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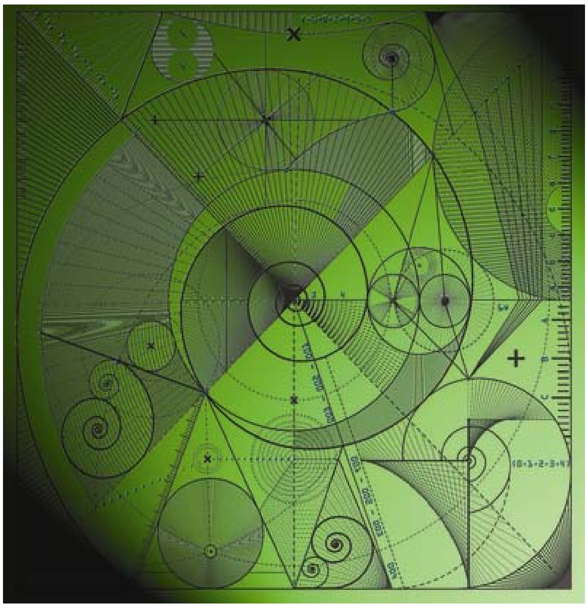

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Variational data assimilation for parameter estimation: application to a simple morphodynamic model

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Abstract Data assimilation is a sophisticated mathematical technique for combining observational data with model predictions to produce state and parameter estimates that most accurately approximate the current and future states of the true system. The technique is commonly used in atmospheric and oceanic modelling, combining empirical observations with model predictions to produce more accurate and well-calibrated forecasts. Here we consider a novel application within a coastal environment and describe how the method can also be used to deliver improved estimates of uncertain morphodynamic model parameters. This is achieved using a technique known as state augmentation. Earlier applications of state augmentation have typically employed the 4D-Var, Kalman filter or ensemble Kalman filter assimilation schemes. Our new method is based on a computationally inexpensive 3D-Var scheme, where the specification of the error covariance matrices is crucial for success. A simple 1D model of bed-form propagation is used to demonstrate the method. The scheme is capable of recovering near perfect parameter values and therefore improves the capability of our model to predict future bathymetry. Such positive results suggest the potential for application to more complex morphodynamic models.

Keywords Data assimilation · Morphodynamic modelling · Parameter estimation · State augmentation

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1 Introduction

Changes to weather patterns, with increasing incidence of coastal flooding in recent years, have led to growing concern over the effects of climate change on flooding and highlighted the importance of accurate knowledge of coastal morphology in natural disaster prediction and management. It is essential that we improve our ability to predict floods; being able to better identify and anticipate flood risk would facilitate the development of suitable strategies for the management of coastal areas and help to limit the damage and distress caused by flooding. Key to this aim is better knowledge and understanding of how the morphology of the coastal zone is evolving over time (Nicholls et al. (2007), Stelling (2000)). Accurate bathymetry immediately prior to a storm event would allow improved flood forecasting using coastal inundation models.

Coastal morphodynamics presents a challenge to modellers. Modelling is difficult because longer term morphological changes are driven by shorter term processes such as waves and tides (Masselink and Hughes (2003)). State of the art models are growing more sophisticated in an attempt to accurately model coastal morphology (e.g. Lesser et al. (2004)). However, in practice, models suffer from uncertainty in their parameters, for example those that arise from parameterization of the sediment transport flux. Inaccurate representation of model parameters will lead to the growth of model error and therefore affect the ability of our model to accurately predict the true system state. A key question in model development is how to estimate these parameters a priori. Generally, parameters are determined theoretically or by calibration of the model against observations, although there are other approaches (e.g. Hill et al. (2003), Knaapen and Hulscher (2003), Vrugt et al. (2005), Wüst (2004)). Here we present a novel approach using a variational data assimilation scheme.

Data assimilation is a technique for combining observational data with model predictions to 1) produce a model state that most accurately approximates the current and future states of the true system and 2) provide estimates of the model parameters. Whilst it is routinely used in atmospheric and oceanic prediction, the possibility of transferring data assimilation techniques to coastal morphodynamic modelling and prediction has only recently been investigated. In a precursor to the current work, Scott and Mason (2007) explored the use of data assimilation for state estimation in estuarine morphodynamic modelling using Morecambe Bay as a study site. A 2DH decoupled morphodynamic model of the bay was enhanced by integrating waterline observations derived from SAR satellite images (Mason et al. (2001)) using a simple optimal interpolation (OI) assimilation scheme. Despite the known deficiencies of the OI algorithm (see e.g. Lorenc (1981)), the method was shown to improve the ability of the model to predict large scale changes in bathymetry over a three year period. In an unrelated study, van Dongeren et al. (2008) used a least squares estimator to assimilate multiple, remotely-sensed information sources into the Delft 3D modelling system. This system did not take account of spatial correlations between model variables and thus only updated model variables where there were co-located observations. Nevertheless, the system showed good skill in estimating the nearshore subtidal bathymetry when applied to two data-rich test sites at Duck, NC, USA and Egmond, The Netherlands.

