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Numerical Methods for Optimal
Control Problems with State Constraints

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Numerical Analysis Report 8/91

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Abstract

In this report we consider numerical methods for solving state constrained optimal control problems. The theoretical aspects of the problem are first examined and necessary conditions for the optimum are stated.

The possibility of using Valentine's Procedure for numerical calculations is considered and dismissed because of the poor numerical results. The tests on an example have shown that the method exhibits numerical instability as the state constraint is approached.

The alternative approach is to use an interior penalty method, a projection method or a combination of these two methods. All of these possibilities have been examined in conjunction with the gradient and conjugate gradient method. The tests were carried out on a simple example with a known analytical solution.

The methods were then applied to a simple linear, two-way generation tidal model. The analytical solution was derived under certain assumptions for comparison purposes. The numerical results have proved disappointing which leads to the conclusion that "reduced" methods should not be applied to the existing tidal power generation model with state constraints.

1. Introduction

In this report we consider numerical methods for solving state constrained optimal control problems. The motivation for the research was provided by the need to apply such methods to models which analyse control strategies in tidal power generation. The research into tidal power generation by the Reading University group has produced a versatile model with a number of special features such as ebb and two-way generation, consideration of expansion losses and inclusion of a pumping option. The model, however, does not include any mechanism to control the water level in the basin. Our initial aim is to adapt the flat-basin model by imposing constraints on the maximum and minimum levels of the water.

The problem is first formulated as an optimal control problem with state and control inequality constraints. Necessary conditions for the optimum are then examined. Various numerical strategies are considered and tested on a simple problem with a known analytical solution.

Later sections of the report deal with applications to the tidal power generation problem and in particular to a specific case where an analytical solution can be determined.

2. The Problem

The problem considered here requires the optimisation of a given cost functional subject to a set of ordinary differential equations and some additional constraints. In mathematical terms this can be stated as:

$$\max_{\underline{u}} J = \int_0^T g(\underline{x}, \underline{u}, t) dt \quad (2.0.1)$$

subject to:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) , \quad \underline{x}(0) = \underline{x}_0 \quad (2.0.2)$$

and

$$\underline{R}(\underline{u}, t) \leq \underline{0} \quad (2.0.3)$$

$$\underline{S}(\underline{x}, t) \leq \underline{0} \quad (2.0.4)$$

where $\underline{x}(t) \in X$ is a state vector belonging to a state space and $\underline{u}(t) \in U$ is a control vector belonging to a control space. X is the set of piecewise smooth functions of time with a finite number of corners, while U is the set of functions which are allowed to undergo jump changes and are restricted to be piecewise continuous only. The functional (2.0.1) is known as the "cost functional" and the system of equations (2.0.2) as the "state equation". The conditions (2.0.3) and (2.0.4) shall be referred to as the "control constraint" and the "state constraint" respectively.

If we temporarily ignore the state constraint (2.0.4) the necessary conditions for the optimum are given by the following theorem (Hadley and Kemp, 1971):

Theorem Let $\underline{u}(t)$ be an admissible control and $\underline{x}(t)$ the corresponding solution to (2.0.2). Then if $(\underline{u}(t), \underline{x}(t))$ yield an absolute maximum of J it is necessary that there exist a number $\lambda_0 \geq 0$ (which without loss of generality may be taken to be 0 or 1) and a vector-valued function $\underline{\lambda}(t)$, the components of which are continuous functions of time with the property that $(\lambda_0, \underline{\lambda}(t)) \neq \underline{0}$ for any t and such that if

$$H(\underline{x}, \underline{u}, \underline{\lambda}, t) = \lambda_0 g(\underline{x}, \underline{u}, t) + \underline{\lambda}^T f(\underline{x}, \underline{u}, t) \quad (2.0.5)$$

then

$$\dot{\underline{x}} = \frac{\partial H}{\partial \underline{\lambda}} (\underline{x}, \underline{u}, \underline{\lambda}, t) = \underline{f}(\underline{x}, \underline{u}, t) ; \underline{x}(0) = \underline{x}_0 \quad (2.0.6)$$

and

$$\dot{\underline{\lambda}} = - \frac{\partial H}{\partial \underline{\lambda}} (\underline{x}, \underline{u}, \underline{\lambda}, t) ; \underline{\lambda}(T) = \underline{0} \quad (2.0.7)$$

Furthermore, if

$$M(\underline{x}, \underline{\lambda}, t) = \sup_{\underline{w} \in U} H(\underline{x}, \underline{w}, \underline{\lambda}, t) \quad (2.0.8)$$

then

$$H(\underline{x}, \underline{u}, \underline{\lambda}, t) = M(\underline{x}, \underline{\lambda}, t) \quad (2.0.9)$$

The last condition is known as the "Maximum Principle".

We now consider the changes which the inclusion of the state constraint (2.0.4) makes to the above theory.

We first need to introduce explicit control dependency into the constraint. This may be achieved by differentiating the constraint with respect to t and substituting $f(\underline{x}, \underline{u}, t)$ for $\dot{\underline{x}}$ a number of times, say q , until $\underline{u}(t)$ is introduced explicitly into the constraint. We now define the Hamiltonian in the following way:

$$H^*(\underline{x}, \underline{u}, \underline{\lambda}, t) = g(\underline{x}, \underline{u}, t) + \underline{\lambda}^T \underline{f}(\underline{x}, \underline{u}, t) + \underline{\mu}^T \underline{S}^{(q)} \quad (2.0.10)$$

where $\underline{S} = \underline{0}$ and $\underline{S}^{(q)} = \underline{0}$ on the constraint boundary, and $\underline{\mu} = \underline{0}$ off the constraint boundary (i.e. where $\underline{S} < \underline{0}$).

The necessary conditions for the optimum are then given by (Bryson and Ho, 1975):

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t), \quad \underline{x}(0) = \underline{x}_0 \quad (2.0.11)$$

$$\dot{\underline{\lambda}} = \begin{cases} -\frac{\partial g}{\partial \underline{x}} - \underline{\lambda}^T \frac{\partial f}{\partial \underline{x}} & \text{when } \underline{S} < \underline{0} \\ -\frac{\partial g}{\partial \underline{x}} - \underline{\lambda}^T \frac{\partial f}{\partial \underline{x}} - \underline{\mu}^T \frac{\partial S^{(q)}}{\partial \underline{x}} & \text{when } \underline{S} = \underline{0} \end{cases} \quad (2.0.12)$$

$$\frac{\partial H^*}{\partial \underline{u}} = \frac{\partial g}{\partial \underline{u}} + \underline{\lambda}^T \frac{\partial f}{\partial \underline{u}} + \underline{\mu}^T \frac{\partial S^{(q)}}{\partial \underline{u}} = \underline{0} \quad \text{when } \underline{S} = \underline{0} \quad (2.0.13)$$

together with the Maximum Principle. It is also necessary for $\underline{\mu}$ to satisfy:

$$\underline{\mu}(t) \leq \underline{0} \quad \text{when } \underline{S} = \underline{0} . \quad (2.0.14)$$

The negativity of $\underline{\mu}$ when $\underline{S} = \underline{0}$ can be interpreted as the requirement that the gradient;

$$\frac{\partial H}{\partial \underline{u}} \equiv \frac{\partial g}{\partial \underline{u}} + \underline{\lambda}^T \frac{\partial f}{\partial \underline{u}} \quad (2.0.15)$$

be such that the improvement can only come by violating the constraints. We also need to satisfy the following conditions at the point where the state constraint is encountered:

$$\underline{N}(\underline{x}, t) \equiv \begin{bmatrix} \underline{S}(\underline{x}, t) \\ \underline{S}^{(1)}(\underline{x}, t) \\ \vdots \\ \underline{S}^{(q-1)}(\underline{x}, t) \end{bmatrix} = \underline{0} \quad (2.0.16)$$

At this point we note that the Lagrange multipliers (adjoint variables)

$\underline{\lambda}(t)$ are discontinuous at the junction points between constrained and unconstrained arcs. In fact they are not unique along the boundary arc $\underline{S} = \underline{0}$. It is a matter of choice whether to force the adjoint to be discontinuous at the point where the constraint boundary is first encountered or at the point where the optimal path leaves the constraint boundary. It is, however, usual to apply the "tangency conditions" (2.0.16) at the entry point and therefore force the discontinuity in $\underline{\lambda}(t)$ to coincide with the point at which the state constraint becomes active, and to allow the adjoint variable to be continuous at the exit.

