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From the classical moment problem to the realizability problem on basic semi-algebraic sets of generalized functions

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To Aljoša and Ingrid

When moments become memories

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Aldo Rota.

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Abstract

We derive necessary and sufficient conditions for an infinite sequence of Radon measures to be realized by, or to be the sequence of moment functions of, a finite measure concentrated on a pre-given basic semi-algebraic subset of the space of generalized functions on $\mathbb{R}^d$. A set of such a kind is given by (not necessarily countable many) polynomial constraints. We get realizability conditions of semidefinite type that can be more easily and efficiently verified, via semidefinite programming, than the well-known Riesz-Haviland type condition. As a consequence, we characterize the support of the realizing measure in terms of its moments functions.

As concrete examples of basic semi-algebraic sets of generalized functions, we present the set of all Radon measures, the set of all bounded Radon measures with Radon-Nikodym density w.r.t. the Lebesgue measure, the set of all probabilities, the set of all subprobabilities and the set of all point configurations. These examples are considered in numerous areas of applications dealing with the description of large complex system.

Our approach is based on a combination of classical results about the moment problem on nuclear spaces and of techniques developed to solve the moment problem on basic semi-algebraic sets of $\mathbb{R}^d$. For this reason, we provide a unified exposition of some aspects of the classical real moment problem which have inspired our main result. Particular importance is given to criteria for existence and uniqueness of the realizing measure on $\mathbb{R}^d$ via the multivariate Carleman condition and the operator-theoretical approach. We also give a formulation of the moment problem on general finite dimensional spaces in duality which makes clear the analogies with the infinite dimensional moment problem on nuclear spaces.

Contents

Introduction

It is still very difficult to describe, by looking at either their microscopic or macroscopic scale measuraments, the physics of some complex or "random" many-body systems such as a liquid composed of molecules or a galaxy made of stars. Nevertheless, to obtain information about one of these systems it is often very helpful to investigate some of its characteristics or physical parameters which are easier to observe, measure and handle mathematically. For example, in the case of a liquid, the characteristics under study are objects like the density, the average distance between the molecules, the pressure, the viscosity, etc.. The knowledge of these quantities provides information about the liquid state which entirely describes the thermodynamical properties of the liquid (see [30]). Another example belongs to spatial statistics and consists in the study of population dynamics in continuous spaces where it is important to understand the evolution of individual births, deaths and movements (see [77]). In this case, the first spatial moment is the mean density, the second parameters is the density of pairs of individuals which measures how an individual correlates with its neighbour, etc.. Similar questions have been widely treated in heterogeneous materials and mesoscopic structures (see [78]), stochastic geometry (see [52]), spatial ecology (see [54]) and neural spike trains (see [13, 37]), just to mention a few.

The main new contribution of this thesis is about the full power moment problem on a pre-given subset $\mathcal{S}$ of $\mathscr{D}'(\mathbb{R}^d)$, the space of all generalized functions on $\mathbb{R}^d$. From a mathematical point of view, the choice of this framework is convenient and general enough to comprehensively include all the applications mentioned above. More precisely, we ask whether certain given generalized functions are in fact the moments of some finite measure concentrated on $\mathcal{S}$. If such a measure exists we say that it *realizes*, on $\mathcal{S}$, the prescribed sequence of generalized functions. Moreover, we investigate how to delineate the support of the realizing measure directly from some positivity properties of its moment functions. To get the main theorem of this thesis we connect some well-known results about the moment problem on nuclear spaces with the techniques recently developed

to treat the classical moment problem on finite dimensional basic semi-algebraic sets.
To be more concrete about the main result, homogeneous polynomials are defined as powers of linear functionals on $\mathscr{D}'(\mathbb{R}^d)$ and their linear continuous extensions. Let us denote by $\mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'(\mathbb{R}^d))$ the set of all polynomials on $\mathscr{D}'(\mathbb{R}^d)$ with coefficients in $\mathcal{C}_c^\infty(\mathbb{R}^d)$, where the latter is the set of all infinite differentiable functions with compact support in $\mathbb{R}^d$. We find a characterization, via moment functions, of measures concentrated on basic *semi-algebraic* subsets of $\mathscr{D}'(\mathbb{R}^d)$, i.e. sets given by polynomial inequalities. Namely, a basic semi-algebraic set $\mathcal{S}$ is of the form

$$\mathcal{S} = \bigcap_{i \in Y} \left\{ \eta \in \mathscr{D}'(\mathbb{R}^d) \middle| \; P_i(\eta) \geq 0 \right\},$$

where Y is an arbitrary index set (not necessarily countable) and each P_i is a polynomial in $\mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'(\mathbb{R}^d))$.

To our knowledge, the infinite dimensional moment problem has been only treated in general on affine subsets (see [8, 5]) and cones (see [73]) of nuclear spaces. Special situations have also been handled (see e.g. [81, 6, 39]). The results concerning nuclear spaces are stated in Chapter 3. In the first two chapters of this work instead we give a review of some classical results about the moment problem on $\mathbb{R}^d$ on which our approach is based. This exposition is mostly based on the Riesz functional and the operator-theoretical approach.

Let us recall that the well-known K-moment problem, where K is a closed subset of $\mathbb{R}^d$, asks when a given sequence of real numbers represents the successive *moments* $\int_{\mathbb{R}^d} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \mu(dx_1, \ldots, dx_d)$, $\alpha_i = 0, 1, \ldots$, of a non-negative measure μ with support contained in K. If such a measure μ exists, we say that the given sequence is *realized* by μ on K. Moreover, if μ is unique we say that it is *determinate* or that the moment problem has a *unique solution*.
The moment problem mainly consists in establishing necessary and sufficient conditions for a sequence to be the moment sequence of a measure μ and to decide whether this measure is unique or not. An obvious necessary condition for the solvability of the moment problem is the non-negativity of a certain form associated with the initial sequence of numbers which are also called putative moments. More precisely, given a sequence y of putative moments, one introduces on the set of all polynomials the so-called Riesz functional L_y, which associates to each polynomial its putative expectation and is solely expressed in terms of the putative moments. If a polynomial P is non-negative on the pre-given set K, then a necessary condition for the realizability of y on K is that $L_y(P)$ is

non-negative as well. This condition alone is also sufficient for the existence of a realizing measure concentrated on $K \subseteq \mathbb{R}^d$ as stated by the Riesz-Haviland theorem (see [64, 32]). The disadvantage of such a type of positivity condition is that it may be rather difficult, and also computationally expensive, to identify all non-negative polynomials on K, especially if the latter is geometrically non-trivial. For this reason, a lot of work was devoted to develop more checkable positivity conditions.

Let us first focus, for reasons that are we going to explain later, on the case when $K = \mathbb{R}$. This is also known as Hamburger's moment problem named after H. Hamburger who was one of the fathers of this rich branch of mathematics (see [29]). A well-known result shows that all non-negative polynomials on $\mathbb{R}$ can be written as the sum of two squares of polynomials (see [58]). On $\mathbb{R}$, it is therefore sufficient for the realizability of a sequence y to require that L_y is non-negative on squares of polynomials, that is, y is *positive semidefinite.* Such a positivity condition, in contrast with the Riesz-Haviland condition, is easy to check by semidefinite programming (see e.g. [45]).
After the discussion about the existence let us look at the uniqueness of the Hamburger moment problem. A sufficient condition for the determinacy of the moment problem on $\mathbb{R}$ is a growth restriction on $y = (y_n)_{n\in\mathbb{N}_0}$ given by the Carleman condition, i.e.

$$\sum_{n=1}^{\infty} y_{2n}^{-\frac{1}{2n}} = +\infty.$$

In fact, if the moments of a measure μ satisfy the Carleman condition, there is no other measure ν having the same moments as μ. The condition was discovered by T. Carleman in his treatise on quasi-analytic functions in 1926 (see [14]). Actually, the weakest known condition that is sufficient for the uniqueness of the measure realizing a positive semidefinite sequence y is that the class $C\{y_n\}$ is quasi-analytic (see [38]). Recall that a quasi-analytic class of functions is a generalization of the class of real analytic functions based upon the following fact. If f is an analytic function on $\mathbb{R}$ and at some point the function f and all its derivatives are zero, then f is identically zero on $\mathbb{R}$. Quasi-analytic classes are larger classes of functions for which this statement is still true. The Denjoy-Carleman theorem gives criteria on the sequence y under which the class $C\{y_n\}$ is quasi-analytic. If the sequence is log-convex, i.e. $y_n^2 \leq y_{n-1}y_{n+1}$ for any $n \in \mathbb{N}$, and if $y_0 = 1$, these criteria are equivalent to the Carleman condition.
For the moment problem on $\mathbb{R}^d$, with $d \geq 2$, things are slighty different. In fact, the positive semidefiniteness of y is not anymore sufficient for realizability as al-

ready D. Hilbert pointed out in the description of his 17th problem (see [33]). However, the positive semidefiniteness of y becomes sufficient if one additionally assumes a restriction on the growth of the moments given by the so-called multivariate Carleman condition. Using the operator-theoretical approach we explain the existence and uniqueness of the moment problem on $\mathbb{R}^d$. We will first start from the case $d = 1$ so that the differences with the multi-dimensional case will be more evident. Let us describe the general procedure we are going to use. By the standard GNS construction (see e.g. [23, Sect. 4]), we set up a Hilbert space starting from $\mathbb{R}[x]$ and from a positive semidefinite sequence y and define an inner product through the Riesz functional L_y. An operator of multiplication by x is introduced too. The latter is symmetric but not self-adjoint. However, by a classical results of J. von-Neumann and A. Galindo (see [63, 24]), there exist self-adjoint extensions on a larger domain. These extensions have a spectral measure which is non-negative and has y as a moment sequence. If the putative moments fulfill the Carleman condition, the operator admits a unique extension and so the realizing measure is determinate.
In dimension $d \geq 2$, the corresponding approach to get existence requires conditions which automatically imply uniqueness, see Chapter 2 for more details. Let us give an idea of why this happens. Symmetric operators of multiplication by x_i, for $i = 1, \ldots, d$, are constructed on the space of polynomials. As in the one dimensional case, these operators have self-adjoint extensions. However, in order to get the existence of a realizing measure on $\mathbb{R}^d$ we have to apply the spectral theorem for several operators in which the essential requirement is that the involved operators strongly commute. The latter means that the associated unitary groups commute. To check this, the unitary groups and their mixed products have to be uniquely determined by the operators and their powers on the set of so-called quasi-analytic vectors, see a result due to A. E. Nussbaum in [56]. This is based on the multivariate Carleman condition which also gives uniqueness of the realizing measure. This argumentation makes clear why, in higher dimensions, one cannot separate existence and uniqueness.

Beyond the case of $K = \mathbb{R}^d$, for a long time the moment problem was only studied for specific subsets K of $\mathbb{R}^d$ rather than general classes of sets. Among these we recall, for $d = 1$, the Stieltjes and the Hausdorff moment problem which seek necessary and sufficient conditions for a sequence of numbers to be the moment sequence of some Borel measure supported on the ray $[0, \infty)$ (see [75, 76]) and on the closed unit interval $[0, 1]$ (see [31]), respectively. However, enormous progress has recently been made for the moment problem on general basic semi-

algebraic sets of $\mathbb{R}^d$. These results have the advantage to encode properties of the support of the realizing measure in positivity conditions stronger than the positive semidefiniteness. Namely, L_y is non-negative on the *quadratic-module* generated by the polynomials $(P_i)_{i \in Y}$ defining the basic semi-algebraic set K, that is the set of all polynomials given by finite sums of the form $\sum_i Q_i P_i$ where Q_i is a sum of squares of polynomials. Semidefinite programming allows then to efficiently treat such positivity conditions.
Let us mention just a few works which inspired the results presented in this thesis (for a more complete overview see [45, 47, 51]). In 1982 C. Berg and P. H. Maserick showed in [10] that for a compact basic semi-algebraic $K \subset \mathbb{R}$ the positivity condition involving the quadratic module is also sufficient. The sketch of their proof is presented in Chapter 1. Concerning the higher dimensional case, a few years later, K. Schmüdgen proved in [68] that for a compact basic semi-algebraic $K \subset \mathbb{R}^d$ a slightly stronger positivity condition, that is, L_y is non-negative on the *pre-ordering* generated by $(P_i)_{i \in Y}$, is sufficient. This result was soon refined by M. Putinar in [61] for Archimedean quadratic modules. Since then, the problem to extend their results to wider classes of K has intensively been studied, (see e.g. [60, 41, 15]). By additionally requiring a growth condition which implies the multivariate Carleman condition, J. B. Lasserre has recently shown in [46] that the non-negativity of L_y on the quadratic module is sufficient for the realizability on a general basic semi-algebraic set $K \subseteq \mathbb{R}^d$. One main ingredient of all these works is to prove the existence of a realizing measure on $\mathbb{R}^d$ (this is the reason why we firstly discussed the "Hamburger moment problem"), which is subsequently shown to have support contained in K. In order to show the latter property and also the determinacy problem, a crucial point in the proof is to show that the moments of a signed measure and the ones of a non-negative measure are equal. Via what we call *splitting procedure*, this equality between moments is replaced by another one which only compares the moments of two non-negative measures. One of these two measures is such that either its support is compact or its sequence of moments satisfies the Carleman condition. We show that also in the case of compact support the Carleman conditions is automatically implied and so the uniqueness of the realizing measure.

Henceforth, let us discuss the infinite dimensional moment problem also called *realizability problem*. Using the central idea of the works about the classical moment problem on basic semi-algebraic sets, we prove that also for a moment problem on an infinite-dimensional basic semi-algebraic set $\mathcal{S}$, the non-negativity of the Riesz functional on the associated quadratic module is sufficient for the realizability of a sequence of putative moments satisfying a certain growth condition.

To better understand the step from the finite to the infinite dimensional case, in Chapter 3 we state the classical moment problem on a general finite dimensional vector space which is in dual pairing with another vector space under a scalar product. The most important theorem of this chapter, and fundamental tool used in the proof of our main result, is due to Y. M. Berezansky, Y. G. Kondratiev and S. N. Šifrin. In particular, it gives an answer to the solvability of the moment problem on Ω', where the latter is the topological dual of a nuclear space Ω given by the projective limit of a family of separable Hilbert spaces. This theorem is the analogue for nuclear spaces of the result about existence and uniqueness for the classical moment problem on $K = \mathbb{R}^d$. The equivalent of the multivariate Carleman condition is a growth condition on the sequence of putative moments. Such a sequence is called *determining* because this property guarantees the uniqueness of the realizing measure as well. However, in the infinite dimensional case, this determinacy condition additionally involves regularity properties and growth restrictions on the moments as functions.

In our case, we will consider Ω to be the space of test functions $\mathcal{C}_c^\infty(\mathbb{R}^d)$ represented as the uncountable intersection of weighted Sobolev spaces and equipped with the associated projective topology. The correspondent space of generalized functions is $\mathscr{D}'_{proj}(\mathbb{R}^d)$. Let us point out that usually $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is endowed with the standard inductive topology and we denote its topological dual space by $\mathscr{D}'_{ind}(\mathbb{R}^d)$. The inductive topology is strictly stronger than the projective one and, as a consequence, $\mathscr{D}'_{proj}(\mathbb{R}^d)$ is strictly smaller than $\mathscr{D}'_{ind}(\mathbb{R}^d)$.

We will prove the existence and the uniqueness of the realizing measure on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ via the theorem due to Berezansky, Kondratiev and Šifrin and we will be able to prove that this measure is actually concentrated on $\mathcal{S}$. Moreover, even solely in the context of the finite dimensional moment problem, the ideas employed in the proof of our main result also extend to basic semi-algebraic sets defined by an uncountable family of polynomials and to the most general bound on the growth of the moments given by the multivariate Carleman condition. To consider these kinds of sets in infinite dimensions, the use of the inductive topology on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is essential as $\mathcal{S}$ is closed in $\mathscr{D}'_{ind}(\mathbb{R}^d)$ with respect to the strong topology and the latter topological space is Radon. Let us emphasize that, in our main result, the only regularity assumption is that the putative moments are Radon measures.

In the last part of this work we use the main theorem to derive realizability results under more concrete circumstances. Essentially, we need to find a representation of the desired support as a basic semi-algebraic subset of the space of generalized functions. The positivity conditions which we obtain depend on the chosen representation of $\mathcal{S}$. In particular, we investigate conditions under which the moment functions can be realized by a finite measure concentrated on the space of all Radon measures on $\mathbb{R}^d$. Furthermore, we show how to characterize, via moment measures, the set of Radon measures with Radon-Nikodym density w.r.t. the Lebesgue measure fulfilling an a priori L^∞-bound, the set of all probabilities and subprobabilities and, eventually, the space of point configurations. These examples demonstrate that, in contrast to the finite dimensional case, a semi-algebraic set defined by uncountably many polynomials leads to very natural and treatable conditions on the moments in the infinite dimensional context.

Even if we have already given most of the contents of the thesis, let us give a brief outline.

Chapter 1 reviews, using the Riesz functional approach, necessary and sufficient conditions for the solvability of the classical moment problem on a subset K of the real line. Particular importance is given to the theorem due to Riesz (which we revisit using a proof different from the original one) and, as consequence, to Hamburger's theorem used as essential tool to get the existence of the realizing measure on basic semi-algebraic subsets of $\mathbb{R}$ (as Berg and Maserick do). We investigate, by a nonstandard proof, the uniqueness of the solution via the Carleman condition and we describe some possible alternative approaches such as the Weierstrass and the monotone class theorem and point out that these are not suitable to be used in the proof of Berg and Maserick. In preparation for the multi-dimensional case, we explain the existence (and uniqueness) of the moment problem on $\mathbb{R}$ via the operator-theoretical approach.

Chapter 2 describes some aspects of the moment problem extended to $\mathbb{R}^d$. It mainly focuses, by using the operator-theoretical approach, on the sufficient conditions for a multi-sequence to be determinate. To get the existence of the realizing measure on $\mathbb{R}^d$ via spectral theorem, an important role is played by the pairwise strong commutativity of the involved operators. This property is guaranteed by a theorem due to Nussbaum which requires the existence of a total set of quasi-analytic vectors for all the involved operators. The proof of Nussbaum's result is rewritten by using a path different from the original one. Moreover, a proof due to Schmüdgen about the moment problem on compact basic semi-algebraic set of $\mathbb{R}^d$ is presented. Furthermore, we show how Lasserre treated the case of basic semi-algebraic set *not* necessarily compact.

Chapter 3 states the classical moment problem on a general finite dimensional vector space which forms, under a scalar product, a dual pair with another vector space. Moreover, the background and the well-known result about the full realizability problem on nuclear spaces are given.

Chapter 4 contains the main contribution of the thesis for the realizability problem on $\mathcal{S}$ basic semi-algebraic subset of the nuclear space $\mathscr{D}'_{proj}(\mathbb{R}^d)$. We consider a sequence $m = (m^{(n)})_{n\in\mathbb{N}_0}$ of putative moment functions consisting of a special class of generalized functions. Indeed, each $m^{(n)}$ is a Radon measure on $\mathbb{R}^{dn}$ and so $m^{(n)} \in \mathscr{D}'_{proj}(\mathbb{R}^{dn})$.

As already said, existence and determinacy criteria for the moment problem are related to the spectral theorem, to the quasi-analyticity of some classes of functions and, for the multi-dimensional case, also to the strong commutativity of certain symmetric operators. For this reason, in Appendix A we collect some results from the theory of quasi-analyticity and in Appendix B some considerations about the spectral theorem. In particular, we clarify the relation between the powers of the operators and the moments of the spectral measure. Every self-adjoint extension of the symmetric multiplication operator, associated with the sequence y of putative moments, produces indeed a measure which realizes y. Appendix C contains a collection of further auxiliar results used throughout this dissertation.

Chapter 1

The one-dimensional power moment problem

In the present chapter we review, using the Riesz functional approach, necessary and sufficient conditions for the solvability of the classical moment problem on a set $K \subseteq \mathbb{R}$. We particularly focus on the results due to Hamburger for $K = \mathbb{R}$ (see [29]) and to Berg and Maserick for K basic semi-algebraic subset of $\mathbb{R}$ (see [10]). In both cases the uniqueness of the solution is investigated.
We also give an operator approach explanation to the existence and uniqueness of the moment problem which will be essential in the following chapters.

1.1 Statement of the problem

Let us recall that

Definition 1.1.1.
The support of a non-negative Borel measure μ on $\mathbb{R}$ is defined as the unique smallest closed set $\mathrm{supp}(\mu) \subseteq \mathbb{R}$ *such that* $\mu(\mathbb{R} \setminus \mathrm{supp}(\mu)) = 0$.

From now on, let $K \subseteq \mathbb{R}$ be *closed*.

Definition 1.1.2 (Moments on K)**.**
Let μ be a non-negative Borel measure μ on $\mathbb{R}$ with support contained in K. The number

$$\int_K x^\alpha \, \mu(dx), \quad \alpha \in \mathbb{N}_0,$$

is called the α^{th}–moment *of μ on K.*

We denote by $\mathcal{M}^*(K)$ the set of all non-negative Borel measures on $\mathbb{R}$ with support contained in K and such that $\int_K |x^\alpha| \, \mu(dx) < \infty$ for all $\alpha \in \mathbb{N}_0$. Note that in particular a measure in $\mathcal{M}^*(K)$ has finite moments of all orders on K.

Remark 1.1.3.
The integrals should be computed on $\mathrm{supp}(\mu)$ *which is contained in* K*. Nevertheless, we prefer to write the integral on* K *for notational convenience.*
Moreover, there is no difference for the integrals if μ *is seen as a measure on* K *or as a measure on* $\mathbb{R}$ *supported on* K*. In fact, any measure on* $(K, \mathcal{B}(K))$ *is in one to one correspondence with a measure on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *supported on* K*. Hence, for the moments of* μ *in* $\mathcal{M}^*(K)$ *we have that*

$$\int_{\mathbb{R}} x^\alpha \, \mu(dx) = \int_K x^\alpha \, \mu(dx) < \infty, \quad \forall \alpha \in \mathbb{N}_0.$$

Let us observe that measures in $\mathcal{M}^*(K)$ are automatically *finite*. In fact, when $\alpha = 0$, we have $\mu(\mathbb{R}) = \mu(K) < \infty$. Thus, they are also Radon measures, i.e. Borel measures finite on compact subsets of $\mathbb{R}$.

Given $\mu \in \mathcal{M}^*(K)$ we are always able to compute the sequence of its moments on K

$$\left(\int_K x^\alpha \, \mu(dx) \right)_{\alpha \in \mathbb{N}_0}$$

which is called K-*moment sequence* of μ.

The moment problem is a sort of inverse problem.

Definition 1.1.4 (Moment problem on K)**.**
Given an infinite sequence of real numbers $y = (y_\alpha)_{\alpha \in \mathbb{N}_0}$*, find* $\mu \in \mathcal{M}^*(K)$ *such that* y_α *is the* α^{th}*-moment of* μ *on* K*, i.e.*

$$y_\alpha = \int_K x^\alpha \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0. \tag{1.1}$$

If such a measure exists we say that the sequence y *is* realized *by* μ*, or that* y *has a* representing *(or* realizing*) measure* μ*, on* K*.*
If the representing measure is unique we say that μ *is* determinate *or that the moment problem has a* unique solution*.*

The two main questions in solving the moment problem are: to find necessary and sufficient conditions for a sequence y to be the moment sequence of a measure $\mu \in \mathcal{M}^*(K)$ and to decide whether this measure is unique or not.

The most well known examples of moment problems are the following.

- The Hamburger moment problem: $K = \mathbb{R}$ and $(y_\alpha)_{\alpha\in\mathbb{N}_0} \subset \mathbb{R}$.
- The Stieltjes moment problem: $K = [0, +\infty)$ and $(y_\alpha)_{\alpha\in\mathbb{N}_0} \subset \mathbb{R}^+$.
- The Hausdorff moment problem: $K = [a, b]$ and $(y_\alpha)_{\alpha\in\mathbb{N}_0} \subset \mathbb{R}$.
- The Toeplitz moment problem: K is the unit circle in $\mathbb{C}$ and $(y_\alpha)_{\alpha\in\mathbb{Z}} \subset \mathbb{C}$.

In this chapter we will only give a brief survey about *real* moment problems. Namely, from now on, we will consider $K \subseteq \mathbb{R}$ and $y = (y_\alpha)_{\alpha\in\mathbb{N}_0} \subset \mathbb{R}$.

To describe the theory behind the moment problem we will make use of the so-called *linear functional approach.*
Given $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ we define the linear Riesz functional L_y on the ring $\mathbb{R}[x]$ of all polynomials with real coefficients as

$$L_y(x^\alpha) := y_\alpha, \quad \alpha \in \mathbb{N}_0. \tag{1.2}$$

Let us notice that, for a polynomial $p(x) := \sum_{\alpha\in I} p_\alpha x^\alpha \in \mathbb{R}[x]$, with I finite subset of $\mathbb{N}_0$, by linearity we have

$$L_y(p) = \sum_{\alpha\in I} p_\alpha y_\alpha.$$

In particular, when y is realized by $\mu \in \mathcal{M}^*(K)$ we have the following

Proposition 1.1.5.
Let y be a sequence realized by $\mu \in \mathcal{M}^*(K)$. Then,

$$L_y(p) = \int_K p(x)\,\mu(dx)$$

for any $p \in \mathbb{R}[x]$.

Proof.
Since y is realized by $\mu \in \mathcal{M}^*(K)$, equation (1.1) holds. Then,

$$\begin{aligned} L_y(p) = \sum_{\alpha\in I} y_\alpha p_\alpha &= \sum_{\alpha\in I} \left(\int_K x^\alpha\,\mu(dx) \right) p_\alpha \\ &= \int_K \left(\sum_{\alpha\in I} p_\alpha x^\alpha \right) \mu(dx) \\ &= \int_K p(x)\,\mu(dx). \end{aligned}$$

$\square$

1.2 Necessary conditions for the solvability of the moment problem

In this section we study necessary conditions for a sequence y to be the K-moment sequence of a measure $\mu \in \mathcal{M}^*(K)$.

From now on, $\mathbb{R}^+_K[x]$ will denote the convex cone of real polynomials which are non-negative on K. When $K = \mathbb{R}$ we simply write $\mathbb{R}^+[x]$ instead of $\mathbb{R}^+_{\mathbb{R}}[x]$ and we also drop the set $\mathbb{R}$ on the symbol of the integrals.

Proposition 1.2.1.
A sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ has a representing non-negative Borel measure μ supported on K only if L_y is non-negative for all non-negative polynomials on K, i.e.,

$$\left(\exists\, \mu \in \mathcal{M}^*(K) \ s.t.\ y_\alpha = \int_K x^\alpha \mu(dx),\ \forall \alpha \in \mathbb{N}_0\right) \Longrightarrow \left(L_y(p) \geq 0,\ \forall p \in \mathbb{R}^+_K[x]\right).$$

Proof.
Assume that y is realized by $\mu \in \mathcal{M}^*(K)$. Then, by Proposition 1.1.5, we have that

$$L_y(p) = \int_K p(x)\, \mu(dx),$$

which is non-negative since μ is supported on K where p is non-negative.

$\square$

The following definition will play a central role in the further discussions.

Definition 1.2.2.
A sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ is called positive semidefinite *if for any sequence $(h_\alpha)_{\alpha\in I} \subset \mathbb{R}$, with I finite subset of $\mathbb{N}_0$,*

$$\sum_{\alpha,\beta\in I} y_{\alpha+\beta} h_\alpha h_\beta \geq 0.$$

Equivalently, y is positive semidefinite if for all $h(x) := \sum_{\alpha\in I} h_\alpha x^\alpha \in \mathbb{R}[x]$, with I finite subset of $\mathbb{N}_0$,

$$L_y(h^2) \geq 0.$$

Some authors use a more general definition of positive semidefiniteness by considering the ring $\mathbb{C}[x]$ of all polynomials with complex coefficients. The equivalence between the definition of positive semidefiniteness in the real and complex case is given by the following proposition.

Proposition 1.2.3.

$$\big(L_y(h^2) \geq 0, \quad \forall h \in \mathbb{R}[x]\big) \Longleftrightarrow \big(L_y(h\overline{h}) \geq 0, \quad \forall h \in \mathbb{C}[x]\big).$$

Proof.
If $h \in \mathbb{C}[x]$ then $h\overline{h} = h_1^2 + h_2^2$ with $h_1, h_2 \in \mathbb{R}[x]$. Therefore, if $L_y(h^2) \geq 0$ for all $h \in \mathbb{R}[x]$, we have that $L_y(h\overline{h}) = L_y(h_1^2) + L_y(h_2^2) \geq 0$ for all $h \in \mathbb{C}[x]$.

Viceversa, if $L_y(h\overline{h}) \geq 0$ for all $h \in \mathbb{C}[x]$, then in particular for all $h \in \mathbb{R}[x]$ we have that $\overline{h} = h$ and so $L_y(h\overline{h}) = L_y(h^2) \geq 0$.

□

A direct consequence of Proposition 1.2.1 is the following.

Corollary 1.2.4.
A sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0}$ has a representing non-negative Borel measure μ supported on K only if y is positive semidefinite, i.e.,

$$\left(\exists\, \mu \in \mathcal{M}^*(K) \ s.t.\ y_\alpha = \int_K x^\alpha \mu(dx),\ \forall \alpha \in \mathbb{N}_0\right) \Longrightarrow \Big(L_y(h^2) \geq 0,\ \forall h \in \mathbb{R}[x]\Big).$$

Proof.
The result directly follows from Proposition 1.2.1 because the polynomial h^2 is non-negative on K.

□

Corollary 1.2.4 for $K = \mathbb{R}$ is the necessary condition of Hamburger's theorem whose sufficient part is stated in Subsection 1.3.2.

Let us see now some additional necessary conditions of positive semidefinite type when the set K is *basic semi-algebraic*, i.e. given by the intersection of the intervals where a certain number of fixed polynomials are non-negative.

Let us make some preliminary considerations.

We define the *shift operator* E on the set of sequences $y = (y_\alpha)_{\alpha \in \mathbb{N}_0}$ in the following way,

$$(Ey)_\alpha := y_{\alpha+1}, \quad \alpha \in \mathbb{N}_0.$$

For a real polynomial $g(x) = \sum\limits_{\beta \in I} g_\beta x^\beta$, with $I \subset \mathbb{N}_0$ finite, $g(E)$ is the polynomial shift operator $g(E) := \sum_{\beta \in I} g_\beta E^\beta$ (where E^0 is the identity operator), i.e.

$$(g(E)y)_\alpha = \sum_{\beta \in I} g_\beta y_{\alpha+\beta}.$$

We will make use of the following lemma several times.

Lemma 1.2.5.
Let y be a sequence and $g \in \mathbb{R}[x]$. Then,

$$L_y(qg) = L_{g(E)y}(q) \tag{1.3}$$

for all $q \in \mathbb{R}[x]$.