The current work is focused on developing a method for using data assimilation to deliver improved morphodynamic model parameter estimates. This can be achieved through *state augmentation*. State augmentation is a conceptually straightforward technique that allows us to estimate and update uncertain model parameters jointly with the model state variables (Jazwinski (1970)) as part of the assimilation process. The same approach can be used in the context of model error or bias estimation. See e.g. Bell et al. (2004), Griffith and Nichols (1996), Griffith and Nichols (2000) , Martin et al. (2002), Dee (2005).

In theory state augmentation can be applied to any of the standard data assimilation methods. The model state vector is augmented with a vector containing the parameters we wish to estimate, the equations governing the evolution of the model state are combined with the equations describing the evolution of these parameters and the chosen assimilation algorithm is simply applied to this new augmented system in the usual way. Navon (1997) and Evensen et al. (1998) review the use of the technique in the context of 4D Var. Yang and Hamrick (2003) use a related scheme to recover parameters for cohesive sediment modelling. State augmentation has also been applied with the Kalman filter (see e.g. Martin et al. (1999)).

In this study we combine the technique with a three dimensional variational assimilation (3D Var) scheme. To the best of the authors' knowledge, state augmentation has not been used with 3D Var before. The crucial difference between standard 3D Var and the other schemes mentioned is that the covariance matrices are not evolved (implicitly or explicitly) by the 3D Var algorithm. It is therefore vital that the cross-covariances between the parameters and the state are given a good a priori specification. 3D Var has other advantages, such as ease of implementation (no model adjoints required); computational robustness (given reasonably specified covariances) and computational efficiency.

The aim of this paper is to demonstrate parameter estimation using 3D Var data assimilation for a simple 1D model of bed-form propagation. The long term objective is to implement a parameter estimation scheme in a full morphodynamic assimilation-forecast system. However, the use of a simple model in the current work allows ideas to be developed, tested and understood without the obfuscating features of a more complex system. Our results show that 3D Var can be used successfully for parameter estimation. The scheme is capable of recovering near perfect parameter values and therefore improves our models capability to predict future bathymetry. Such positive results suggest the potential for application to more complex morphodynamic models.

This paper is organized as follows. In section 2 we explain state augmentation and formulate the data assimilation problem for the augmented system. Our simple 1D model is introduced in section 3. In section 4 we discuss the roles of the observation and background error covariance matrices giving particular attention to the cross correlations between the background errors in the state and parameter estimates. The experimental design is described in section 5 followed by the main results. Finally, in section 6 we summarise the conclusions from this work.

2 Data assimilation

In this paper we shall consider the discrete, linear, time-invariant system model

$$\mathbf{z}_{k+1} = \mathbf{M}(\mathbf{p})\mathbf{z}_k, \qquad k = 0, \ldots, N-1, \tag{1}$$

where the vector $\mathbf{z}_k \in \mathbb{R}^m$ represents the model state at time t_k and $\mathbf{M} \in \mathbb{R}^{m\times m}$ is a constant, non-singular matrix describing the dynamic evolution of the state from time t_k to time t_{k+1}.

Although data assimilation techniques can be applied to any general system model, the model (1) offers a simple framework within which we can explain/ present the theory of the approach.

The model (1) depends on parameters whose values are imprecisely known. We use the vector $\mathbf{p} \in \mathbb{R}^q$ to represent these parameters, where q is the number of unknown parameters. We assume that $\mathbf{p}$ is constant, that is, the parameters are not altered by the forecast model from one time step to the next. The evolution model for the parameters can therefore be written as

$$\mathbf{p}_{k+1} = \mathbf{p}_k, \qquad k = 0, \dots, N-1. \tag{2}$$

2.1 State augmentation

In this section we formulate the data assimilation problem in terms of an augmented system. The procedure for basic state estimation is identical and can be derived by simply omitting the parameter vector from what follows (Smith et al. (2008)).

We augment the state vector $\mathbf{z}$ with a vector $\mathbf{p}$ containing the parameters we wish to estimate, giving the *augmented state vector*

$$\mathbf{w} = \begin{pmatrix} \mathbf{z} \\ \mathbf{p} \end{pmatrix}, \tag{3}$$

where $\mathbf{z} \in \mathbb{R}^m$, $\mathbf{p} \in \mathbb{R}^q$, and $\mathbf{w} \in \mathbb{R}^{m+q}$.