With inequality constrained optimal control problems we often encounter corners - the points at which the control undergoes discontinuous changes. Let t^- denote the time just before and t^+ the time just after the corner. Then the following conditions hold:

(i) problems with control constraints only

$$\begin{aligned}\underline{\lambda}(t^-) &= \underline{\lambda}(t^+) \\ H(t^-) &= H(t^+) \\ \frac{\partial H}{\partial \underline{u}}(t^-) &= \frac{\partial H}{\partial \underline{u}}(t^+)\end{aligned}\tag{2.0.17}$$

(ii) problems with state and control constraints. At the entry:

$$\begin{aligned}\underline{\lambda}^T(t^-) &= \underline{\lambda}^T(t^+) + \underline{\pi}^T \frac{\partial \underline{N}}{\partial \underline{x}} \\ H^*(t^-) &= H^*(t^+) - \underline{\pi}^T \frac{\partial \underline{N}}{\partial t} \\ \frac{\partial H^*}{\partial \underline{u}}(t^-) &= \frac{\partial H^*}{\partial \underline{u}}(t^+)\end{aligned}\tag{2.0.18}$$

where $\underline{\pi}$ is a constant vector of Lagrange multipliers. At the exit point the same conditions as in (i) hold.

3. Numerical Methods

We have considered a number of numerical techniques to solve the above class of problems. They broadly fall into the following categories:

(i) transformation techniques, (ii) penalty methods and (iii) projection methods.

3.1. Method 1 A Transformation Technique

Initially, an attempt was made to adapt the transformation technique proposed by Jacobson and Lele (1969). They consider the following state constrained problem:

$$\max J = \int_0^T g(\underline{x}, u, t) dt \quad (3.1.1)$$

subject to

$$\dot{\underline{x}} = \underline{f}(\underline{x}, u, t) \quad \underline{x}(0) = \underline{x}_0 \quad (3.1.2)$$

and

$$S(\underline{x}, t) \leq 0 \quad (3.1.3)$$

There are no other constraints present and the final time T is given explicitly. Valentine's device is used to convert the state constraint into an equality constraint by the introduction of a slack variable $\alpha(t)$:

$$S(\underline{x}, t) + \frac{1}{2} \alpha^2(t) = 0 \quad (3.1.4)$$

Differentiating (3.1.4) q times the following set of equations is obtained

$$\begin{aligned} S^{(1)}(\underline{x}, t) + \alpha \alpha_1 &= 0 \\ S^{(2)}(\underline{x}, t) + \alpha_1^2 + \alpha \alpha_2 &= 0 \\ &\vdots \\ S^{(q)}(\underline{x}, t) + \{\text{terms involving } \alpha_{q-1}, \dots, \alpha_1\} + \alpha \alpha_q &= 0 \end{aligned} \tag{3.1.5}$$

where

$$\begin{aligned} \dot{\alpha} &= \alpha_1 \\ \dot{\alpha}_1 &= \alpha_2 \\ &\vdots \\ \dot{\alpha}_{q-1} &= \alpha_q \end{aligned} \tag{3.1.6}$$

It is assumed that the q^{th} equation in (3.1.5) contains u explicitly and we can obtain the following expression for u :

$$u = G(\underline{x}, \alpha, \alpha_1, \dots, \alpha_{q-1}, \alpha_q, t) \tag{3.1.7}$$

This can be used to eliminate u from the problem. The transformed problem is now

$$\max J = \int_0^T g(\underline{x}, G(\underline{x}, \alpha, \alpha_1, \dots, \alpha_q, t), t) dt \tag{3.1.8}$$

subject to

$$\begin{aligned}
 \dot{\underline{x}} &= f(\underline{x}, G(\underline{x}, \alpha, \alpha_1, \dots, \alpha_q, t), t), & \underline{x}(t_0) &= \underline{x}_0 \\
 \dot{\alpha} &= \alpha_1, & \alpha(t_0) &= \alpha^0 \\
 \dot{\alpha}_1 &= \alpha_2, & \alpha_1(t_0) &= \alpha_1^0 \\
 & \vdots & & \\
 \dot{\alpha}_{q-1} &= \alpha_q, & \alpha_{q-1}(t_0) &= \alpha_{q-1}^0
 \end{aligned} \tag{3.1.9}$$

where the initial conditions are chosen in the following way:

$$\begin{aligned}
 \alpha(t_0) &= \pm \sqrt{-2S(\underline{x}(t_0), t_0)} \\
 \alpha_1(t_0) &= -S_1(\underline{x}(t_0), t_0)/\alpha(t_0) \\
 \alpha_2(t_0) &= -[S_2(\underline{x}(t_0), t_0) + \alpha_1^2(t_0)]/\alpha(t_0)
 \end{aligned} \tag{3.1.10}$$

etc.

In the transformed problem α_q is the new control variable while $\underline{x}, \alpha, \alpha_1, \dots, \alpha_{q-1}$ are the state variables.

This method cannot cope with the case where there are more state constraints than control variables.

In our applications we want to be able to include the control constraints as well as the state constraints. Under the proposed transformation the control constraint

$$R(u, t) \leq 0 \tag{3.1.11}$$

becomes

$$R(G(\underline{x}, \alpha, \alpha_1, \dots, \alpha_{q-1}, t), t) \leq 0 \tag{3.1.12}$$

To test the feasibility of this procedure the transformation was applied in conjunction with the projected gradient method to the following problem:

$$\max \int_0^1 (x_1 + x_2 + u) dt \quad (3.1.13)$$

subject to

$$\dot{x}_1 = x_2 \quad x_1(0) = 0 \quad (3.1.13)$$

$$\dot{x}_2 = 4u - 8t \quad x_2(0) = 0 \quad (3.1.15)$$

$$0 \leq u \leq 1 \quad (3.1.16)$$

$$x_2 \leq 0.5 \quad (3.1.17)$$

The procedure has shown signs of extreme numerical instability. We note that in (3.1.7) α_q , the control, is always multiplied by α which becomes very small as the state constraint is approached. This causes inaccuracies in the numerical calculations.

This approach has been abandoned.

3.2 Method 2 Interior Penalty Function

Optimal control problems with inequality constraints are often converted into unconstrained form by means of penalty functions. The type of penalty function considered here was suggested by Lasdon, Waren and Rice (1967), and works from inside the constraint set, with the penalty increasing as the boundary is approached. In particular, if the problem (2.0.1) to (2.0.4) is considered then it may be converted into an unconstrained form by adding a penalty function P to the objective (2.0.1), where

$$P = r \left[\sum_{i=1}^k \int_0^T \frac{dt}{S_i(\underline{x}, t)} + \sum_{i=1}^{\ell} \int_0^T \frac{dt}{R_i(\underline{u}, t)} \right] \quad (3.2.1)$$

and r is a small positive scalar. A sequence of unconstrained problems is then solved for different values of r . It was shown by Lasdon, Waren and Rice (1967) that under some general assumptions the maximum point of the penalised problem approaches the solution of the inequality constrained problem as $r \rightarrow 0$.

Whatever solution method for the unconstrained (penalised) problem is adopted, the following must be ensured:

- (a) the starting point must lie inside the feasible region,
- (b) any subsequent iterates must lie within the feasible region.

We have considered solution techniques based on gradient and conjugate gradient methods.

(i) **Gradient Method**

The algorithm generates a sequence of admissible controls $\{\underline{u}^k\}$ for which the values of the functional $J^k = J(\underline{u}^k)$ are monotonically non-decreasing. Let \underline{u}^k be an optimal control with the corresponding state and adjoint variables \underline{x}^k and $\underline{\lambda}^k$ satisfying (2.0.6) and (2.0.7) respectively. The new approximation is made as follows:

$$\underline{u}^{k+1} = \underline{u}^k + s^k \underline{\gamma}^k \quad (3.2.2)$$

where $\underline{\gamma}^k = \frac{\partial H(\underline{u}^k)}{\partial \underline{u}}$ denotes the gradient of the Hamiltonian and s^k is the length of a step in the gradient direction chosen so that the value of the cost functional is increased and \underline{u}^{k+1} and \underline{x}^{k+1} stay within the feasible region.