Proof.
Let $q(x) = \sum_{\alpha \in J} q_\alpha x^\alpha$, with $J \subset \mathbb{N}_0$ finite. Then,

$$(qg)(x) = \sum_{\alpha \in J} \sum_{\beta \in I} q_\alpha g_\beta x^{\alpha+\beta}$$

and

$$\begin{aligned} L_y(qg) = L_y\left(\sum_{\alpha \in J} \sum_{\beta \in I} q_\alpha g_\beta x^{\alpha+\beta}\right) &= \sum_{\alpha \in J} \sum_{\beta \in I} q_\alpha g_\beta L_y(x^{\alpha+\beta}) \\ &= \sum_{\alpha \in J} \sum_{\beta \in I} q_\alpha g_\beta y_{\alpha+\beta} \\ &= \sum_{\alpha \in J} q_\alpha \sum_{\beta \in I} g_\beta y_{\alpha+\beta} \\ &= \sum_{\alpha \in J} q_\alpha (g(E)y)_\alpha \\ &= L_{g(E)y}(q). \end{aligned}$$

□

Case of one polynomial

Let $K \subseteq \mathbb{R}$ be the basic semi-algebraic set given by a fixed real polynomial g, i.e. the set where g is non-negative. Namely,

$$K := \{x \in \mathbb{R} : g(x) \geq 0\}. \tag{1.4}$$

Proposition 1.2.6.
A sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ has a representing non-negative Borel measure μ on K only if y and $g(E)y$ are positive semidefinite, i.e.,

$$\left(\exists\, \mu \in \mathcal{M}^*(K) \text{ s.t. } y_\alpha = \int_K x^\alpha \mu(dx),\ \forall \alpha \in \mathbb{N}_0\right) \Longrightarrow \begin{pmatrix} L_y(\,h^2) \ \geq 0, \ \forall h \in \mathbb{R}[x] \\ L_y(h^2 g) \geq 0, \ \forall h \in \mathbb{R}[x] \end{pmatrix}.$$

Proof.
Assume that y is realized by $\mu \in \mathcal{M}^*(K)$. By Corollary 1.2.4, we have that $L_y(h^2) \geq 0$ for all $h \in \mathbb{R}[x]$. Moreover, by (1.3) and Proposition 1.1.5, we get

$$L_{g(E)y}(h^2) = L_y(h^2 g) = \int_K h^2(x) g(x)\, \mu(dx),$$

which is non-negative since μ is supported on K where g is non-negative. Hence, the sequences y and $g(E)y$ are positive semidefinite.

□

Let us note that $\mathbb{R}$ can be thought as a basic semi-algebraic set of the form (1.4) by taking $g(x) = c$, with $c > 0$. In this sense, Proposition 1.2.6 coincides with Corollary 1.2.4 for $K = \mathbb{R}$.

Case of several polynomials

Let $K \subseteq \mathbb{R}$ be the basic semi-algebraic set given by the fixed real polynomials $g_1, \ldots, g_m$, i.e. the set where all g_j's are non-negative. Namely,

$$K := \bigcap_{j=1}^{m} \{x \in \mathbb{R} : g_j(x) \geq 0\}. \tag{1.5}$$

Proposition 1.2.7.
A sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ has a representing non-negative Borel measure μ on K only if y and $g_j(E)y$, for $j = 1, \ldots, m$, are positive semidefinite, i.e.,

$$\left(\exists\, \mu \in \mathcal{M}^*(K) \text{ s.t. } y_\alpha = \int_K x^\alpha \mu(dx),\ \forall \alpha \in \mathbb{N}_0\right) \Longrightarrow \begin{pmatrix} L_y(\,h^2\,) \ \geq 0, \ \forall h \in \mathbb{R}[x] \\ L_y(h^2 g_j) \geq 0, \ \forall h \in \mathbb{R}[x] \\ \hfill j=1, \ldots, m \end{pmatrix}.$$

Proof.
The conclusion follows applying the same procedure described in the proof of Proposition 1.2.6 to each g_j.

□

Remark 1.2.8.
The latter theorem also holds if in (1.5) we consider an uncountable number of polynomials g_j. Note that in this case K is still closed as intersection of closed sets.

1.3 Sufficient conditions for the existence and uniqueness of a realizing measure

In this section we prove Riesz's theorem which characterizes the realizability of a sequence y of real numbers by a measure $\mu \in \mathcal{M}^*(K)$ through a condition involving the non-negativity of the functional L_y on the set of non-negative polynomials on K.
Successively, we investigate conditions for the uniqueness of the solution of the moment problem. In particular, we present Carleman's condition which ensures the uniqueness of the Hamburger moment problem. The relative questions in the case of K basic semi-algebraic are also treated.

1.3.1 Riesz's existence result

Theorem 1.3.1 (Riesz, [64]).
A sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ has a representing non-negative Borel measure μ supported on K if L_y is non-negative for all non-negative polynomials on K, i.e.,

$$\left(\exists\, \mu\in\mathcal{M}^*(K)\ s.t.\ y_\alpha=\int_K x^\alpha \mu(dx),\ \forall\alpha\in\mathbb{N}_0\right)\Longleftarrow\left(L_y(p)\geq 0,\ \forall p\in\mathbb{R}^+_K[x]\right).$$

Proof.
Let V be the vector space of all the real-valued continuous functions polynomially bounded on K. Namely,

$$V := \left\{f\in\mathcal{C}(K)|\ \exists\, m_f\in\mathbb{N},\, c\in\mathbb{R}^+ \text{ s.t. } |f(x)|\leq c(1+|x|^{m_f}),\ \forall\, x\in K\right\}.$$

Let V_0 be the ring $\mathbb{R}[x]$ of all polynomials with real coefficients. It is easy to see that V is a vector lattice which is dominated by its subspace V_0. Then, by Theorem C.0.6, L_y can be extended to a (not necessarily unique) non-negative linear functional, which we also call L_y, on V. If we can show that condition (Dec) in Theorem C.0.7 holds on V, then we have that there exists a non-negative measure μ on $(K, \sigma(V))$ such that

$$L_y(f) = \int_K f(x)\mu(dx)$$

for any $f \in V$.
In particular, if $f(x) = x^\alpha$, we have for all $\alpha \in \mathbb{N}_0$

$$y_\alpha = L_y(x^\alpha) = \int_K x^\alpha \mu(dx),$$

which means that the sequence y is realized by $\mu \in \mathcal{M}^*(K)$.

Let us notice that the σ-algebra generated by V coincides with the Borel σ-algebra on K. That is,

$$\sigma(V) \equiv \mathcal{B}(K).$$

Let us prove this statement in two steps.
Step I: $\sigma(V) \subseteq \mathcal{B}(K)$.
Every $f \in V$ is continuous on K and trivially measurable w.r.t. the σ-algebra $\mathcal{B}(K)$. In other words, $\mathcal{B}(K)$ is a σ-algebra w.r.t. which all $f \in V$ are measurable. Hence, $\sigma(V) \subseteq \mathcal{B}(K)$ because $\sigma(V)$ is the smallest σ-algebra w.r.t. which all $f \in V$ are measurable.
Step II: $\sigma(V) \supseteq \mathcal{B}(K)$.
Let us first prove that $\sigma(V)$ contains the collection τ_K of all open sets in K, i.e. $\sigma(V) \supseteq \tau_K$.
Let us recall that $\sigma(V) := \{f^{-1}(S) | \ S \in \mathcal{B}(\mathbb{R}), f \in V\}$ and that, by definition of Borel σ-algebra, $\tau_K \subseteq \mathcal{B}(K)$. Therefore, if we take $A \in \tau_K$ (and so $A \in \mathcal{B}(K) \subseteq \mathcal{B}(\mathbb{R})$) we can trivially write A as $Id^{-1}(A)$ where $Id : x \mapsto x$ is the identity polynomial with domain K. Hence, $A \in \sigma(V)$ and so $\sigma(V) \supseteq \tau_K$.
Since $\mathcal{B}(K)$ is the smallest σ-algebra containing τ_K, we get our conclusion.

It only remains to verify that L_y satisfies condition (Dec) in Theorem C.0.7. Let $(f_n)_{n\in\mathbb{N}_0}$ be a monotonically decreasing sequence in V which converges pointwise to 0, and let $\epsilon > 0$. Since $f_0 \geq f_n \geq 0$ for each $n \in \mathbb{N}$ and since $f_0 \in V$, there exists a non-negative integer m_0 and a constant c such that

$$f_n(x) \leq c(1 + |x|^{m_0}), \quad \forall\, n \in \mathbb{N}_0, \ \forall\, x \in K.$$

W.l.o.g. we can always assume that m_0 is even because if m_0 is odd there always exists a constant $c' > c$ such that

$$f_0(x) \leq c(1+|x|^{m_0}) \leq c'(1+x^{m_0+1}), \quad \forall\, x \in K.$$

Let us define now the sets

$$K_n := \left\{x \in K | \ f_n(x) \geq \epsilon\big(1+q(x)\big)\right\},$$

where $q(x) = x^2(1+x^{m_0})$.

Each K_n is closed in K w.r.t. the topology τ_K because f_n and q are continuous. Moreover, K_n is bounded because

$$\begin{aligned} K_n &\subseteq \left\{x \in K | \ c(1+x^{m_0}) \geq \epsilon\big(1+q(x)\big)\right\} = \left\{x \in K \ \middle| \ \frac{1+q(x)}{1+x^{m_0}} \leq \frac{c}{\epsilon}\right\} \\ &\subseteq \left\{x \in K \ \middle| \ \frac{q(x)}{1+x^{m_0}} \leq \frac{c}{\epsilon}\right\} = \left\{x \in K \ \middle| \ x^2 \leq \frac{c}{\epsilon}\right\}. \end{aligned}$$

By Proposition C.0.3, K_n is then compact.

Since $(f_n)_{n\in\mathbb{N}_0}$ decreases to zero pointwise, we have that

$$\bigcap_{n\in\mathbb{N}_0} K_n = \emptyset. \tag{1.6}$$

In fact, if there was $\overline{x} \in K_n$ for all $n \in \mathbb{N}_0$ we would have

$$f_n(\overline{x}) \geq \epsilon\big(1+q(\overline{x})\big) \geq \epsilon, \quad \forall n \in \mathbb{N}_0,$$

which contradicts the assumption that $(f_n)_{n\in\mathbb{N}_0}$ decreases to zero.
Since K_n is compact and (1.6) holds, there exists a non-negative integer $N \in \mathbb{N}$ such that $K_n = \emptyset$ for $n \geq N$, i.e.

$$f_n(x) \leq \epsilon\big(1+q(x)\big), \quad \forall\, n \geq N, \ \forall\, x \in K.$$

The latter condition, by non-negativity and linearity of L_y, implies that for $n \geq N$ we have $0 \leq L_y(f_n) \leq \epsilon\big(1+L_y(q)\big)$. By the arbitrarity of ϵ, we conclude that $\lim_{n\to\infty} L_y(f_n) = 0$.

□

Let us notice that in Theorem 1.3.1 the realizing measure is not unique because of the non-unique way to extend L_y via Theorem C.0.6. Moreover, the space V of all continuous functions polynomially bounded on K is contained in $L^1(\mu)$ and

$|x|^\alpha \in V$, for all $\alpha \in \mathbb{N}_0$, so we have that the realizing measure has finite moments of any order.

Remark 1.3.2.
Theorem 1.3.1 is not very practical since the problem of characterizing non-negative polynomials on a general set K is not very easy. Nevertheless, as we are going to see in the following sections, when K is basic semi-algebraic we get positive semidefinite type conditions efficiently checkable through a semidefinite programming which is a technique of convex optimization (see [44]).

1.3.2 Hamburger's existence result

Let us introduce the following preliminary result.

Lemma 1.3.3 (Pólya and Szego, [59]).
A polynomial $p \in \mathbb{R}[x]$ is non-negative on $\mathbb{R}$ if and only if it can be written as a sum of squares of other polynomials, i.e.,

$$\Big(p(x) \geq 0, \quad \forall x \in \mathbb{R}\Big) \Longleftrightarrow \Big(p(x) = h_1^2(x) + h_2^2(x), \quad h_1, h_2 \in \mathbb{R}[x]\Big).$$

Proof.
($\Leftarrow$) Trivially, a sum of squares of polynomials is non-negative on $\mathbb{R}$.
($\Rightarrow$) Let us suppose that a polynomial $p \in \mathbb{R}[x]$ with highest degree term $p_{2d}x^{2d}$ is non-negative on $\mathbb{R}$. Then p is of the form

$$p(x) = p_{2d} \prod_{j=1}^{r} (x - \lambda_j)^{2m_j} \prod_{l=1}^{h} \Big(x - (a_l + ib_l)\Big)\Big(x - (a_l - ib_l)\Big), \qquad (1.7)$$

where λ_j, $j = 1, \ldots, r$, are the real roots of even multiplicity $2m_j$ (no multiplicity can be odd otherwise in a neighborhooh of the corresponding real root the polynomial would change sign) and $a_l \pm ib_l$, $l = 1, \ldots, h$, are the complex roots in conjugate pairs.
Since

$$\Big(x - (a_l + ib_l)\Big)\Big(x - (a_l - ib_l)\Big) = (x - a_l)^2 + b_l^2,$$

we can write (1.7) as

$$p(x) = p_{2d} \prod_{j=1}^{r} (x - \lambda_j)^{2m_j} \prod_{l=1}^{h} \big((x - a_l)^2 + b_l^2\big). \qquad (1.8)$$

Note that the leading coefficient p_{2d} needs to be positive.

The product $\prod_{l=1}^{h}\left((x-a_l)^2+b_l^2\right)$ gives rise to a polynomial written as sum of two squares. In fact, and in general, if $A=f^2+g^2$ and $B=t^2+k^2$ where $f,g,t,k\in\mathbb{R}[x]$ then

$$\begin{aligned}
AB &= \left(f^2+g^2\right)\left(t^2+k^2\right)\\
&= f^2t^2+f^2k^2+g^2t^2+g^2k^2\\
&= \left(f^2t^2+2ftgk+g^2k^2\right)+\left(f^2k^2-2ftgk+g^2t^2\right)\\
&= (\underbrace{ft+gk}_{=:R})^2+(\underbrace{fk-gt}_{=:S})^2=R^2+S^2
\end{aligned}$$

and clearly $R,S\in\mathbb{R}[x]$.
By repeating the latter procedure $h-1$ times in (1.8), we get that

$$\begin{aligned}
p(x) &= p_{2d}\prod_{j=1}^{r}(x-\lambda_j)^{2m_j}\left(R^2+S^2\right)\\
&= \underbrace{\left[\sqrt{p_{2d}}\prod_{j=1}^{r}(x-\lambda_j)^{m_j}\,R\right]^2}_{=:h_1(x)}+\underbrace{\left[\sqrt{p_{2d}}\prod_{j=1}^{r}(x-\lambda_j)^{m_j}\,S\right]^2}_{=:h_2(x)}\\
&= h_1^2(x)+h_2^2(x).
\end{aligned}$$

□

Remark 1.3.4.
The representations (1.7)*, and as a consequence* (1.8)*, of p are not possible when we deal with polynomials in more variables. In fact, the fundamental theorem of algebra does not hold for polynomials in more variables. Hence, for $d\geq 2$, a non-negative polynomial on $\mathbb{R}^d$ does not necessarily have a sum of squares representation.*
This was known already to David Hilbert in 1888 (see [33]) although his proof was non-constructive. A first concrete example was given by Motzkin only in 1967 (see [53]). The Motzkin polynomial

$$s(x_1,x_2)=1-3x_1^2x_2^2+x_1^2x_2^4+x_1^4x_2^2$$

is non-negative on $\mathbb{R}^2$ but it cannot be written as sum of squares. The non-negativity follows from the standard inequality

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc},\qquad a,b,c\geq 0,$$

relating the arithmetic mean and the geometric mean, by taking $a = 1$, $b = x_1^2 x_2^4$, *and* $c = x_1^4 x_2^2$.
To show that s *cannot be written as sum of square we work by contradiction. Let us suppose that the polynomials* s *can be actually written as sum of squares, i.e.* $s(x_1, x_2) = \sum_i f_i^2(x_1, x_2)$ *for some polynomials* $f_i \in \mathbb{R}[x_1, x_2]$. *Since* s *has degree* 6, *each* f_i *can have degree at most* 3. *This means that* s *is given by a real linear combination of*

$$1,\ x_1,\ x_2,\ x_1^2,\ x_1x_2,\ x_2^2,\ x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3.$$

However, the term x_1^3 *does not appear in some* f_i *because otherwise* x_1^6 *would appear in* s *with positive coefficient. Similarly,* x_2^3 *does not appear. Arguing in the same way, the terms* x_1^2, x_2^2, x_1 *and* x_2 *do not appear either. For these reasons,* f_i *has the form*

$$f_i = a_i + b_i x_1 x_2 + c_i x_1^2 x_2 + d_i x_1 x_2^2.$$

Then we would have that $\sum_i b_i^2 = -3$ *which is a contradiction.*

We can get as corollary of Riesz's theorem the following important result (for the original proof see [29]).

Theorem 1.3.5 (Hamburger, [29]).
A sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0}$ *has a representing non-negative Borel measure* μ *supported on* $\mathbb{R}$ *if* y *is positive semidefinite, i.e.,*

$$\left(\exists\, \mu \in \mathcal{M}^*(\mathbb{R})\ s.t.\ y_\alpha = \int x^\alpha \mu(dx),\ \forall \alpha \in \mathbb{N}_0\right) \Longleftarrow \left(L_y(h^2) \geq 0,\ \forall h \in \mathbb{R}[x]\right).$$

Proof.
Let us notice that whenever y is positive semidefinite we also have that

$$L_y(\tilde{h}) \geq 0, \quad \forall \tilde{h} \in \mathbb{R}^+[x]. \tag{1.9}$$

In fact, by Lemma 1.3.3, any polynomial $\tilde{h} \in \mathbb{R}^+[x]$ can be written as sum of squares. Namely,

$$\tilde{h} = h_1^2 + h_2^2$$

for some $h_1, h_2 \in \mathbb{R}[x]$. Then

$$L_y(\tilde{h}) = L_y(h_1^2) + L_y(h_2^2),$$

which is non-negative because $L_y(h_1^2) \geq 0$ and $L_y(h_2^2) \geq 0$, by the positive semidefiniteness of y.

Condition (1.9) implies, by Theorem 1.3.1 for $K = \mathbb{R}$, that there exists $\mu \in \mathcal{M}^*(\mathbb{R})$ realizing the sequence y.

$\square$

Remark 1.3.6.
The positive semidefiniteness condition for the moment problem can be also formulated using the Hankel matrices.
If $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ is a sequence of real numbers, the Hankel matrix $H(y)$ is defined as

$$H(y)(\alpha, \beta) := y_{\alpha+\beta-2}$$

for all $\alpha, \beta \in \mathbb{N}$.
Let us recall that, for a real symmetric square matrix H, the notation $H \succeq 0$ stands for H being positive semidefinite.
Then, Hamburger's theorem together with its necessary part (Corollary 1.2.4 for $K = \mathbb{R}$), is reformulated as follows: "a sequence of real numbers y is realized on $\mathbb{R}$ if and only if $H(y) \succeq 0$". In other words,

$$\Big(L_y(h^2) \geq 0, \quad \forall h \in \mathbb{R}[x]\Big) \Longleftrightarrow \Big(H(y) \succeq 0\Big).$$

Since it is more convenient to work with finite dimensional matrices, the following truncated Hankel matrix $H_n(y)$, $n \in \mathbb{N}_0$, is introduced.

$$H_n(y)(\alpha, \beta) := y_{\alpha+\beta-2},$$

with $\alpha, \beta \in \mathbb{N}$ such that $1 \leq \alpha, \beta \leq (n+1)$.
Then Hamburger's theorem (together with Corollary 1.2.4 for $K = \mathbb{R}$) can be rephrased as "a sequence of real numbers y is realized on $\mathbb{R}$ if and only if $H_n(y) \succeq 0$ for all $n \in \mathbb{N}_0$".

For a more detailed survey about the moment problem on $K \subset \mathbb{R}$ via Hankel matrices see [43] and [47].

1.3.3 Uniqueness of the solution via Carleman's criterion

Definition 1.3.7 (Carleman's condition)**.**
We say that a sequence $(y_n)_{n\in\mathbb{N}_0}$ of real numbers, with $y_{2n} \geq 0$ for all n, satisfies

Carleman's condition if

$$\sum_{n=1}^{\infty} y_{2n}^{-\frac{1}{2n}} = \infty. \tag{1.10}$$

Theorem 1.3.8 (Carleman, [14]).
Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$ have the same moments y_α, i.e.

$$\int x^\alpha \mu(dx) = y_\alpha = \int x^\alpha \nu(dx), \qquad \forall \alpha \in \mathbb{N}_0. \tag{1.11}$$

If $(y_\alpha)_{\alpha\in\mathbb{N}_0}$ satisfies Carleman's condition (1.10) *then $\mu = \nu$.*

Proof.
By assumptions, $(y_\alpha)_{\alpha\in\mathbb{N}_0}$ satisfies (1.10), i.e.

$$\sum_{\alpha=1}^{\infty} \frac{1}{\sqrt[\alpha]{\sqrt{y_{2\alpha}}}} = \infty.$$

By Remark A.0.20, the sequence $(y_{2\alpha})_{\alpha\in\mathbb{N}_0}$ is log-convex and so is the sequence $(\sqrt{y_{2\alpha}})_{\alpha\in\mathbb{N}_0}$. By Denjoy-Carleman's Theorem A.0.21 (w.l.o.g. we can assume $y_0 = 1$, see Remark A.0.22), the class $C\{\sqrt{y_{2\alpha}}\}$ is then quasi-analytic (see Definition A.0.17 and Definition A.0.18).
Let us consider the Fourier-Stieltjes transforms of the finite measures μ and ν, namely

$$F_1(t) := \int e^{-ixt}\mu(dx) \quad \text{and} \quad F_2(t) := \int e^{-ixt}\nu(dx), \quad t \in \mathbb{R}.$$

The function $F_1(t) - F_2(t)$ is in the class $C\{\sqrt{y_{2\alpha}}\}$. In fact, it is infinitely differentiable on $\mathbb{R}$ and since

$$\frac{d^\alpha}{dt^\alpha}F_1(t) = \int (-ix)^\alpha e^{-ixt}\mu(dx) \quad \text{and} \quad \frac{d^\alpha}{dt^\alpha}F_2(t) = \int (-ix)^\alpha e^{-ixt}\nu(dx),$$

we have that

$$\begin{aligned}
\left|\frac{d^\alpha}{dt^\alpha}\big(F_1(t)-F_2(t)\big)\right| &\leq \left|\frac{d^\alpha}{dt^\alpha}\big(F_1(t)\big)\right|+\left|\frac{d^\alpha}{dt^\alpha}\big(F_2(t)\big)\right| \\
&= \left|\int(-ix)^\alpha e^{-ixt}\mu(dx)\right|+\left|\int(-ix)^\alpha e^{-ixt}\nu(dx)\right| \\
&\leq \int|x|^\alpha\mu(dx)+\int|x|^\alpha\nu(dx) \\
&\leq c_\mu\Big(\int x^{2\alpha}\mu(dx)\Big)^{\frac{1}{2}}+c_\nu\Big(\int x^{2\alpha}\nu(dx)\Big)^{\frac{1}{2}} \qquad (1.12) \\
&= c_\mu(y_{2\alpha})^{\frac{1}{2}}+c_\nu(y_{2\alpha})^{\frac{1}{2}} \qquad (1.13) \\
&= (c_\mu+c_\nu)\cdot\sqrt{y_{2\alpha}},
\end{aligned}$$

where in (1.12) and (1.13) we have made use of Cauchy-Schwarz's inequality ($c_\mu=\sqrt{\mu(\mathbb{R})}$, $c_\nu=\sqrt{\nu(\mathbb{R})}$) and (1.11), respectively.
Moreover,

$$\frac{d^\alpha}{dt^\alpha}\big(F_1(0)-F_2(0)\big)=0$$

because

$$\frac{d^\alpha}{dt^\alpha}F_1(0)=(-i)^\alpha\int x^\alpha\mu(dx)=(-i)^\alpha\int x^\alpha\nu(dx)=\frac{d^\alpha}{dt^\alpha}F_2(0).$$

By the quasi-analitycity of the class $C\{\sqrt{y_{2\alpha}}\}$, the function F_1-F_2 is then zero everywhere on $\mathbb{R}$. Consequentely $F_1=F_2$, i.e. μ and ν have the same Fourier-Stieltjes transforms.
Let $\mathscr{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$ (see [63, Vol. I, Sect. V.3]). Then, by Levy's inversion theorem (see [63, Vol. II, p. 3]), we have that

$$\int f(x)\mu(dx)=\int f(x)\nu(dx) \qquad (1.14)$$

for any $f\in\mathscr{S}(\mathbb{R})$.

In fact, if $\widehat{f}$ denotes the Fourier transform of $f \in \mathscr{S}(\mathbb{R})$

$$\begin{aligned}
\int f(x)\mu(dx) &= \int \left(\int \widehat{f}(t)e^{itx}dt \right) \mu(dx) \\
&= \int \widehat{f}(t) \left(\int e^{ixt}\mu(dx) \right) dt \\
&= \int \widehat{f}(t)\overline{F_1(t)}dt \\
&= \int \widehat{f}(t)\overline{F_2(t)}dt \\
&= \int \widehat{f}(t) \left(\int e^{ixt}\nu(dx) \right) dt \\
&= \int \left(\int \widehat{f}(t)e^{itx}dt \right) \nu(dx) \\
&= \int f(x)\nu(dx).
\end{aligned}$$

Since every characteristic function of a set $A \in \mathcal{B}(\mathbb{R})$ is limit of compactly supported functions in $\mathscr{S}(\mathbb{R})$, we have that (1.14) holds also for the characteristic functions $\mathbb{1}_A$, i.e.

$$\mu(A) = \nu(A)$$

for all $A \in \mathcal{B}(\mathbb{R})$.
Then $\mu = \nu$.

$\square$

Theorem 1.3.8 and Theorem 1.3.5 allow us to write the following result which brings together existence and uniqueness of the moment problem on $\mathbb{R}$.

Theorem 1.3.9.
A sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0}$ has a unique *representing non-negative Borel measure μ supported on $\mathbb{R}$ if y is positive semidefinite and satisfies Carleman's condition, i.e. if*

- $L_y(h^2) \geq 0, \quad \forall h \in \mathbb{R}[x],$
- $\sum_{\alpha=1}^{\infty} {y_{2\alpha}}^{-\frac{1}{2\alpha}} = \infty,$

then $\exists! \, \mu \in \mathcal{M}^*(\mathbb{R})$ such that

$$y_\alpha = \int x^\alpha \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Proof.
Since the sequence y is positive semidefinite then, by Theorem 1.3.5, y is realized by a measure $\mu \in \mathcal{M}^*(\mathbb{R})$. We want to show that this measure is also unique. Let us assume that there exists another measure $\nu \in \mathcal{M}^*(\mathbb{R})$ which realizes y, i.e. $y_\alpha = \int x^\alpha \, \nu(dx)$ for all $\alpha \in \mathbb{N}_0$. In other words, we are assuming that there exists another measure ν having the same moments of μ. Then, by Theorem 1.3.8, $\mu = \nu$.

□

Remark 1.3.10.
Let us note that in the previous theorem we did not specify that the sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ *was such that* $y_{2\alpha} \geq 0$, *for all* $\alpha \in \mathbb{N}_0$, *as required in Definition 1.3.7. In fact, this condition automatically holds whenever the sequence* y *is positive semidefinite because, in particular for the polynomial* $x^{2\alpha}$, *we do have*

$$y_{2\alpha} = L_y(x^{2\alpha}) \geq 0.$$

In conclusion, we state some results which will be useful in the next section.

Lemma 1.3.11.
If $\mu \in \mathcal{M}^*(\mathbb{R})$ *has compact support* C *then the sequence of its moments* $(y_n)_{n\in\mathbb{N}_0}$ *satisfies Carleman's condition* (1.10).

Proof.
W.l.o.g. we can assume that $C = [a_1, a_2]$ with $a_1, a_2 \in \mathbb{R}$. Let $c := \mu(C)$ (c is finite and non-negative as well as μ). For $a := \max\{|a_1|, |a_2|\}$ we have

$$y_{2n} \leq a^{2n} c, \quad \forall n \in \mathbb{N}_0. \tag{1.15}$$

In fact,

$$y_{2n} := \int_C x^{2n} \mu(dx) \leq \int_C a^{2n} \mu(dx) = a^{2n} c.$$

Let us observe that $a = 0$ or $c = 0$ only happen when $C = \{0\}$ or $C = \emptyset$, respectively. In both cases, $y_{2n} \equiv 0$ for all $n \in \mathbb{N}$ and (1.10) is trivially true.
For all the other cases, a^{2n} and c are always different from zero and by (1.15) follows that

$$\frac{1}{(a^{2n}c)^{\frac{1}{2n}}} \leq \frac{1}{(y_{2n})^{\frac{1}{2n}}}.$$

Hence,

$$\frac{1}{a} \sum_{n=1}^{\infty} c^{-\frac{1}{2n}} \leq \sum_{n=1}^{\infty} y_{2n}^{-\frac{1}{2n}}.$$

Since $\lim_{n\to\infty} c^{-\frac{1}{2n}} = 1 \neq 0$, the series of non-negative terms on the left-hand side diverges and so does the series $\sum_{n=1}^{\infty} y_{2n}^{-\frac{1}{2n}}$.

□

Proposition 1.3.12.
Let $\mu, \eta \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int x^\alpha \mu(dx) = \int x^\alpha \eta(dx), \qquad \forall \alpha \in \mathbb{N}_0.$$

If μ has compact support then $\mu = \eta$.

Proof.
By Lemma 1.3.11 the moment sequence of μ satisfies Carleman's condition. By Theorem 1.3.8, $\mu = \eta$.

□

1.3.4 Berg and Maserick's solution on basic semi-algebraic sets

In the following we derive sufficient conditions for the solvability of the moment problem when K is a basic semi-algebraic set. Let us make some preliminary considerations for the proof of the main result which is Theorem 1.3.14.

Lemma 1.3.13.
Let $C \subseteq \mathbb{R}$ be closed. Suppose that $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ is realized by $\mu \in \mathcal{M}^*(C)$, i.e.

$$y_\alpha = \int_C x^\alpha \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Then, for any $g \in \mathbb{R}[x]$, we have that

$$(g(E)y)_\alpha = \int_C x^\alpha g(x) \, \mu(dx), \quad \alpha \in \mathbb{N}_0.$$

Proof.
By (1.3) and by Lemma 1.1.5 for $K = C$ we have that

$$(g(E)y)_\alpha = L_{g(E)y}(x^\alpha) = L_y(x^\alpha g) = \int_C x^\alpha g(x) \, \mu(dx).$$

□

Let $K \subset \mathbb{R}$ be the basic semi-algebraic set given by a fixed real polynomial g. Namely,

$$K := \{x \in \mathbb{R} : g(x) \geq 0\}.$$

For the sake of brevity, we will sometimes write $\{g \geq 0\}$.

Theorem 1.3.14 (Berg-Maserick, [10]).
A sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0}$ has a unique representing non-negative Borel measure μ on K if y and $g(E)y$ are positive semidefinite and K is compact, i.e. if

- $L_y(\, h^2 \,) \geq 0,\ L_y(h^2 g) \geq 0, \quad \forall h \in \mathbb{R}[x],$
- *K is compact,*

then $\exists!\, \mu \in \mathcal{M}^*(K)$ such that

$$y_\alpha = \int_K x^\alpha \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Remark 1.3.15.
Let us recall that $g(E)y$ positive semidefinite means that $L_{g(E)y}(h^2) \geq 0$ which is equivalent to write $L_y(h^2 g) \geq 0$ since $L_{g(E)y}(h^2) = L_y(h^2 g)$ by (1.3).

Proof. (of Theorem 1.3.14)
The conditions $L_y(h^2) \geq 0$ and $L_y(h^2 g) \geq 0$ for all $h \in \mathbb{R}[\mathbf{x}]$ imply, by Hamburger's Theorem 1.3.5, that y and $g(E)y$ are both realized on $\mathbb{R}$, i.e. there exist two non-negative measures $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$y_\alpha = \int x^\alpha \, \mu(dx) \quad \text{and} \quad (g(E)y)_\alpha = \int x^\alpha \, \nu(dx), \quad \forall \alpha \in \mathbb{N}_0. \tag{1.16}$$

We are going to show now that μ and ν are related to each other and that actually $\mu \in \mathcal{M}^*(K)$.