This allows us to write equations (1) and (2) as the equivalent augmented system model

$$\mathbf{w}_{k+1} = \tilde{\mathbf{M}}\mathbf{w}_k, \tag{4}$$

where

$$\tilde{\mathbf{M}} = \begin{pmatrix} \mathbf{M}(\mathbf{p}) & 0 \\ 0 & I \end{pmatrix} \in \mathbb{R}^{(m+q)\times(m+q)}.$$

We suppose that we have a set of r observations to assimilate and that these are related to the model state by the equations

$$\mathbf{y}_k = \mathbf{h}(\mathbf{z}_k) + \boldsymbol{\varepsilon}_k^o, \qquad k = 0, \dots, N-1, \tag{5}$$

where $\mathbf{y_k} \in \mathbb{R}^r$ is a vector of r observations at time t_k, $\mathbf{h} : \mathbb{R}^m \longrightarrow \mathbb{R}^r$ is a nonlinear observation operator that maps from model to observation space, and $\boldsymbol{\varepsilon}_k^o \in \mathbb{R}^r$ is a random vector representing the observation errors. If we have direct observations, $\mathbf{h}$ is simply an interpolation operator for interpolating variables from the model grid to observation locations. Often, the model variables we wish to analyse cannot be observed directly and instead we have observations of another measurable quantity. In this case $\mathbf{h}$ will also include transformations based on physical relationships that convert the model variables to the observations.

We can write (5) in terms of the augmented state vector as

$$\mathbf{y}_k = \tilde{\mathbf{h}}(\mathbf{w}_k) + \boldsymbol{\varepsilon}_k^o, \tag{6}$$

where $\tilde{\mathbf{h}} : \mathbb{R}^{m+q} \longrightarrow \mathbb{R}^r$, and

$$\tilde{\mathbf{h}}(\mathbf{w}) = \tilde{\mathbf{h}} \begin{pmatrix} \mathbf{z} \\ \mathbf{p} \end{pmatrix} = \mathbf{h}(\mathbf{z}).$$

We also suppose that we have a *background* state $\mathbf{w}_0^b \in \mathbb{R}^{m+q}$, that includes *a priori* estimates of the initial system state $\mathbf{z}_0$ and parameters $\mathbf{p}_0$. This is a best guess estimate obtained (for example) from a previous assimilation run or a recent bathymetric survey.

The aim of data assimilation is to combine the measured observations $\mathbf{y}$ with the model predictions $\mathbf{w}^b$ in order to derive a model state $\mathbf{w}^a \in \mathbb{R}^{m+q}$ that most accurately describes the true state of the system $\mathbf{w}^t$. This optimal estimate is called the *analysis*.

A wide variety of data assimilation schemes exist (e.g. Kalnay (2003), Lewis et al. (2006)). In this study we apply a standard method based on statistical estimation theory known as *three dimensional variational data assimilation* (3D Var).

2.2 Three Dimensional Variational assimilation

The 3D Var method (e.g. Courtier et al. (1998)) is based on a maximum a posteriori estimate approach and derives the analysis by seeking a state that minimises a cost function measuring the misfit between the model state $\mathbf{w}$ and the background $\mathbf{w}^b$ and observations $\mathbf{y}$,

$$\begin{aligned} J(\mathbf{w}) = {} & (\mathbf{w} - \mathbf{w}^b)^{\mathbf{T}}\tilde{\mathbf{B}}^{-1}(\mathbf{w} - \mathbf{w}^b) \\ & + (\mathbf{y} - \tilde{\mathbf{h}}(\mathbf{w}))^{\mathbf{T}}\mathbf{R}^{-\mathbf{1}}(\mathbf{y} - \tilde{\mathbf{h}}(\mathbf{w})). \end{aligned} \tag{7}$$

The matrices $\tilde{\mathbf{B}} \in \mathbb{R}^{(m+q)\times(m+q)}$ and $\mathbf{R} \in \mathbb{R}^{r\times r}$ are the covariance matrices of the background and observation errors. They represent the errors associated with the background and observations and determine the relative weighting of $\mathbf{w}^b$ and $\mathbf{y}$ in the analysis. If it is assumed that the background errors are small relative to the observation errors then the analysis will be close to the background state. Conversely, if it is assumed that the background errors are relatively large the analysis will lie closer to the observations.

The minimising state can be found using the gradient of the cost function with respect to $\mathbf{w}$. The 3D Var method does this numerically using a gradient descent algorithm. For this work we employ a quasi-Newton method (Gill et al. (1981)).

Although the technique of state augmentation is straightforward in theory, practical implementation of the approach relies strongly on the relationships between the parameters and state components being well defined and assumes that we have sufficient knowledge to reliably describe them. Since it is not possible to observe the parameters themselves, the parameter updates are only influenced by the observations through the cross covariances that describe the correlations between the error of the model state estimate and the error of the model parameter estimate (Martin (2000)). Successful parameter estimation will therefore only be possible if these cross correlations are adequately specified. We will consider ways of defining the error covariance matrices in section 4 but first we introduce our model.