(ii) **Conjugate Gradient Method**

In this case a new approximation is made according to the formula:

$$\underline{u}^{k+1} = \underline{u}^k + s^k \underline{\Gamma}^k \quad (3.2.3)$$

where

$$\underline{\Gamma}^{k+1} = \underline{\gamma}^{k+1} + \beta^k \underline{\Gamma}^k \quad (3.2.4)$$

$$\beta^k = (\underline{\gamma}^{k+1}, \underline{\gamma}^{k+1}) / (\underline{\gamma}^k, \underline{\gamma}^k) \quad (3.2.5)$$

and

$$(\underline{\gamma}^k, \underline{\gamma}^k) = \sum_i \int_0^T \gamma_i^k(t) \gamma_i^k(t) dt \quad (3.2.6)$$

The step length s^k and γ^k are chosen in the same way as in the previous method.

3.3 Method 3 Based on Projected Gradient Algorithm

The basis of this method is the projected gradient algorithm. As in the previous method we are generating a sequence of admissible controls $\{\underline{u}^k\}$ for which the values of the cost functional are non-decreasing. The new approximation to the optimal control is chosen as

$$\underline{u}^{k+1} = \mathcal{P}(\underline{u}^k + s^k \underline{\gamma}^k) \quad (3.3.1)$$

where \mathcal{P} is the L_2 projection operator on U . In cases where the constraints on the state variable are present, they also act as implicit constraints on the control variable. Let U_c denote a set of controls which satisfy the control constraints and U_s the set of controls which

produce the states that satisfy the state constraints. Then the admissible space of controls is $U = U_c \cap U_s$ and the projection operator must be constructed accordingly.

We consider the following algorithm:

- Step 1 Choose $\underline{u}^0 \in U_c, s^0, \underline{\gamma}^0$
- Step 2 $\underline{u}^{k+1} = \mathcal{P}_c(\underline{u}^k + s^k \underline{\gamma}^k)$
- Step 3 calculate \underline{x}^{k+1}
- Step 4 project \underline{x}^{k+1} to satisfy the state constraint
- Step 5 recalculate \underline{u}^{k+1} to correspond to \underline{x}^{k+1}
- Step 6 if no improvement in J^{k+1} set $s^k = \frac{s^k}{2}$ and go to step 2.
- Step 7 if $|J^{k+1} - J^k| < \epsilon_1$, or $s^k < \epsilon_2$ go to step 13
- Step 8 calculate the adjoint variable $\underline{\lambda}^{k+1}$
- Step 9 calculate $\underline{\mu}^{k+1}$
- Step 10 calculate the gradient $\underline{\gamma}^{k+1}$. On the constraint $\underline{\gamma}^{k+1} = \underline{0}$
if $\underline{\mu} < 0$
- Step 11 $s^{k+1} = 1$
- Step 12 go to step 2
- Step 13 STOP

Remarks

(i) In step 5 we assume that the readjusted control immediately lies in U_c and therefore no further projections are necessary.

(ii) Steps 4 and 5 ensure that the pair $(\underline{u}^{k+1}, \underline{x}^{k+1})$ satisfies all given constraints. In practice, because of the discretization, the state variable could be violating the state constraint by a small amount proportional to the step-size.

(iii) Steps 9 and 10 make use of the condition that

$$\underline{\mu}(t) \leq \underline{0} \quad \text{on} \quad \underline{S} = \underline{0}$$

to speed up the procedure.

(iv) The procedure terminates if $|J^{k+1} - J^k| < \epsilon_1$.

The procedure aborts if we cannot make any further improvement in the given gradient direction.

(v) The state equation is solved numerically (STEP 3) by forward integration using the Trapezium Rule. The adjoint equation is solved by backward integration using the same method (STEP 8).

Another variant of this method has been considered. In step 2 of the algorithm instead of taking a step in the gradient direction, a step in the $\underline{\Gamma}^k$ direction is taken, as described in the section dealing with conjugate gradient methods.

3.4 Method 4 Combination of Projected Gradient and Penalty Methods

In this method the state constraint is imposed via an interior penalty. The projected gradient method is then applied to the state-penalised problem in order to satisfy the control constraint.

3.5 The Test Example

The above methods have been tested on the following example:

$$\max_u \int_0^1 \left\{ x_1 \cos 2\pi t - \frac{1}{4\pi^2} u \right\} dt \quad (3.5.1)$$

subject to

$$\dot{x}_1 = x_2, \quad x_1(0) = 0 \quad (3.5.2)$$

$$\dot{x}_2 = u, \quad x_2(0) = 0 \quad (3.5.3)$$

$$-1 \leq u \leq 1 \quad (3.5.4)$$

$$-\frac{1}{8} \leq x_2 \leq \frac{1}{8} \quad (3.5.5)$$

The problem has an analytical solution which can be obtained using the necessary conditions (2.0.11)-(1.0.18). The solution is illustrated in fig. 1. Note that the switches in control occur at $t = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}$ and $\frac{3}{4}$. The adjoint variable λ_1 corresponding to the first state equation is continuous while λ_2 corresponding to the second state equation has discontinuities at $t = \frac{1}{8}$ and $t = \frac{5}{8}$. The value of the cost functional is calculated as $J = 1.2583 \times 10^{-2}$.

Method 2

The results were calculated using both gradient and conjugate gradient methods. A number of penalty parameters were used which affected the number of iterations. Fig. 2 gives the results obtained for the gradient method with a penalty parameter of 10^{-6} and $\epsilon_1 = 10^{-10}$. The value of the cost functional was obtained as $J = 1.2210 \times 10^{-2}$ and the number of iterations was 709. Fig. 3 gives the results for the conjugate gradient method with the same parameters. In this case $J = 1.2305 \times 10^{-2}$ and the number of iterations was 451. It must be pointed out, however, that the conjugate gradient method requires more work per iteration than the gradient method. In either case the switching points are reasonably well approximated.

Method 3

Fig. 4 gives the results for the gradient method. The method aborted

after 200 iterations and the constraints were violated. Fig. 5 gives the results for the conjugate gradient method. In this case $J = 1.2355 \times 10^{-2}$ and the method converged after 208 iterations. The switching points were not accurately calculated by either of these methods.

Method 4

The results were calculated using the conjugate gradient method and are displayed in fig. 6. The penalty parameter was 10^{-6} and J was calculated as $J = 1.2260 \times 10^{-2}$ in 181 iterations. The position of the switching points was calculated reasonably accurately.

The results are summarised in table 1.

The results indicate that Method 4 is possibly the best one to use. It appears to combine the best features of Methods 2 and 3.

TABLE 1

Method	Number of iterations	Cost functional value	Stopping criterion	pen.
Method 2/gradient	709	1.2210×10^{-2}	10^{-10}	10^{-6}
Method 2/conjugate gradient	451	1.2305×10^{-2}	10^{-10}	10^{-6}
Method 3/gradient	200*	—	aborted	—
Method 3/conjugate gradient	208	1.2355×10^{-2}	10^{-10}	—
Method 4/conjugate gradient	181	1.2260×10^{-2}	10^{-10}	10^{-6}
Analytical solution		1.2583×10^{-2}		

4. Application to Tidal Power Generation

4.1 The Problem

The model used is an adaptation of the simplest linear model investigated by the Reading University Group incorporating two-way generation and no separate sluices. It can be stated as follows

$$\max E = \int_0^T e(X(h)u(t), h) dt \quad (4.1.1)$$

subject to

$$\dot{\eta}(t) = -\frac{1}{\tilde{A}} X(h)u(t) \quad (4.1.2)$$

$$\eta(0) = \eta_0 \quad (4.1.3)$$

$$u(t) \in [0, 1] \quad (4.1.4)$$

where

$$X(h) \equiv \begin{cases} -Q_1(h) & h \leq 0 \\ Q_2(h) & h > 0 \end{cases} \quad (4.1.5)$$

and

$$e(X(h)u(t), h(t)) \equiv \begin{cases} -Q_1 u h & h \leq 0 \\ Q_2 u h & h > 0 \end{cases} \quad (4.1.6)$$

Here

$\eta(t)$ is the basin level above some datum point

$f(t)$ is the tidal level

$h(t)$ is the head difference defined as $h(t) \equiv \eta(t) - f(t)$

T is the tidal period

\tilde{A} is the basin surface area

$Q_1(h)$ is the maximum sluicing capacity of the turbines for head h

$Q_2(h)$ is the maximum flow for turbine use for head h .