The integral representation of y in (1.16) implies, by Lemma 1.3.13, that

$$(g(E)y)_\alpha = \int x^\alpha g(x) \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0. \tag{1.17}$$

We can write (1.17) as

$$(g(E)y)_\alpha = \int_K x^\alpha g(x) \, \mu(dx) + \int_{\mathbb{R} \setminus K} x^\alpha g(x) \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Hence, by (1.16), we have that for any $\alpha \in \mathbb{N}_0$

$$\begin{aligned}\int_K x^\alpha g(x)\,\mu(dx) &= (g(E)y)_\alpha - \int_{\mathbb{R}\setminus K} x^\alpha g(x)\,\mu(dx)\\ &= \int x^\alpha\,\nu(dx) - \int_{\mathbb{R}\setminus K} x^\alpha g(x)\,\mu(dx),\end{aligned}$$

which can be written as

$$\int x^\alpha \mathbb{1}_K(x) g(x)\,\mu(dx) = \int x^\alpha \left(\nu(dx) - \mathbb{1}_{\mathbb{R}\setminus K}(x) g(x)\,\mu(dx)\right).$$

The latter shows that the two *non-negative* measures on $\mathbb{R}$

$$\mathbb{1}_K g\,d\mu \quad \text{and} \quad d\nu - \mathbb{1}_{\mathbb{R}\setminus K} g\,d\mu \tag{1.18}$$

have the same moments.
Since the measure $\mathbb{1}_K g\,d\mu$ has compact support, then the two measures in (1.18) have to coincide by Proposition 1.3.12, i.e.

$$\mathbb{1}_K g\,d\mu = d\nu - \mathbb{1}_{\mathbb{R}\setminus K} g\,d\mu.$$

Then

$$\mathbb{1}_K g\,d\mu + \mathbb{1}_{\mathbb{R}\setminus K} g\,d\mu = d\nu$$

and hence

$$d\nu = g\,d\mu.$$

Thus, the signed measure $g\,d\mu$ is actually non-negative as well as ν. This implies that $\mu(\mathbb{R} \setminus \{g \geq 0\}) = 0$ and so $\operatorname{supp}(\mu) \subseteq \{g \geq 0\} =: K$.
Then, μ has compact support too and so, by Proposition 1.3.12, if there is another measure which realizes y this must be equal to μ.

□

Case of several polynomials

Let $K \subset \mathbb{R}$ be the basic semi-algebraic set given by the fixed real polynomials $g_1, \ldots, g_m$. Namely,

$$K := \bigcap_{j=1}^{m} \{x \in \mathbb{R} : g_j(x) \geq 0\}.$$

Theorem 1.3.16 (Berg-Maserick, [10]).
A sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0}$ has a unique representing non-negative Borel measure

μ on K if y and $g_j(E)y$, for $j = 1, \dots, m$, are positive semidefinite and K is compact, i.e. if

- *$L_y(h^2) \geq 0$, $L_y(h^2 g_j) \geq 0$, $\quad \forall h \in \mathbb{R}[x]$ with $j = 1, \dots, m$,*
- *K is compact,*

then $\exists!\, \mu \in \mathcal{M}^*(K)$ such that

$$y_\alpha = \int_K x^\alpha \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Proof.
Assume that $L_y(h^2) \geq 0$, $L_y(h^2 g_j) \geq 0$ for all $h \in \mathbb{R}[x]$ and for $j = 1, \dots, m$. Let us also suppose that at least one of the sets $\{g_j \geq 0\}$ is compact. For instance, let $\{g_1 \geq 0\}$ be compact.
For each $j = 1, \dots, m$, the condition $L_y(h^2 g_j) \geq 0$ means that $L_{g_j(E)y}(h^2) \geq 0$ (for all $h \in \mathbb{R}[x]$), and then by Hamburger's Theorem 1.3.5 we have that there exists a non-negative measure $\nu_j \in \mathcal{M}^*(\mathbb{R})$ realizing the sequence $g_j(E)y$, i.e.

$$(g_j(E)y)_\alpha = \int x^\alpha \, \nu_j(dx), \quad \forall \alpha \in \mathbb{N}_0, \quad j = 1, \dots, m. \tag{1.19}$$

In particular this is true for $j = 2, \dots, m$.
In the rest of this proof, whenever it is not specified, we intend $j = 2, \dots, m$ and $\alpha \in \mathbb{N}_0$.
By Theorem 1.3.14, $L_y(h^2) \geq 0$ and $L_y(h^2 g_1) \geq 0$, for all $h \in \mathbb{R}[x]$, imply that there exists a non-negative measure $\mu \in \mathcal{M}^*\big(\{g_1 \geq 0\}\big)$ such that

$$y_\alpha = \int_{\{g_1 \geq 0\}} x^\alpha \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0. \tag{1.20}$$

We want to show that actually $\mu \in \mathcal{M}^*(K)$.

The integral representation of y in (1.20) implies, by Lemma 1.3.13, that we also have

$$(g_j(E)y)_\alpha = \int_{\{g_1 \geq 0\}} x^\alpha g_j(x) \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

We can trivially write the latter as

$$(g_j(E)y)_\alpha = \int_A x^\alpha g_j(x) \, \mu(dx) + \int_B x^\alpha g_j(x) \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0,$$

where $A := \{g_j \geq 0\} \cap \{g_1 \geq 0\}$ and $B := \{g_j < 0\} \cap \{g_1 \geq 0\}$.

Hence, by (1.19), we have that for any $\alpha \in \mathbb{N}_0$

$$\begin{aligned}\int_A x^\alpha g_j(x)\,\mu(dx) &= (g_j(E)y)_\alpha - \int_B x^\alpha g_j(x)\,\mu(dx)\\ &= \int x^\alpha\,\nu_j(dx) - \int_B x^\alpha g(x)\,\mu(dx),\end{aligned}$$

which can be rewritten as

$$\int x^\alpha \mathbb{1}_A(x) g_j(x)\,\mu(dx) = \int x^\alpha \left(\nu_j(dx) - \mathbb{1}_B(x) g_j(x)\,\mu(dx)\right).$$

The latter implies that the two *non-negative* measures on $\mathbb{R}$

$$\mathbb{1}_A g_j\,d\mu \quad \text{and} \quad d\nu_j - \mathbb{1}_B g_j\,d\mu$$

have the same moments. Therefore, by Proposition 1.3.12, they coincide since the first measure has compact support. In fact, since $\mathrm{supp}(\mu) \subseteq \{g_1 \geq 0\}$, we have that

$$\mathrm{supp}\,(\mathbb{1}_A g_j\,d\mu) \subseteq \Big(\{g_j \geq 0\} \cap \{g_1 \geq 0\}\Big) \subseteq \{g_1 \geq 0\}$$

and so $\mathrm{supp}\,(\mathbb{1}_A g_j\,d\mu)$, as closed subset of the compact set $\{g_1 \geq 0\}$, is compact. Hence,

$$d\nu_j = g_j\,d\mu, \quad j = 2, \ldots, m.$$

Each signed measure $g_j\,d\mu$ has to be non-negative as well as ν_j, then $\mu(\mathbb{R} \setminus \{g_j \geq 0\}) = 0$ for all $j = 2, \ldots, m$. Hence, $\mathrm{supp}(\mu) \subseteq \{g_j \geq 0\}$ for all $j = 1, \ldots, m$. It follows that

$$\begin{aligned}\mu(\mathbb{R} \setminus K) = \mu\left(\mathbb{R} \setminus \bigcap_{j=1}^m \{g_j \geq 0\}\right) &= \mu\left(\bigcup_{j=1}^m \mathbb{R} \setminus \{g_j \geq 0\}\right)\\ &\leq \sum_{j=1}^m \mu(\mathbb{R} \setminus \{g_j \geq 0\}) = 0,\end{aligned} \tag{1.21}$$

which implies that $\mathrm{supp}(\mu) \subseteq K$. Then, μ has compact support too and so by Proposition 1.3.12 it is unique.

The case when all the sets $\{g_j \geq 0\}$ are all non-compact can be roughly proved as follows. The main idea is to reduce the problem to the case where at least one polynomial is non-negative on a compact set. W.l.o.g., this can be done "normalizing" in a certain way the first two polynomials g_1 and g_2 (supposed $\{g_1 \geq 0\} \neq \{g_2 \geq 0\}$) in order to get another polynomial g_2^1 which is non-negative on a compact set $A \subset \mathbb{R}$. Then, taking into account the "equivalent" system of polynomials $g_2^1, g_3, \ldots, g_m$ (equivalent in the sense that the intersection of the sets where $g_2^1, g_3, \ldots, g_m$ are non-negative is still K) the assertion follows by similar arguments of the first part of this proof.

□

Remark 1.3.17.
Note that by the σ-subadditivity of μ, the latter theorem also holds if in (1.21) we consider a countable number of polynomials g_j.
If instead K is a basic semi-algebraic set given by an uncountable number of polynomials g_j we have to use the inner-regularity *of the measure μ. We analyze this case in details and in a more general setting in Chapter 4.*

Remark 1.3.18.
With the same notation as in Remark 1.3.6, in Theorem 1.3.14 we have that

$$\Big(L_y(h^2 g) \geq 0, \quad \forall h \in \mathbb{R}[x]\Big) \Longleftrightarrow \Big(H_n\big(g(E)y\big) \succeq 0, \quad \forall n \in \mathbb{N}_0\Big).$$

It is then straightforward to reformulate Theorem 1.3.16 in terms of Hankel matrices.

1.4 Alternative approaches to the uniqueness problem

In this section we are going to study the problem whether a measure $\mu \in \mathcal{M}^*(\mathbb{R})$ is uniquely determined by its moments with an approach alternative to the one used in Subsection 1.3.3. In particular, we will proceed by analyzing different cases depending on the form of supp(μ).

As first step we consider measures with finite support.
Let δ_t be the Dirac measure concentrated in $t \in \mathbb{R}$.

Lemma 1.4.1.
Let $x_0, \ldots, x_n \in \mathbb{R}$ with $n \in \mathbb{N}_0$. A measure μ has finite support $\{x_0, \ldots, x_n\}$ if and only if

$$\mu(A) = \sum_{i=0}^{n} \mu(\{x_i\})\delta_{x_i}(A)$$

for any Borel set $A \subseteq \mathbb{R}$.

Proof.
($\Rightarrow$) Let $\{x_0, \ldots, x_n\}$ be the support of the measure μ and let A be a measurable

subset of $\mathbb{R}$. Then,

$$\mu(A) = \mu(A \cap \{x_0, \dots, x_n\}) = \sum_{x_i \in A} \mu(\{x_i\}) = \sum_{i=0}^{n} \mu(\{x_i\})\delta_{x_i}(A).$$

($\Leftarrow$) Let us assume that $\mu(A) = \sum_{i=0}^{n} \mu(\{x_i\})\delta_{x_i}(A)$ for any measurable subset A of $\mathbb{R}$. Clearly, the measurable set $\{x_0, \dots, x_n\}$ is in $\text{supp}(\mu)$. On the other hand, for $(\mathbb{R} \setminus \{x_0, \dots, x_n\})$ we have that

$$\mu(\mathbb{R} \setminus \{x_0, \dots, x_n\}) = \sum_{i=0}^{n} \mu(\{x_i\})\delta_{x_i}(\mathbb{R} \setminus \{x_0, \dots, x_n\})$$

which is zero because, for all $i = 0, \dots, n$, we have that $\delta_{x_i}(\mathbb{R} \setminus \{x_0, \dots, x_n\})$ is zero. Then, by Definition 1.1.1, we conclude that $\text{supp}(\mu) \equiv \{x_0, \dots, x_n\}$.

$\square$

Proposition 1.4.2.
Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int x^\alpha \mu(dx) = \int x^\alpha \nu(dx), \qquad \forall \alpha \in \mathbb{N}_0.$$

If both measures have finite support then $\mu = \nu$.

Proof.
Let us call $(x_i)_{i=0}^n$, with $n \in \mathbb{N}_0$, the points of the union of $\text{supp}(\mu)$ and $\text{supp}(\nu)$.

By Lemma 1.4.1 we can write μ and ν as

$$\mu = \sum_{i=0}^{n} c_i \delta_{x_i} \quad \text{and} \quad \nu = \sum_{i=0}^{n} b_i \delta_{x_i}$$

where, for any $i = 0, \dots, n$, $c_i = \mu(\{x_i\})$ and $b_i = \nu(\{x_i\})$ are real numbers.

Let us call $(y_\alpha^\mu)_{\alpha \in \mathbb{N}_0}$ and $(y_\alpha^\nu)_{\alpha \in \mathbb{N}_0}$ the moment sequence of μ and ν, respectively. Since for any $\alpha \in \mathbb{N}_0$

$$y_\alpha^\mu := \int x^\alpha d\,\mu = \sum_{i=0}^{n} c_i x_i^\alpha \quad \text{and} \quad y_\alpha^\nu := \int x^\alpha d\,\nu = \sum_{i=0}^{n} b_i x_i^\alpha$$

and since we are assuming $y_\alpha^\mu = y_\alpha^\nu$, we have that

$$\sum_{i=0}^{n} c_i x_i^\alpha = \sum_{i=0}^{n} b_i x_i^\alpha$$

which can be rewritten as

$$\sum_{i=0}^{n} x_i^\alpha (c_i - b_i) = 0. \tag{1.22}$$

Note that the matrix associated to the homogeneous system (1.22) in the variables $(c_i - b_i)$, for $i = 0, \ldots, n$, is the Vandermonde matrix

$$V(x_0, \ldots, x_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix}$$

whose determinant is given by

$$det(V(x_0, \ldots, x_n)) = \prod_{0 \leq i < j \leq n} (x_j - x_i).$$

In this case, $det(V(x_0, \ldots, x_n))$ is always non-zero since all the x_i's are distinct. It follows that the system (1.22) has only the trivial solution $c_i - b_i = 0$, i.e. $c_i = b_i$, for $i = 0, \ldots, n$. Hence, $\mu = \nu$.

□

As second step, we generalize Proposition 1.4.2 to measures having compact support. To this aim let us show the following lemma.

Lemma 1.4.3.
Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int x^\alpha \mu(dx) = \int x^\alpha \nu(dx), \qquad \forall \alpha \in \mathbb{N}_0. \tag{1.23}$$

If both measures have compact support K then

$$\int f(x)\mu(dx) = \int f(x)\nu(dx), \qquad \forall f \in \mathcal{C}(K).$$

Proof.
By (1.23) follows that

$$\int p(x)\mu(dx) = \int p(x)\nu(dx), \qquad \forall p \in \mathbb{R}[x]. \tag{1.24}$$

In fact, if $p(x) = \sum\limits_{\alpha\in I} p_\alpha x^\alpha$, with $I \subset \mathbb{N}_0$ finite, then

$$\int p(x)\mu(dx) = \sum_{\alpha\in I} p_\alpha \int x^\alpha \mu(dx) = \sum_{\alpha\in I} p_\alpha \int x^\alpha \nu(dx) = \int p(x)\nu(dx).$$

Since K is compact, for any $f \in \mathcal{C}(K)$ there exists a sequence of real polynomials $(P_n)_{n\in\mathbb{N}_0}$ which uniformly converges to f (Stone-Weierstrass theorem, [66, Theorem 7.24]). Moreover, all P_n's are measurable and uniformly bounded (i.e. $|P_n| < M$ for some $M \in \mathbb{R}$).
By the dominated convergence theorem and by (1.24), we then have that

$$\begin{aligned}
\int f(x)\mu(dx) &= \lim_{n\to\infty} \int P_n(x)\mu(dx) \\
&= \lim_{n\to\infty} \int P_n(x)\nu(dx) \qquad (1.25) \\
&= \int f(x)\nu(dx).
\end{aligned}$$

□

Proposition 1.4.4.
Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int x^\alpha \mu(dx) = \int x^\alpha \nu(dx), \qquad \forall \alpha \in \mathbb{N}_0.$$

If both measures have compact support K then $\mu = \nu$.

Proof.
By Lemma 1.4.3, we have that

$$\int f(x)\mu(dx) = \int f(x)\nu(dx), \qquad \forall f \in \mathcal{C}(K).$$

By Riesz-Markov's Theorem C.0.5 we then have that $\mu = \nu$.

□

Remark 1.4.5.
Berg and Maserick's theorems cannot make use of Proposition 1.4.4 (instead of Proposition 1.3.12). In fact, in the proofs of Theorem 1.3.14 and Theorem 1.3.16, we compare two measures on $\mathbb{R}$ and only one of them is known to have compact support. In Proposition 1.4.4 and in Lemma 1.4.3 instead both the measures have compact support and we cannot avoid this condition because it is essential in (1.25).

For the problem on a non compact interval, uniqueness is instead a more delicate question. In the following we are going to show that, in the non-compact case, the assumption of two measures having the same moments needs to be substituted by a stronger condition involving the continuous and bounded functions on the support of the measures.

For our purpose, we will make use of the following functional form of the monotone class theorem which, in dealing with integrals, is often useful.

Theorem 1.4.6 (Functional monotone class theorem, [36] p. 37)**.**
Let $\mathcal{K}$ be a collection of bounded real-valued functions on K that is closed under products (i.e. if $f, g \in \mathcal{K}$ then $fg \in \mathcal{K}$), and let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{K}$. Let $\mathcal{H} \supset \mathcal{K}$ be a vector space (over $\mathbb{R}$) of bounded real-valued functions on K such that

(a) $\mathcal{H}$ contains the constant functions

(b) if $(f_n)_{n\in\mathbb{N}_0} \subset \mathcal{H}$ with

$$\sup_{n\in\mathbb{N}_0} \sup_{k\in\mathbb{R}} |f_n(k)| < \infty \quad \text{and} \quad 0 \leq f_0 \leq f_1 \leq \cdots \leq f_n \leq \dots ,$$

then $f := \lim_{n\to\infty} f_n \in \mathcal{H}$.

Under these conditions, $\mathcal{H}$ contains the class $\mathcal{H}_K^{bm}$ of all bounded $\mathcal{B}$-measurable real-valued function on K.

Proposition 1.4.7.
Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int f(x)\mu(dx) = \int f(x)\nu(dx), \qquad \forall f \in \mathcal{C}^b(K),$$

where K, not necessarily compact, is the support of both measures. Then

$$\int h(x)\mu(dx) = \int h(x)\nu(dx), \qquad \forall h \in \mathcal{H}_K^{bm},$$

where $\mathcal{H}_K^{bm}$ is the class of all bounded $\mathcal{B}(K)$-measurable functions on K.
In particular, $\mu = \nu$.

Proof.
Let $\mathcal{K}$ be the class $\mathcal{C}^b(K)$ of all continuous and bounded functions f on K. The class $\mathcal{K}$ satisfies all the assumptions in Theorem 1.4.6. Note that $\mathcal{B} = \mathcal{B}(K)$. In

fact, K is metrizable[1] and we have that the Borel σ-algebra $\mathcal{B}(K)$ on K and the Baire σ-algebra $\mathcal{B}_0(K)$ on K coincide, namely

$$\mathcal{B} = \sigma(\mathcal{K}) = \sigma(\mathcal{C}^b(K)) = \mathcal{B}_0(K) \equiv \mathcal{B}(K).$$

Take $\mathcal{H}$ to be the class of all bounded $\mathcal{B}$-measurable function h on K such that

$$\int h(x)\mu(dx) = \int h(x)\nu(dx). \tag{1.26}$$

The class $\mathcal{H}$ satisfies conditions *(a)* and *(b)* of Theorem 1.4.6. In fact, the constants are polynomial of zero degree, bounded on K, $\mathcal{B}$-measurable and such that (1.26) holds for them (see assumptions) and *(b)* holds because of Lebesgue's monotone convergence theorem. Then, by Theorem 1.4.6, we have that $\mathcal{H} \supseteq \mathcal{H}_K^{bm}$. In particular, the characteristic functions of any $A \in \mathcal{B}(K)$ are in $\mathcal{H}_K^{bm}$ and so in $\mathcal{H}$. Hence, $\mu = \nu$.

□

1.5 Existence and uniqueness via operator theory

In this section we are going to give an operator-theoretical explanation to the existence (and uniqueness) of the Hamburger moment problem.
For classical notations, definitions and results of spectral theory we address to Appendix B.

Let $\mathcal{H}$ be a real Hilbert space with its inner product given by the bilinear form $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$. If $T : \mathcal{D}(T) \mapsto \mathcal{H}$ is a symmetric unbounded operator on $\mathcal{H}$ and $v \in \mathcal{H}$ is such that $v \in \bigcap_{n=1}^{\infty} \mathcal{D}(T^n)$, then the sequence $\Big(\langle v, T^n v\rangle\Big)_{n\in\mathbb{N}_0}$ is positive semidefinite. In fact, for any finite sequence of real numbers $(h_0, h_1, \ldots, h_n)$ we have that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} h_i h_j \langle v, T^{i+j} v\rangle = \langle \sum_{i=0}^{n} h_i T^i v, \sum_{j=0}^{n} h_j T^j v\rangle = \left\| \sum_{i=0}^{n} h_i T^i v \right\|^2 \geq 0.$$

If, in addition, T was self-adjoint and v is such that $T^n v \in \mathcal{D}(T)$, for any $n \in \mathbb{N}_0$,

[1]Every subset of a metric space is metrizable. It is enough to take the restriction of the metric on the space.

then we could use the spectral theorem (see Corollary B.2.2) to show that there exists a non-negative measure $\mu_v \in \mathcal{M}^*(\mathbb{R})$ (depending on v) such that

$$\langle v, T^n v\rangle = \int x^n \mu_v(dx), \quad \forall n \in \mathbb{N}_0.$$

It actually sufficies to require that T admits some self-adjoint extension $\widetilde{T}$ on a bigger space $\mathcal{D}(\widetilde{T}) \supset \mathcal{D}(T)$ so that, under the same assumptions on v, we have

$$\langle v, T^n v\rangle = \langle v, \widetilde{T}^n v\rangle = \int x^n \mu_v(dx), \quad \forall n \in \mathbb{N}_0.$$

We can then formulate the moment problem on $\mathbb{R}$ in the following version.
Given a sequence of real number $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$, find a symmetric operator T and a vector v such that T admits a self-adjoint extension $\widetilde{T}$ and in turn a spectral measure $\mu_v \in \mathcal{M}^*(\mathbb{R})$ associated to v such that

$$y_\alpha = \langle v, T^\alpha v\rangle = \langle v, \widetilde{T}^\alpha v\rangle = \int x^\alpha \mu_v(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

It is then fundamental to understand under which condition it is possible to have self-adjoint extensions of a symmetric operator.
A first answer to this problem is given by the following simple and useful criterion due to von Neumann.

Lemma 1.5.1 (von Neumann (see [63] Vol. II, p. 319), Galindo, [24])**.**
On a real Hilbert space every symmetric operator has self-adjoint extensions.

We will apply the latter to give an operator version of the Hamburger moment problem in which only the existence of some self-adjoint extensions is needed.

Theorem 1.5.2 (Hamburger)**.**
A sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ has a representing non-negative Borel measure μ supported on $\mathbb{R}$ if y is positive semidefinite.

Proof.
Let us consider the space $\mathbb{R}[x]$ of all real polynomials on $\mathbb{R}$ with the following bilinear form defined by

$$\langle p, q\rangle = \langle \sum_{\alpha=0}^{\alpha_1} p_\alpha x^\alpha, \sum_{\beta=0}^{\beta_1} q_\beta x^\beta\rangle := \sum_{\alpha=0}^{\alpha_1}\sum_{\beta=0}^{\beta_1} p_\alpha q_\beta y_{\alpha+\beta}, \quad p, q \in \mathbb{R}[x].$$

Note that $\langle p, q\rangle \equiv L_y(pq)$, where L_y is the linear functional on $\mathbb{R}[x]$ defined in (1.2).

Since the sequence y is assumed to be positive semidefinite, the form is non-negative. In fact, $\langle p, p\rangle = L_y(p^2) \geq 0$ for any $p \in \mathbb{R}[x]$. However, the form is *not* an inner product because $\langle p, p\rangle = 0$ does not necessarily imply $p = 0$. Indeed, if we consider the sequence $y = (1, 0, 0\dots)$ we have that y is positive semidefinite and for $\alpha \geq 1$ we have $\langle x^\alpha, x^\alpha\rangle = L_y(x^{2\alpha}) = y_{2\alpha} = 0$ without being x^α identically equal to zero.
Let then $N := \{h \in \mathbb{R}[x] : L_y(h^2) = 0\}$ and let $\mathcal{H}_y$ be the Hilbert space obtained by completing $\mathbb{R}[x]/N$ w.r.t. the inner product $\langle\cdot,\cdot\rangle$, i.e. $\mathcal{H}_y := \overline{\mathbb{R}[x]/N}$. On $\mathcal{H}_y$ the inner product will be denoted again by $\langle\cdot,\cdot\rangle$. Let us introduce the following operator

$$\begin{aligned} X: \quad & \mathbb{R}[x] && \to \mathbb{R}[x] \\ & h(x) = \sum_{\alpha=0}^{\alpha_1} h_\alpha x^\alpha && \mapsto (Xh)(x) := \sum_{\alpha=0}^{\alpha_1} h_\alpha x^{\alpha+1}\,. \end{aligned}$$

Note that X is symmetric and by Schwarz inequality it maps N in N. In fact,

$$\langle X\,h_1, h_2\rangle = L_y(xh_1\,h_2) = L_y(h_1\,xh_2) = \langle h_1, X\,h_2\rangle$$

and

$$\langle Xh, Xh\rangle = \left|\langle X^2h, h\rangle\right| \leq \langle X^2h, X^2h\rangle^{\frac{1}{2}}\langle h, h\rangle^{\frac{1}{2}}$$

for all $h_1, h_2 \in \mathbb{R}[x]$ and $h \in N$. In other words, we can write

$$\begin{aligned} X: \ \mathbb{R}[x]/N & \to \mathbb{R}[x]/N \\ h & \mapsto Xh := x\,h\,, \end{aligned}$$

where we made an abuse of notation on X and denoted by h the class $[h]$. By Lemma 1.5.1, X admits some self-adjoint extension, call it $\widetilde{X}$. By spectral theorem (see Corollary B.2.2), then there exists a non-negative measure $\mu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\langle 1, \widetilde{X}^\alpha 1\rangle = \int x^\alpha \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Note that $1 \in \mathcal{D}(X)$ and is such that $\widetilde{X}^\alpha 1 \in \mathcal{D}(\widetilde{X})$ for all $\alpha \in \mathbb{N}_0$.
In other words, we have that y is realized by μ. In fact,

$$\int x^\alpha \mu(dx) = \langle 1, \widetilde{X}^\alpha 1\rangle = \langle 1, X^\alpha\,1\rangle = \langle 1, \underbrace{X\cdots X}_{\alpha-\text{times}}\cdot 1\rangle = L_y(x^\alpha) = y_\alpha, \quad \forall \alpha \in \mathbb{N}_0.$$

□

Let us note that as soon as the operator X has a unique self-adjoint extension the moment problem is determinate.
Nussbaum in [56] showed that if $T : \mathcal{D}(T) \mapsto \mathcal{H}$, symmetric unbounded operator on a Hilbert space $\mathcal{H}$, has a total set of *vectors of uniqueness* then T can be extented to a self-adjoint operator. According to Nussbaum, a vector $v \in \mathcal{H}$ such that the moment sequence $\Big(\langle v, T^n v\rangle\Big)_{n\in\mathbb{N}_0}$ is determinate is called *vector of uniqueness*. We are going to use instead the equivalent definition in [63, vol. II, Definition 2, p. 201].

First, let us introduce some preliminar notions.

Definition 1.5.3.
A symmetric operator T is called essentially self-adjoint *if its closure $\overline{T}$ is self-adjoint.*

We have the following fact.

Theorem 1.5.4 ([63] Vol. I, p. 256)**.**
Let B be a symmetric operator on its domain. Then B is essentially self-adjoint if and only if B has a unique self-adjoint extension.

Definition 1.5.5 (C^∞-vectors)**.**
A vector $v \in \mathcal{H}$ is called a C^∞-vector *if v belongs to the domain*

$$\mathcal{D}^\infty(T) := \bigcap_{n=1}^{\infty} \mathcal{D}(T^n).$$

The reason for this terminology lies in Proposition B.5.1.

Definition 1.5.6 (Vector of uniqueness)**.**
Let $v \in \mathcal{D}^\infty(T)$ with T symmetric operator on $\mathcal{H}$. Let us define the set

$$\mathcal{D}_v := \left\{ \sum_{n=0}^{N} t_n T^n v | \ t_n \in \mathbb{R}, N \in \mathbb{N} \right\}$$

and the operator

$$\begin{aligned} T_v : \quad & \mathcal{D}_v & \to \mathcal{D}_v \\ & \sum_{n=0}^{N} t_n T^n v & \mapsto \sum_{n=0}^{N} t_n T^{n+1} v, \end{aligned}$$

i.e.

$$T_v = T|_{\mathcal{D}_v}.$$

The vector v is called vector of uniqueness *for T if and only if the operator T_v is essentially self-adjoint on $\mathcal{D}_v$ (as an operator on the Hilbert space $\overline{\mathcal{D}_v}$).*
Let $V_u(T)$ denote the set of all vectors of uniqueness for T.

Finally, a subset S of $\mathcal{H}$ is *total* if the set of all finite linear combinations of elements of S is dense in $\mathcal{H}$.

Lemma 1.5.7 (Nussbaum, [56] (see also [63] Vol. II, p. 201))**.**
Let T be a symmetric operator and suppose $\mathcal{D}(T)$ contains a total set of vectors of uniqueness. Then T is essentially self-adjoint.

In the same paper, Nussbaum shows that certain classes of vectors, namely the quasi-analytic vectors, are always vectors of uniqueness and so he can conclude Theorem 1.5.9.

Definition 1.5.8 (Quasi-analytic vector)**.**
A vector $v \in \mathcal{D}^\infty(T)$ is called quasi-analytic *vector for T if*

$$\sum_{n=1}^{\infty} ||T^n v||^{-\frac{1}{n}} = \infty.$$

Let $\mathcal{D}^{qa}(T)$ denote the set of all quasi-analytic vectors for T.

The following result is a generalization of the classical analytic vector theorem due to Nelson (see [55]).

Theorem 1.5.9 (Nussbaum, [56] (see also [69] p. 149))**.**
Let T be a symmetric operator on a Hilbert space $\mathcal{H}$ and suppose $\mathcal{D}(T)$ contains a total set of quasi-analytic vectors. Then T is essentially self-adjoint.

Note that T is densely defined. In fact, if we call $\mathcal{D}$ the total set of quasi-analitic vectors contained in $\mathcal{D}(T)$, we have

$$\mathcal{D} \subset \mathcal{D}^{\mathrm{qa}}(T) \subset \mathcal{D}(T)$$

which implies that

$$\mathcal{H} = \overline{Span\,\mathcal{D}} \subseteq \overline{Span\,\mathcal{D}^{\mathrm{qa}}(T)} \subseteq \overline{\mathcal{D}(T)} \subseteq \mathcal{H},$$

where the first equality holds by assumption.