3 The model

For the purpose of demonstrating the data assimilation technique we consider the simple case of a model with a single unknown parameter. For this we use the one-dimensional linear advection model described in Smith et al. (2007)

$$\frac{\partial z}{\partial t} + A\frac{\partial z}{\partial x} = 0, \tag{8}$$

where $z(x,t)$ is the bathymetry or bed height, A is the (constant) advection velocity and t is the time.

As discussed in Smith et al. (2007) we can use the method of characteristics to derive an analytic solution to this equation valid at discrete points (x_i, t_k). Given initial data

$$z(x,0) = f(x), \qquad -\infty < x < \infty, \tag{9}$$

the solution at time $t \geq 0$ is simply

$$z(x,t) = f(x - At). \tag{10}$$

We assume that the true value of A is unknown. Since A is constant, the parameter evolution equation is given by

$$\frac{dA}{dt} = 0. \tag{11}$$

Equation (11) together with the model equation (8) constitute our augmented state system model (4).

The purpose of this study is to explore the application of the state augmentation technique within the framework of this simplistic model before moving on to a more complex morphodynamic model. The advantage of the linear advection equation (8) is that it can be solved analytically and therefore provides a reference solution against which we can assess the performance of our scheme.

We use Gaussian initial data for (9) to give a smooth, isolated bell-shaped bedform. The solution (10) is such that as time increases the bed propagates undistorted across the model domain with constant speed. We wish to investigate whether, given an uncertain initial bathymetry and unknown advection velocity, and using observations taken from the true solution, we are able to construct an augmented data assimilation scheme that will produce a more accurate estimate of the true velocity, thereby improving the ability of our model to predict the true system state.

4 Error covariances

Error covariances play an important role in variational data assimilation. Before we can implement our 3D Var algorithm we need to specify the error covariance matrices $\tilde{\mathbf{B}}$ and $\mathbf{R}$.

We are assuming that our model structure is perfect, i.e. with the correct choice of parameter the model equations provide an exact representation of the dynamical system. Obviously, this assumption is unrealistic. In practice it is impossible to describe the true system behaviour completely and model predictions will also contain errors as a result of uncertain parameters and inaccurate initial and boundary conditions. In addition, the observations we wish to assimilate are likely to incorporate some kind of error, however small. Our assimilation scheme needs to take account of the errors that arise as a result of these imperfections. The precision of the analysis is dependent on the precision of the background and observations: correct specification of the error covariance matrices $\tilde{\mathbf{B}}$ and $\mathbf{R}$ is therefore crucial to the success of the scheme. If we can ensure that these matrices are an appropriate representation of the true error statistics, our data assimilation algorithm will produce optimal results.

4.1 Observation error covariance

The observation error covariance matrix $\mathbf{R}$ gives a statistical description of the errors in $\mathbf{y}$. These errors originate from instrument error, errors in the forward model $\mathbf{h}$ and representativeness errors (observing scales that cannot be represented in the model) (Daley (1991)). For simplicity we assume that the observation errors are spatially and temporally uncorrelated and take $\mathbf{R}$ to be a constant diagonal matrix with error variance σ_o^2.

4.2 Background error covariance

The matrix $\tilde{\mathbf{B}} \in \mathbb{R}^{(m+q)\times(m+q)}$ is the background error covariance matrix for the augmented system, and can be written as

$$\tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{B_{zz}} & \mathbf{B_{zp}} \\ (\mathbf{B_{zp}})^{\mathbf{T}} & \mathbf{B_{pp}} \end{pmatrix}. \tag{12}$$

Here $\mathbf{B_{zz}} \in \mathbb{R}^{m\times m}$ is the state background error covariance matrix. This matrix represents our uncertainty in the background state estimate $\mathbf{z}^b$. $\mathbf{B_{pp}} \in \mathbb{R}^{q\times q}$ is the covariance matrix of the errors in the parameter vector $\mathbf{p}^b$ and $\mathbf{B_{zp}} \in \mathbb{R}^{m\times q}$ is the covariance matrix for the cross correlations between the background errors in the state and parameter vectors. If we assume that these errors are unbiased, we can define

$$\mathbf{B_{zz}} = E\left(\boldsymbol{\varepsilon}_b \boldsymbol{\varepsilon}_b^T\right), \; \mathbf{B_{pp}} = E\left(\boldsymbol{\varepsilon}_p \boldsymbol{\varepsilon}_p^T\right), \; \mathbf{B_{zp}} = E\left(\boldsymbol{\varepsilon}_b \boldsymbol{\varepsilon}_p^T\right) \tag{13}$$

where $\boldsymbol{\varepsilon}_b = \mathbf{z}^b - \mathbf{z}^t$ and $\boldsymbol{\varepsilon}_p = \mathbf{p}^b - \mathbf{p}^t$.