In our calculations we have made the following assumptions:

$$f(x) = \cos 2\pi t \quad (4.1.7)$$

$$- Q_1(h) = Q_2(h) = h \quad (4.1.8)$$

which are consistent with the assumptions made by (Andrews, Nichols and Xu, 1990). Under these assumptions the problem becomes:

$$\max E = \int_0^T u h^2 dt \quad (4.1.9)$$

subject to

$$\dot{\eta} = - \frac{1}{A} h u \quad (4.1.10)$$

$$\eta(0) = \eta_0 \quad (4.1.11)$$

$$u(t) \in [0,1] \quad (4.1.12)$$

In addition we require $\eta(t)$ to be between some given bounds i.e.

$$\eta_{\min} \leq \eta \leq \eta_{\max} \quad (4.1.13)$$

which is a state constraint; a feature not previously considered.

We note that the form of the boundary condition (4.1.11) is different from the one previously considered by the Reading University Group. Their investigations were based on the periodic boundary condition $\eta(0) = \eta(T)$.

4.2 Analytical Solution

A problem can be solved analytically provided a particular control pattern is assumed. First of all we map the interval $[0,T]$ onto the

interval $[0,1]$ by a simple change of variable. The problem is now equivalent to

$$\max_u \int_0^1 u h^2 dt \quad (4.2.1)$$

Subject to

$$\dot{\eta} = -\frac{T}{\tilde{A}} h u = -\frac{1}{\tilde{A}} h u \quad (4.2.2)$$

$$\eta(0) = \eta_0 \quad (\text{given}) \quad (4.2.3)$$

where $h = \eta - f$, $f = \cos 2\pi t$ and $A = \frac{\tilde{A}}{T}$. We solve the problem assuming the following control pattern:

$$u = \begin{array}{ll} 1 & 0 \leq t \leq \tau_1 \\ 0 & \tau_1 < t \leq \tau_2 \\ 1 & \tau_2 < t \leq \tau_3 \\ 0 & \tau_3 < t \leq \tau_4 \\ 1 & \tau_4 < t \leq 1 \end{array} \quad (4.2.4)$$

This control pattern is also suggested by some numerical experiments.

If $u = 1$ then

$$\eta = C e^{-\frac{1}{\tilde{A}} t} + \phi (2\pi A \sin 2\pi t + \cos 2\pi t) \quad (4.2.5)$$

$$\lambda = D e^{\frac{1}{\tilde{A}} t} + CA e^{-\frac{1}{\tilde{A}} t} + 4\pi A^2 \phi \sin 2\pi t \quad (4.2.6)$$

where C and D are constants and $\phi = \frac{1}{4\pi^2 A^2 + 1}$.

Since $\eta(0) = \eta_0$ then $\lambda(1) = 0$ and therefore

$$\lambda = -CA e^{\frac{1}{\tilde{A}} (t-2)} + CA e^{-\frac{1}{\tilde{A}} t} + 4\pi A^2 \phi \sin 2\pi t \quad \tau_4 \leq t \leq 1. \quad (4.2.7)$$

Let $H = uh^2 - \lambda hu/A$. We require H to be continuous at τ_4 , i.e.

$$H(\tau_{4-}) = H(\tau_{4+}) \quad (4.2.8)$$

We note that $u = 0$ for $\tau_3 \leq t < \tau_4$ which implies $H(\tau_{4-}) = 0$. Thus

$$h^2(\tau_4) - \lambda(\tau_4) h(\tau_4) / A = 0 \quad (4.2.9)$$

or

$$A(\eta_{\min} - \cos 2\pi\tau_4) = \lambda(\tau_4) \quad (4.2.10)$$

We also know that $\eta(\tau_4) = \eta_{\min}$ and therefore

$$C e^{-\frac{1}{A}\tau_4} + \phi(2\pi A \sin 2\pi\tau_4 + \cos 2\pi\tau_4) = \eta_{\min} \quad (4.2.11)$$

which defines the constant C for $\tau_4 \leq t \leq 1$.

From (4.2.10) and (4.2.7)

$$\lambda(\tau_4) = -CA e^{\frac{1}{A}(\tau_4-2)} + CA e^{-\frac{1}{A}\tau_4} + 4\pi A^2 \phi \sin 2\pi\tau_4 = A(\eta_{\min} - \cos 2\pi\tau_4) \quad (4.2.12)$$

(4.2.11) and (4.2.12) define τ_4 and provided $\tilde{A} = 3.33 \times 10^3$, $T = 4.32 \times 10^4$ and $\eta_{\min} = -0.5$ the solution is found to be $\tau_4 = 0.9015$ regardless of the starting point η_0 .

We now determine switching points τ_2 and τ_3 . We know that

$$\eta(\tau_2) = \eta_{\max} \quad \text{and} \quad \eta(\tau_3) = \eta_{\min} . \quad (4.2.13)$$

We also require $H(\tau_{3-}) = H(\tau_{3+})$ and $H(\tau_{2-}) = H(\tau_{2+})$ which gives the following conditions

$$\lambda(\tau_3) = h(\tau_3)A = (\eta_{\min} - \cos 2\pi\tau_3) A \quad (4.2.14)$$

$$\lambda(\tau_2) = h(\tau_2)A = (\eta_{\max} - \cos 2\pi\tau_2) A .$$

Thus we have four equations in four unknowns C , D , τ_2 and τ_3 . These reduce to the following pair

$$\begin{aligned} [\eta_{\max} - \phi(2\pi A \sin 2\pi\tau_2 + \cos 2\pi\tau_2)] e^{\frac{1}{A} \tau_2} = \\ [\eta_{\min} - \phi(2\pi A \sin 2\pi\tau_3 + \cos 2\pi\tau_3)] e^{\frac{1}{A} \tau_3} \end{aligned} \quad (4.2.15)$$

$$\begin{aligned} e^{-\frac{1}{A} \tau_2} (A(1-\phi) \cos 2\pi\tau_2 + 2\pi A^2 \phi \sin 2\pi\tau_2) = \\ e^{-\frac{1}{A} \tau_3} (A(1-\phi) \cos 2\pi\tau_3 + 2\pi A^2 \phi \sin 2\pi \tau_3) . \end{aligned} \quad (4.2.16)$$

These can be no further reduced and a numerical solution is required.

This was surprisingly difficult to calculate and it was found to consist of two pairs of solutions which were then repeated periodically. One of the pairs was found not to lie in the interval $[0,1]$ and was therefore rejected. The other pair (which was more difficult to find numerically) was $\tau_2 = 0.4647$ and $\tau_3 = 0.5512$. It was found that the equations were ill-conditioned. A change in the solution of the order 10^{-6} produced a

change in the residual of the order 10^8 . It is worth noting that the positions of τ_2 and τ_3 are not dependent on the choice of boundary conditions but mainly on the bounds η_{\max} and η_{\min} and the forcing function.

To determine τ_1 we use the initial condition $\eta(0) = \eta_0$ and the fact that $\eta(\tau_1) = \eta_{\max}$. This gives the equation:

$$(\eta_0 - \gamma)e^{-\frac{1}{A}\tau_1} + \gamma(2\pi A \sin 2\pi \tau_1 + \cos 2\pi \tau_1) = \eta_{\max} \quad (4.2.17)$$

and for $\eta_{\max} = 0.5$, $\eta_{\min} = -0.5$ and $\eta_0 = 0.25$ we find $\tau_1 = 0.0316$.

4.3 Numerical Results

All of the methods already discussed have been applied to the tidal power generation problem and all of them have given unsatisfactory results. Fig. 7 illustrates the control pattern obtained by the penalty method with conjugate gradients, while fig. 6 gives the control calculated using the projection method with conjugate gradients. Neither of these are even close to the analytical solution. However, given the ill-conditioned nature of the problem, this is perhaps not surprising. It remains to conclude that the "reduced" methods such as ones discussed in this report should not be used to solve the existing tidal power generation model with state constraints.