Proof. (of Theorem 1.5.9)
First of all, we want to prove that any quasi-analytic vector v for T is a vector of

uniqueness for T, i.e. the operator T_v introduced in Definition 1.5.6 is essentially self-adjoint on $\mathcal{D}_v$. Note that T_v is symmetric as well as T. Then, by Lemma 1.5.1 there exist self-adjoint extensions of T_v. Let us consider two of them, namely $\widetilde{T_v}$ and $\widehat{T_v}$. Moreover, since $v \in \mathcal{D}_v$, we have that $T_v^n v \in \mathcal{D}(\widetilde{T_v})$ and $T_v^n v \in \mathcal{D}(\widehat{T_v})$ for any $n \in \mathbb{N}_0$. So by spectral theorem (see Corollary B.2.2), we know that there exist two measures $\mu_v, \nu_v \in \mathcal{M}^*(\mathbb{R})$ such that

$$\langle v, \widetilde{T_v}^n v \rangle = \int x^n \, \mu_v(dx)$$

and

$$\langle v, \widehat{T_v}^n v \rangle = \int x^n \, \nu_v(dx),$$

respectively.
Since for any $v \in \mathcal{D}_v$ we have $\widetilde{T_v} v = \widehat{T_v} v = T_v v = T v$, then

$$\langle v, \widetilde{T_v}^n v \rangle = \langle v, T^n v \rangle = \langle v, \widehat{T_v}^n v \rangle, \quad \forall n \in \mathbb{N}_0,$$

and so

$$\int x^n \, \mu_v(dx) = \int x^n \, \nu_v(dx), \quad \forall n \in \mathbb{N}_0.$$

If we set $y_n := \int x^n \, \mu_v(dx)$, we have that the sequence $(y_n)_{n \in \mathbb{N}_0}$ satisfies Carleman's condition $\sum_{n=1}^{\infty} {y_{2n}}^{-\frac{1}{2n}} = \infty$. In fact,

$$||T^n v||^2 = \langle T^n v, T^n v \rangle = \langle v, T^{2n} v \rangle = \int x^{2n} \mu_v(dx) = y_{2n}$$

and so

$$\sum_{n=1}^{\infty} ||T^n v||^{-\frac{1}{n}} = \sum_{n=1}^{\infty} y_{2n}^{-\frac{1}{2n}},$$

which diverges because v is quasi-analytic.
By Theorem 1.3.8, we can conclude that the two measures μ_v and ν_v coincide since they have the same moments satisfying Carleman's condition, i.e. $\mu_v = \nu_v$. Since all the self-adjoint extensions of T_v are equal we can conclude, by Theorem 1.5.4, that T_v is essentially self-adjoint on $\mathcal{D}_v$ and so that $v \in V_u(T)$.

To sum up, we have proved that

$$\mathcal{D}^{\mathrm{qa}}(T) \subset V_u(T). \tag{1.27}$$

If we call $\mathcal{D}$ the set of quasi-analitic vectors which is contained in $\mathcal{D}(T)$, the relation (1.27) becomes

$$\mathcal{D} \subset \mathcal{D}^{\mathrm{qa}}(T) \subset V_u(T)$$

which implies that

$$\mathcal{H} = \overline{Span\,\mathcal{D}} \subseteq \overline{Span\,\mathcal{D}^{\mathrm{qa}}(T)} \subseteq \overline{Span\,V_u(T)} \subseteq \mathcal{H},$$

where the first equality holds by assumption. Hence, $\overline{Span\,V_u(T)} = \mathcal{H}$. Therefore, by Lemma 1.5.7, T is essentially self-adjoint.

□

Note that Carleman's Theorem 1.3.8 is only a result on quasi-analytic functions. In fact, its proof does not involve any technique from operator, nor moment, theory. For this reason we could use it in the previous proof.

We can finally present the operator-theoretical proof of Theorem 1.3.9 which, for convenience, we report here.

Theorem 1.5.10.
A sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0}$ has a unique representing non-negative Borel measure μ supported on $\mathbb{R}$ if y is positive semidefinite and satisfies Carleman's condition, i.e. if

- $L_y(h^2) \geq 0, \quad \forall h \in \mathbb{R}[x],$
- $\sum\limits_{\alpha=1}^{\infty} {y_{2\alpha}}^{-\frac{1}{2\alpha}} = \infty,$

then $\exists!\, \mu \in \mathcal{M}^*(\mathbb{R})$ such that

$$y_\alpha = \int x^\alpha \, \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Proof.
Let L_y be the linear functional on $\mathbb{R}[x]$ defined in (1.2), and let $\mathcal{H}_y$ be the canonical Hilbert space associated with the positive semidefinite sequence y, i.e. $\mathcal{H}_y := \overline{\mathbb{R}[x]/N}$ where $N := \{h \in \mathbb{R}[x] : L_y(h^2) = 0\}$ (for details see proof of Theorem 1.5.2). Let us introduce the following operator

$$\begin{aligned} X :\ \ \mathbb{R}[x]/N &\to \mathbb{R}[x]/N \\ h &\mapsto Xh := x\,h\,. \end{aligned}$$

The operator X is symmetric. Moreover, we do have (and we will prove this at the end of this proof) that the set $\mathcal{D} = \{x^k,\, k \in \mathbb{N}_0\}$ is a total set of quasi-analitic

vectors for X. Then, by Theorem 1.5.9, X is essentially self-adjoint. Moreover, by Theorem 1.5.4, the closure $\overline{X}$ is the only self-adjoint extension of X.
By spectral theorem (see Corollary B.2.2), then there exists a unique non-negative measure $\mu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\langle 1, \overline{X}^\alpha 1\rangle = \int x^\alpha \mu(dx), \quad \forall \alpha \in \mathbb{N}_0.$$

Note that $1 \in \mathcal{D}(X)$ and is such that $\overline{X}^\alpha 1 \in \mathcal{D}(\overline{X})$ for any $\alpha \in \mathbb{N}_0$.
In other words, we have that y is realized by μ, in fact

$$\int x^\alpha \mu(dx) = \langle 1, \overline{X}^\alpha 1\rangle = \langle 1, X^\alpha 1\rangle = \langle 1, \underbrace{X \cdots X}_{\alpha-\text{times}} \cdot 1\rangle = L_y(x^\alpha) = y_\alpha, \quad \forall \alpha \in \mathbb{N}_0.$$

It remains to prove that the set of powers x^k, $k \in \mathbb{N}_0$, is a total set of quasi-analytic vectors for X.
W.l.o.g. let us suppose that $y_0 = 1$. Moreover, note that the sequence $(y_{2\alpha})_{\alpha\in\mathbb{N}_0}$ is log-convex (see Remark A.0.20).
Since for any $\alpha \in \mathbb{N}$

$$||X^\alpha x^k||^2 = \langle X^\alpha x^k, X^\alpha x^k\rangle = L_y(x^{\alpha+k}x^{\alpha+k}) = L_y(x^{2\alpha+2k}) = y_{2\alpha+2k},$$

we have that

$$||X^\alpha x^k|| = (y_{2\alpha+2k})^{\frac{1}{2}}, \quad \forall k \in \mathbb{N}_0.$$

Hence,

$$\sum_{\alpha=1}^{\infty} ||X^\alpha x^k||^{-\frac{1}{\alpha}} = \sum_{\alpha=1}^{\infty} (y_{2\alpha+2k})^{-\frac{1}{2\alpha}}, \quad \forall k \in \mathbb{N}_0. \tag{1.28}$$

In (1.28), the left-hand side series diverges because, by Theorem A.0.30, Carleman's condition implies that the series on the right-hand side diverges too.
To sum up, we have shown that

$$\{x^k |\ k \in \mathbb{N}_0\} \subset \mathcal{D}^{\mathrm{qa}}(X) \subseteq \mathbb{R}[x]/N.$$

Since $Span\{x^k |\ k \in \mathbb{N}_0\} = \mathbb{R}[x]$ we have that $\{x^k |\ k \in \mathbb{N}_0\}$ is total in $\mathcal{H}_y$.

□

Note that, although the uniqueness of the realizing measure could have been directly derived by using Theorem 1.3.8, we gave the previous alternative proof to be used as a model scheme for the analogous result in higher dimension (see Chapter 2).

Chapter 2

The multi-dimensional power moment problem

In this chapter we are going to show and review some aspects of the moment problem extended to higher dimension. We will mainly focus on the sufficient conditions for a multi-sequence to be determinate. This will be done using the operator-theoretical approach and the results of the classical moment problem contained in the previous chapter.
In particular, we present the proof due to Schmüdgen ([68]) of a conjecture of Berg and Maserick (see [10, p. 495] and [9, p. 119]) about the moment problem on K compact basic semi-algebraic set of $\mathbb{R}^d$. Furthermore, we show how a similar result has been provided by Lasserre (see [46]) for the case when K is a basic semi-algebraic set *not* necessarily compact.

2.1 Preliminaries and statement of the problem

Let $\mathbb{R}[\mathbf{x}]$ denote the ring of all real polynomials in the variable $\mathbf{x} := (x_1, \ldots, x_d)$ in $\mathbb{R}^d$, whereas $\Sigma[\mathbf{x}]$ denotes its subset of sums of squares (s.o.s.) polynomials. For every $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ (the set of the d-tuples of non-negative integers), let us introduce the following notations, $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ (where x_j^0 is understood to be 1) and $|\alpha| := \alpha_1 + \cdots + \alpha_d$. For an arbitrary set $K \subseteq \mathbb{R}^d$, $\mathbb{R}^+_K[\mathbf{x}]$ denotes the convex cone of polynomials which are non-negative on K. As usual, we will write $\mathbb{R}^+[\mathbf{x}]$ instead of $\mathbb{R}^+_{\mathbb{R}^d}[\mathbf{x}]$. A polynomial $p \in \mathbb{R}[\mathbf{x}]$, considered as a function $\mathbb{R}^d \to \mathbb{R}$, is written as

$$p(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}_0^d} p_\alpha \mathbf{x}^\alpha, \tag{2.1}$$

(with $p_\alpha \neq 0$ for finitely many α).

Most of the definitions and theorems we made use of in the previous chapter continue to hold also in higher dimensions. We will directly refer to them whenever we need throughout this section without rewriting the statement and the proof for the multi-dimensional case.
Nevertheless, it is worth to rewrite the following basic definitions.

Let $K \subseteq \mathbb{R}^d$ be *closed*.

Definition 2.1.1 (Moments on K)**.**
Let μ be a non-negative Borel measure μ on $\mathbb{R}^d$ with support contained in K. The number

$$\int_K \mathbf{x}^\alpha \, \mu(d\mathbf{x}), \quad \alpha \in \mathbb{N}_0^d,$$

*is called the $\alpha^{th}-$*moment *of μ on K.*
Explicitly, the number

$$\int_K x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} \mu(dx_1, dx_2, \ldots, dx_d), \quad \alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{N}_0,$$

is the $(\alpha_1, \alpha_2, \ldots, \alpha_d)-th$ moment of μ.

With the same notation of the previous chapter, given $\mu \in \mathcal{M}^*(K)$ (where this time μ is a measure on $\mathcal{B}(\mathbb{R}^d)$, $\mathbf{x} \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$) we are always able to compute the multi-sequence of its moments on K

$$\left(\int_K \mathbf{x}^\alpha \, \mu(d\mathbf{x}) \right)_{\alpha \in \mathbb{N}_0^d},$$

which is called K-*moment multi-sequence* (or d-sequence) of μ.
The next example will help us to understand better the previous definition.

Example 2.1.2.
Let $K \subseteq \mathbb{R}^2$ and $\mu \in \mathcal{M}^*(\mathbb{R}^2)$ supported on K. The K-moment 2-sequence of μ is

$$\begin{bmatrix} \int_K x_1^0 x_2^0 \mu(dx_1, dx_2) & \int_K x_1^0 x_2^1 \mu(dx_1, dx_2) & \int_K x_1^0 x_2^2 \mu(dx_1, dx_2) & \cdots \\ \int_K x_1^1 x_2^0 \mu(dx_1, dx_2) & \int_K x_1^1 x_2^1 \mu(dx_1, dx_2) & \int_K x_1^1 x_2^2 \mu(dx_1, dx_2) & \cdots \\ \int_K x_1^2 x_2^0 \mu(dx_1, dx_2) & \int_K x_1^2 x_2^1 \mu(dx_1, dx_2) & \int_K x_1^2 x_2^2 \mu(dx_1, dx_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For example, the numbers

$$\int_K x_1^2 \mu(dx_1, dx_2), \quad \int_K x_2^2 \mu(dx_1, dx_2) \quad \text{and} \quad \int_K x_1 x_2^2 \mu(dx_1, dx_2)$$

are the $(2,0)-th$, *the* $(0,2)-th$ *and the* $(1,2)-th$ *moments of* μ, *respectively.*

The multi-dimensional moment problem is instead the inverse problem.

Definition 2.1.3 (Moment problem on K)**.**
Given an infinite multi-sequence of real numbers $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$, *find* $\mu \in \mathcal{M}^*(K)$ *such that* y_α *is the* $\alpha^{th}-$*moment of* μ *on* K, *i.e.*

$$y_\alpha = \int_K \mathbf{x}^\alpha \, \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}_0^d. \tag{2.2}$$

If such a measure exists we say that the multi-sequence y *is* realized *by* μ, *or that* y *has a* representing *(or* realizing*) measure* μ, *on* K.
If the representing measure is unique we say that μ *is* determinate *or that the moment problem has a* unique solution.

Example 2.1.4 (Moment problem on $K \subseteq \mathbb{R}^2$)**.**
Let $K \subseteq \mathbb{R}^2$. *Given a 2-sequence (matrix) of real numbers*

$$y = (y_\alpha)_{\alpha \in \mathbb{N}_0^2} = \begin{bmatrix} y_{(0,0)} & y_{(0,1)} & y_{(0,2)} & \cdots \\ y_{(1,0)} & y_{(1,1)} & y_{(1,2)} & \cdots \\ y_{(2,0)} & y_{(2,1)} & y_{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

find a non-negative measure $\mu \in \mathcal{M}^*(K)$ *such that*

$$y_{(\alpha_1,\alpha_2)} = \int_K x_1^{\alpha_1} x_2^{\alpha_2} \mu(dx_1, dx_2), \quad \forall \alpha_1, \alpha_2 \in \mathbb{N}_0,$$

i.e. such that $y_{(\alpha_1,\alpha_2)}$ *is the* $(\alpha_1, \alpha_2)-th$ *moment of* μ.

Given $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ we define the linear Riesz's functional L_y on $\mathbb{R}[\mathbf{x}]$ as

$$L_y(\mathbf{x}^\alpha) := y_\alpha, \quad \alpha \in \mathbb{N}_0^d. \tag{2.3}$$

For a polynomial as in (2.1), by linearity we have

$$L_y(p) = \sum_{\alpha \in \mathbb{N}_0^d} p_\alpha y_\alpha.$$

In the following we are going to make use several times of the term $L_y(x_i^{2k})$, with $i = 1, \ldots, d$ and $k \in \mathbb{N}_0$, which is understood to be $L_y(x_1^0 \cdots x_i^{2k} \cdots x_d^0)$, i.e. $y_{(0,\ldots,2k,\ldots,0)}$ with $2k$ at the i-th place.

Remark 2.1.5.
To a multi-sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ is associated an infinite real symmetric matrix $M(y)$, called moment matrix*. Its truncated version $M_r(y)$, with $r \in \mathbb{N}_0$, is defined as the submatrix of $M(y)$ whose rows and columns are indexed in $\mathbb{N}_{\leq r}^d := \{\alpha \in \mathbb{N}_0^d : |\alpha| \leq r\}$, i.e.*

$$M_r(y)(\alpha, \beta) := L_y(\mathbf{x}^\alpha \mathbf{x}^\beta) = y_{\alpha+\beta}, \quad \forall \alpha, \beta \in \mathbb{N}_{\leq r}^d.$$

Then, the condition of positive semidefiniteness of y can be given in terms of moment matrices. Namely,

$$\begin{aligned} \Big(L_y(h^2) \geq 0, \quad \forall h \in \mathbb{R}[\mathbf{x}]\Big) &\iff \Big(M(y) \succeq 0\Big) \\ &\iff \Big(M_r(y) \succeq 0, \quad \forall r \in \mathbb{N}_0\Big). \end{aligned}$$

Let us notice that when $d = 1$ the moment matrix coincides with the Hankel matrix defined in Remark 1.3.6. In fact, given $y = (y_0, y_1, y_2, y_3, y_4 \ldots)$ we have

$$M_0(y) = [y_0], \quad M_1(y) = \begin{bmatrix} y_0 & y_1 \\ y_1 & y_2 \end{bmatrix}, \quad M_2(y) = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix}, \quad \ldots.$$

When $d = 2$ the truncated moment matrices of

$$y = \begin{bmatrix} y_{(0,0)} & y_{(0,1)} & y_{(0,2)} & \cdots \\ y_{(1,0)} & y_{(1,1)} & y_{(1,2)} & \cdots \\ y_{(2,0)} & y_{(2,1)} & y_{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

are

$$M_0(y) = [\underbrace{y_{(0,0)}}_{(0,0)(0,0)}], \quad M_1(y) = \begin{bmatrix} \underbrace{y_{(0,0)}}_{(0,0)(0,0)} & \underbrace{y_{(1,0)}}_{(0,0)(1,0)} & \underbrace{y_{(0,1)}}_{(0,0)(0,1)} \\ \underbrace{y_{(1,0)}}_{(1,0)(0,0)} & \underbrace{y_{(2,0)}}_{(1,0)(1,0)} & \underbrace{y_{(1,1)}}_{(1,0)(0,1)} \\ \underbrace{y_{(0,1)}}_{(0,1)(0,0)} & \underbrace{y_{(1,1)}}_{(0,1)(1,0)} & \underbrace{y_{(0,2)}}_{(0,1)(0,1)} \end{bmatrix}, \quad \ldots .$$

For $d \geq 3$ *the moment matrices approach is even more natural because it allows to rearrange a given multi-sequence (an* ∞^d*-hypercube) of numbers in terms of an infinite matrix.*

For $i = 1, \ldots, d$, we denote by E_i the *shift operator* on the set of multi-sequences $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ defined as follows.

$$(E_i y)_\alpha := y_{\alpha + \delta^{(i)}},$$

where $\delta^{(i)} = (0, \ldots, 1, \ldots, 0)$ with 1 at the i-th entry. More generally, for a real polynomial $p(\mathbf{x}) = \sum_{\beta \in \mathbb{N}_0^d} p_\beta \mathbf{x}^\beta$, with $p_\beta \neq 0$ for finitely many β, $p(E)$ is the polynomial shift operator $p(E) := \sum_{\beta \in \mathbb{N}_0^d} p_\beta E^\beta$, where $E^\beta = E_1^{\beta_1} \cdots E_d^{\beta_d}$, i.e.

$$(p(E)y)_\alpha = \sum_{\beta \in \mathbb{N}_0^d} p_\beta y_{\alpha+\beta}, \quad \alpha \in \mathbb{N}_0^d. \tag{2.4}$$

2.2 Existence and uniqueness of the realizing measure

2.2.1 Riesz-Haviland's solution

Theorem 1.3.1 was subsequently extended to higher dimensions by Haviland.

Theorem 2.2.1 (Riesz-Haviland, [32])**.**
A multi-sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ *has a representing non-negative Borel measure* μ *supported on* K *if* L_y *is non-negative for all non-negative polynomials on* K*, i.e.*

$$\left(\exists\, \mu \in \mathcal{M}^*(K) \ s.t.\ y_\alpha = \int_K \mathbf{x}^\alpha \mu(d\mathbf{x}),\ \forall \alpha \in \mathbb{N}_0^d\right) \Longleftarrow \left(L_y(p) \geq 0,\ \forall p \in \mathbb{R}_K^+[\mathbf{x}]\right).$$

The proof of the latter theorem is a straightforward generalization of Theorem 1.3.1. The only difference is that in this case we have to deal with the euclidean norm $|\mathbf{x}|$ of the vector $\mathbf{x}$.

As we have already observed in Remark 1.3.2, the difficulty in using Theorem 2.2.1 consists in the challenging problem to characterize all the non-negative polynomials on a set $K \subseteq \mathbb{R}^d$. Nevertheless, also in this case, when K is of a particular form we get positive semidefinite type conditions which can be easily checked by semidefinite programs.

2.2.2 Nussbaum's operator theoretical approach to the existence and uniqueness problem on $\mathbb{R}^d$

The next theorem was stated in the following form by Berg in [9] but it has been already proved by Nussbaum in [56]. It gives sufficient conditions for a positive semidefinite sequence to be a moment sequence. Similar results were obtained also by Shohat and Tamarkin in [72, p. 21], by Devinatz in [20] and by Eskin in [21].

Theorem 2.2.2.
Let $d \geq 2$.
If $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ is a multi-sequence such that

- $L_y(h^2) \geq 0, \quad \forall h \in \mathbb{R}[\mathbf{x}]$,
- $\sum\limits_{k=1}^{\infty} L_y(x_i^{2k})^{-\frac{1}{2k}} = \infty, \quad \forall\, i = 1, \ldots, d$ *(Multi-variate Carleman's condition),*

then there exists a unique non-negative Borel measure μ on $\mathbb{R}^d$ with finite moments of any order which realizes the sequence y.

We will see that, in contrast with Theorem 1.3.9, the condition of positive semidefiniteness of y solely does *not* allow us anymore to prove, with analogue techniques, the existence of a realizing measure on $\mathbb{R}^d$ when $d \geq 2$. In other words, we cannot prove the equivalent of Hamburger's Theorem 1.3.5 for higher dimensions. This is because non-negative polynomials on $\mathbb{R}^d$ are not always s.o.s. (see Remark 1.3.4) and so we cannot pass through Riesz-Haviland's theorem as we used to do in the one-dimensional case.
For this reason, we are going to use the operator-theoretical approach and, in particular, the spectral theorem for more than one self-adjoint operators (see Corollary B.4.5) in which an important role is played by the pairwise strong

commutativity of the involved operators (namely the closure of some X_j's). The strong commutativity of such operators is guaranteed by Theorem 2.2.3 which indeed requires the existence of a total set of quasi-analytic vectors for all X_j's and some analysis of quasi-analytic functions.

For sake of semplicity, we will directly prove Theorem 2.2.2 for the case of $d = 2$ operators.

Proof. (of Theorem 2.2.2 for $d = 2$)
Let L_y be the linear functional on $\mathbb{R}[\mathbf{x}]$ defined in (2.3) with $d = 2$, and let $\mathcal{H}_y$ be the canonical Hilbert space associated with the positive semidefinite sequence y, i.e. $\mathcal{H}_y := \overline{\mathbb{R}[\mathbf{x}]/N}$ where $N := \{h \in \mathbb{R}[\mathbf{x}] : L_y(h^2) = 0\}$ (for more details in one dimension see proof of Theorem 1.5.2). On $\mathcal{H}_y$ the inner product will be denoted again by $\langle \cdot, \cdot \rangle$. For $j = 1, 2$, we introduce the following operators

$$
\begin{aligned}
X_j : \quad \mathbb{R}[\mathbf{x}]/N \quad & \to \mathbb{R}[\mathbf{x}]/N \\
h(x_1, x_2) \quad & \mapsto (X_j h)(x_1, x_2) := x_j\, h(x_1, x_2)\,.
\end{aligned}
$$

Let us note that

- X_1 and X_2 are symmetric.
- If $\mathcal{D} := \{x_1^m x_2^n | m, n \in \mathbb{N}_0\}$ we have that $X_j \mathcal{D} \subset \mathcal{D}$ for $j = 1, 2$.
- $X_1 X_2\, h = X_2 X_1\, h$ for all $h \in \mathcal{D}$.
- $\mathcal{D}$ is total in $\mathcal{H}_y$.
- $\mathcal{D}$ is a set of quasi-analytic vectors for both X_1 and X_2. We will prove this at the end of the proof.

Then, by Theorem 2.2.3, $\overline{X_1}$ and $\overline{X_2}$ are strongly commuting self-adjoint operators. This also means that X_1 and X_2 are essentially self-adjoint and so $\overline{X_1}$ and $\overline{X_2}$ are the only possible extensions (see Definition 1.5.3 and Theorem 1.5.4).
By spectral theorem for more operators (see Corollary B.4.5), there exists a unique non-negative measure $\mu \in \mathcal{M}^*(\mathbb{R}^2)$ such that

$$
\langle 1, \underbrace{\overline{X_1} \cdots \overline{X_1}}_{\alpha_1-\text{times}} \underbrace{\overline{X_2} \cdots \overline{X_2}}_{\alpha_2-\text{times}} \cdot 1 \rangle = \int_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} \mu(dx_1, dx_2), \quad \forall\, (\alpha_1, \alpha_2) \in \mathbb{N}_0^2.
$$

Note that 1 is such that the hypotheses of Corollary B.4.5 are satisfied.

In other words, we have that y is realized by μ on $\mathbb{R}^2$, in fact

$$\int_{\mathbb{R}^2} \mathbf{x}^\alpha \mu(d\mathbf{x}) = \langle 1, \overline{\mathbf{X}}^\alpha \cdot 1\rangle = \langle 1, \mathbf{X}^\alpha \cdot 1\rangle = L_y(\mathbf{x}^\alpha) = y_\alpha, \quad \forall \alpha \in \mathbb{N}_0^2.$$

It remains to prove that $\mathcal{D}$ is a set of quasi-analytic vectors for both X_1 and X_2. W.l.o.g. let us suppose that $y_{(0,0)} = 1$. Moreover, note that the sequences $(y_{(2\alpha_1,0)})_{\alpha_1\in\mathbb{N}_0}$ and $(y_{(0,2\alpha_2)})_{\alpha_2\in\mathbb{N}_0}$ are log-convex (see Remark A.0.20).
Let us recall that a polynomial h is a quasi-analytic vector for both X_1 and X_2 if and only if $\sum_{k=1}^{\infty} ||{X_1}^k h||^{-\frac{1}{k}} = \infty$ and $\sum_{k=1}^{\infty} ||{X_2}^k h||^{-\frac{1}{k}} = \infty$.
Let us first prove that the powers $x_1^m x_2^n$, with $m, n \in \mathbb{N}_0$, are quasi-analytic vectors for the operator X_1.
In fact, for $m, n \in \mathbb{N}_0$, by using Cauchy-Schwarz's inequality and the fact that $||x_1^{2k+2m}||^2 = L_y(x_1^{2(2k+2m)}) = y_{(2(2k+2m),0)}$ and $||x_2^{2n}||^2 = L_y(x_2^{2(2n)}) = y_{(0,2(2n))}$,

$$||{X_1}^k x_1^m x_2^n||^2 = \langle x_1^{2k+2m}, x_2^{2n}\rangle \leq ||x_1^{2k+2m}|| \cdot ||x_2^{2n}|| = \left(y_{(2(2k+2m),0)}\right)^{\frac{1}{2}} \left(y_{(0,2(2n))}\right)^{\frac{1}{2}}$$

and hence

$$\sum_{k=1}^{\infty} ||{X_1}^k x_1^m x_2^n||^{-\frac{1}{k}} \geq \sum_{k=1}^{\infty} \left(\left(y_{(2(2k+2m),0)} \cdot y_{(0,2(2n))}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2k}}.$$

In the latter, the left-hand side series diverges because, by Theorem A.0.30 together with Lemma A.0.26 and Lemma A.0.28, the multi-variate Carleman condition implies that the series on the right-hand side diverges too.
Similarly, we get that

$$\sum_{k=1}^{\infty} ||{X_2}^k x_1^m x_2^n||^{-\frac{1}{k}} = \infty.$$

$\square$

Theorem 2.2.3 (Nussbaum, [56] (see also [69] p. 153)).
Let A and B be two symmetric operators in a Hilbert $\mathcal{H}$ and $\mathcal{D}$ a set of vectors in $\mathcal{H}$ which are quasi-analytical for both A and B and such that $A\mathcal{D} \subset \mathcal{D}$, $B\mathcal{D} \subset \mathcal{D}$, $AB\phi = BA\phi$ for all $\phi \in \mathcal{D}$. If the set $\mathcal{D}$ is total in $\mathcal{H}$, namely

$$\overline{Span\mathcal{D}} = \mathcal{H},$$

then $\overline{A}$ and $\overline{B}$ are strongly commuting self-adjoint operators.

Remark 2.2.4.
Note that the assumptions $A\mathcal{D} \subset \mathcal{D}$, $B\mathcal{D} \subset \mathcal{D}$, $AB\phi = BA\phi$ for all $\phi \in \mathcal{D}$ also

imply that, for any $m, n \in \mathbb{N}_0$, $A^m B^n \phi = B^n A^m \phi$ for all $\phi \in \mathcal{D}$. To prove this, let us proceed by induction on n.

Let $m \in \mathbb{N}_0$ be fixed and suppose that, for all $j \leq n-1$, $A^m B^j \psi = B^j A^m \psi$ for all $\psi \in \mathcal{D}$. Then, since $\phi, B\phi \in \mathcal{D}$,

$$\begin{aligned} A^m B^n \phi = A^m B^{n-1}(B\phi) &= B^{n-1} A^m (B\phi) \\ &= B^{n-1}(A^m B\phi) = B^{n-1}(BA^m \phi) = B^n A^m \phi. \end{aligned}$$

Proof. (of Theorem 2.2.3)

Let us first note that since $\mathcal{D} \subset (\mathcal{D}^{\mathrm{qa}}(A) \cap \mathcal{D}^{\mathrm{qa}}(B))$ we also have that $\mathcal{D} \subset \mathcal{D}^{\mathrm{qa}}(A)$ and $\mathcal{D} \subset \mathcal{D}^{\mathrm{qa}}(B)$. Since $\overline{Span\, \mathcal{D}} = \mathcal{H}$, by Theorem 1.5.9, the operators A and B are essentially self-adjoint, i.e. $\overline{A}$ and $\overline{B}$ are self-adjoint. Let us show that they also strongly commute (see Definition B.4.3). For this aim, let us consider the complexification of the real Hilbert $\mathcal{H}$ which we call $\mathcal{H}$ again. Moreover, given $\phi \in \mathcal{D}$ (note that $\phi \in \mathcal{D}^\infty(A)$ and $\phi \in \mathcal{D}^\infty(B)$), let us consider the functions

$$\begin{aligned} f_{\overline{A}}: \ & \mathbb{R} && \to \mathcal{H} \\ & a && \mapsto f_{\overline{A}}(a) := e^{ia\overline{A}}\phi \end{aligned}$$

and

$$\begin{aligned} f_{\overline{B}}: \ & \mathbb{R} && \to \mathcal{H} \\ & b && \mapsto f_{\overline{B}}(b) := e^{ib\overline{B}}\phi \end{aligned}$$

which, by Proposition B.5.1, are $\mathcal{C}^\infty(\mathbb{R})$-maps. Let us also define and consider

$$\begin{aligned} F_1: \ & \mathbb{R}^2 && \to \mathbb{C} \\ & (a,b) && \mapsto \langle f_{\overline{B}}(b)\phi, \overline{f_{\overline{A}}(a)}\phi \rangle = \langle e^{ib\overline{B}}\phi, e^{-ia\overline{A}}\phi \rangle \end{aligned}$$

and

$$\begin{aligned} F_2: \ & \mathbb{R}^2 && \to \mathbb{C} \\ & (a,b) && \mapsto \langle e^{ia\overline{A}}\phi, e^{-ib\overline{B}}\phi \rangle\,. \end{aligned}$$

The functions F_1 and F_2 are $\mathcal{C}^\infty(\mathbb{R}^2)$-maps since $f_{\overline{A}}, f_{\overline{B}} \in \mathcal{C}^\infty(\mathbb{R})$. Moreover, for all $\alpha_1, \alpha_2 \in \mathbb{N}_0$

$$\begin{aligned}
\frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_1(a,b) &= \frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \left[\langle e^{ib\overline{B}}\phi, (-i)^{\alpha_1} \overline{A}^{\alpha_1} e^{-ia\overline{A}}\phi\rangle \right] \\
&= \langle i^{\alpha_2} \overline{B}^{\alpha_2} e^{ib\overline{B}}\phi, (-i)^{\alpha_1} \overline{A}^{\alpha_1} e^{-ia\overline{A}}\phi\rangle \\
&= i^{\alpha_2+\alpha_1} \langle \overline{B}^{\alpha_2} e^{ib\overline{B}}\phi, \overline{A}^{\alpha_1} e^{-ia\overline{A}}\phi\rangle
\end{aligned}$$

and, similarly,

$$\frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_2(a,b) = i^{\alpha_2+\alpha_1} \langle \overline{A}^{\alpha_1} e^{ia\overline{A}}\phi, \overline{B}^{\alpha_2} e^{-ib\overline{B}}\phi\rangle.$$

If we evaluate the derivates of F_1 and F_2 in $(a,b) = (0,0)$ we get that

$$\begin{aligned}
\frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_1(0,0) &= i^{\alpha_2+\alpha_1} \langle \overline{B}^{\alpha_2}\phi, \overline{A}^{\alpha_1}\phi\rangle, \\
\frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_2(0,0) &= i^{\alpha_2+\alpha_1} \langle \overline{A}^{\alpha_1}\phi, \overline{B}^{\alpha_2}\phi\rangle = i^{\alpha_2+\alpha_1} \langle \overline{B}^{\alpha_2}\phi, \overline{A}^{\alpha_1}\phi\rangle,
\end{aligned}$$

where in the last equality we have made use of the self-adjointness of the operators $\overline{A}$ and $\overline{B}$ and the fact that their powers also commute on $\mathcal{D}$ because so do A and B (see Remark 2.2.4 and remember that $A = \overline{A}$ and $B = \overline{B}$ on $\mathcal{D}$).
Since the derivates of F_1 and F_2 in $(0,0)$ are equal, we get that

$$\frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} \left(F_1 - F_2\right)(0,0) = 0. \tag{2.5}$$

Moreover, we have that

$$\begin{aligned}
&\left| \frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} \left(F_1 - F_2\right)(a,b) \right| \\
&= \left| \frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_1(a,b) - \frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_2(a,b) \right| \\
&\leq \left| \frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_1(a,b) \right| + \left| \frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} F_2(a,b) \right| \\
&= \left| \langle \overline{B}^{\alpha_2} e^{ib\overline{B}}\phi, \overline{A}^{\alpha_1} e^{-ia\overline{A}}\phi\rangle \right| + \left| \langle \overline{A}^{\alpha_1} e^{ia\overline{A}}\phi, \overline{B}^{\alpha_2} e^{-ib\overline{B}}\phi\rangle \right| \\
&\leq ||\overline{B}^{\alpha_2} e^{ib\overline{B}}\phi|| \cdot ||\overline{A}^{\alpha_1} e^{-ia\overline{A}}\phi|| + ||\overline{A}^{\alpha_1} e^{ia\overline{A}}\phi|| \cdot ||\overline{B}^{\alpha_2} e^{-ib\overline{B}}\phi|| \qquad (2.6) \\
&= 2\, ||\overline{B}^{\alpha_2}\phi|| \cdot ||\overline{A}^{\alpha_1}\phi||. \qquad (2.7)
\end{aligned}$$

Let us observe that to get (2.6) we made use of Cauchy-Schwarz's inequality and for (2.7) the fact that whenever C is a self-adjoint operator, $e^{\pm icC}$ is unitary, i.e.