Specification of the background error covariance matrix is one of the key parts of the assimilation problem. The correlations in $\tilde{\mathbf{B}}$ govern the spreading and smoothing of observational information and are therefore fundamental in determining the nature of the resulting analysis. Background errors arise from errors in both the initial conditions and model errors. Since, by the nature of the problem, these errors are not known exactly they have to be approximated in some manner.

Formulation of the background error covariance can be made considerably easier by specifying the error correlations as analytic functions. A number of correlation models have been proposed (see Daley (1991) for further discussion on this). An approach commonly used by the numerical weather prediction (NWP) community is the NMC method (Parrish and Derber (1992)) which uses the difference between forecasts that verify at the same time. The literature gives various other methods, including using innovation (observation minus background) statistics and studying differences in background fields using ensemble techniques. Fisher (2003) provides a useful review of current NWP techniques.

State covariance A standard approach used in state estimation is to assume that the background error covariances are homogeneous and isotropic. $\mathbf{B_{zz}}$ is then equal to the product of the estimated error variance and a correlation matrix defined using a pre-specified correlation function. Although this method is somewhat crude it makes the data assimilation problem far more tractable.

To characterise the background errors in the state vector $\mathbf{B_{zz}} = \{b_{ij}\}$ we use the correlation function (Rodgers (2000))

$$b_{ij} = \sigma_b^2 \rho^{|i-j|}, \qquad i, j = 1, \ldots, m. \tag{14}$$

Element b_{ij} defines the covariance between components i and j of the error vector $\boldsymbol{\varepsilon}_b$. Here $\rho = \exp(-\Delta x/L)$ where Δx is the model grid spacing and L is a correlation length scale and σ_b^2 is the state background error variance.

Parameter covariance For our simple model (8) we only have a single unknown parameter, the parameter vector $\mathbf{p}^b$ is therefore scalar. We approximate the true advection velocity A with $\tilde{A}$ where $\tilde{A} = A + \varepsilon_A$. Setting $\boldsymbol{\varepsilon}_p = \varepsilon_A$ in (13) we have

$$\mathbf{B_{pp}} = E(\varepsilon_A^2) = \sigma_A^2, \tag{15}$$

where σ_A^2 is the parameter error variance.

Cross covariances In order to define the matrix $\mathbf{B_{zp}}$ for the augmented system we need to consider the relationship between the errors in the parameter estimates $\boldsymbol{\varepsilon}_p$ and the errors in the state background $\boldsymbol{\varepsilon}_b$. As they depend on the same data, we expect them to be correlated.

One possible method for calculating these covariance matrices is by averaging the statistics over the assimilation window, using our knowledge of the truth and background states. However, since in reality the true solution is not known, this is difficult to do in practice. For simplicity we would like these matrices to be of a functional form similar to that used for the state background error covariance matrix (14). Successful parameter estimation relies upon these correlations being suitably specified, so it is important to ensure that the choice of function is appropriate to the particular model application.

Since ε_A is scalar, the cross covariance matrix $\mathbf{B_{zp}}$ will be a vector of length m. From (13) we have

$$\mathbf{B_{zp}} = E\left(\boldsymbol{\varepsilon}_b \boldsymbol{\varepsilon}_p^T\right) = E\left(\varepsilon_A \boldsymbol{\varepsilon}_b\right) = \begin{bmatrix} E(\varepsilon_A \varepsilon_b(x_1,t)) \\ E(\varepsilon_A \varepsilon_b(x_2,t)) \\ \vdots \\ E(\varepsilon_A \varepsilon_b(x_m,t)) \end{bmatrix}. \tag{16}$$

Here $\varepsilon_b(x_i,t)$ is the ith component of the vector $\boldsymbol{\varepsilon}_b$, representing the background error associated with $\mathbf{z}^{\boldsymbol{b}}$ at the grid point x_i at time t. Element $b_{zp}(i) = E(\varepsilon_A \varepsilon_b(x_i,t))$ defines the covariance between ε_A and $\varepsilon_b(x_i,t)$.

To determine a suitable form for $\mathbf{B_{zp}}$ we first seek an approximation to the background error $\boldsymbol{\varepsilon}_b$. We assume that our model is perfect and begin by considering a single realisation. The background error $\varepsilon_b(x,t)$, at a particular point x and time t, will be a combination of error in the initial condition and error in the parameter estimate. There are four possibilities: (i) known initial bathymetry and known advection velocity; (ii) unknown initial bathymetry and known advection velocity; (iii) known initial bathymetry and unknown advection velocity; (iv) unknown initial bathymetry and unknown advection velocity. Here we consider case (iv) but note that solutions for the other three cases can be derived in a similar manner (Smith et al. (2008)).

We define

$$\tilde{z}(x,t) = z(x,t) + \varepsilon_b(x,t) \tag{17}$$

and

$$\tilde{f}(x) = f(x) + \varepsilon_b(x,0) \tag{18}$$

where $\tilde{z}(x,t)$ is our approximation to the true bathymetry $z(x,t)$ and $\tilde{f}(x)$ is our estimate of the true initial state $f(x) = z(x,0)$.