5. A Possible Way Forward

Another class of methods may possibly give a solution for the tidal power generation problem. These methods proceed by discretising the cost

functional (using, say, the trapezium rule), the state equation (using some finite difference method) and the constraints. The discrete problem may take the form:

$$\max_{\underline{u}_i} \hat{J} = h \sum_{i=1}^{N-1} g(\underline{x}_i, \underline{u}_i, t_i) + \frac{h}{2} (g(\underline{x}_0, \underline{u}_0, t_0) + g(\underline{x}_N, \underline{u}_N, t_N)) \quad (5.0.1)$$

subject to

$$\underline{x}_{i+1} = \underline{x}_i + \frac{h}{2} [f(\underline{x}_i, \underline{u}_i, t_i) + f(\underline{x}_{i+1}, \underline{u}_{i+1}, t_{i+1})] \quad (5.0.2)$$

$$R(\underline{u}_i, t_i) \leq 0 \quad (5.0.3)$$

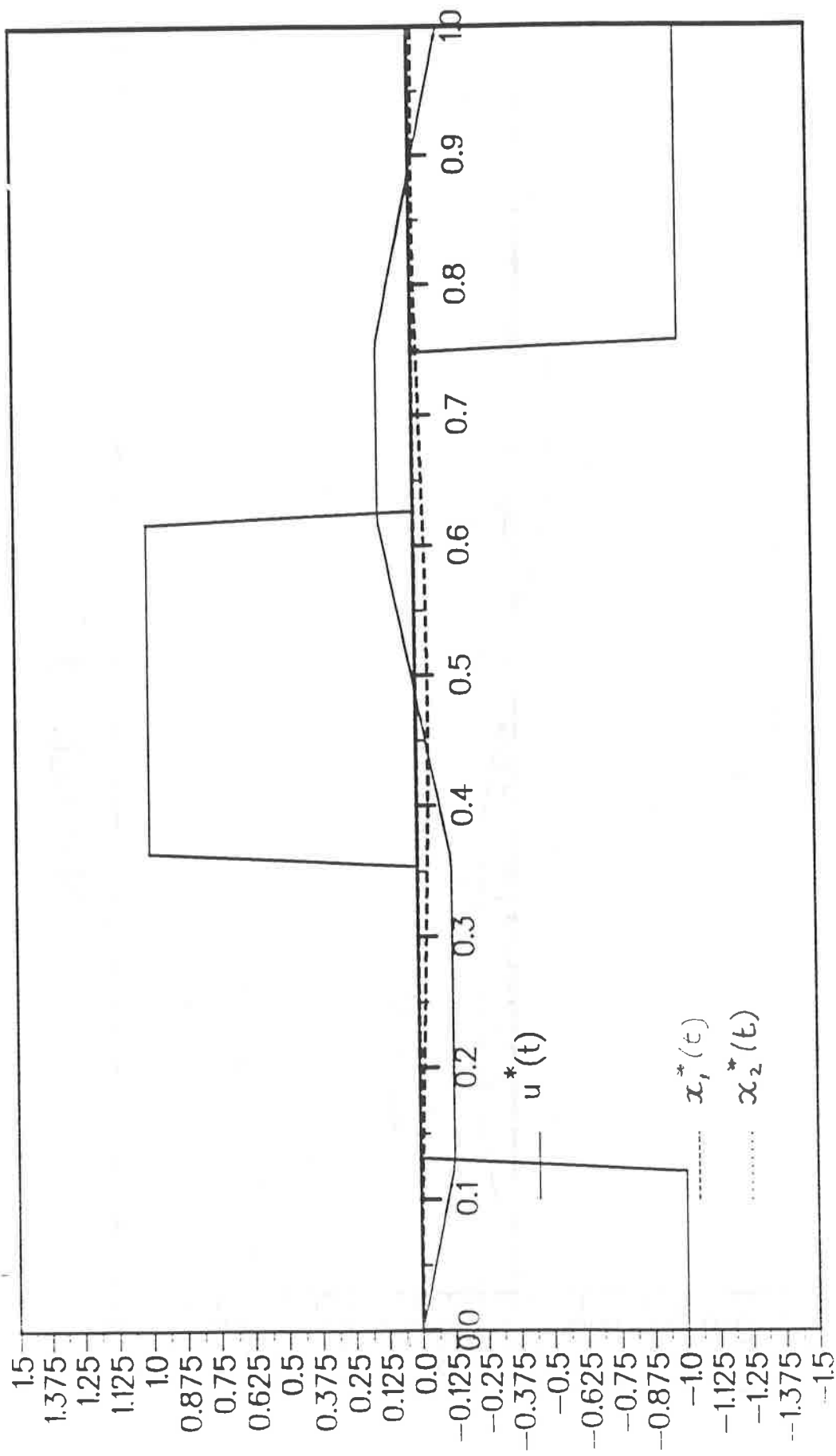
$$S(\underline{x}_i, t_i) \leq 0 \quad (5.0.4)$$

where $\underline{u}_i \cong \underline{u}(t_i)$, $\underline{x}_i \cong \underline{x}(t_i)$ and $h = 1/N$. This is an optimization problem where a maximum of the given objective function is sought subject to some non-linear equality and inequality constraints. Problems of this kind can be solved by the package LANCELOT [3]. This has been applied to the tidal problem and the preliminary results have been encouraging although the run times were quite high.

This approach merits further investigation.