$e^{\pm icC}$ is a bounded operator such that $||e^{\pm icC}v|| = ||v||$. If we now set

$$y_{(1,m)} := ||\overline{A}^m\phi|| \quad \text{and} \quad y_{(2,m)} := ||\overline{B}^m\phi||,$$

we can re-write the relation (2.7) in the following more suitable way

$$\left|\frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}}\frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}}\left(F_1 - F_2\right)(a, b)\right| \leq 2y_{(2,\alpha_2)}y_{(1,\alpha_1)}.$$

The previous relation and (2.5) imply, by Theorem A.0.32, that $F_1 - F_2 \equiv 0$ on $\mathbb{R}^2$ (note that, by symmetry of the operators, the sequences $\left(y_{(1,m)}\right)_{m\in\mathbb{N}_0}$ and $\left(y_{(2,m)}\right)_{m\in\mathbb{N}_0}$ are log-convex, and w.l.o.g. we can suppose $||\phi|| = 1$).
Since $F_1 \equiv F_2$ on $\mathbb{R}^2$ we have

$$\langle e^{ib\overline{B}}\phi, e^{-ia\overline{A}}\phi\rangle = \langle e^{ia\overline{A}}\phi, e^{-ib\overline{B}}\phi\rangle, \quad \forall a, b \in \mathbb{R}, \forall \phi \in \mathcal{D},$$

which also holds for all $\phi \in \mathcal{H}$ since $\mathcal{D}$ is total in $\mathcal{H}$ and the operators $e^{ia\overline{A}}$ and $e^{ia\overline{B}}$ are continuous. The latter equality then becomes

$$\langle e^{ia\overline{A}}e^{ib\overline{B}}\phi, \phi\rangle = \langle e^{ib\overline{B}}e^{ia\overline{A}}\phi, \phi\rangle, \quad \forall a, b \in \mathbb{R}, \forall \phi \in \mathcal{H},$$

or, equivalently,

$$\langle \left(e^{ia\overline{A}}e^{ib\overline{B}} - e^{ib\overline{B}}e^{ia\overline{A}}\right)\phi, \phi\rangle = 0, \quad \forall a, b \in \mathbb{R}, \forall \phi \in \mathcal{H}.$$

By polarization identity[1], we get

$$\langle \left(e^{ia\overline{A}}e^{ib\overline{B}} - e^{ib\overline{B}}e^{ia\overline{A}}\right)\psi_1, \psi_2\rangle = 0, \quad \forall a, b \in \mathbb{R}, \forall \psi_1, \psi_2 \in \mathcal{H}. \tag{2.8}$$

If in (2.8) we put $\psi_2 = \left(e^{ia\overline{A}}e^{ib\overline{B}} - e^{ib\overline{B}}e^{ia\overline{A}}\right)\psi_1$ we get that

$$\left|\left|\left(e^{ia\overline{A}}e^{ib\overline{B}} - e^{ib\overline{B}}e^{ia\overline{A}}\right)\psi_1\right|\right|^2 = 0, \quad \forall \psi_1 \in \mathcal{H}.$$

Then necessarily

$$\left(e^{ia\overline{A}}e^{ib\overline{B}} - e^{ib\overline{B}}e^{ia\overline{A}}\right)\psi_1 = 0, \quad \forall \psi_1 \in \mathcal{H},$$

[1] If T is an operator on a complex Hilbert space H and $x, y \in H$, then

$$\langle x, Ty\rangle = \frac{1}{4}\langle x+y, T(x+y)\rangle - \frac{1}{4}\langle x-y, T(x-y)\rangle - \frac{i}{4}\langle x+iy, T(x+iy)\rangle + \frac{i}{4}\langle x-iy, T(x-iy)\rangle.$$

and, as consequence,

$$e^{ia\overline{A}}e^{ib\overline{B}} = e^{ib\overline{B}}e^{ia\overline{A}}$$

for all $a, b \in \mathbb{R}$. In other words, $\overline{A}, \overline{B}$ strongly commute.

$\square$

From Theorem 2.2.2 the following holds.

Corollary 2.2.5.
Let $\mu, \eta \in \mathcal{M}^*(\mathbb{R}^d)$ have the same moments y_α, i.e.

$$\int \mathbf{x}^\alpha \mu(d\mathbf{x}) = y_\alpha = \int \mathbf{x}^\alpha \eta(d\mathbf{x}), \qquad \forall \alpha \in \mathbb{N}_0^d.$$

If $(y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ satisfies multi-variate Carleman's condition then $\mu = \eta$.

2.2.3 Schmüdgen's and Lasserre's solution on basic semi-algebraic sets

Let $g_j \in \mathbb{R}[\mathbf{x}]$, $j = 0, 1, \dots, m$, with $g_0(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{R}^d$, and let $K \subseteq \mathbb{R}^d$ be the basic closed semi-algebraic set given by

$$K := \bigcap_{j=1}^{m} \{\mathbf{x} \in \mathbb{R}^d : g_j(\mathbf{x}) \geq 0\}.$$

Let us introduce also the following subsets of the ring $\mathbb{R}[\mathbf{x}]$.
The *quadratic module*

$$Q_{g_0,\dots,g_m} := \left\{ \sum_{j=0}^{m} \sigma_j g_j : \sigma_j \in \Sigma[\mathbf{x}], j = 0, \dots, m \right\} \tag{2.9}$$

and the *preordering set*

$$P_{g_0,\dots,g_m} := \left\{ \sum_{J \subseteq \{1,\dots,m\}} \sigma_J g_J : \sigma_J \in \Sigma[\mathbf{x}], J \subseteq \{1, \dots, m\} \right\}, \tag{2.10}$$

where for every $J \subseteq \{1, \dots, m\}$ we set $g_J := \prod_{k \in J} g_k$, with the convention $g_\emptyset := 1$.

Note that the preordering set is closed under the sum and the multiplication of its elements whereas the quadratic module is closed only under the sum.

A first generalization of Theorem 1.3.16 to the multi-dimensional case is given by the following theorem.

Theorem 2.2.6 (Schmüdgen, [68]).
A multi-sequence $y = (y_\alpha)_{\alpha\in\mathbb{N}_0^d}$ has a unique representing non-negative Borel measure μ on K if y and $(g_{j_1}\dots g_{j_n})(E)y$, for all possible choices $j_1,\dots,j_n$ of pairwise different numbers from the set $\{1,\dots,m\}$, are positive semidefinite and K is compact, i.e. if

(a) $L_y(h^2) \geq 0,\ L_y(h^2 g_{j_1}\cdots g_{j_n}) \geq 0,\ \forall h \in \mathbb{R}[\mathbf{x}], \forall\{j_1,\dots,j_n\}\subset\{1,\dots,m\}$ *with* $j_i \neq j_k$ *for* $i \neq k$,

(b) K *is compact,*

then $\exists!\, \mu \in \mathcal{M}^*(K)$ *such that*

$$y_\alpha = \int_K \mathbf{x}^\alpha\, \mu(d\mathbf{x}), \quad \forall\alpha \in \mathbb{N}_0^d.$$

Note that the conditions in *(a)* can be replaced with the condition $L_y(p) \geq 0$ for all $p \in P_{g_0,\dots,g_m}$.

Proof. (of Theorem 2.2.6)
Let L_y be the linear functional on $\mathbb{R}[\mathbf{x}]$ defined in (2.3), and let $\mathcal{H}_y$ be the canonical Hilbert space associated with the positive semidefinite sequence y, i.e. $\mathcal{H}_y = \overline{\mathbb{R}[\mathbf{x}]/N}$ where $N := \{h \in \mathbb{R}[\mathbf{x}] : L_y(h^2) = 0\}$ (for more details in one dimension see proof of Theorem 1.5.2). We denote by $||\cdot||$ the norm on $\mathcal{H}_y$ given by the product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}_y$ (it will be clear from the context if the same symbol is used for the norm of the operators).
For $j = 1,\dots,d$, let us introduce the following operator (we work directly with the representing elements of a class)

$$\begin{aligned} X_j :\ \mathbb{R}[\mathbf{x}]/N &\to \mathbb{R}[\mathbf{x}]/N \\ h &\mapsto X_j h := x_j h. \end{aligned}$$

Since K is compact, and so bounded, there exists a positive constant ϱ such that $\varrho^2 - |\mathbf{x}|^2 > 0$ for all $\mathbf{x} \in K$. For the multiplication operator X_j it is shown in [68] that

$$||x_j h|| \leq \varrho||h||, \quad \forall h \in \mathbb{R}[\mathbf{x}]. \tag{2.11}$$

To prove (2.11) Schmüdgen makes use of the Positivstellensatz ([12, Corollaire 4.4.3, (ii)], cf. [74]) for the polynomial $\varrho^2 - |\mathbf{x}|^2$ (strictly positive on K), namely the fact that there exist two polynomials $G, H \in P_{g_0,\dots,g_m}$ such that $(\varrho^2 - |\mathbf{x}|^2)G = 1 + H$.

The operator X_j is bounded and symmetric. In fact,

$$||X_j|| = \sup_{\substack{h\in\mathbb{R}[\mathbf{x}]/N\\ h\neq 0}} \frac{||X_j h||}{||h||} = \sup_{\substack{h\in\mathbb{R}[\mathbf{x}]/N\\ h\neq 0}} \frac{||x_j h||}{||h||} \leq \varrho \sup_{\substack{h\in\mathbb{R}[\mathbf{x}]/N\\ h\neq 0}} \frac{||h||}{||h||} = \varrho < \infty,$$

and

$$\langle X_j h_1, h_2\rangle = L_y(x_j h_1 h_2) = L_y(h_1 x_j h_2) = \langle h_1, X_j h_2\rangle,$$

for all $h_1, h_2 \in \mathbb{R}[\mathbf{x}]/N$.
Moreover, it is easy to see that the operators X_j, $j = 1,\dots,d$, pairwise commute on $\mathbb{R}[\mathbf{x}]/N$, i.e. $X_{j_1}X_{j_2} = X_{j_2}X_{j_1}$ for $j_1 \neq j_2$. Since X_j is bounded, by the B.L.T. Theorem (see [63, Vol. I, p. 9]) we have that X_j has a unique bounded extension to $\mathcal{H}_y$, namely the closure $\overline{X_j}$ which is self-adjoint. The extended operators $\overline{X_j}$, $j = 1,\dots,d$, pairwise commute as well as X_j, $j = 1,\dots,d$. Then, by spectral theorem (see Corollary B.4.2), there exists a non-negative measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ with $supp(\mu) \subseteq \sigma(\overline{X_1},\dots,\overline{X_d}) \subseteq B_{||\overline{X_1}||}(0) \times \cdots \times B_{||\overline{X_d}||}(0) \subseteq [-\varrho,\varrho] \times \cdots \times [-\varrho,\varrho] =: Q$ such that

$$\langle 1, \underbrace{\overline{X_1}\cdots\overline{X_1}}_{\alpha_1-\text{times}}\cdots\underbrace{\overline{X_d}\cdots\overline{X_d}}_{\alpha_d-\text{times}}\cdot 1\rangle = \int_Q x_1^{\alpha_1}\cdots x_d^{\alpha_d}\mu(dx_1,\dots,dx_d). \tag{2.12}$$

In other words, we have that y is realized by μ on Q, in fact

$$\int_Q \mathbf{x}^\alpha \mu(d\mathbf{x}) = \langle 1, \overline{\mathbf{X}}^\alpha \cdot 1\rangle = \langle 1, \mathbf{X}^\alpha \cdot 1\rangle = L_y(\mathbf{x}^\alpha) = y_\alpha, \quad \forall \alpha \in \mathbb{N}_0^d.$$

The latter also means that the functional L_y has a representation as integral. In particular, by the positive semidefinitess of $(g_j(E)y)$ we have that for any $j \in \{1,\dots,m\}$ and for any $h \in \mathbb{R}[\mathbf{x}]$

$$0 \leq L_y(g_j h^2) = \int_Q g_j(\mathbf{x})h^2(\mathbf{x})\mu(d\mathbf{x}).$$

Since Q is compact, by Stone-Weierstrass approximation theorem ([66, Theorem 7.24]), we also have that for any $f \in \mathcal{C}(Q)$

$$0 \leq \int_Q g_j(\mathbf{x})f^2(\mathbf{x})\mu(d\mathbf{x}),$$

or equivalently, for any $f \in \mathcal{C}^+(Q)$

$$0 \leq \int_Q g_j(\mathbf{x}) f(\mathbf{x}) \mu(d\mathbf{x}).$$

If we set $L(f) := \int_Q g_j(\mathbf{x}) f(\mathbf{x}) \mu(d\mathbf{x})$, the integral representation of L is unique by Riesz-Markov's Theorem C.0.5. Then necessarily the signed measure $g_j(\mathbf{x})\mu(d\mathbf{x})$ has to be non-negative. Since the latter condition has to be true for all $j \in \{1, \ldots, m\}$, we can conclude that

$$supp(\mu) \subseteq \{\mathbf{x} \in \mathbb{R}^d : g_j(\mathbf{x}) \geq 0, \text{ for } j = 1, \ldots, m\} := K.$$

$\square$

Remark 2.2.7.
Note that the compactness of K implies a restriction on the growth of the even i-entries of the given sequence y, i.e. $L_y(x_i^{2k}) \leq \varrho^{2k}$. In fact, by (2.11)

$$L_y(x_i^{2k}) = \langle x_i^k, x_i^k \rangle = ||x_i^k||^2 = ||x_i x_i^{k-1}||^2 \leq \varrho^2 ||x_i^{k-1}||^2 \leq \cdots \leq \varrho^{2k}. \tag{2.13}$$

Bounds of this type, and even more general, will frequently appear in the next section. Moreover, by (2.13) *we have that*

$$L_y(x_i^{2k})^{-\frac{1}{2k}} \geq \varrho^{-1}$$

and so

$$\sum_{k=1}^{\infty} L_y(x_i^{2k})^{-\frac{1}{2k}} \geq \sum_{k=1}^{\infty} \frac{1}{\varrho} = \infty,$$

i.e. multi-variate Carleman's condition holds.

Theorem 2.2.6 was soon refined by Putinar for Archimedean quadratic module, i.e. for quadratic modules Q such that $N - |\mathbf{x}|^2 \in Q$ for some $N \in \mathbb{N}$.

Theorem 2.2.8 (Putinar, [62])**.**
A multi-sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ has a unique representing non-negative Borel measure μ on K if K is compact, if $Q_{g_0,\ldots,g_m}$ is Archimedean and if L_y is non-negative on $Q_{g_0,\ldots,g_m}$, i.e.

$$L_y(h^2) \geq 0, \; L_y(h^2 g_j) \geq 0, \quad \forall h \in \mathbb{R}[\mathbf{x}], \; j = 1, \ldots, m.$$

Putinar's theorem is the right equivalent generalization of Theorem 1.3.16 by Berg and Maserick to the multi-dimensional case. In the case of $d = 1$ the assumption of Archimedian quadratic module is omitted because each quadratic module is automatically Archimedian. In fact, by Theorem 2.15 in [43], $Q_{g_0,\dots,g_m}$ is Archimedian if there exists $q \in Q_{g_0,\dots,g_m}$ such that $\{q \geq 0\}$ is compact and in the proof of Theorem 1.3.16 we said that this property can be always proved to be true.

Case of severals polynomials and K non-compact: Lasserre

Lasserre generalized the results of the previous paragraph to closed basic semi-algebraic sets not necessarily compact assuming a bound on $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ depending on a weight multi-sequence $w = (w_\alpha)_{\alpha \in \mathbb{N}_0^d}$, where $w_\alpha := (2\lceil \frac{|\alpha|}{2} \rceil)!$

Theorem 2.2.9 (Lasserre, [46])**.**
A multi-sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ has a representing non-negative Borel measure μ on K if L_y is non-negative on $Q_{g_0,\dots,g_m}$ and $\sup_{\alpha \in \mathbb{N}_0^d} \frac{|y_\alpha|}{w_\alpha} \leq M$ for some $M > 0$, i.e. if

(A) $L_y(h^2) \geq 0$, $L_y(h^2 g_j) \geq 0$, $\quad \forall h \in \mathbb{R}[\mathbf{x}]$, $j = 1, \dots, m$,

(B) $\sup_{\alpha \in \mathbb{N}_0^d} \frac{|y_\alpha|}{w_\alpha} \leq M$ for some $M > 0$,

then $\exists!\, \mu \in \mathcal{M}^*(K)$ such that

$$y_\alpha = \int_K \mathbf{x}^\alpha \, \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}_0^d.$$

Note that the conditions in *(A)* are equivalent to the condition $L_y(p) \geq 0$ for all $p \in Q_{g_0,\dots,g_m}$.

For sake of simplicity, we prove Theorem 2.2.9 in the case of one polynomial. The proof for several polynomials is then a straighforward consequence. This simplification will also help us to understand better the analogies and the dissimilarities with Theorem 1.3.14 due to Berg and Maserick.

For this reason, let us assume that the set K, non compact, is the set where a fixed polynomial g is non-negative.

We will make use of the following lemmas.

Lemma 2.2.10 (Lasserre, [46])**.**
Let $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ such that $\sup_{\alpha \in \mathbb{N}_0^d} \frac{|y_\alpha|}{w_\alpha} \leq M$ for some $M > 0$. Then, for all $i = 1, \dots, d$, and all $k \in \mathbb{N}_0$,

$$L_y(x_i^{2k}) \leq M(2k)!\,.$$

Moreover, y satisfies the multi-variate Carleman condition

$$\sum_{k=1}^{\infty} L_y(x_i^{2k})^{-\frac{1}{2k}} = \infty, \quad i = 1, \ldots, d.$$

Proof.
For any $k \in \mathbb{N}_0$ and $i = 1, \ldots, d$, let us set $y_{2k} = y_{(0,\ldots,2k,\ldots,0)}$ and $w_{2k} = w_{(0,\ldots,2k,\ldots,0)}$ with $2k$ at the i-th entry. Then,

$$\frac{|y_{2k}|}{w_{2k}} \leq \sup_{\alpha \in \mathbb{N}_0^d} \frac{|y_\alpha|}{w_\alpha} \leq M \quad \Longrightarrow \quad |y_{2k}| \leq M w_{2k},$$

and so

$$L_y(x_i^{2k}) = y_{2k} \leq |y_{2k}| \leq M w_{2k} = M \left(2 \left\lceil \frac{2k}{2} \right\rceil\right)! = M(2k)! \ . \tag{2.14}$$

Moreover, y satisfies the multi-variate Carleman condition. In fact, by applying in (2.14) the following bound (given by Stirling's formula)

$$n! \leq n^{\left(n+\frac{1}{2}\right)} e^{1-n}, \quad n \in \mathbb{N},$$

we get that

$$L_y(x_i^{2k})^{\frac{1}{2k}} \leq (M(2k)!)^{\frac{1}{2k}} \leq M^{\frac{1}{2k}} (2k)(2k)^{\frac{1}{4k}} e^{\frac{1}{2k}-1} \leq C \cdot 2k$$

where $k \geq k_0$ is sufficiently large so that $M^{\frac{1}{2k}}(2k)^{\frac{1}{4k}} e^{\frac{1}{2k}-1} \leq C$ with C positive constant. Then,

$$\sum_{k=1}^{\infty} L_y(x_i^{2k})^{-\frac{1}{2k}} \geq \frac{1}{C} \sum_{k=k_0}^{\infty} \frac{1}{2k} = +\infty.$$

$\square$

Similarly, the following can be proved.

Lemma 2.2.11 (Lasserre, [46] Lemma 5.1)**.**
Let μ be a non-negative Borel measure whose sequence of moments $y = (y_\alpha)_{\alpha \in \mathbb{N}_0^d}$ is such that for all $i = 1, \ldots, d$, and all $k \in \mathbb{N}_0$, $L_y(x_i^{2k}) \leq M(2k)!$ for some M. Let $p \in \mathbb{R}[\mathbf{x}]$ be such that $L_y(x_i^{2t} p) \geq 0$ for all $i = 1, \ldots, d$, and all $t \in \mathbb{N}_0$. Then

the sequence $p(E)y$ satisfies the multi-variate Carleman condition

$$\sum_{k=1}^{\infty} L_{p(E)y}(x_i^{2k})^{-\frac{1}{2k}} = \infty, \quad i = 1, \dots, d.$$

In Chapter 4, an extension of the previous lemma will be proved in the infinite dimensional case.

Proof. (of Theorem 2.2.9 for one polynomial)
By Lemma 2.2.10 we have that for all $i = 1, \dots, d$, and all $k \in \mathbb{N}_0$, $L_y(x_i^{2k}) \leq M(2k)!$ for some $M > 0$ and y satisfies the multi-variate Carleman condition. The conditions $L_y(h^2) \geq 0$ for all $h \in \mathbb{R}[\mathbf{x}]$ and $\sum_{k=1}^{\infty} L_y(x_i^{2k})^{-\frac{1}{2k}} = +\infty$ for all $i = 1, \dots, d$, imply, by Theorem 2.2.2, that there exists a unique non-negative measure $\mu \in \mathcal{M}^*(\mathbb{R}^d)$ which realizes y, i.e.

$$y_\alpha = \int \mathbf{x}^\alpha \, \mu(d\mathbf{x}), \quad \alpha \in \mathbb{N}_0^d. \tag{2.15}$$

Moreover, the conditions $L_y(h^2 g) \geq 0$, for all $h \in \mathbb{R}[\mathbf{x}]$, and $L_y(x_i^{2k}) \leq M(2k)!$ (for all $i = 1, \dots, d$, and all $k \in \mathbb{N}_0$) imply, by Lemma 2.2.11, that $g(E)y$ satisfies the multi-variate Carleman condition $\sum_{k=1}^{\infty} L_{g(E)y}(x_i^{2k})^{-\frac{1}{2k}}$. Hence, by Theorem 2.2.2, $g(E)y$ is realizable on $\mathbb{R}^d$, i.e. there exists a unique non-negative measures $\nu \in \mathcal{M}^*(\mathbb{R}^d)$ such that

$$(g(E)y)_\alpha = \int \mathbf{x}^\alpha \, \nu(d\mathbf{x}), \quad \alpha \in \mathbb{N}_0^d.$$

The integral representation of y in (2.15) implies, by Lemma 1.3.13, that we also have

$$(g(E)y)_\alpha = \int \mathbf{x}^\alpha g(\mathbf{x}) \, \mu(d\mathbf{x}), \quad \alpha \in \mathbb{N}_0^d.$$

Then, for any $\alpha \in \mathbb{N}_0^d$,

$$\int \mathbf{x}^\alpha g(\mathbf{x}) \, \mu(d\mathbf{x}) = \int \mathbf{x}^\alpha \, \nu(d\mathbf{x}),$$

or, equivalently,

$$\int_{\Gamma^+ \cup \Gamma^-} \mathbf{x}^\alpha g(\mathbf{x}) \, \mu(d\mathbf{x}) = \int \mathbf{x}^\alpha \, \nu(d\mathbf{x}),$$

where $\Gamma^+ = \{\mathbf{x} : g(\mathbf{x}) \geq 0\}$ and $\Gamma^- = \{\mathbf{x} : g(\mathbf{x}) < 0\}$.

The latter can be written as

$$\int \mathbf{x}^\alpha \mathbb{1}_{\Gamma^+}(\mathbf{x}) g(\mathbf{x})\, \mu(d\mathbf{x}) = \int \mathbf{x}^\alpha \Big(\nu(d\mathbf{x}) - \mathbb{1}_{\Gamma^-}(\mathbf{x}) g(\mathbf{x})\, \mu(d\mathbf{x}) \Big)$$

which shows that the two *non-negative* measures on $\mathbb{R}^d$

$$\mathbb{1}_{\Gamma^+} g\, d\mu \quad \text{and} \quad d\nu - \mathbb{1}_{\mathbb{R}^d \setminus \Gamma^+} g\, d\mu \tag{2.16}$$

have the same moments.
Moreover, the moments of the measure $\mathbb{1}_{\Gamma^+} g\, d\mu$ satisfy the multi-variate Carleman condition. In fact, $\mathbb{1}_{\Gamma^+}\, d\mu \leq d\mu$ and for all $i = 1, \ldots, d$, and $k \in \mathbb{N}_0$

$$\int x_i^{2k} \mathbb{1}_{\Gamma^+}(\mathbf{x})\, \mu(d\mathbf{x}) \leq \int x_i^{2k}\, \mu(d\mathbf{x}) = L_y(x_i^{2k}).$$

The conclusion follows by Lemma 2.2.11.
By Corollary 2.2.5 follows that the two measures in (2.16) must be equal and this, in turn, implies that $g\, d\mu = d\nu$, i.e. the signed measure $g\, d\mu$ is actually non-negative as well as ν. Then, $\mathrm{supp}(\mu) \subseteq \{g \geq 0\} =: K$.

$\square$

Theorem 1.3.16, due to Berg and Maserick, and Theorem 2.2.9, due to Lasserre, as well as Theorem 2.2.6 due to Schmüdgen, have similarities and in their proofs there are three main stages which is worth to point out.

1. Bound on the sequence y.
   Having a bound on the growth of the sequence y is important to ensure the existence of a realizing measure μ on $\mathbb{R}^d$ since, in more than one dimension, this bound is necessary for the applicability of the spectral theorem.
   In the case of $d > 1$, Lasserre assumes directly the bound.
   Schmüdgen, instead, bounds the multi-sequence y via compactness of K (which forces the associated operators X_j to be bounded as well) before the existence of the realizing measure is established (see Remark 2.2.7).
   When $d = 1$, one does not actually need the bound since the existence of the realizing measure is given by Hamburger's theorem which can be also proved without spectral theorem. Nevertheless, the compactness of $K \subseteq \mathbb{R}^d$ pushes, a posteriori, the sequence y *realized* by μ to have the bound $|y_\alpha| \leq c a^{|\alpha|}$ (where c, a are positive constant and $\alpha \in \mathbb{N}_0^d$). In fact,

$$|y_\alpha| := \left| \int_K \mathbf{x}^\alpha \mu(d\mathbf{x}) \right| \leq \int_K |\mathbf{x}^\alpha|\, \mu(d\mathbf{x}) \leq \max_{\mathbf{x} \in K} |\mathbf{x}^\alpha| \int_K \mu(d\mathbf{x}) \leq a^{|\alpha|} c.$$

2. Existence and uniqueness of the realizing measure μ.

   A common root in all theorems is the centrality of Carleman's type conditions which, as we have shown in Lemma 1.3.11 and in Lemma 2.2.10, are derived and used by Berg and Maserick, as well as by Lasserre. In particular, in Theorem 1.3.16 Carleman's condition is obtained from the compactness of K whereas in Theorem 2.2.9 the multi-variate Carleman's condition is a direct consequence of the a priori bound on y.

   Berg and Maserick in Theorem 1.3.16 use Hamburger's Theorem 1.3.5, which does not require the knowledge of any bound on y, in order to guarantee the existence of a non-negative measure on $\mathbb{R}$ which realizes y. Moreover, they obtain the Carleman condition a posteriori using the compact support of the measure. Lasserre, instead, ensures existence *and* uniqueness of the non-negative realizing measure on $\mathbb{R}^d$ *at the same time* via Theorem 2.2.2 which he can use because the multi-variate Carleman condition is a direct consequence of the a priori bound on y he assumes.

   Instead, the use of the spectral theory is explicit in Schmüdgen.

3. Trick to get the support of μ.

   In all theorems (the same could be applied to Schmüdgen's proof) we arrive to a common point where the moments of a signed measure and the ones of a non-negative measure are equal, i.e.

$$\int \mathbf{x}^\alpha g(\mathbf{x})\,\mu(d\mathbf{x}) = \int \mathbf{x}^\alpha\,\nu(d\mathbf{x}), \quad \alpha \in \mathbb{N}_0^d. \tag{2.17}$$

   This does not allow to conclude that the two measure are equal (it does not even if both measures are non-negative). To overcome this problem, a sort of "splitting procedure" is used to rewrite (2.17) in terms of two *non-negative* measures depending on $gd\mu$ and ν. One of these non-negative measure is such that either its support is compact or its sequence of moments satisfies Carleman's condition (note that in the first case Carleman's condition is implied again by Lemma 1.3.11). To sum up, it is possible to rewrite (2.17) as

$$s_\alpha := \int \mathbf{x}^\alpha \eta_{g\mu}(d\mathbf{x}) = \int \mathbf{x}^\alpha\,\rho_{g\mu,\nu}(d\mathbf{x}), \quad \alpha \in \mathbb{N}_0^d,$$

   where $\eta_{g\mu}$ and $\rho_{g\mu,\nu}$ are non-negative measures (depending on $g\mu$ and $g\mu, \nu$, respectively) with $s = (s_\alpha)_{\alpha\in\mathbb{N}_0^d}$ satisfying the multivariate Carleman condition.