From (10) we have the solution

$$\tilde{z}(x,t) = \tilde{f}(x - \tilde{A}t), \qquad t \geq 0. \tag{19}$$

Using (17)

$$\begin{aligned} \varepsilon_b(x,t) &= \tilde{z}(x,t) - z(x,t) \\ &= \tilde{f}(x - \tilde{A}t) - f(x - At) \\ &= \tilde{f}(x - At - \varepsilon_A t) - f(x - At). \end{aligned} \tag{20}$$

Assuming that $\varepsilon_A t$ is small and that $f(x)$ is a continuous, differentiable function we can expand (20) in a Taylor series about $\tilde{f}(x-At)$, yielding

$$\begin{aligned}\varepsilon_b(x,t) &= \tilde{f}(x-At-\varepsilon_A t) - f(x-At)\\ &= \left[\tilde{f}(x-At) - \varepsilon_A t\tilde{f}'(x-At) + \frac{\varepsilon_A^2}{2!}t^2\tilde{f}''(x-At) - \dots\right]\\ &\quad - f(x-At)\\ &= \varepsilon_b(x-At,0) - \varepsilon_A t\tilde{f}'(x-At) + O\left((\varepsilon_A t)^2\right). \qquad (21)\end{aligned}$$

If we further assume that the errors $\varepsilon_b(x,0)$ are smooth we can use (18) to rewrite (21) as

$$\varepsilon_b(x,t) = \varepsilon_b(x-At,0) - \varepsilon_A t f'(x-At) + \dots \qquad (22)$$

As time increases the higher order terms dominate. For sufficiently small t the background error at the point x at time t will be linearly related to the value of the derivative of the initial state at the starting point $x_0 = x - At$.

Conventional 3D Var schemes assume that the background error covariances are stationary so that the structure of $\tilde{\mathbf{B}}$ is fixed for all time. Initial experiments with the augmented system concluded that whilst this approach is sufficient for estimation of the state background error covariance matrix $\mathbf{B_{zz}}$, it does not provide an adequate representation of the state-parameter cross covariances (Smith et al. (2008)).

The approximation (22) derived above is based on a scenario where there is no data assimilation, and therefore assumes that the form of the background state $\tilde{f}(x)$ and estimated advection velocity $\tilde{A}$ remain the same for all time. With data assimilation both the background and parameter estimates will change as the model bathymetry is updated at each new analysis time. Thus we will have a different $\tilde{f}(x)$ and $\tilde{A}$ at the start of each new model integration. As $\tilde{f}(x)$ and $\tilde{A}$ change so too will the errors ε_b and ε_A and the correlation between them.

We take account of the fact that the background-parameter error cross covariances will change by making the matrix $\mathbf{B_{zp}}$ time dependent. Motivated by our practical experiments we use the following approximation

$$\varepsilon_b(x_i,t) = -\varepsilon_A f'(x_i - At), \qquad i = 1,\dots,m. \qquad (23)$$

Using definition (13), we multiply by ε_A and take the expected value over many realisations, to give

$$\begin{aligned}b_{zp}(i) &= E\left(-\varepsilon_A^2 f'(x_i-At)\right)\\ &= -E\left(\varepsilon_A^2\right) f'(x_i-At)\\ &= -\tilde{\sigma}_A^2 f'(x_i-At). \qquad (24)\end{aligned}$$

where $\tilde{\sigma}_A^2$ is the estimated parameter error variance.

Since by the nature of the problem both $f'(x)$ and A are unknown, we cannot evaluate $f'(x-At)$. Instead we substitute f' with $\tilde{f}'$ and replace $(x-At)$ with $(x-\hat{\gamma})$, where $\hat{\gamma}$ is a time dependent value chosen such that the the covariances are centred on the maximum value of the current background state. The matrix $\mathbf{B_{zp}}$ entries then become

$$b_{zp}(i) = -\tilde{\sigma}_A^2\tilde{f}'(x_i-\hat{\gamma}), \qquad i = 1,\dots,m. \qquad (25)$$

5 Results

We assume that the evolution of the true bathymetry can be described by the linear advection model (8). Given an approximate velocity $\tilde{A}$ and starting from a perturbed initial state we wish to examine whether our augmented data assimilation scheme is able to deliver both an accurate model bathymetry and an accurate estimate of the true advection velocity A. The analytic solution (10) is used to evaluate the performance of the method.

For the purpose of these experiments, we assume that the true bathymetry is Gaussian shaped and set the true advection velocity at a physically realistic value of $A = 0.002\ ms^{-1}$. In the example illustrated, the initial model bathymetry is also taken to be a Gaussian but is rescaled so that it is slightly shorter and wider than the true initial state. We over-estimate the advection velocity, setting $\tilde{A} = 0.02\ ms^{-1}$.