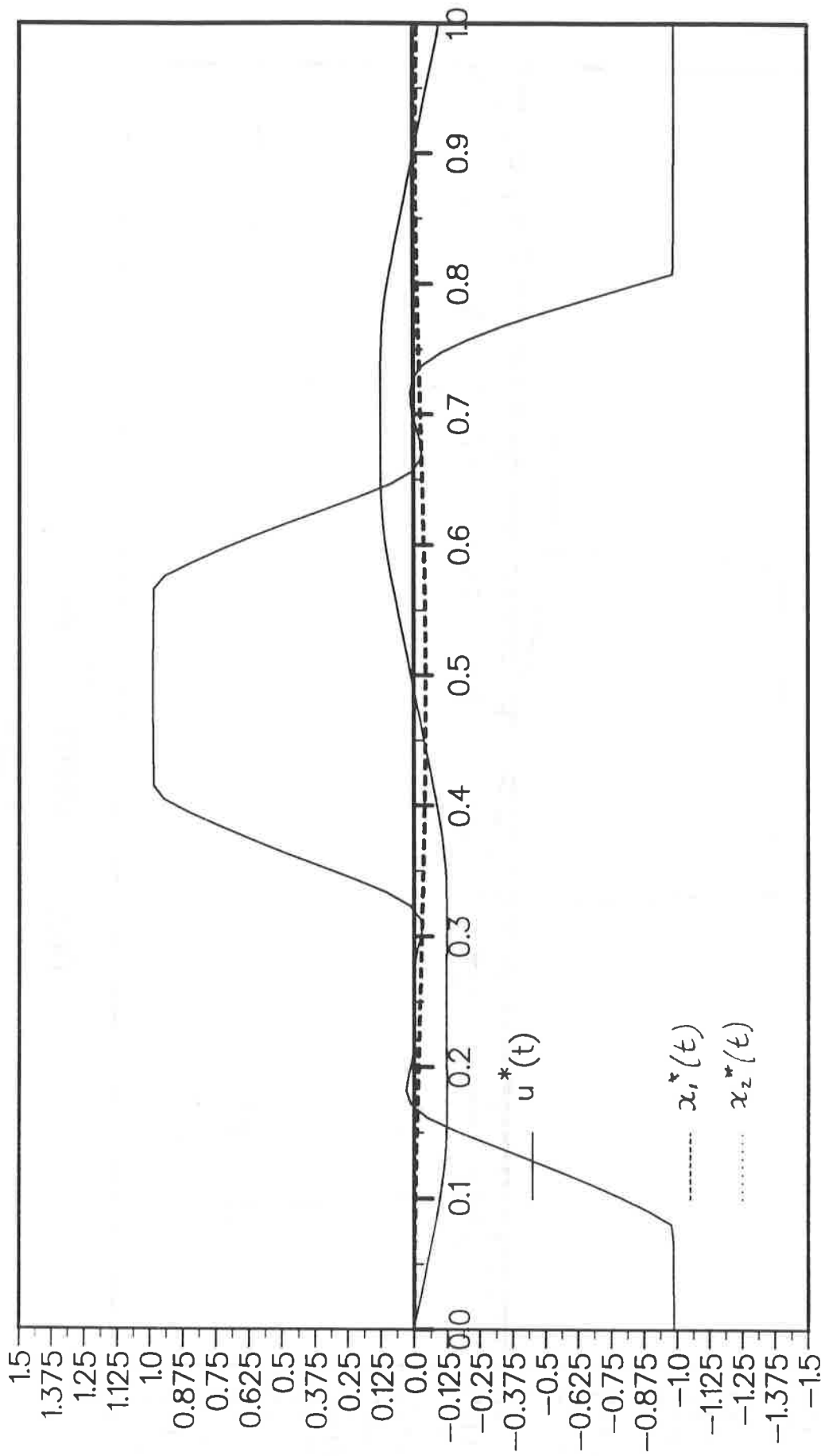
References

1. ANDREWS, T.P., NICHOLS, N.K. and XU, Z.: "The Form of Optimal Controllers for Tidal-Power-Generation Schemes". Numerical Analysis Report 8/90, Department of Mathematics, University of Reading (1990).
2. BRYSON, A.E. and HO, Y.C.: "Applied Optimal Control", Halstead Press(1975).
3. GOULD, N.I.M., CONN, A.R. and TOINT, Ph. L.: LANCELOT
4. GRUVER, W.A. and SACKS, E.: "Algorithmic Methods in Optimal Control, Pitman (1975).
5. HADLEY, G. and KEMP, M.C.: "Variational Methods in Economics, North-Holland/American Elsevier (1971).
6. JACOBSON, H.J. and LELE, M.M.: "A Transformation Technique for Optimal Control Problems with a State Variable Inequality Constraint", IEEE Trans. Automat. Control 14, (1969) pp 457-464.
7. LASDON, S.L., WARREN, A.D., RICE, R.K.: "An Interior Penalty Method for Inequality Constrained Optimal Control Problems", IEEE Trans. Automat. Control 12, (1967), pp 388-395.