Remark 2.2.12.
Theorem 2.2.9 continues to hold if we consider a countable *number of polynomials* g_j. *This consideration allows us, for example, to solve the full moment problem on discrete and non-compact sets like the one of the natural numbers. In fact,* $\mathbb{N}_0$ *can be written as the intersection of infinitely many polynomials as follows*

$$\mathbb{N}_0 = \bigcap_{j \in \mathbb{N}_0} \{x \in \mathbb{R} |\, g_j(x) \geq 0\} \cap \{x \in \mathbb{R} |\, g(x) = x \geq 0\}$$

where $g_j(x) = x^2 - (2j+1)x + (j^2 + j)$ *for* $j \in \mathbb{N}_0$.
By using the inner-regularity of the realizing measure μ*, Theorem 2.2.9 also holds for an* uncountable *number of polynomials* g_j*. We analyze this case in details for a more general problem in Chapter 4.*

Chapter 3

Generalized power moment problem

3.1 Generalized power moment problem on finite dimensional spaces

In this section we are going to present the classical moment problem on a general finite dimensional vector space W which is in dual pairing with another vector space V under a scalar product. This will help to better understand the next section and Chapter 4, where a moment problem for particular infinite dimensional spaces is studied.

Dual pairing

Let V and W be two vector spaces over $\mathbb{R}$ and suppose that the function

$$\begin{aligned} \langle \cdot, \cdot \rangle : \quad V \times W & \to \mathbb{R} \\ (v, x) & \mapsto \langle v, x \rangle \,, \end{aligned}$$

is a bilinear form on $V \times W$ which is non-degenerate, i.e.

- $(\langle v, x \rangle = 0, \ \forall v \in V) \Longrightarrow x = 0$,
- $(\langle v, x \rangle = 0, \ \forall x \in W) \Longrightarrow v = 0$.

Then, the spaces V and W are said to be a *pair in duality* with respect to the bilinear form $\langle \cdot, \cdot \rangle$ also called *scalar product between* V and W.

Example 3.1.1.
Let V be any vector space and let us consider the special case of $W = V'$ (the space of linear functionals on V). Then by defining

$$\langle v, f \rangle := f(v), \quad f \in V', v \in V,$$

we have a bilinear form on $V \times V'$ which is non-degenerate.

The following propositions are of particular importance.
From now on, (V, W) is a pair of vector spaces in duality with respect to a bilinear form $\langle \cdot, \cdot \rangle$.

Proposition 3.1.2 ([26] p. 65)**.**
The map

$$\begin{aligned} \varphi : \ V &\to W' \\ v &\mapsto l_v \end{aligned}$$

where

$$l_v(x) := \langle v, x \rangle, \quad x \in W, \tag{3.1}$$

is injective and linear.

Proposition 3.1.3 ([26] p. 76)**.**
Assume that W has finite dimension. Then the injection $\varphi : V \to W'$ defined by (3.1) *is surjective and hence a linear isomorphism.*
In particular, V has finite dimension and $\dim V = \dim W$.

The latter proposition implies that, whenever W has finite dimension, any linear functional on W is of the form l_v for some v in V. More precisely, if $L \in W'$ then there exists $v \in V$ such that

$$L(x) = l_v(x) = \langle v, x \rangle,$$

for all $x \in W$.

Remark 3.1.4.
If V and W are in duality with respect to $\langle \cdot, \cdot \rangle$ then φ, as in (3.1)*, identifies V with W'. By the symmetry of $\langle \cdot, \cdot \rangle$, also W and V are in duality with respect to the same product. Then, by repeating the steps as above, we conclude that the analogous mapping of φ identifies W and V'.*

Dual pairing of tensor products

Let us recall the following definition.

Definition 3.1.5 (Tensor product)**.**
If $A_1, \dots, A_n$ are linear spaces (over $\mathbb{R}$), their tensor product is the linear space $A_1 \otimes \cdots \otimes A_n$ together with a n-linear form

$$\begin{aligned} \otimes : \quad A_1 \times \cdots \times A_n &\to A_1 \otimes \cdots \otimes A_n \\ (a_1, \dots, a_n) &\mapsto a_1 \otimes \cdots \otimes a_n \, , \end{aligned}$$

such that, for any vector space U and any n-linear form

$$f^{(n)} : A_1 \times \cdots \times A_n \to U,$$

there exists a unique (up to isomorphisms) linear *map*

$$\tilde{f} : A_1 \otimes \cdots \otimes A_n \to U$$

*with $\tilde{f} \circ \otimes = f^{(n)}$ (*Universal Property*).*

In other words, from the latter definition the following diagram commutes.

$$\begin{tikzcd} A_1 \times \cdots \times A_n \arrow[r, "f^{(n)}"] \arrow[d, "\otimes"'] & U \\ A_1 \otimes \cdots \otimes A_n \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

Suppose now that, for $i = 1, \dots, n$, the pair of spaces $(V[i], W[i])$ is in duality with respect to the bilinear form $\langle \cdot, \cdot \rangle_i$. We have the following fact (for more details see [27, p. 33]): the pair $(V[1] \otimes \cdots \otimes V[n], W[1] \otimes \cdots \otimes W[n])$ is in duality with respect to the non-degenerate bilinear form

$$\langle v[1] \otimes \cdots \otimes v[n], x[1] \otimes \cdots \otimes x[n] \rangle_\otimes := \langle v[1], x[1] \rangle_1 \cdots \langle v[n], x[n] \rangle_n. \qquad (3.2)$$

In particular, if we consider n copies of the pair in duality (V, W), the spaces $V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n-\text{times}}$ and $W^{\otimes n} := \underbrace{W \otimes \cdots \otimes W}_{n-\text{times}}$ form a pair in duality with respect to the product

$$\langle v_1 \otimes \cdots \otimes v_n, x_1 \otimes \cdots \otimes x_n \rangle = \langle v_1, x_1 \rangle \cdots \langle v_n, x_n \rangle. \tag{3.3}$$

Note that in the bilinear form on $V^{\otimes n} \times W^{\otimes n}$ we dropped the subscript symbol of tensor for simplicity.

Construct now the map

$$\begin{aligned} \phi : \quad V^{\otimes n} &\to (W^{\otimes n})' \\ v^{(n)} &\mapsto l_{v^{(n)}} \end{aligned}$$

where

$$l_{v^{(n)}}(x_1 \otimes \cdots \otimes x_n) := \langle v^{(n)}, x_1 \otimes \cdots \otimes x_n \rangle, \quad x_1 \otimes \cdots \otimes x_n \in W^{\otimes n}. \tag{3.4}$$

Corollary 3.1.6.
If W is a finite dimensional vector space, the map $\phi : V^{\otimes n} \to (W^{\otimes n})'$ defined as in (3.4) *is a linear isomorphism.*

The latter means that any linear functional on $W^{\otimes n}$ is of the form $l_{v^{(n)}}$ for some $v^{(n)}$ in $V^{\otimes n}$. More precisely, if $L \in (W^{\otimes n})'$ then there exists $v^{(n)} \in V^{\otimes n}$ such that

$$L(x_1 \otimes \cdots \otimes x_n) = l_{v^{(n)}}(x_1 \otimes \cdots \otimes x_n) = \langle v^{(n)}, x_1 \otimes \cdots \otimes x_n \rangle,$$

for any $x_1 \otimes \cdots \otimes x_n \in W^{\otimes n}$.

Polynomials

Let us introduce the notion of polynomial in this general setting. In the following, the spaces are considered to be finite dimensional.

Definition 3.1.7.
A map $p : W \mapsto \mathbb{R}$ is called homogeneous polynomial of degree n if there exists a symmetric n-linear form $f^{(n)} : W^{\times n} \mapsto \mathbb{R}$ such that $p(x) = f^{(n)}(x, x, \ldots, x)$ for any $x \in W$.

If $\{e_1, \ldots, e_d\}$ is a vector basis for W then the previous definition coincides with the classical definition of polynomial.

In fact, let us consider the following map

$$
\begin{aligned}
I: \qquad \mathbb{R}^d &\to W \\
(x_1, \ldots, x_d) &\mapsto x := \sum_{i=1}^{d} x_i e_i,
\end{aligned}
\tag{3.5}
$$

where in $\mathbb{R}^d$ we consider the canonical basis. Note that the map I is a non canonical isomorphism in the sense that it does depend on the choice of the basis on W. Consider the following diagram.

$$
\begin{tikzcd}
\mathbb{R}^d \arrow[r, "I"] & W \arrow[r, "p"] \arrow[d, "\times n"'] & \mathbb{R} \\
& W^{\times n} \arrow[ur, "f^{(n)}"'] &
\end{tikzcd}
$$

The vector $(x_1, \ldots, x_d)$ is mapped under I to $x = \sum_{i=1}^{d} x_i e_i \in W$ and then

$$
\begin{aligned}
p(x) &= f^{(n)}\left(\sum_{i_1=1}^{d} x_{i_1} e_{i_1}, \sum_{i_2=1}^{d} x_{i_2} e_{i_2}, \ldots, \sum_{i_n=1}^{d} x_{i_n} e_{i_n}\right) \\
&= \sum_{i_1,\ldots,i_n=1}^{d} x_{i_1} x_{i_2} \cdots x_{i_n} f^{(n)}(e_{i_1}, e_{i_2}, \ldots, e_{i_n}).
\end{aligned}
\tag{3.6}
$$

The numbers $f^{(n)}(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$ are called the *coefficients* of the polynomial p in the *indeterminates* $x_1, x_2, \ldots, x_d$. In other words, (3.6) is a polynomial in the classical sense.

Homogeneous polynomials can be conveniently expressed using symmetric tensors. In fact, the latters allow arguments about n-linear maps to be carried out in terms of linear maps only (see Universal Property in Definition 3.1.5). The diagram which we have to keep in mind is the following.

$$
\begin{tikzcd}
W \arrow[r, "p"] \arrow[d, "\times n"'] & \mathbb{R} \\
W^{\times n} \arrow[r, "\otimes n"'] \arrow[ur, "f^{(n)}"] & W^{\otimes n} \arrow[u, dashed, "\tilde{f}"']
\end{tikzcd}
$$

If $x \in W$ then

$$\begin{aligned} p(x) &= f^{(n)}(x, \ldots, x) \\ &= \tilde{f}(x \otimes \cdots \otimes x) \\ &= \tilde{f}(x^{\otimes n}). \end{aligned}$$

Hence, we can give the definition of homogeneous polynomial via tensor product.

Definition 3.1.8.
A map $p : W \mapsto \mathbb{R}$ is called homogeneous polynomial of degree n if there exists a linear form $\tilde{f} : W^{\otimes n} \mapsto \mathbb{R}$ such that $p(x) = \tilde{f}(x^{\otimes n})$ for any $x \in W$.

Definition 3.1.8 says that a polynomial p is given by a linear functional on $W^{\otimes n}$, i.e. $\tilde{f} \in (W^{\otimes n})'$.
In turn this implies, by Corollary 3.1.6, that there exists $v^{(n)} \in V^{\otimes n}$ such that

$$p(x) = f^{(n)}(x, \ldots, x) = \tilde{f}(x^{\otimes n}) = \langle v^{(n)}, x^{\otimes n} \rangle.$$

The latter is a general representation of homogeneous polynomials of degree n whenever we have two spaces in dual pairing with respect to a bilinear form $\langle \cdot, \cdot \rangle$.

We can then define a generic polynomial of degree N on W as

$$P(x) = \sum_{n=0}^{N} \langle v^{(n)}, x^{\otimes n} \rangle,$$

where $x \in W$ and $v^{(n)} \in V^{\otimes n}$, for $n = 0, \ldots, N$, with the convention $\langle v^{(0)}, x^{\otimes 0} \rangle = v^{(0)} \in \mathbb{R}$.

Remark 3.1.9.
If $\{e_i\}_{i=1}^{d}$ is a basis for W and the latter is in dual pairing with V then, for each $j = 1 \ldots, d$, the vectors $\bar{e}_j \in V$ defined such that

$$\langle \bar{e}_j, e_i \rangle = \delta_{i,j} = \begin{cases} 1, & \textit{if } i = j, \\ 0, & \textit{otherwise}, \end{cases}$$

form a basis for V (called dual basis*).*

In general, if $\{(a_j)_i\}_{i=1}^{d}$ is basis for A_j then $\{a_{1i_1} \otimes \cdots \otimes a_{ni_n}\}_{i_1,\ldots,i_n=1}^{d}$ is a basis for $A_1 \otimes \cdots \otimes A_n$.

Hence, $(e_i \otimes e_j)_{i,j=1}^{d}$ is a basis for the tensor product $W^{\otimes 2}$ in dual pairing *with $V^{\otimes 2}$. Then, the set of vectors $(\bar{e}_i \otimes \bar{e}_j)_{i,j=1}^d$ is the dual basis for $V^{\otimes 2}$ and*

$$\langle \bar{e}_{i'} \otimes \bar{e}_{j'}, e_i \otimes e_j \rangle = \delta_{i',i}\ \delta_{j',j}.$$

The moment tensor of a measure

Let μ be a measure on W such that the map

$$\begin{aligned} \widetilde{m}^{(n)}: \quad & V^{\times n} && \to \mathbb{R} \\ & (v_1, \ldots, v_n) && \mapsto \int_W \langle v_1, x\rangle \cdots \langle v_n, x\rangle \mu(dx) \end{aligned}$$

is well-defined. Note that the function $\widetilde{m}^{(n)}$ is symmetric w.r.t. permutations of its variables $(v_1, \ldots, v_n)$.
Since $\widetilde{m}^{(n)}$ is n-linear it can be seen as a linear map on the tensor product $V^{\otimes n}$, i.e. (with abuse of notation)

$$\begin{aligned} \widetilde{m}^{(n)}: \quad & V^{\otimes n} && \to \mathbb{R} \\ & v^{(n)} && \mapsto \int_W \langle v^{(n)}, x^{\otimes n}\rangle \mu(dx)\,. \end{aligned}$$

Note that $\widetilde{m}^{(n)} \in (V^{\otimes n})'$. Then, by duality (the result of Remark 3.1.4 holds also for tensors), there exists $m^{(n)} \in W^{\otimes n}$ such that

$$\langle v^{(n)}, m^{(n)}\rangle = \int_W \langle v^{(n)}, x^{\otimes n}\rangle \mu(dx)\,,$$

for any $v^{(n)} \in V^{\otimes n}$.
The function $m^{(n)}$ is called the n-th *moment tensor* of μ.

We now show how the moment tensors of μ are related to the classical definition of moments which we gave on $\mathbb{R}^d$.

Let us consider the maps in (3.5). Let $\mu_\#$ be the image measure of μ under the map I^{-1} (see Definition C.0.8).
Since $\mu_\#$ is a measure on $\mathbb{R}^d$, its moments of order n, with $n \in \mathbb{N}$, are given by

$$m^{(n)}_{i_1,\ldots,i_n} = \int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_n} \mu_\#(dx_1, \ldots, dx_d),$$

where $i_1, \ldots, i_n$ are selected from the set $\{1, \ldots, d\}$ in order to form sequences of size n (whose elements are not necessarily distinct) such that the order of their elements is not taken into account, i.e. two sequences $i_1, \dots, i_n$ and $\sigma(i_1, \dots, i_n)$, of which one can be obtained from the other by permuting its terms, are the same and $m^{(n)}_{i_1,\dots,i_n} = m^{(n)}_{\sigma(i_1,\dots,i_n)}$. Let us define the function

$$\begin{aligned} f: \quad \mathbb{R}^d \quad &\to \mathbb{R} \\ (x_1 \dots, x_d) \quad &\mapsto x_{i_1} \cdots x_{i_n}, \end{aligned}$$

with $\{i_1, \dots, i_n\} \in \{1, \dots, d\}$ as above.
Then, by Definition C.0.8, we have that

$$m^{(n)}_{i_1,\dots,i_n} = \int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_n} \mu_\#(dx_1, \dots, dx_d) = \int_W f(I^{-1}(x))\mu(dx), \tag{3.7}$$

if the second integral above exists. Let us note that if $x = \sum\limits_{i=1}^{d} x_i e_i \in W$ and we consider the dual basis $\{\bar{e}_1, \dots, \bar{e}_d\}$ in V we have that

$$\langle \bar{e}_j, x \rangle = \langle \bar{e}_j, \sum_{i=1}^{d} x_i e_i \rangle = \sum_{i=1}^{d} x_i \langle \bar{e}_j, e_i \rangle = x_j, \quad j = 1, \dots, d.$$

Then $I^{-1}(x) = (x_1, \dots, x_d)$ can be written as $(\langle \bar{e}_1, x\rangle, \dots, \langle \bar{e}_d, x \rangle)$ and so

$$f(I^{-1}(x)) = \langle \bar{e}_{i_1}, x \rangle \cdots \langle \bar{e}_{i_n}, x \rangle = \langle \bar{e}_{i_1} \otimes \cdots \otimes \bar{e}_{i_n}, x^{\otimes n} \rangle,$$

where in the last equality we made use of (3.3). So the second integral in (3.7) exists.
Relation (3.7) then becomes

$$\begin{aligned} m^{(n)}_{i_1,\dots,i_n} = \int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_n} \mu_\#(dx_1, \dots, dx_d) \quad &= \quad \int_W \langle \bar{e}_{i_1} \otimes \cdots \otimes \bar{e}_{i_n}, x^{\otimes n} \rangle \mu(dx) \\ &= \quad \langle \bar{e}_{i_1} \otimes \cdots \otimes \bar{e}_{i_n}, m^{(n)} \rangle. \end{aligned} \tag{3.8}$$

As element of $W^{\otimes n}$, $m^{(n)}$ can be written in terms of a basis $(e_{j_1} \otimes \cdots \otimes e_{j_n})^d_{j_1,\dots,j_n=1}$ as

$$m^{(n)} = \sum_{j_1,\dots,j_n=1}^{d} \lambda_{j_1,\dots,j_n} \, e_{j_1} \otimes \cdots \otimes e_{j_n}, \tag{3.9}$$

for some coefficients $(\lambda_{j_1,\dots,j_n})_{j_1,\dots,j_n}$.
The coefficients $(\lambda_{j_1,\dots,j_n})_{j_1,\dots,j_n}$ are the moments $\left(m^{(n)}_{j_1,\dots,j_n}\right)$ of order n of $\mu_\#$. In

fact, substituting (3.9) in (3.8) we get

$$\begin{aligned} m^{(n)}_{i_1,\dots,i_n} &= \sum_{j_1,\dots,j_n=1}^{d} \lambda_{j_1,\dots,j_n} \langle \bar{e}_{i_1} \otimes \cdots \otimes \bar{e}_{i_n}, e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle \\ &= \sum_{j_1,\dots,j_n=1}^{d} \lambda_{j_1,\dots,j_n} \langle \bar{e}_{i_1}, e_{j_1} \rangle \cdots \langle \bar{e}_{i_n}, e_{j_n} \rangle \\ &= \sum_{j_1,\dots,j_n=1}^{d} \lambda_{j_1,\dots,j_n} \, \delta_{i_1,j_1} \cdots \delta_{i_n,j_n} = \lambda_{i_1,\dots,i_n} \ . \end{aligned}$$

So we have that

$$m^{(n)} = \sum_{j_1,\dots,j_n=1}^{d} \left(\int_{\mathbb{R}^d} x_{j_1} \cdots x_{j_n} \mu_{\#}(dx_1, \dots, dx_d) \right) e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

Statement of the generalized power moment problem

We can sum up the previous subsection in the following definition.
Let W be a finite dimensional space in dual paring with V with respect to $\langle \cdot, \cdot \rangle$. Let μ be a measure supported on W such that for every $n \in \mathbb{N}_0$ and $v^{(n)} \in V^{\otimes n}$

$$\int_W \langle v^{(n)}, x^{\otimes n} \rangle \mu(dx) < +\infty,$$

with the convention, $\langle v^{(0)}, x^{\otimes 0} \rangle := v^{(0)}$ with $v^{(0)} \in \mathbb{R}$.

Definition 3.1.10 (Moment tensor)**.**
The $n-th$ moment tensor *of μ is defined as the symmetric function $m^{(n)}_\mu \in W^{\otimes n}$ such that*

$$\langle v^{(n)}, m^{(n)}_\mu \rangle = \int_W \langle v^{(n)}, x^{\otimes n} \rangle \mu(dx), \tag{3.10}$$

for all $v^{(n)} \in V^{\otimes n}$.

By (3.10), a measure μ always gives rise to its moment tensor $m^{(n)}_\mu$.
The tensor moment problem is a sort of inverse problem.

Definition 3.1.11 (Moment problem on W)**.**
Given a sequence $(m^{(n)})_{n \in \mathbb{N}_0}$ of symmetric functions in $W^{\otimes n}$ with $n \in \mathbb{N}_0$, find a measure μ on W such that

$$m^{(n)} = m^{(n)}_\mu \quad \textit{for } n = 0, 1, \dots$$

i.e. such that $m^{(n)}$ is the $n-th$ moment tensor of μ for $n = 0, 1, \dots$.

If such a measure μ exists we say that $(m^{(n)})_{n\in\mathbb{N}_0}$ *realized* by μ on W.
If we require that the measure μ has support contained in a measurable subset $\mathcal{S}$ of W then we can reformulate the previous definition as we have done at the beginning of Chapter 1.

3.2 Generalized power problem on nuclear spaces

In the following we will consider all the spaces as being separable and real.

Let us consider a family $(H_k)_{k\in K}$ of Hilbert spaces (K is an index set containing 0). Suppose that $\Omega = \bigcap_{k\in K} H_k$ is dense in each H_k and equip this linear space with the following topology. A neighborhood base about zero in Ω is understood to be a collection of sets

$$U_{k_1,\dots,k_m;\varepsilon_1,\dots,\varepsilon_m} = \left\{ f \in \Omega : \ ||f||_{H_{k_1}} < \varepsilon_1, \dots, ||f||_{H_{k_m}} < \varepsilon_m \right\},$$

where $k_1,\dots,k_m \in K$ and $\varepsilon_1 > 0,\dots,\varepsilon_m > 0$ with $m \in \mathbb{N}$.
The linear topological space Ω, constructed as above, is called the *projective limit* of the spaces H_k.

From now on, we will assume that the norms are directed by topological imbedding, i.e.

$$\forall\ k_1, k_2 \in K\ \exists\, k_3\ :\ H_{k_3} \subseteq H_{k_1}\ ,\ H_{k_3} \subseteq H_{k_2}\ \text{(topologically)}.$$

This implies that each neighborhood $U_{k_1,\dots,k_m;\varepsilon_1,\dots,\varepsilon_m}$ contains a neighborhood $U_{k;\varepsilon}$ for some $k \in K$ and $\varepsilon > 0$. Therefore, a neighborhood base about zero for Ω can be directly given by the collections of sets $U_{k;\varepsilon}$ with $k \in K$ and $\varepsilon > 0$.

Let us also assume that Ω is *nuclear*, i.e. for each $k \in K$ there exists $k' \in K$ such that $H_{k'} \subseteq H_k$, and this imbedding is *quasi-nuclear* according to the following.

Definition 3.2.1 (Quasi-nuclear operator and imbedding)**.**
Let H_1 and H_2 be two Hilbert spaces and suppose that H_1 is separable.
A continuous linear operator $T : H_1 \to H_2$ is called Hilbert-Schmidt operator or a quasi-nuclear operator if $\sum_{i=1}^{\infty} ||Te_i||^2_{H_2} < \infty$ for some orthonormal basis $(e_i)_{i=1}^{\infty}$ in H_1.

An imbedding $H_1 \subseteq H_2$ is said to be quasi-nuclear if the imbedding operator $O : H_1 \to H_2$ is quasi-nuclear.

Let us denote by Ω' the *topological* dual space of Ω.
W.l.o.g. we assume that each H_k is imbedded topologically (i.e. densily and continuously) into H_0. Then, the inner product on H_0 determines a dual pairing between Ω and Ω' which, however, differs from the canonical one introduced in (3.1). In fact, as $H_0 \supseteq \Omega$, each element $\chi \in H_0$ gives rise to a continuous linear functional l_χ in the following way. We consider the map

$$\begin{aligned} \varphi : \quad H_0 &\to \Omega' \\ \chi &\mapsto l_\chi \end{aligned}$$

where

$$l_\chi(f) := \langle f, \chi \rangle_{H_0}, \quad f \in \Omega.$$

Identifying χ with l_χ, we get an imbedding of H_0 in the space Ω'. (The identification is unambiguous: if $l_\chi = 0$, then $\chi = 0$.) If Ω' is endowed with the weak topology, then the imbedding $\varphi : H_0 \to \Omega'$ is obviously continuous. We have constructed the chain

$$\Omega' \supseteq H_0 \supseteq \Omega.$$

For $\eta \in \Omega'$ and $f \in \Omega$, we denote by $\langle f, \eta \rangle$ the extension of $\langle f, \chi \rangle_{H_0}$ by continuity as $\chi \to \eta$ with $\chi \in H_0$ (for more details, see [3, Chapter 1] and [5, Vol. I, Chapter 1]).

Consider the $n-$th ($n \in \mathbb{N}_0$) tensor power $\Omega^{\otimes n}$ of the space Ω which is defined as the projective limit of $H_k^{\otimes n}$; in particular, for $n = 0$, $H_k^{\otimes n} = \mathbb{R}$. Then its dual space is

$$\left(\Omega^{\otimes n}\right)' = \bigcup_{k \in K} \left(H_k^{\otimes n}\right)' = \bigcup_{k \in K} (H_k')^{\otimes n}$$

which we can equip with the weak topology.

The spaces $\Omega^{\otimes n}$ and $\left(\Omega^{\otimes n}\right)'$ are a pair in duality with respect to the product induced by $\langle \cdot, \cdot \rangle$ on $\Omega \times \Omega'$. Namely,

$$\langle f_1 \otimes \cdots \otimes f_n, \eta_1 \otimes \cdots \otimes \eta_n \rangle = \langle f_1, \eta_1 \rangle \cdots \langle f_n, \eta_n \rangle.$$

Remark 3.2.2.
The bilinear form that establishes duality between $\Omega^{\otimes n}$ and $(\Omega^{\otimes n})'$ is actually the inner product on $H_0^{\otimes n}$.

Let us consider a finite Borel measure μ on Ω' (μ is called *generalized process*) such that the map

$$\begin{aligned} \widetilde{m}^{(n)}: \qquad \Omega^{\times n} \qquad &\to \mathbb{R} \\ (f_1, \dots, f_n) \quad &\mapsto \int_{\Omega'} \langle f_1, \eta \rangle \cdots \langle f_n, \eta \rangle \mu(d\eta) \end{aligned}$$

is well-defined and continuous on $\Omega^{\times n}$.
Since $\widetilde{m}^{(n)}$ is n-linear it can be seen as a linear map on the tensor product $\Omega^{\otimes n}$ due to the kernel theorem (see [7, Theorem 6.2, p. 163]). Then there exists $m^{(n)} \in (\Omega^{\otimes n})'$ such that

$$\langle f_1 \otimes \cdots \otimes f_n, m^{(n)} \rangle = \int_{\Omega'} \langle f_1, \eta \rangle \cdots \langle f_n, \eta \rangle \mu(d\eta)\,,$$

for any $f_1, \dots, f_n \in \Omega$.
The function $m^{(n)}$ is called the n-th *generalized moment function* of μ.

Let us formalize better what we have done so far and let us introduce the main objects involved in the generalized power moment problem.
A generalized process is a finite measure μ defined on the Borel $\sigma-$algebra on Ω'. Moreover, we say that a generalized process μ is *concentrated on* a measurable subset $\mathcal{S} \subseteq \Omega'$ if $\mu\left(\Omega' \setminus \mathcal{S}\right) = 0$.

Definition 3.2.3 (Finite $n-$th local moment).
Given $n \in \mathbb{N}_0$, a generalized process μ on Ω' has finite $n-$th local moment *(or local moment of order n) if for every $f \in \Omega$ we have*

$$\int_{\Omega'} |\langle f, \eta \rangle|^n \mu(d\eta) < \infty.$$

The latter condition implies that the functional

$$\begin{aligned} \Omega^{\times n} \qquad &\to \mathbb{R} \\ (f_1, \dots, f_n) \quad &\mapsto \int_{\Omega'} \langle f_1 \otimes \cdots \otimes f_n, \eta^{\otimes n} \rangle \mu(d\eta) \end{aligned} \tag{3.11}$$

is a well-defined linear functional on $\Omega^{\times n}$. In fact, since μ has $n-$th finite local moment, for any $f_1, \dots, f_n \in \Omega$ we get

$$\begin{aligned}\left|\int_{\Omega'} \langle f_1 \otimes \cdots \otimes f_n, \eta^{\otimes n}\rangle \mu(d\eta)\right| &\leq \int_{\Omega'} \prod_{i=1}^{n} |\langle f_i, \eta\rangle| \mu(d\eta) \\ &\leq \prod_{i=1}^{n} \left(\int_{\Omega'} |\langle f_i, \eta\rangle|^n \mu(d\eta)\right)^{\frac{1}{n}} < \infty, \quad (3.12)\end{aligned}$$

where we made use of the generalization of Hölder's inequality.
The functionals in (3.11) are the *moments* of μ. In the following, we will require slightly more regularity on the moments but this assumption is easy to check in most of applications.

Definition 3.2.4 ($n-$th generalized moment function)**.**
*Given $n \in \mathbb{N}_0$, a generalized process μ on Ω' has $n-$*th generalized moment function in the sense of Ω' *if μ has finite $n-th$ local moment and if for all n the functional $f \mapsto \int_{\Omega'} |\langle f, \eta\rangle|^n \mu(d\eta)$ is continuous.*

This means that there exists a symmetric functional $m_\mu^{(n)} \in (\Omega^{\otimes n})'$ such that

$$\langle f_1 \otimes \cdots \otimes f_n, m_\mu^{(n)}\rangle = \int_{\Omega'} \langle f_1 \otimes \cdots \otimes f_n, \eta^{\otimes n}\rangle \mu(d\eta). \quad (3.13)$$

In fact, by the assumption of continuity and by (3.12) the multilinear functional (3.11) is continuous and so we can apply the kernel theorem. By convention, $\langle f_0, \eta^{\otimes 0}\rangle := f_0$ with $f_0 \in \mathbb{R}$.