The assimilation process was carried out sequentially, as described in Smith et al. (2008), with a new set of observations being assimilated every hour. The model was sampled on a regular grid with a spacing of $\Delta x = 1.0m$. Observations were generated from the true solution at intervals of $25\Delta x$. They are assumed to be perfect and without any noise. We therefore weight in their favour, setting the observation and background error variances to be $\sigma_o^2 = 0.1$ and $\sigma_b^2 = 1.0$ respectively. At the end of each assimilation cycle the analysis was integrated forward using the model to obtain the background state for the next analysis time.

The benefits of data assimilation are illustrated by comparing model runs performed with and without the parameter estimation scheme. Figures 1 and 2 show the results produced when the model is run both with and without data assimilation over a 24 hour period. With no data assimilation (figure 1), the model bathymetry (dashed blue line) diverges away from the true bathymetry (solid red line). The shape of the initial bathymetry remains unaltered and the inaccurate advection velocity produces a phase error that grows with time. After 24 hours the predicted model state is far from the true model state. Running the model with the augmented data assimilation scheme produces the results shown in figure 2. The red dot-dash line represents the true bathymetry; observations are given by circles, the background state by the dashed blue line and the analysis by the solid green line. The difference in the results is obvious. The model is able to produce an accurate representation of the true bathymetry after just 4 assimilation cycles. At 24 hours it is almost impossible to distinguish between the predicted model bathymetry and the truth bathymetry.

Figures 3(a) and 3(b) show the updating of the advection velocity for the above example and for a second test case in which we use an initial guess of $\tilde{A} = 0\ ms^{-1}$. The accuracy of the parameter estimate increases with time as the assimilation cycle is repeated and more observations are processed.

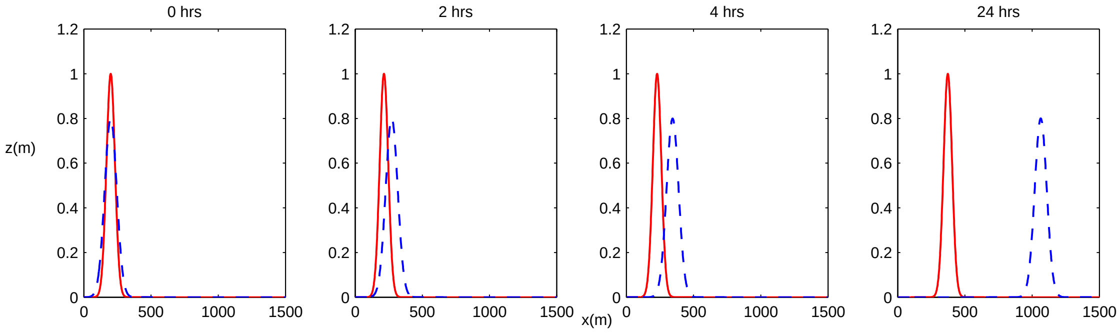

Fig. 1 Model run without data assimilation: the solid red line represents the true bathymetry $\mathbf{z^t}$ and the dashed blue line represents the predicted model bathymetry $\mathbf{z^b}$

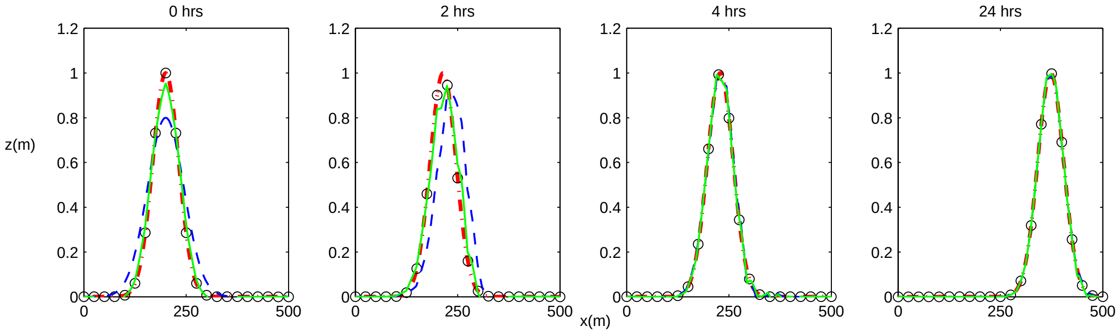

Fig. 2 Model run with data assimilation: the red dot-dash line represents the true bathymetry $\mathbf{z^t}$, observations $\mathbf{y}$ are given by circles, the background $\mathbf{z^b}$ by the dashed blue line and the analysis $\mathbf{z^a}$ by the solid green line. Note the change in spatial scale from figure 1 above.