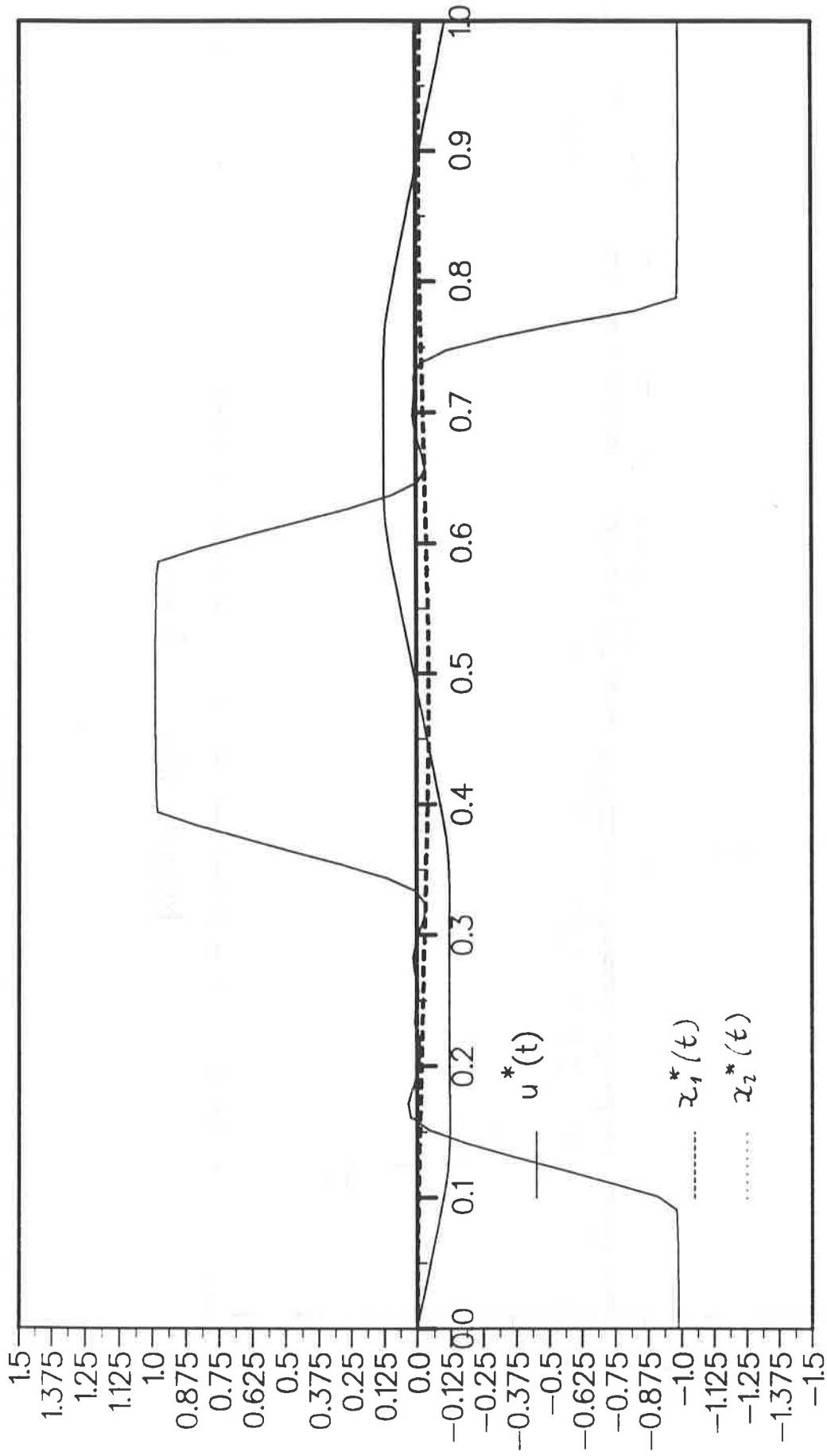
Normalised Time

Figure 1



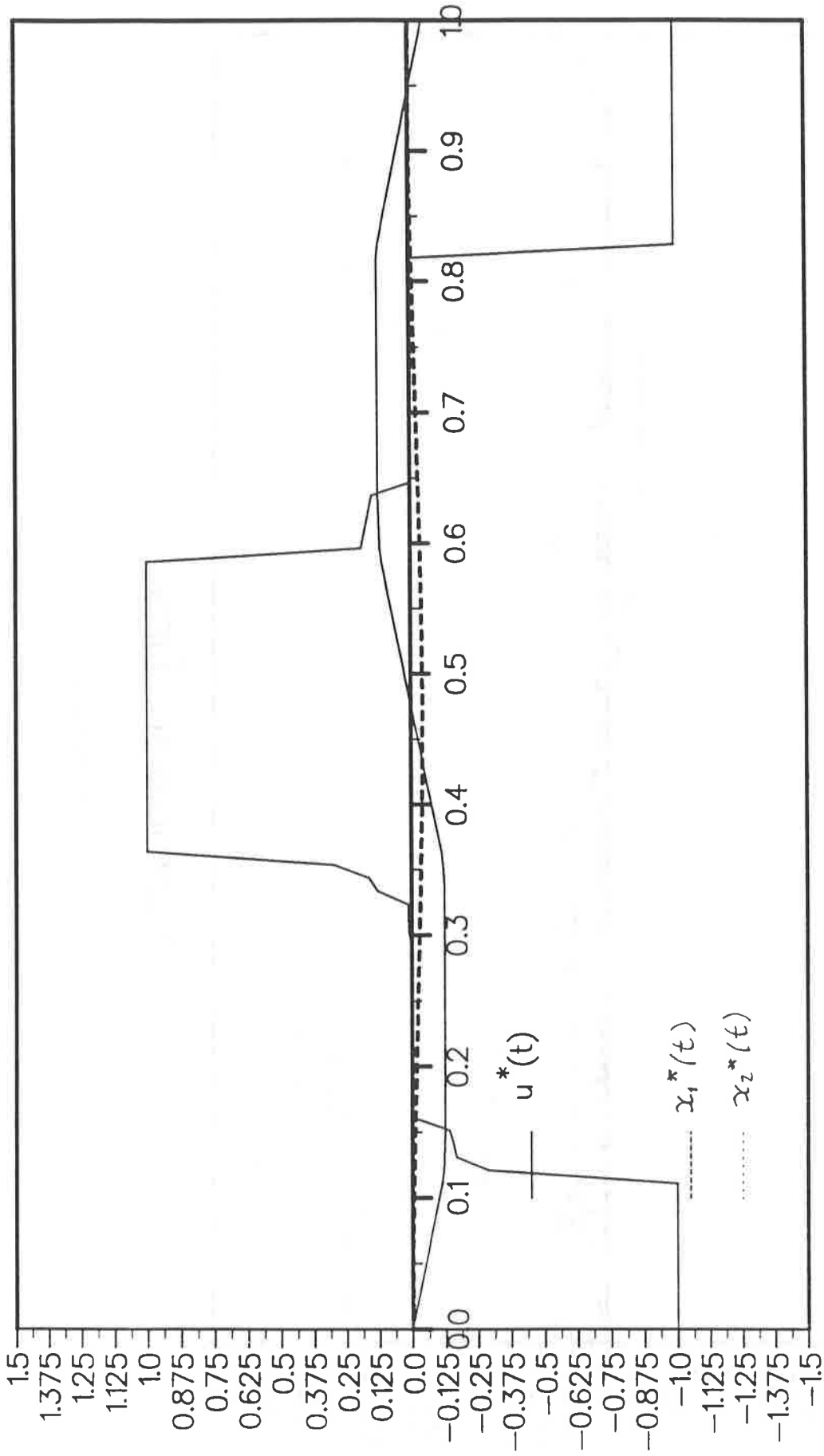
Normalised Time

Figure 2



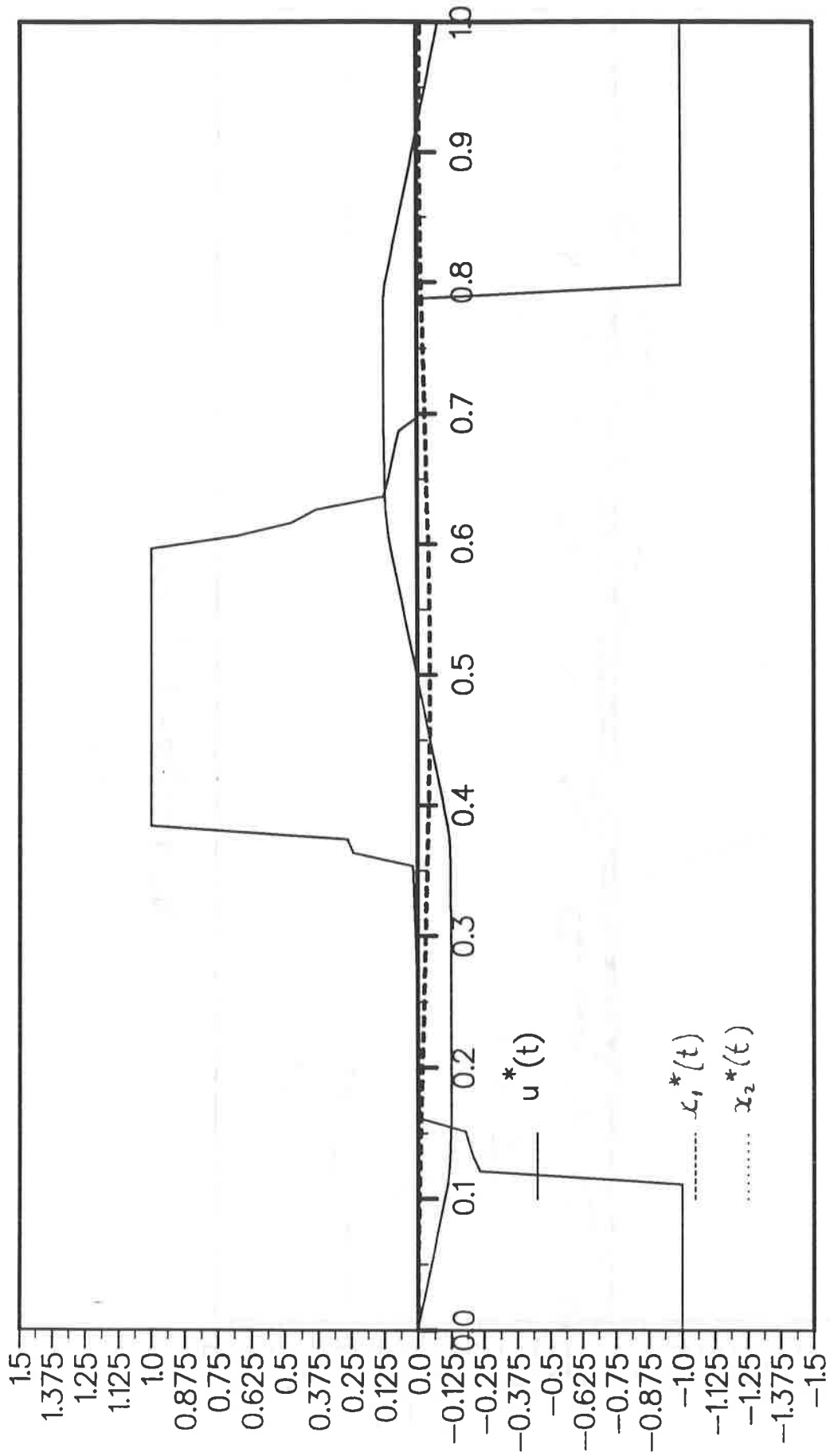
Normalised Time

Figure 3