Proposition 3.2.5.
If μ is a generalized process on Ω' with generalized moment functions (in the sense of Ω') of any order, then for any $n \in \mathbb{N}$ and for any $f^{(n)} \in \Omega^{\otimes n}$ we have

$$\int_{\Omega'} \langle f^{(n)}, \eta^{\otimes n}\rangle \mu(d\eta) < \infty$$

and

$$\langle f^{(n)}, m_\mu^{(n)}\rangle = \int_{\Omega'} \langle f^{(n)}, \eta^{\otimes n}\rangle \mu(d\eta). \quad (3.14)$$

Proof. ($n = 2$)
Let us consider $f^{(2)} \in \Omega^{\otimes 2}$. Then, we can write

$$f^{(2)} = \sum_{i,j=1}^{\infty} \varphi_i \otimes \psi_j, \quad (3.15)$$

for some $\varphi_i, \psi_j \in \Omega$. Let $f_p^{(2)} := \sum_{i,j=1}^{p} \varphi_i \otimes \psi_j$. Since $f_p^{(2)} \to f^{(2)}$ as $p \to \infty$ by (3.15), we have that the sequence $\left(f_p^{(2)}\right)_{p\in\mathbb{N}}$ is a Cauchy sequence in $\Omega^{\otimes 2}$, i.e. $\forall k \in K$ we have that $\|f_p^{(2)} - f_q^{(2)}\|_{H_k^{\otimes 2}} \to 0$ as $p, q \to \infty$.
Consequently, the sequence $\left(f_p^{(2)}\right)_{p\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega', \mu)$. In fact,

$$\begin{aligned}
0 &\leq \left|\int_{\Omega'} \left(\langle f_p^{(2)}, \eta^{\otimes 2}\rangle - \langle f_q^{(2)}, \eta^{\otimes 2}\rangle\right)^2 \mu(d\eta)\right| \\
&= \left|\int_{\Omega'} \left(\langle f_p^{(2)} - f_q^{(2)}, \eta^{\otimes 2}\rangle\right)^2 \mu(d\eta)\right| \\
&= \left|\int_{\Omega'} \langle \left(f_p^{(2)} - f_q^{(2)}\right)^{\otimes 2}, \eta^{\otimes 4}\rangle \mu(d\eta)\right| \\
&= \left|\langle \left(f_p^{(2)} - f_q^{(2)}\right)^{\otimes 2}, m_\mu^{(4)}\rangle\right| \\
&\leq \|m_\mu^{(4)}\|_{(H_k^{\otimes 4})'} \cdot \left\|\left(f_p^{(2)} - f_q^{(2)}\right)^{\otimes 2}\right\|_{H_k^{\otimes 4}} \to 0, \text{ as } p, q \to \infty,
\end{aligned}$$

where in the last equality and inequality we used (3.13) and the assumption $m_\mu^{(4)} \in (\Omega')^{\otimes 4}$ (i.e. there exists $k \in K$ such that $m_\mu^{(4)} \in (H_k^{\otimes 4})'$), respectively.
Then, since $L^2(\Omega', \mu)$ is complete, there exists $F \in L^2(\Omega', \mu)$ with $\langle f_p^{(2)}, \eta^{\otimes 2}\rangle \to F(\eta)$ in $L^2(\Omega', \mu)$. This implies that there exists a subsequence $(f_{p_k}^{(2)})_{k\in\mathbb{N}}$ such that $\langle f_{p_k}^{(2)}, \eta^{\otimes 2}\rangle \to F(\eta)$, $\mu-$a.e. in Ω'. On the other hand, by (3.15), we know also that for all $\eta \in \Omega'$

$$\langle f_p^{(2)}, \eta^{\otimes 2}\rangle \to \langle f^{(2)}, \eta^{\otimes 2}\rangle.$$

Then, necessarily, we have that $\langle f^{(2)}, \eta^{\otimes 2}\rangle \equiv F(\eta) \in L^2(\Omega', \mu)$. This means that

$$\left(\int_{\Omega'} \langle f^{(2)}, \eta^{\otimes 2}\rangle^2 \mu(d\eta)\right)^{\frac{1}{2}} < \infty$$

and, since μ is finite, we get that

$$\int_{\Omega'} \langle f^{(2)}, \eta^{\otimes 2}\rangle \mu(d\eta) \leq \left(\int_{\Omega'} \langle f^{(2)}, \eta^{\otimes 2}\rangle^2 \mu(d\eta)\right)^{\frac{1}{2}} (\mu(\Omega'))^{\frac{1}{2}} < \infty.$$

In conclusion, we proved that $\langle f_p^{(2)}, \eta^{\otimes 2}\rangle \to \langle f^{(2)}, \eta^{\otimes 2}\rangle$ in $L^1(\Omega', \mu)$ and so

$$\langle f_p^{(2)}, m_\mu^{(2)}\rangle = \int_{\Omega'} \langle f_p^{(2)}, \eta^{\otimes 2}\rangle \mu(d\eta) \to \int_{\Omega'} \langle f^{(2)}, \eta^{\otimes 2}\rangle \mu(d\eta).$$

Moreover, by (3.15) we have

$$\langle f_p^{(2)}, m_\mu^{(2)} \rangle \to \langle f^{(2)}, m_\mu^{(2)} \rangle.$$

Hence, by uniqueness of the limit, we get (3.14) for $n = 2$.

$\square$

For a generalized processes μ the moment functions $m_\mu^{(n)}$ are given by (3.14). The moment problem, which in the infinite dimensional context is often called the *realizability problem*, addresses exactly the inverse question.

Problem 3.2.6 (Realizability problem on $\mathcal{S} \subseteq \Omega'$)**.**
Given a sequence $(m^{(n)})_{n=0}^\infty$ of symmetric functions with $m^{(n)} \in (\Omega^{\otimes n})'$ for any $n \in \mathbb{N}_0$, find a generalized process μ with finite local moments of any order and concentrated on a measurable subset $\mathcal{S}$ of Ω' such that

$$m^{(n)} = m_\mu^{(n)} \quad \textit{for } n = 0, 1, \ldots$$

i.e. $m^{(n)}$ is the $n-$th moment function of μ for $n = 0, 1 \ldots$.

If such a measure μ does exist we say that $(m^{(n)})_{n=0}^\infty$ is *realized* by μ on $\mathcal{S}$.

An obvious positivity property which is necessary for a sequence $(m^{(n)})_{n=0}^\infty$, as above, to be the moment sequence of some measure on Ω' is the following.

Definition 3.2.7 (Positive semidefinite sequence)**.**
Let $m = (m^{(n)})_{n=0}^\infty$ where $m^{(n)} \in (\Omega^{\otimes n})'$ and $m^{(n)}$ is a symmetric function of its n variables. The sequence m is said to be positive semidefinite *if for any $f^{(j)} \in \Omega^{\otimes j}$ we have*

$$\sum_{i,j=0}^{\infty} \langle (f^{(i)} \otimes f^{(j)}), m^{(i+j)} \rangle \geq 0.$$

The latter is a generalization of the classical notion of positive semidefiniteness given in Definition 1.2.2.

Note that, as we work with real spaces, the involution on Ω considered in [5] is here chosen to be the identity.

Let us introduce the property of determining sequence which essentially is a growth restriction on the sequence of the $m^{(n)}$'s. We will show that this condition gives uniqueness of the realizing measure.

Definition 3.2.8 (Determining sequence)**.**
Let $m = (m^{(n)})_{n=0}^\infty$ where $m^{(n)} \in (\Omega^{\otimes n})'$ and $m^{(n)}$ is a symmetric functional of its

n variables. The sequence m is said to be determining *if for any $n \in \mathbb{N}$ and any $f_1, \dots, f_{2n} \in \Omega$ we have*

$$\left|\left\langle f_1 \otimes \cdots \otimes f_{2n}, m^{(2n)} \right\rangle\right| \leq \tilde{m}_n^2 \prod_{l=1}^{2n} \|f_l\|_{H_{k(m)}} \text{ for some } k(m) \in K, \tag{3.16}$$

where $(\tilde{m}_n)_{n=0}^{\infty}$ is a sequence of finite positive real numbers such that the class $C\{\tilde{m}_n\}$ is quasi-analytical (see Definition A.0.18).

The condition for a sequence to be determining can be given in a more general formulation than (3.16) (see [5, Vol. II, p. 54]). We chose this definition because it is easier to handle and, as we are going to see, it will show better the analogy with the classical Carleman's condition.

Let us state now the main result known in literature for the full realizability problem on such a kind of space Ω' (cf. [5, Vol. II, Theorem 2.1, p. 54]).

Theorem 3.2.9.
Let $m = (m^{(n)})_{n=0}^{\infty}$ where $m^{(n)} \in (\Omega^{\otimes n})'$ and $m^{(n)}$ is a symmetric function of its n variables. If m is a positive semidefinite sequence which is also determining, then there exists a unique non-negative measure μ on Ω', with generalized moment functions in the sense of Ω' of any order, such that for any $f^{(n)} \in \Omega^{\otimes n}$

$$\left\langle f^{(n)}, m^{(n)} \right\rangle = \int_{\Omega'} \left\langle f^{(n)}, \eta^{\otimes n} \right\rangle \mu(d\eta). \tag{3.17}$$

Remark 3.2.10.
The steps of the proof of Theorem 3.2.9 are similar, but considerably more difficult, to those we studied in the proof of Theorem 2.2.2. Starting from a positive semidefinite sequence, a Hilbert space $\mathcal{H}_m$ is constructed. A countable family of unbounded commuting operators on $\mathcal{H}_m$ is introduced. As in the classical moment problem, the domains of these operators are showed to contain a total subset of quasi-analytic vectors. The existence of the latter set allows to extend this family of operators to self-adjoint commuting operators on $\mathcal{H}_m$. The spectral theorem for infinitly countable unbounded self-adjoint operators (see [5, Vol. I, p. 314]) is then used to prove that there exists a spectral measure $\widetilde{\mu}$ on $\mathbb{R}^{\mathbb{N}_0}$. In the remaining part of the proof is shown that the sequence m is of the form (3.17).

Remark 3.2.11.
The proof of Theorem 3.2.9 shows that the measure μ is actually concentrated on one of the Hilbert spaces $H'_{k'(m)}$ for some index $k'(m) \in K$ depending on the sequence m (see [5, Vol. II, Remark 1, p. 72]).

Chapter 4

Concrete conditions for realizability of moment measures via quadratic modules

In the following we are going to apply Theorem 3.2.9 for the special realizability problem when $\Omega = \mathscr{D}_{proj}(\mathbb{R}^d)$, which is the projective limit of a family of weighted Sobolev spaces $H_k := W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$ and it is nuclear (see Theorem 4.1.12). Hence, $\Omega^{\otimes n} = \mathscr{D}_{proj}(\mathbb{R}^{dn})$ and so the sequence m consists of generalized functions, i.e. $m^{(n)} \in \mathscr{D}'_{proj}(\mathbb{R}^{dn})$. We will actually take the $m^{(n)}$'s in $\mathcal{R}(\mathbb{R}^{dn})$, which is a subset of $\mathscr{D}'_{proj}(\mathbb{R}^{dn})$ consisting of all Radon measures on $\mathbb{R}^d$. Theorem 3.2.9 gives a solution for the full realizability problem on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ whenever the sequence m is positive semidefinite and determining. Using this result, we will show how to get necessary and sufficient conditions on such a m to be the moment sequence of a measure concentrated on a basic semi-algebraic $\mathcal{S} \subseteq \mathscr{D}'_{proj}(\mathbb{R}^d)$.

4.1 The space of generalized functions

Let us first recall some standard general notations.
For $Y \subseteq \mathbb{R}^d$ let us denote by $\mathcal{B}(Y)$ the Borel σ-algebra on Y, by $\mathcal{C}_c(Y)$ the space of all real-valued continuous functions on $\mathbb{R}^d$ with compact support contained in Y and by $\mathcal{C}_c^\infty(Y)$ its subspace of all infinitely differentiable functions. Moreover, $\mathcal{C}_c^+(Y)$ and $\mathcal{C}_c^{+,\infty}(Y)$ will denote the cones consisting of all non-negative functions in $\mathcal{C}_c(Y)$ and $\mathcal{C}_c^\infty(Y)$, respectively.

We will denote by Ω_τ the space Ω endowed with a topology τ and by Ω'_τ its topological dual space. The suffix will be dropped whenever the topology is clear from the context.

For any $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we define $\mathbf{r}^\alpha := r_1^{\alpha_1} \cdots r_d^{\alpha_d}$. Moreover, for any $\beta \in \mathbb{N}_0^d$ the symbol D^β denotes the weak partial derivative $\frac{\partial^{|\beta|}}{\partial r_1^{\beta_1} \cdots \partial r_d^{\beta_d}}$ where $|\beta| := \sum_{i=1}^{d} \beta_i$.

The spaces $\mathscr{D}_{ind}(\mathbb{R}^d)$ and $\mathscr{D}_{proj}(\mathbb{R}^d)$ are obtained by endowing $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with two different topologies which both make the latter into a complete locally convex vector space.

4.1.1 Topological structures on $\mathcal{C}_c^\infty(\mathbb{R}^d)$

The first topology on the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is constructed as follows (see [63, Section V.4, vol. I] and Section C.1 for more details and definitions).

Definition 4.1.1.
Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an increasing family of relatively compact open sets such that $\mathbb{R}^d = \bigcup_{n\in\mathbb{N}} \Lambda_n$. Let us consider the space $\mathcal{C}_c^\infty(\overline{\Lambda_n})$ of all infinitely differentiable functions on $\mathbb{R}^d$ with compact support contained in $\overline{\Lambda_n}$ and let us endow $\mathcal{C}_c^\infty(\overline{\Lambda_n})$ with the Frechét topology generated by the seminorms

$$\|\varphi\|_\beta := \left\|D^\beta \varphi\right\|_\infty = \max_{\mathbf{r}\in\overline{\Lambda_n}} \left|D^\beta \varphi(\mathbf{r})\right|, \quad \beta \in \mathbb{N}_0^d.$$

Then as sets

$$\mathcal{C}_c^\infty(\mathbb{R}^d) = \bigcup_{n\in\mathbb{N}} \mathcal{C}_c^\infty(\overline{\Lambda_n}).$$

We denote by $\mathscr{D}_{ind}(\mathbb{R}^d)$ the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ endowed with the inductive limit topology τ_{ind} induced by this construction.
A neighbourhood base for τ_{ind} about zero is given by

$$\begin{aligned} \mathscr{B}_{\tau_{ind}}(0) : \;=\; & \big\{O \subset \mathcal{C}_c^\infty(\mathbb{R}^d) : O \text{ balanced, absorbing and convex,} \\ & \text{s.t. } O \cap \mathcal{C}_c^\infty(\overline{\Lambda_n}) \text{ is open in } \mathcal{C}_c^\infty(\overline{\Lambda_n})\big\}. \end{aligned} \tag{4.1}$$

The previous definition is independent of the choice of the Λ_n's.

Remark 4.1.2.
Let us define on $\mathcal{C}_c^\infty(\overline{\Lambda_n})$ the following family of seminorms

$$\|\varphi\|_{\leq a} := \sum_{\substack{|\beta|\leq a \\ \beta\in\mathbb{N}_0^d}} \max_{\mathbf{r}\in\overline{\Lambda_n}} \left|D^\beta \varphi(\mathbf{r})\right|, \quad a \in \mathbb{N}_0. \tag{4.2}$$

The two families of seminorms $(\|\cdot\|_\beta)_{\beta\in\mathbb{N}_0^d}$ *and* $(\|\cdot\|_{\leq a})_{a\in\mathbb{N}_0}$ *are equivalent. The latter family has the advantage to be* directed, *which in this case means that if* $a \leq b$ *then* $\|\varphi\|_{\leq a} \leq \|\varphi\|_{\leq b}$ *for all* $\varphi \in \mathcal{C}_c^\infty(\overline{\Lambda_n})$.
In the following, we will choose the family of seminorms more convenient for our aims.

The space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ can be also endowed with a projective limit topology τ_{proj} in the following way, (see [3, Chapter I, Section 3.10] for more details).

Definition 4.1.3.
Let I *be the set of all* $k = (k_1, k_2(\mathbf{r}))$ *such that* $k_1 \in \mathbb{N}_0$, $k_2 \in \mathcal{C}^\infty(\mathbb{R}^d)$ *with* $k_2(\mathbf{r}) \geq 1$ *for all* $\mathbf{r} \in \mathbb{R}^d$. *For each* $k \in I$, *let us introduce a norm on* $\mathcal{C}_c^\infty(\mathbb{R}^d)$ *by setting*

$$\|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} := \max_{\mathbf{r}\in\mathbb{R}^d}\left(k_2(\mathbf{r})\sum_{\substack{|\beta|\leq k_1\\ \beta\in\mathbb{N}_0^d}}\left|(D^\beta\varphi)(\mathbf{r})\right|\right).$$

Denote by $\mathscr{D}_k(\mathbb{R}^d)$ *the completion of* $\mathcal{C}_c^\infty(\mathbb{R}^d)$ *w.r.t. the norm* $\|\cdot\|_{\mathscr{D}_k(\mathbb{R}^d)}$. *Then as sets*

$$\mathcal{C}_c^\infty(\mathbb{R}^d) = \bigcap_{k\in I}\mathscr{D}_k(\mathbb{R}^d).$$

We denote by $\mathscr{D}_{proj}(\mathbb{R}^d)$ *the space* $\mathcal{C}_c^\infty(\mathbb{R}^d)$ *endowed with the projective limit topology* τ_{proj} *induced by this construction.*
A neighbourhood base for τ_{proj} *about zero is given by*

$$\mathscr{B}_{\tau_{proj}}(0) := \left\{U_{k;\varepsilon} \subset \mathcal{C}_c^\infty(\mathbb{R}^d) : k \in I,\, 0 < \varepsilon \in \mathbb{R}\right\}, \tag{4.3}$$

with

$$U_{k;\varepsilon} := \left\{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d) : \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} < \varepsilon\right\}.$$

As sets, $\mathscr{D}_{ind}(\mathbb{R}^d)$ and $\mathscr{D}_{proj}(\mathbb{R}^d)$ are the same but the topologies τ_{ind} and τ_{proj} are not equivalent. In fact, the following relation holds.

Proposition 4.1.4.
$\tau_{proj} \subseteq \tau_{ind}$

Proof.
To show $\tau_{proj} \subseteq \tau_{ind}$ we need to prove that

$$\forall\, U_{k;\varepsilon} \in \mathscr{B}_{\tau_{proj}}(0),\ \exists\, O \in \mathscr{B}_{\tau_{ind}}(0) :\ O \subseteq U_{k;\varepsilon}. \tag{4.4}$$

For convenience, in Definition 4.1.1 we will take as Λ_n the open ball $B_n(0)$ of center $0 \in \mathbb{R}^d$ and radius $n \in \mathbb{N}$, and we will consider on $\mathcal{C}_c^\infty(\overline{B_n(0)})$ the family of seminorms defined in (4.2).

Let us fix $k \in I$ and $\varepsilon > 0$. We show that (4.4) is satisfied for $O = U_{k;\varepsilon}$. In fact, the set $U_{k;\varepsilon}$ is balanced, absorbing and convex because it is defined by the seminorm $\|\cdot\|_{\mathscr{D}_k(\mathbb{R}^d)}$ (see Proposition C.1.7). Moreover, $U_{k;\varepsilon} \cap \mathcal{C}_c^\infty(\overline{B_n(0)})$ is open in $\mathcal{C}_c^\infty(\overline{B_n(0)})$.

To prove the latter we need to show that for any $\varphi \in U_{k;\varepsilon} \cap \mathcal{C}_c^\infty(\overline{B_n(0)})$

$$\exists\, 0 < \varepsilon' \in \mathbb{R},\ \exists\, a \in \mathbb{N}_0 \text{ s.t. } B^a_{\varepsilon'}(\varphi) \subseteq U_{k;\varepsilon} \cap \mathcal{C}_c^\infty(\overline{B_n(0)}), \tag{4.5}$$

where $B^a_{\varepsilon'}(\varphi) := \left\{\psi \in \mathcal{C}_c^\infty(\overline{B_n(0)}) : \|\psi - \varphi\|_{\leq a} < \varepsilon'\right\}$.

Fixed $\varphi \in U_{k;\varepsilon} \cap \mathcal{C}_c^\infty(\overline{B_n(0)})$, let us choose:

- $a \in \mathbb{N}_0$ such that $a \geq k_1$.
  This implies that $\|f\|_{\leq a} \geq \|f\|_{\leq k_1}$ for all $f \in \mathcal{C}_c^\infty(\overline{B_n(0)})$.
- $\varepsilon' := \frac{\varepsilon - \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)}}{\max_{\mathbf{r} \in \overline{B_n(0)}} k_2(\mathbf{r})}$.
  Note that the assumptions on k_2 guarantee that $0 < \max_{\mathbf{r} \in \overline{B_n(0)}} k_2(\mathbf{r}) < \infty$.

If $\psi \in B^a_{\varepsilon'}(\varphi)$ then we have

$$\begin{aligned}
\|\psi\|_{\mathscr{D}_k(\mathbb{R}^d)} &\leq \|\psi - \varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} + \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} \\
&= \max_{\mathbf{r} \in \overline{B_n(0)}} \left(k_2(\mathbf{r}) \sum_{|\beta| \leq k_1} \left|(D^\beta(\psi - \varphi))(\mathbf{r})\right| \right) + \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} \\
&\leq \left(\max_{\mathbf{r} \in \overline{B_n(0)}} k_2(\mathbf{r}) \right) \sum_{|\beta| \leq k_1} \max_{\mathbf{r} \in \overline{B_n(0)}} \left|(D^\beta(\psi - \varphi))(\mathbf{r})\right| + \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} \\
&= \left(\max_{\mathbf{r} \in \overline{B_n(0)}} k_2(\mathbf{r}) \right) \|\psi - \varphi\|_{\leq k_1} + \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} \\
&\leq \left(\max_{\mathbf{r} \in \overline{B_n(0)}} k_2(\mathbf{r}) \right) \|\psi - \varphi\|_{\leq a} + \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} \\
&\leq \left(\max_{\mathbf{r} \in \overline{B_n(0)}} k_2(\mathbf{r}) \right) \varepsilon' + \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)} = \varepsilon.
\end{aligned}$$

Hence, we proved that if $\psi \in B^a_{\varepsilon'}(\varphi)$ then $\psi \in U_{k;\varepsilon} \cap \mathcal{C}_c^\infty(\overline{B_n(0)})$. Therefore, (4.5) holds.

$\square$

From Proposition 4.1.4, it follows that $\mathscr{D}'_{proj}(\mathbb{R}^d) \subseteq \mathscr{D}'_{ind}(\mathbb{R}^d)$. The latter inclusion is actually strict.
For instance, let us consider the case $d = 1$ and the function

$$\eta(\varphi) := \sum_{\beta=0}^{\infty} D^{\beta}(\delta_{\beta}(\varphi)) = \sum_{\beta=0}^{\infty} D^{\beta}\varphi(\beta)$$

for any $\varphi \in \mathcal{C}_c^{\infty}(\mathbb{R})$. We can prove that η is in $\mathscr{D}'_{ind}(\mathbb{R})$ but it does not belong to $\mathscr{D}'_{proj}(\mathbb{R})$.
Step I: $\eta \in \mathscr{D}'_{ind}(\mathbb{R})$.
Recall that $\eta \in \mathscr{D}'_{ind}(\mathbb{R})$ if and only if for every compact set $\Lambda \subset \mathbb{R}$ there exists a positive constant C and an integer m such that, for any $\varphi \in \mathcal{C}_c^{\infty}(\Lambda)$, we have that $|\eta(\varphi)| \leq C||\varphi||_{\leq m}$ (see [63, Vol. I, p. 148]).
Let $m \in \mathbb{N}$ such that $\Lambda \subseteq [-m, m]$. Take $C = 1$. Then, for any $\varphi \in \mathcal{C}_c^{\infty}(\Lambda)$ we have that

$$\begin{aligned} |\eta(\varphi)| = \left|\sum_{\beta=0}^{\infty} D^{\beta}(\varphi(\beta))\right| &= \left|\sum_{\beta=0}^{m} D^{\beta}(\varphi(\beta))\right| \\ &\leq \sum_{\beta=0}^{m} \left|D^{\beta}(\varphi(\beta))\right| \\ &\leq \sum_{\beta=0}^{m} \max_{r\in\Lambda} \left|D^{\beta}(\varphi(r))\right| \\ &= ||\varphi||_{\leq m}. \end{aligned}$$

Step II: $\eta \notin \mathscr{D}'_{proj}(\mathbb{R})$.
Recall that $\eta \in \mathscr{D}'_{proj}(\mathbb{R})$ if and only if there exists $k \in I$ such that $\eta \in \mathscr{D}'_k(\mathbb{R})$. We then have to show that for every $k \in I$ and for all $C > 0$ there exists $\psi \in \mathcal{C}_c^{\infty}(\mathbb{R})$ such that

$$|\eta(\psi)| > C\, \|\psi\|_{\mathscr{D}_k(\mathbb{R})}. \tag{4.6}$$

For any $k_1 \in \mathbb{N}_0$ and for any $\lambda \in \mathbb{R}$, let us consider a function $\varphi \in \mathcal{C}_c^{\infty}((k_1, k_1+2))$, with $k_1 + 1 \in supp(\varphi)$, which we define via its $(k_1 + 1)$-translated (along the opposite orientation of the r-axis) function

$$\varphi_{k_1,\lambda}(r) := (\chi f_{k_1,\lambda})(r)$$

where

$$\chi \in \mathcal{C}_c^{\infty}(\mathbb{R}) \text{ and } \chi(r) := \begin{cases} 1 & \text{if } |r| \leq \frac{1}{4} \\ 0 & \text{if } |r| \geq \frac{1}{2}, \end{cases}$$

and $f_{k_1,\lambda} : [-1,1] \to \mathbb{R}$ is defined as follows.

$$f_{k_1,\lambda}(r) := \int_{-1}^{r} f_{k_1-1,\lambda}(t)dt \quad \text{with} \quad f_{0,\lambda}(r) = \sin(\lambda r).$$

Let us notice that $\varphi_{k_1,\lambda} \in \mathcal{C}_c^\infty(\mathbb{R})$ because $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ and $f_{k_1,\lambda} \in \mathcal{C}^\infty((-1,1))$. In particular, $0 \in supp(\varphi_{k_1,\lambda})$ and $supp(\varphi_{k_1,\lambda}) \subset [-\frac{1}{2}, \frac{1}{2}]$.
Moreover, for $k_1 \geq 1$ we have that $Df_{k_1,\lambda}(r) = f_{k_1-1,\lambda}(r)$ and so, in general, for $\beta \leq k_1+1$, $D^\beta f_{k_1,\lambda}(r) = f_{k_1-\beta,\lambda}(r)$ with $f_{-1,\lambda}(r) := Df_{0,\lambda}(r)$.
Then, for any λ we have that

$$\begin{aligned}
|\eta(\varphi(r))| = \left|D^{k_1+1}(\varphi(k_1+1))\right| &= \left|D^{k_1+1}(\varphi_{k_1,\lambda}(0))\right| \\
&= \left|D^{k_1+1}(\chi f_{k_1,\lambda}(0))\right| \\
&= \left|\sum_{j=0}^{k_1+1} \binom{k_1+1}{j} D^{k_1+1-j}\chi(0)\, D^j f_{k_1,\lambda}(0)\right| \\
&= \left|D^{k_1+1} f_{k_1,\lambda}(0)\right| \\
&= |f_{-1,\lambda}(0)| \\
&= |Df_{0,\lambda}(0)| \\
&= |\lambda \cos(\lambda \cdot 0)| \\
&= |\lambda|\,.
\end{aligned}$$

Furthermore, since $|f_{0,\lambda}| \leq 1$, we get that $|f_{k_1,\lambda}| \leq 2^{k_1}$ for $k_1 \geq 0$. In fact,

$$|f_{k_1,\lambda}(r)| \leq \int_{-1}^{r} |f_{k_1-1,\lambda}(t)|dt \leq \int_{-1}^{r} 2^{k_1-1}dt \leq 2^{k_1}.$$

Moreover, we have that for any $\beta, i \in \mathbb{N}_0$ with $i \leq \beta \leq k_1$

$$2^{k_1-\beta+i}\binom{\beta}{i} \leq \sum_{j=0}^{\beta} 2^{k_1-\beta+j}\binom{\beta}{j} = 2^{k_1}\sum_{j=0}^{\beta}\left(\frac{1}{2}\right)^{\beta-j}\binom{\beta}{j} = 3^\beta 2^{k_1-\beta} \leq 6^{k_1}.$$

Then, for $k = (k_1, k_2(r)) \in I$, we have that

$$
\begin{aligned}
\|\varphi\|_{\mathscr{D}_k(\mathbb{R})} &= \max_{r\in\mathbb{R}} \left(k_2(r) \sum_{\beta=0}^{k_1} \left| (D^\beta \varphi_{k_1,\lambda})(r+k_1+1) \right| \right) \\
&= \max_{r\in\mathbb{R}} \left(k_2(r-k_1-1) \sum_{\beta=0}^{k_1} \left| (D^\beta \varphi_{k_1,\lambda})(r) \right| \right) \\
&= \max_{r\in\mathbb{R}} \left(k_2(r-k_1-1) \sum_{\beta=0}^{k_1} \left| (D^\beta (\chi f_{k_1,\lambda}))(r) \right| \right) \\
&= \max_{r\in\mathbb{R}} \left(k_2(r-k_1-1) \sum_{\beta=0}^{k_1} \left| \sum_{j=0}^{\beta} \binom{\beta}{j} D^j \chi(r)\, D^{\beta-j} f_{k_1,\lambda}(r) \right| \right) \\
&= \max_{r\in\mathbb{R}} \left(k_2(r-k_1-1) \sum_{\beta=0}^{k_1} \left| \sum_{j=0}^{\beta} \binom{\beta}{j} D^j \chi(r)\, f_{k_1-\beta+j,\lambda}(r) \right| \right) \\
&\leq \max_{r\in\mathbb{R}} \left(k_2(r-k_1-1) \sum_{\beta=0}^{k_1} \sum_{j=0}^{\beta} 2^{k_1-\beta+j} \binom{\beta}{j} \left| D^j \chi(r) \right| \right) \\
&\leq 6^{k_1}(k_1+1) \max_{r\in\mathbb{R}} \left(k_2(r-k_1-1) \sum_{j=0}^{k_1} \left| D^j \chi(r) \right| \right) \\
&= 6^{k_1}(k_1+1) \max_{r\in\mathbb{R}} \left(k_2(r) \sum_{j=0}^{k_1} \left| D^j \chi(r+k_1+1) \right| \right) \\
&= 6^{k_1}(k_1+1) \, \|\chi_{k_1}\|_{\mathscr{D}_k(\mathbb{R})} ,
\end{aligned}
$$

where χ_{k_1} is the (k_1+1)-translation (along the orientation of the r-axis) of χ. Therefore, (4.6) is satisfied by taking $\psi = \varphi$ and λ such that

$$
|\lambda| > C\, 6^{k_1}(k_1+1)\, \|\chi_{k_1}\|_{\mathscr{D}_k(\mathbb{R})} .
$$

4.1.2 Measurability of $\mathscr{D}'_{proj}(\mathbb{R}^d)$ in $\mathscr{D}'_{ind}(\mathbb{R}^d)$

Let us equip the space $\mathscr{D}'_{proj}(\mathbb{R}^d)$ with the weak topology τ_w^{proj}, i.e. the smallest topology such that the mappings

$$
\begin{aligned}
\Phi_f : \quad \mathscr{D}'_{proj}(\mathbb{R}^d) &\to \mathbb{R} \\
\eta \qquad &\mapsto \langle f, \eta \rangle := \eta(f)
\end{aligned}
\tag{4.7}
$$

are continuous for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

In the same way, we equip the space $\mathscr{D}'_{ind}(\mathbb{R}^d)$ with the weak topology τ_w^{ind}, i.e. the smallest topology such that the mappings

$$\begin{aligned} \Psi_f: \quad \mathscr{D}'_{ind}(\mathbb{R}^d) &\to \mathbb{R} \\ \eta \quad &\mapsto \langle f, \eta \rangle := \eta(f) \end{aligned} \tag{4.8}$$

are continuous for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

In this section we consider the relation between the spaces $\left(\mathscr{D}'_{proj}(\mathbb{R}^d), \tau_w^{proj}\right)$ and $\left(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_w^{ind}\right)$ and their associated Borel $\sigma-$algebras. Let us denote by $\breve{\tau}_w^{ind}$ the relative topology given by τ_w^{ind} on $\mathscr{D}'_{proj}(\mathbb{R}^d)$, which is defined as follows

$$\breve{\tau}_w^{ind} := \left\{ U \cap \mathscr{D}'_{proj}(\mathbb{R}^d) : U \in \tau_w^{ind} \right\}.$$

Proposition 4.1.5.
The topology τ_w^{proj} coincides with $\breve{\tau}_w^{ind}$ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$.