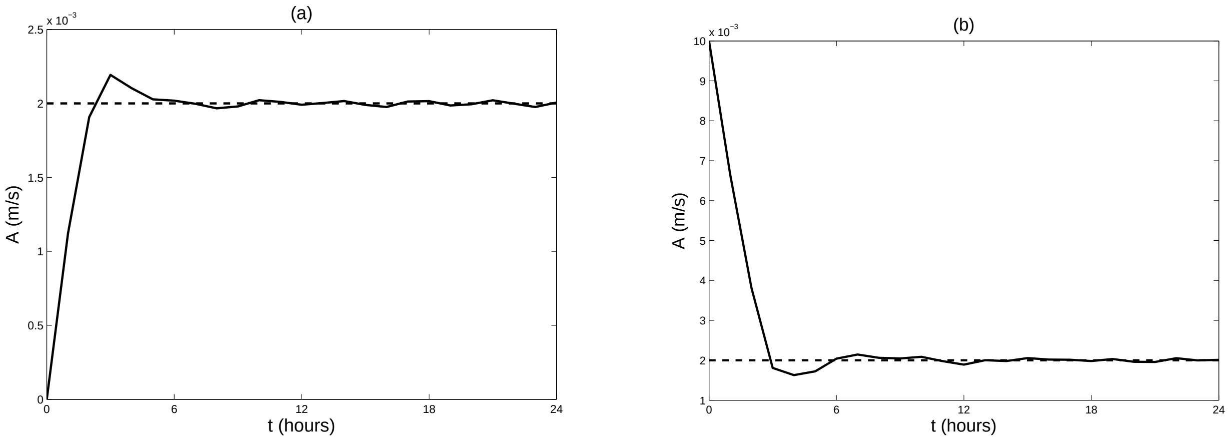

Fig. 3 Updating of parameter A for initial estimates (a) $\tilde{A} = 0.0\ m/s$ and (b) $\tilde{A} = 0.02\ m/s$

In both cases, the scheme converges after around 6 hours, managing to successfully recover the true value of A with a final value of 0.002 ms^{-1} to 3 d.p. As a result the model becomes a much better approximation and thus produces more accurate estimates of the true bathymetry.

The experiments were repeated for a range of both over and under estimated A values, with varying background guesses and observation combinations. Unsurprisingly, the speed of convergence varies depending on the quality of the background state, the location and spatial frequency of the observations and the time between successive assimilations. We do not show results here but refer the reader to Smith et al. (2008) where results from a similar set of experiments can be found. Generally, we found that the lower the spatial and temporal frequency of the observations the longer the scheme takes to reach the correct A value. The quality of the analysis is also affected. If the observations become too infrequent the parameter estimates fail to converge. This raises the issue of observability; whether the available observations contain sufficient information for us to be able to reconstruct the model state (Barnett and Cameron (1990)). We will not discuss the concept any further here but note that it is a question that will need to be addressed in future work.

6 Conclusions

This work is motivated by the problem of parameter estimation in morphodynamic modelling. In this paper, we have presented a novel approach using data assimilation. To our knowledge, the technique of state augmentation has not been used with 3D Var before. Here we have successfully combined the two methods and constructed a scheme that is capable of recovering near-perfect parameter values, therefore improving the ability of our model to predict future bathymetry. To date the technique has only been developed and tested using simplified 1D models but the results of this study indicate that there is great potential for the use of data assimilation based morphodynamic parameter estimation.

The quality of the analysis is highly dependent on the accuracy of the information fed into the assimilation algorithm. In the above experiments we assumed perfect observations drawn from the true solution. We weighted heavily in their favour because we were confident of their accuracy. In reality, observational data are noisy and distributed unevenly in space and time. One way of simulating such errors is to add random noise to the observations. This would then allow us to examine the extent to which over/ under estimation of observation error affects the accuracy of our results.

A key issue that this study has highlighted is the importance of the correct specification of the covariances between the background and parameter errors. In order to yield accurate approximations of both the bathymetry and the model parameters, we must ensure that these correlations are well defined. In this work we have relied heavily on our a priori knowledge of the parameter and of the behaviour of the solution, but in practice we may not have this type of information. Future studies should give more consideration to this and examine alternate methods for estimating these covariances.

The long term aim of this work is to implement the scheme in a full morphodynamic assimilation-forecast system. The results of this study are extremely positive and demonstrate that the state augmentation technique could be a useful tool in identifying uncertain morphodynamic model parameters. Here we have used a very idealized model; further investigation using more realistic models is required in order to assess the practical utility of the method. We are now testing the approach in a 1D non-linear advection model that has two uncertain parameters that need to be set. This has presented further challenges. In addition to the difficulties arising from the non-linearity of the model equations and defining the correlations between the state and model parameter errors, consideration also has to be given to the relationship between the parameters and the possibility of non-uniqueness of solutions, i.e. the parameters do not converge to a single deterministic set of values, but rather there exists a range of complementary combinations that produce the same model behaviour.

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