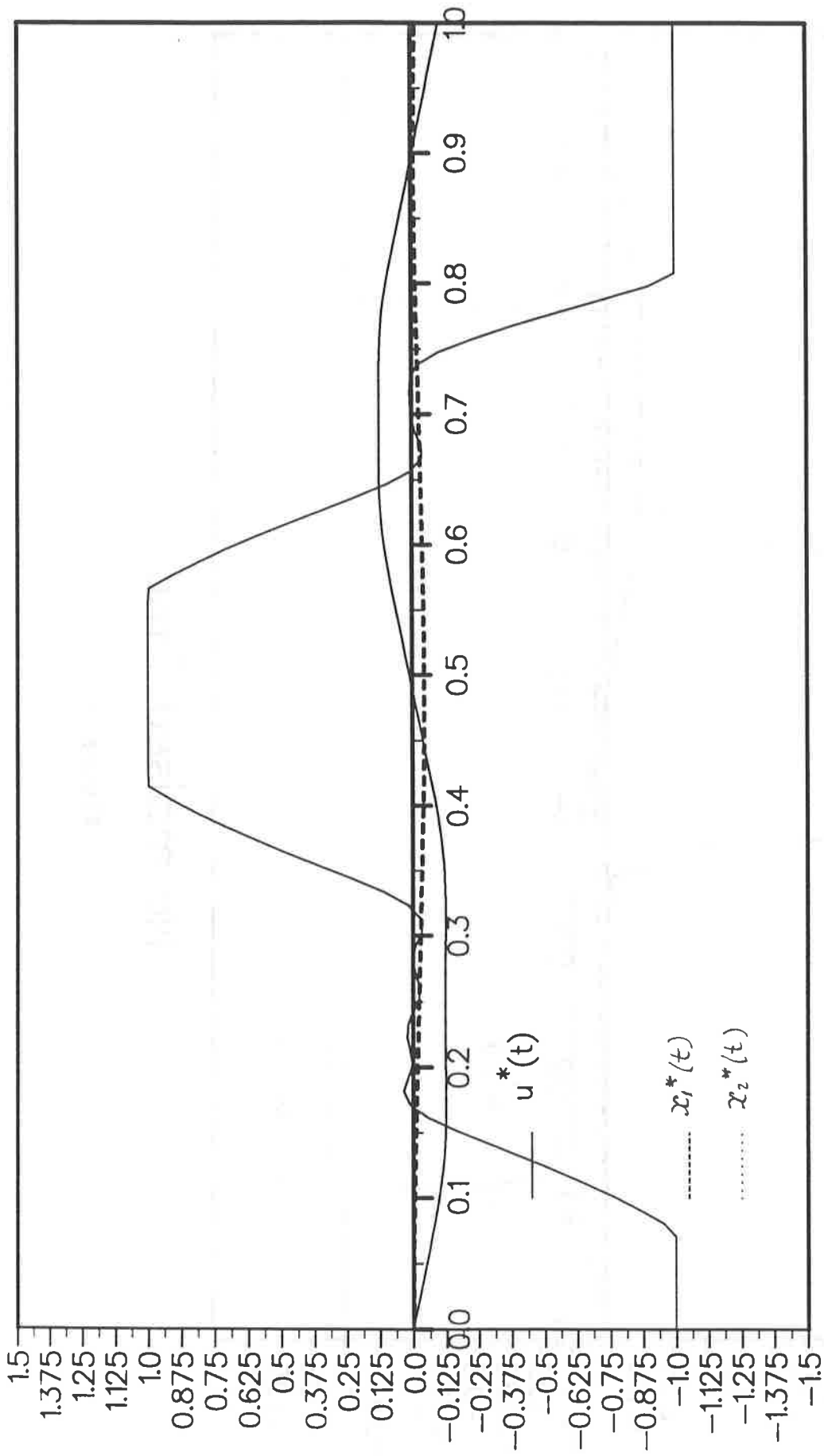
Normalised Time

Figure 4



Normalised Time

Figure 5



Normalised Time

Figure 6

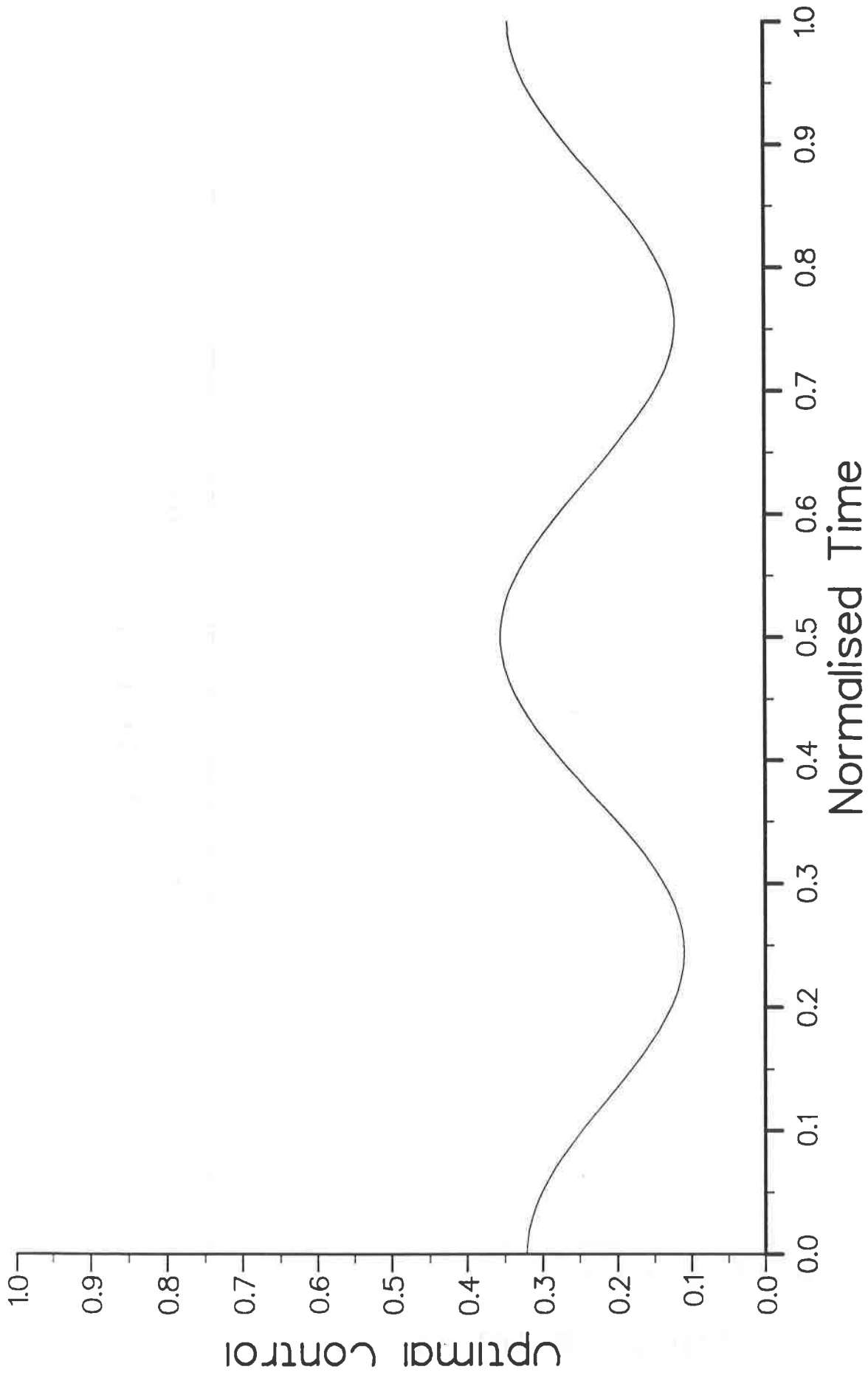


Figure 7

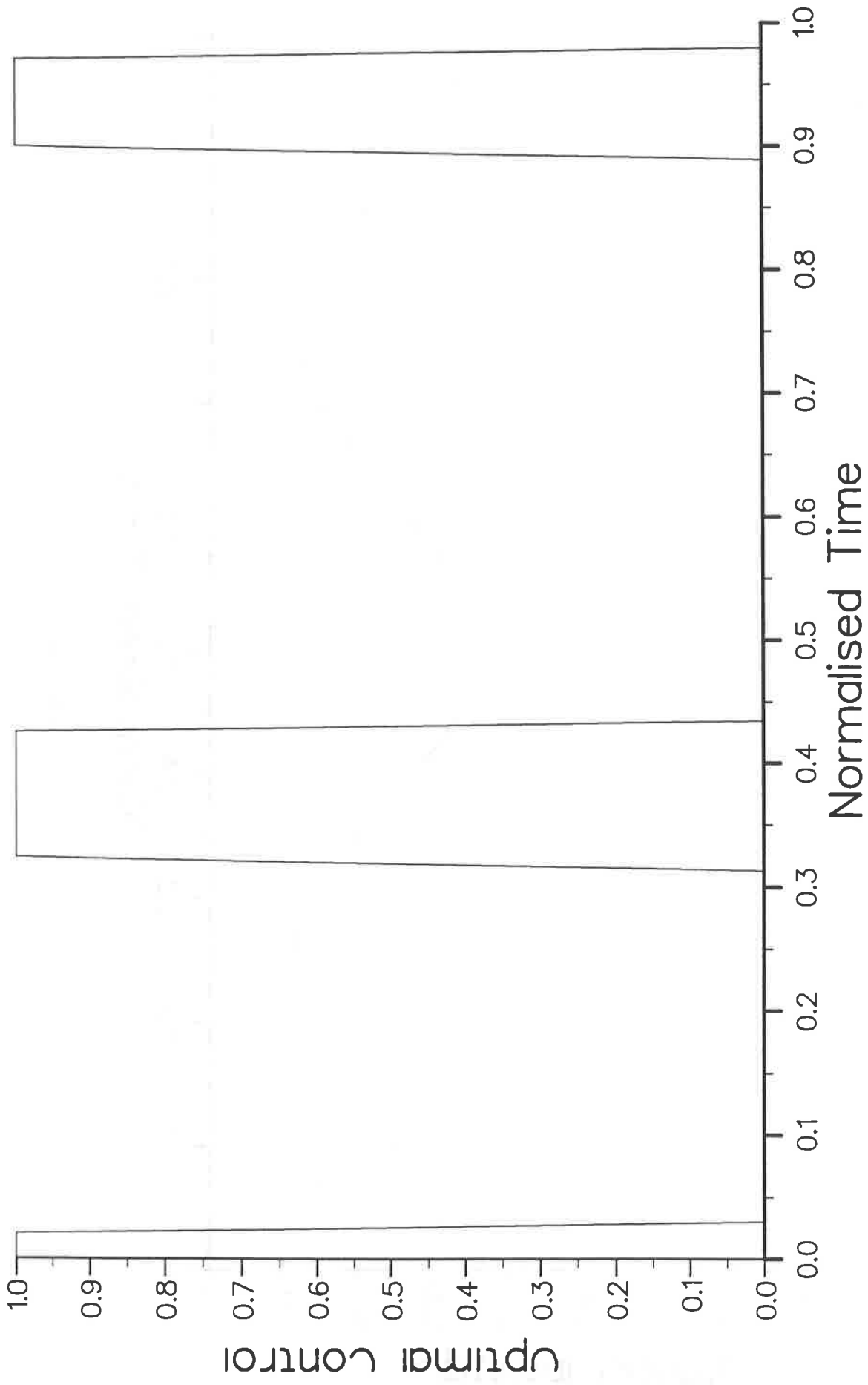


Figure 8