)$$

Of course, for the piecewise constant representation we are considering at present, (13) reduces to $\mathbf{u}^n_{i-1/2} = \frac{1}{2}(\mathbf{u}^n_{i-1} + \mathbf{u}^n_i)$. The mid-points are evolved according to the conservation law (1) with initial data $\mathbf{u}^n_i(x)$ and integrals in the flux definition (12) are evaluated using suitable quadrature. (The evolved mid–cell values will remain continuous for small enough time.) For the piecewise constant case, the mid–cell values remain constant, giving a flux $\mathbf{f}^n_i = \mathbf{f}(\mathbf{u}^n_i)$ and the resulting scheme becomes

$$\mathbf{u}^{n+1}_{i-1/2} = \frac{1}{2}(\mathbf{u}^n_{i-1} + \mathbf{u}^n_i) - \frac{\Delta t}{\Delta x}(\mathbf{f}(\mathbf{u}^n_i) - \mathbf{f}(\mathbf{u}^n_{i-1})),$$

which is a staggered Lax–Friedrichs scheme.

To extend this scheme to higher order accuracy, the MUSCL approach is used, replacing the piecewise constant data by a piecewise linear representation, suitably limited to ensure TVD. For the higher order case, the integrals in (13) are evaluated for the piecewise linear representation, which will evolve in time, and the second order accurate mid-point quadrature rule is used to evaluate the integral in (12).

The method is extended to non-staggered grids in [21].

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