Proof.
Let us preliminarily recall that

- The relative topology $\breve{\tau}_w^{ind}$ is the smallest topology such that the inclusion map $i : \mathscr{D}'_{proj}(\mathbb{R}^d) \hookrightarrow \mathscr{D}'_{ind}(\mathbb{R}^d)$ is continuous.
- For any $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, if Φ_f is defined as in (4.7) then

$$\Psi_f, \text{ as in (4.8), fulfills } \Phi_f = \Psi_f \circ i. \tag{4.9}$$

Step I: $\tau_w^{proj} \subseteq \breve{\tau}_w^{ind}$
Let Φ_f be the function defined in (4.7).
Hence, by (4.9), Φ_f is also continuous w.r.t. $\breve{\tau}_w^{ind}$ because Ψ_f is continuous w.r.t. τ_w^{ind} and i is continuous w.r.t. $\breve{\tau}_w^{ind}$.
Since τ_w^{proj} is the smallest topology such that the mappings Φ_f for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ in (4.7) are continuous, then we have the conclusion.
Step II: $\breve{\tau}_w^{ind} \subseteq \tau_w^{proj}$
The inclusion map i is continuous w.r.t. τ_w^{proj} if and only if for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ we have $\Psi_f \circ i$ is continuous w.r.t. τ_w^{proj}. By (4.9) we get that $\Phi_f = \Psi_f \circ i$ and Φ_f is continuous w.r.t. τ_w^{proj} by the definition of the latter topology. Hence, i is also continuous w.r.t. τ_w^{proj}.
Since $\breve{\tau}_w^{ind}$ is the smallest topology such that i is continuous, we have the conclusion.

□

Corollary 4.1.6.
The $\sigma-$algebra generated by τ_w^{proj} coincides with the one generated by $\breve{\tau}_w^{ind}$ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$, i.e. $\sigma(\tau_w^{proj}) = \sigma(\breve{\tau}_w^{ind})$.

Proposition 4.1.7.
The $\sigma-$algebra $\sigma(\tau_w^{ind}) \cap \mathscr{D}'_{proj}(\mathbb{R}^d)$ coincides with $\sigma(\breve{\tau}_w^{ind})$.

Proof.
Step I: $\sigma(\breve{\tau}_w^{ind}) \subseteq \sigma(\tau_w^{ind}) \cap \mathscr{D}'_{proj}(\mathbb{R}^d)$
The $\sigma-$algebra generated by $\breve{\tau}_w^{ind}$ is the smallest $\sigma-$algebra containing the topology $\breve{\tau}_w^{ind}$, i.e. the smallest $\sigma-$algebra such that the sets $O \cap \mathscr{D}'_{proj}(\mathbb{R}^d)$ are measurable for any $O \in \tau_w^{ind}$.
Hence, it sufficies to show that, for any $O \in \tau_w^{ind}$, the sets $O \cap \mathscr{D}'_{proj}(\mathbb{R}^d)$ are measurable w.r.t. the $\sigma-$algebra generated by τ_w^{ind} restricted to $\mathscr{D}'_{proj}(\mathbb{R}^d)$.
This is true because a set $O \in \tau_w^{ind}$ is trivially measurable w.r.t. the $\sigma-$algebra generated by τ_w^{ind} and therefore, $O \cap \mathscr{D}'_{proj}(\mathbb{R}^d)$ belongs to the $\sigma-$algebra generated by τ_w^{ind} restricted to $\mathscr{D}'_{proj}(\mathbb{R}^d)$.
Step II: $\sigma(\tau_w^{ind}) \cap \mathscr{D}'_{proj}(\mathbb{R}^d) \subseteq \sigma(\breve{\tau}_w^{ind})$
The $\sigma-$algebra generated by τ_w^{ind} restricted to $\mathscr{D}'_{proj}(\mathbb{R}^d)$ is the smallest $\sigma-$algebra which makes the inclusion map $i : \mathscr{D}'_{proj}(\mathbb{R}^d) \hookrightarrow \mathscr{D}'_{ind}(\mathbb{R}^d)$ measurable.
Hence, it remains to show that the inclusion map i is measurable w.r.t. the $\sigma-$algebra generated by $\breve{\tau}_w^{ind}$.
This is true because the inclusion map i results to be continuous w.r.t. $\breve{\tau}_w^{ind}$ and therefore i is also measurable w.r.t. the $\sigma-$algebra generated by $\breve{\tau}_w^{ind}$.

□

Corollary 4.1.8.
The $\sigma-$algebra $\sigma(\tau_w^{ind}) \cap \mathscr{D}'_{proj}(\mathbb{R}^d)$ coincides with $\sigma(\tau_w^{proj})$.

Let us recall some properties of $\mathscr{D}'_{ind}(\mathbb{R}^d)$ (for the definitions of Polish, Lusin and Radon spaces see Definitions C.3.5, C.3.10 and C.3.13, respectively).
The space $(\mathcal{C}_c^\infty(\mathbb{R}^d), \tau_{ind})$ is Lusin, because every Frechét separable space is Polish, and so Lusin, and the inductive limit of countably many Lusin spaces is Lusin (see [71, Examples, p. 115]).
By [71, Corollary 1, p. 115], the space $\left(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_c^{ind}\right)$, where τ_c^{ind} the topology of compact convergence, is Lusin. Let us consider the strong topology τ_s^{ind} on $\mathscr{D}'_{ind}(\mathbb{R}^d)$. The space $\left(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind}\right)$ is Lusin. In fact, τ_s^{ind} coincides with τ_c^{ind} (see [71, p. 115]). Then there exists τ' with $\tau_s^{ind} \subset \tau'$ such that $\left(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau'\right)$ is Polish. Hence, since $\tau_w^{ind} \subset \tau_s^{ind}$, we have that $\left(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_w^{ind}\right)$ is also Lusin.

Since every Lusin space is a Radon space (see [71, Theorem 9, p. 122]), the following proposition holds.

Proposition 4.1.9.
$\left(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_w^{ind}\right)$ is a Radon space, i.e. every finite Borel measure on $\mathscr{D}'_{ind}(\mathbb{R}^d)$ is inner regular (see Definition C.3.1).

We were not able to find in literature analogous results about whether the space $\left(\mathscr{D}'_{proj}(\mathbb{R}^d), \tau_w^{proj}\right)$ is a Radon space or not. Moreover, this property does not follow applying to $\mathscr{D}'_{proj}(\mathbb{R}^d)$ the same techniques as the ones used in [71] for $\mathscr{D}'_{ind}(\mathbb{R}^d)$.

4.1.3 The space $\mathscr{D}_{proj}(\mathbb{R}^d)$ as projective limit of weighted Sobolev spaces

In Definition 4.1.3 we introduced the topological space $\mathscr{D}_{proj}(\mathbb{R}^d)$ as the projective limit of the spaces $\mathscr{D}_k(\mathbb{R}^d)$, for all $k \in I$. Here we show that $\mathscr{D}_{proj}(\mathbb{R}^d)$ can be constructed in a similar way starting from a collection of weighted Sobolev spaces. This construction is more convenient for our purposes because it writes $\mathscr{D}_{proj}(\mathbb{R}^d)$ as projective limit of Hilbert spaces. Moreover, it is possible to prove that $\mathscr{D}_{proj}(\mathbb{R}^d)$ is nuclear.

Let us recall the notion of weighted Sobolev space $W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$ for an integer index k_1 and a positive continuous weight function k_2 on $\mathbb{R}^d$. The space $W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$ is defined as the completion of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with respect to the following weighted norm

$$\|\varphi\|_{W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})} := \left(\sum_{\substack{|\beta| \le k_1 \\ \beta \in \mathbb{N}_0^d}} \int_{\mathbb{R}^d} \left|(D^\beta \varphi)(\mathbf{r})\right|^2 k_2(\mathbf{r}) d\mathbf{r} \right)^{\frac{1}{2}}. \tag{4.10}$$

Note that, although the functions φ are real-valued, we prefer to write $|\cdot|^2$ instead of $(\cdot)^2$.

Definition 4.1.10 (Condition (D))**.**
We say that the set $K_0 \subseteq I$ satisfies Condition (D) *if:*
"For any pair $k = (k_1, k_2(\mathbf{r})) \in K_0$ there exists $k' = (k'_1, k'_2(\mathbf{r})) \in K_0$ such that

- *$k'_1 \ge k_1 + l$ (where l is the smallest integer greater than $\frac{d}{2}$)*

- $k_2'(\mathbf{r}) \geq \left(\max_{|\beta|\leq l} |(D^\beta q)(\mathbf{r})| \right)^2$, $\forall \mathbf{r} \in \mathbb{R}^d$, *for some function* $q(\mathbf{r}) \in \mathcal{C}^l(\mathbb{R}^d)$ *chosen such that*

$$q^2(\mathbf{r}) \geq k_2(\mathbf{r}), \forall \mathbf{r} \in \mathbb{R}^d \quad \textit{and} \quad \int_{\mathbb{R}^d} \frac{k_2(\mathbf{r})}{q^2(\mathbf{r})} d\mathbf{r} < \infty.$$

Note that the function $q(\mathbf{r})$ *depends on* $k_2(\mathbf{r})$ *and* $k_2'(\mathbf{r})$*."*

Condition (D) is sufficient for the space $\underset{(k_1,k_2(\mathbf{r}))\in K_0}{\operatorname{proj\,lim}} W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$ to be nuclear (see [3, p. 79]). In fact, we have that

Proposition 4.1.11.
If K_0 *fulfills* Condition (D), *then for every* $k = (k_1, k_2(\mathbf{r})) \in K_0$ *there exists* $k' = (k_1', k_2'(\mathbf{r})) \in K_0$ *such that the embedding*

$$W_2^{k_1'}(\mathbb{R}^d, k_2'(\mathbf{r})d\mathbf{r}) \hookrightarrow W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$$

is quasi-nuclear.

Moreover, the following theorem shows that if we consider $K_0 = I$ then the projective limit of the corresponding weighted Sobolev spaces is not only nuclear but also coincides with $\mathscr{D}_{proj}(\mathbb{R}^d)$ (see [3, Theorem 3.9, p. 78] for the proof of this result).

Theorem 4.1.12.
Let I *be the set of all pairs* $k = (k_1, k_2(\mathbf{r}))$ *such that* $k_1 \in \mathbb{N}_0$ *and* $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ *with* $k_2(\mathbf{r}) \geq 1$, *for all* $\mathbf{r} \in \mathbb{R}^d$. *The space* $\mathscr{D}_{proj}(\mathbb{R}^d)$ *coincides with the projective limit of all the spaces* $W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$ *with* $k = (k_1, k_2(\mathbf{r})) \in I$ *and it is nuclear.*

Let us note that the set of index I always fulfills Condition (D).
In fact, for any $(k_1, k_2(\mathbf{r})) \in I$ let $k_1' \geq k_1 + l$ (where l is the smallest integer greater than $\frac{d}{2}$) and $k_2'(\mathbf{r}) := 1 + \sum_{|\beta|\leq l} \left|(D^\beta q)(\mathbf{r})\right|^2$, where $q(\mathbf{r}) = (k_2(\mathbf{r})p(\mathbf{r}))^{\frac{1}{2}}$ for some $p(\mathbf{r}) \geq 1$ such that $\int_{\mathbb{R}^d} p(\mathbf{r})^{-1} d\mathbf{r} < \infty$. Note that, $\int_{\mathbb{R}^d} \frac{k_2(\mathbf{r})}{q^2(\mathbf{r})} d\mathbf{r} = \int_{\mathbb{R}^d} p(\mathbf{r})^{-1} d\mathbf{r} < \infty$ and for all $\mathbf{r} \in \mathbb{R}^d$ we have $q^2(\mathbf{r}) = k_2(\mathbf{r})p(\mathbf{r}) \geq k_2(\mathbf{r})$ and

$$k_2'(\mathbf{r}) := 1 + \sum_{|\beta|\leq l} \left|(D^\beta q)(\mathbf{r})\right|^2 \geq \sum_{|\beta|\leq l} \left|(D^\beta q)(\mathbf{r})\right|^2 \geq \left(\max_{|\beta|\leq l} |(D^\beta q)(\mathbf{r})| \right)^2.$$

Hence, since $k_2'(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$ and $k_2' \in \mathcal{C}^\infty(\mathbb{R}^d)$ because so is q, we have that $(k_1', k_2'(\mathbf{r})) \in I$.

Hence, the fact that $\underset{(k_1,k_2(\mathbf{r}))\in I}{\mathrm{proj\,lim}}\ W_2^{k_1}(\mathbb{R}^d, k_2(\mathbf{r})d\mathbf{r})$ is nuclear directly follows by Proposition 4.1.11.

Let us prove a useful inequality.

Proposition 4.1.13.
Given $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$, let $k' = (k'_1, k'_2(\mathbf{r}))$ be a pair such that $\frac{d}{2} < k'_1 \in \mathbb{N}$ and $k'_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $k'_2(\mathbf{r}) \geq |(D^\kappa k_2)(\mathbf{r})|^2$ for all $|\kappa| \leq k'_1$. Then there exists a finite constant $C > 0$ such that

$$k_2(\mathbf{r})|\varphi(\mathbf{r})| \leq C\|\varphi\|_{W_2^{k'_1}(\mathbb{R}^d, k'_2(\mathbf{r})d\mathbf{r})} .$$

Proof.
Let us fix an integer $k'_1 > \frac{d}{2}$ and denote by $B_1(\mathbf{r})$ the open ball of radius 1 about the point $\mathbf{r} \in \mathbb{R}^d$. According to the Sobolev embedding theorem, for any $u \in W_2^{k'_1}(B_1(\mathbf{r}))$, where $W_2^{k'_1}(B_1(\mathbf{r})) := W_2^{k'_1}(B_1(\mathbf{r}), 1)$, we have that

$$\|u\|_{\mathcal{C}(\overline{B_1(\mathbf{r})})} \leq c_1\|u\|_{W_2^{k'_1}(B_1(\mathbf{r}))} ,$$

where c_1 is a positive constant independent of $\mathbf{r}$.
Then for any $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$|u(\mathbf{r})| \leq \|u\|_{\mathcal{C}(\overline{B_1(\mathbf{r})})} \leq c_1\|u\|_{W_2^{k'_1}(B_1(\mathbf{r}))} \leq c_1\|u\|_{W_2^{k'_1}(\mathbb{R}^d)}. \tag{4.11}$$

Since $k_2\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ then, replacing $u(\mathbf{r})$ with $(k_2\varphi)(\mathbf{r})$ in (4.11), we get that

$$\begin{aligned} |k_2(\mathbf{r})\varphi(\mathbf{r})| &\leq c_1\|k_2\varphi\|_{W_2^{k'_1}(\mathbb{R}^d)} \\ &= c_1\left(\sum_{|\nu|\leq k'_1}\int_{\mathbb{R}^d}|(D^\nu(k_2\varphi))(\mathbf{r})|^2\,d\mathbf{r}\right)^{\frac{1}{2}} \\ &\leq c_1\left(\sum_{|\nu|\leq k'_1}\int_{\mathbb{R}^d}\left|\sum_{|\kappa|\leq k'_1}\sum_{|\lambda|\leq k'_1}c_{\nu\kappa\lambda}(D^\kappa k_2)(\mathbf{r})(D^\lambda\varphi)(\mathbf{r})\right|^2 d\mathbf{r}\right)^{\frac{1}{2}}, \end{aligned} \tag{4.12}$$

where the $c_{\nu\kappa\lambda}$'s are the coefficients obtained from Leibniz's formula applied in the last equality. Using Cauchy-Schwarz's inequality in the right-hand side of

(4.12), we get

$$\begin{aligned}
|k_2(\mathbf{r})\varphi(\mathbf{r})| &\leq c_1 \left(\sum_{|\nu|\leq k_1'} \int_{\mathbb{R}^d} c_2 \sum_{|\kappa|\leq k_1'} \sum_{|\lambda|\leq k_1'} c^2_{\nu\kappa\lambda} \left|(D^\kappa k_2)(\mathbf{r})\right|^2 \left|(D^\lambda \varphi)(\mathbf{r})\right|^2 d\mathbf{r} \right)^{\frac{1}{2}} \\
&\leq c_1 c_2^{\frac{1}{2}} \left(\sum_{|\nu|\leq k_1'} \int_{\mathbb{R}^d} \sum_{|\kappa|\leq k_1'} \sum_{|\lambda|\leq k_1'} c^2_{\nu\kappa\lambda} \left|(D^\kappa k_2)(\mathbf{r})\right|^2 \left|(D^\lambda \varphi)(\mathbf{r})\right|^2 d\mathbf{r} \right)^{\frac{1}{2}} \\
&\leq c_1 c_2^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \sum_{|\lambda|\leq k_1'} \left(\sum_{|\nu|\leq k_1'} \sum_{|\kappa|\leq k_1'} c^2_{\nu\kappa\lambda} \right) k_2'(\mathbf{r}) \left|(D^\lambda \varphi)(\mathbf{r})\right|^2 d\mathbf{r} \right)^{\frac{1}{2}} \\
&\leq c_1 c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \sum_{|\lambda|\leq k_1'} k_2'(\mathbf{r}) \left|(D^\lambda \varphi)(\mathbf{r})\right|^2 d\mathbf{r} \right)^{\frac{1}{2}} = C\|\varphi\|_{W_2^{k_1'}(\mathbb{R}^d, k_2'(\mathbf{r})d\mathbf{r})},
\end{aligned}$$

where $c_3 := \max\limits_{|\lambda|\leq k_1'} \left(\sum\limits_{|\nu|\leq k_1'} \sum\limits_{|\kappa|\leq k_1'} c^2_{\nu\kappa\lambda} \right)$ and $C := c_1 c_2^{\frac{1}{2}} c_3^{\frac{1}{2}}$.

□

4.1.4 The space of Radon measure $\mathcal{R}(\mathbb{R}^d)$

By $\mathcal{R}(\mathbb{R}^d)$ we denote the set of all Radon measures (i.e. all non-negative Borel measures which are finite on compact sets) on $\mathbb{R}^d$. Namely,

$$\mathcal{R}(\mathbb{R}^d) = \left\{ \eta : \mathcal{B}(\mathbb{R}^d) \to [0, +\infty] \text{ meas.} |\ \eta(\Lambda) < +\infty, \forall \Lambda \in \mathcal{B}(\mathbb{R}^d), \Lambda \text{ compact} \right\}.$$

Proposition 4.1.14.
The following embedding holds

$$\mathcal{R}(\mathbb{R}^d) \subset \mathscr{D}'_{proj}(\mathbb{R}^d).$$

Proof.
For any $\eta \in \mathcal{R}(\mathbb{R}^d)$, we want to show that the functional in (4.34) is an element of $\mathscr{D}'_{proj}(\mathbb{R}^d)$. In other words, we need to prove that for any $\eta \in \mathcal{R}(\mathbb{R}^d)$ there exist $k = (k_1, k_2(\mathbf{r})) \in I$ and a finite positive constant c such that for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ we have

$$|\langle \varphi, \eta \rangle| \leq c\|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)},$$

(see Definition 4.1.3 for the notations).
Let us consider a partition of unity $\{\chi_n\}_{n=2}^\infty$ of $\mathbb{R}^d$ such that for any $n \in \mathbb{N}$ with

$n \geq 2$ we have $supp(\chi_n) \subset B_n(0) \setminus B_{n-2}(0)$. Recall that for each $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ there exists $N \in \mathbb{N}_0$ such that $\varphi \in \mathcal{C}_c^\infty(\overline{B_N(0)})$. Therefore, for $k_1 = 0$, for some real number $b > 1$ and for some $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that for all $2 \leq n \in \mathbb{N}$ we have $k_2(\mathbf{r}) \geq \eta(supp(\chi_n)) \cdot |\chi_n(\mathbf{r})| \cdot b^n$ on $supp(\chi_n)$, the following holds.

$$\begin{aligned}
|\langle \varphi, \eta \rangle| &= \left| \langle \sum_{n=2}^{\infty} \chi_n \varphi, \eta \rangle \right| = \left| \sum_{n=2}^{N+1} \langle \chi_n \varphi, \eta \rangle \right| \leq \sum_{n=2}^{N+1} |\langle \chi_n \varphi, \eta \rangle| \\
&\leq \sum_{n=2}^{N+1} \eta(supp(\chi_n)) \cdot \max_{\mathbf{r} \in supp(\chi_n)} |\chi_n(\mathbf{r}) \varphi(\mathbf{r})| \\
&= \sum_{n=2}^{N+1} \eta(supp(\chi_n)) \max_{\mathbf{r} \in supp(\chi_n)} (|\chi_n(\mathbf{r})| \cdot |\varphi(\mathbf{r})|) \\
&\leq \sum_{n=2}^{N+1} \max_{\mathbf{r} \in supp(\chi_n)} \left(\frac{k_2(\mathbf{r})}{b^n} |\varphi(\mathbf{r})| \right) \\
&= \sum_{n=2}^{N+1} b^{-n} \max_{\mathbf{r} \in supp(\chi_n)} (k_2(\mathbf{r}) |\varphi(\mathbf{r})|) \\
&\leq \left(\sum_{n=2}^{\infty} b^{-n} \right) \max_{\mathbf{r} \in \mathbb{R}^d} (k_2(\mathbf{r}) |\varphi(\mathbf{r})|) = \left(\frac{1}{b(b-1)} \right) \|\varphi\|_{\mathscr{D}_k(\mathbb{R}^d)}.
\end{aligned}$$

□

For further topological and measurable properties of $\mathcal{R}(\mathbb{R}^d)$ see Section C.2.

4.2 Realizability problem on a basic semi-algebraic subset of $\mathscr{D}'_{proj}(\mathbb{R}^d)$

Let $\mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right)$ be the set of all polynomials on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ of the form

$$P(\eta) := \sum_{j=0}^{N} \langle p^{(j)}, \eta^{\otimes j} \rangle, \tag{4.13}$$

where $p^{(0)} \in \mathbb{R}$ and $p^{(j)} \in \mathcal{C}_c^\infty(\mathbb{R}^{dj})$, $j = 1, \dots, N$ with $N \in \mathbb{N}$.
We denote by $\Sigma(\mathscr{D}'_{proj}(\mathbb{R}^d))$ the subset of all polynomials in $\mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right)$ which can be written as sum of squares of polynomials.

A subset $\mathcal{S}$ of $\mathscr{D}'_{proj}(\mathbb{R}^d)$ is said to be *basic semi-algebraic* if it can be written as

$$\mathcal{S} = \bigcap_{i \in Y} \left\{ \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) | \; P_i(\eta) \geq 0 \right\}, \tag{4.14}$$

where Y is an index set and $P_i \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right)$.
Note that we do not require that the index set Y is necessarily countable.

Moreover, if $\mathscr{P}_{\mathcal{S}}$ is the set of all the polynomials P_i's defining $\mathcal{S}$, we can define the quadratic module $\mathcal{Q}(\mathscr{P}_{\mathcal{S}})$ associated to the representation (4.14) of $\mathcal{S}$ as the convex cone in $\mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'_{proj}(\mathbb{R}^d))$ given by

$$\mathcal{Q}(\mathscr{P}_{\mathcal{S}}) := \bigcup_{\substack{Y_0 \subset Y \\ |Y_0| < \infty}} \left\{ \sum_{i \in Y_0} Q_i P_i \; : \; Q_i \in \Sigma(\mathscr{D}'_{proj}(\mathbb{R}^d)) \right\}.$$

W.l.o.g. we assume that $0 \in Y$ and we define P_0 as the polynomial such that $P_0(\eta) = 1$ for all $\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d)$.

Proposition 4.2.1.
The semi-algebraic set $\mathcal{S}$ defined in (4.14) *is closed in* $(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind})$.

Proof.
The main step is to prove that the polynomials P_i defining $\mathcal{S}$ are continuous w.r.t. τ_s^{ind}. Each P_i is of the form in (4.13) and so it can be written as

$$P_i(\eta) = \sum_{j=0}^{N(i)} F_j(\eta),$$

where $F_j(\eta) := \langle p_i^{(j)}, \eta^{\otimes j} \rangle$ for $j = 0, \ldots, N(i)$. Therefore, to show the continuity of P_i w.r.t. τ_s^{ind}, it is enough to prove that all the mappings F_j's are continuous w.r.t. τ_s^{ind}.
Note that $F_0(\eta) = p_i^{(0)} \in \mathbb{R}$ and so it is trivially continuous. The function $F_1(\eta) = \langle p_i^{(1)}, \eta \rangle$, with $p_i^{(1)} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, is continuous w.r.t. τ_w^{ind} by definition of weak topology. Hence, F_1 is also continuous w.r.t. τ_s^{ind}.
It remains to show that F_j is continuous for $j = 2, \ldots, N(i)$. Let us prove it only for $j = 2$ since the other cases follow similarly.
The mapping

$$\begin{aligned} F_2 : \; (\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind}) &\to \mathbb{R} \\ \eta \qquad &\mapsto F_2(\eta) := \langle p^{(2)}, \eta^{\otimes 2} \rangle, \quad \text{with } p^{(2)} \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}), \end{aligned}$$

is continuous w.r.t. τ_s^{ind} because it can be written as the composition of five continuous mappings, i.e.

$$\begin{tikzcd}
(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind}) \arrow[r, "a"] \arrow[ddddr, bend right, "F_2"'] & (\mathscr{D}'_{ind}(\mathbb{R}^d) \times \mathscr{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind} \times \tau_s^{ind}) \arrow[d, "b"] \\
& (\mathscr{D}'_{ind}(\mathbb{R}^d) \otimes \mathscr{D}'_{ind}(\mathbb{R}^d), \tau_\pi) \arrow[d, "c"] \\
& (\overline{\mathscr{D}'_{ind}(\mathbb{R}^d) \otimes \mathscr{D}'_{ind}(\mathbb{R}^d)}, \tau_\pi) \arrow[d, "d"] \\
& (\mathscr{D}'_{ind}(\mathbb{R}^{2d}), \tau_s^{ind}) \arrow[d, "e"] \\
& \mathbb{R}
\end{tikzcd}$$

where:

- The map a defined as

$$a(\eta) := (\eta, \eta), \quad \forall \eta \in \mathscr{D}'_{ind}(\mathbb{R}^d),$$

  is continuous by definition of Cartesian product.

- Since we considered the algebraic tensor product $\mathscr{D}'_{ind}(\mathbb{R}^d) \otimes \mathscr{D}'_{ind}(\mathbb{R}^d)$ endowed with the $\pi-$topology τ_π (see [79, Definition 43.2]), the map b defined as

$$b((\eta_1, \eta_2)) := \eta_1 \otimes \eta_2, \quad \forall \eta_1, \eta_2 \in \mathscr{D}'_{ind}(\mathbb{R}^d),$$

  is continuous.

- The map c is the natural embedding of $\mathscr{D}'_{ind}(\mathbb{R}^d) \otimes \mathscr{D}'_{ind}(\mathbb{R}^d)$ in its completion $\overline{\mathscr{D}'_{ind}(\mathbb{R}^d) \otimes \mathscr{D}'_{ind}(\mathbb{R}^d)}$ w.r.t. τ_π and hence c is continuous.

- The map d is the isomorphism given by Theorem 51.7 in [79] and hence it is continuous.

- The map e defined as

$$e(\zeta) := \langle p^{(2)}, \zeta \rangle, \quad \forall \zeta \in \mathscr{D}'_{ind}(\mathbb{R}^{2d}),$$

  where $p^{(2)} \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, is continuous w.r.t. the weak topology τ_w^{ind} on the space $\mathscr{D}'_{ind}(\mathbb{R}^{2d})$. Hence, it is also continuous w.r.t. the strong topology τ_s^{ind}

on $\mathscr{D}'_{ind}(\mathbb{R}^{2d})$.

By continuity of P_i, it follows that the set $\left\{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) \mid P_i(\eta) \geq 0\right\}$ is closed in $(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind})$. Consequently, $\mathcal{S}$ is also closed in $(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind})$.

□

Proposition 4.2.2.
The semi-algebraic set $\mathcal{S}$ defined in (4.14) is measurable in $(\mathscr{D}'_{ind}(\mathbb{R}^d), \sigma(\tau_w^{ind}))$.

Proof.
By Proposition 4.2.1 we have that $\mathcal{S} \in \sigma(\tau_s^{ind})$. Furthermore, we know that $\mathscr{D}'_{ind}(\mathbb{R}^d)$ endowed with τ_s^{ind} is a Lusin space and so Suslin (see Definition C.3.11). This guarantees that $\sigma(\tau_w^{ind})$ and $\sigma(\tau_s^{ind})$ coincide (see Proposition C.3.12). Hence, $\mathcal{S} \in \sigma(\tau_w^{ind})$.

□

Let us consider Problem 3.2.6 for $\mathcal{S}$ given by (4.14). To solve this problem we are going to make use of a version of the Riesz linear functional for the moment problem on $\mathscr{D}'_{proj}(\mathbb{R}^d)$.

Definition 4.2.3.
Given a sequence $m = (m^{(n)})_{n=0}^{\infty}$ with $m^{(n)} \in \mathscr{D}'_{proj}(\mathbb{R}^{dn})$ we define its associated functional L_m as follows.

$$\begin{aligned} L_m : \quad & \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right) && \to \mathbb{R} \\ & p(\eta) = \sum_{n=0}^{N} \langle p^{(n)}, \eta^{\otimes n}\rangle && \mapsto L_m(p) := \sum_{n=0}^{N} \langle p^{(n)}, m^{(n)}\rangle. \end{aligned}$$

The following is the main theorem of this chapter.

Theorem 4.2.4.
Let $m = (m^{(n)})_{n=0}^{\infty}$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. Assume that m fulfills the weighted Carleman's type condition

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)}}} = \infty, \tag{4.15}$$

for some $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$.

Then m is realized by a unique non-negative finite measure μ on $\mathcal{S}$ with

$$\int_{\mathcal{S}} \langle \frac{1}{k_2}, \eta\rangle^n \mu(d\eta) < \infty, \quad \forall\, n \in \mathbb{N}_0, \tag{4.16}$$

if and only if the following inequalities hold

$$L_m(h^2) \geq 0,\ L_m(P_i h^2) \geq 0,\ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right),\ \forall i \in Y, \tag{4.17}$$

and for any $n \in \mathbb{N}_0$ we have

$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} < \infty. \tag{4.18}$$

Remark 4.2.5.
Note that the condition in (4.17) *can be replaced with conditions on the quadratic module $\mathcal{Q}(\mathscr{P}_{\mathcal{S}})$, i.e. $L_y(P) \geq 0$ for all $P \in \mathcal{Q}(\mathscr{P}_{\mathcal{S}})$.*

Before proving Theorem 4.2.4 we need to show some preliminary results.

Lemma 4.2.6.
Let $m = (m^{(n)})_{n=0}^\infty$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. If m is realized by a non-negative finite measure μ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ and m satisfies (4.15)*, then for all $n \in \mathbb{N}_0$ we have that*

$$m_n := \int_{\mathbb{R}^{nd}} \frac{m^{(n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{n} k_2(\mathbf{r}_l)} < \infty,$$

where in particular $m_0 := \langle 1, m^{(0)} \rangle$.

Proof.
First of all let us note that by the realizability assumption follows that

$$m_0 = \langle 1, m^{(0)} \rangle = \mu(\mathscr{D}'_{proj}(\mathbb{R}^d)).$$

Hence, $m_0 < \infty$ since the realizing measure μ is assumed to be finite.
Moreover, since we assume that (4.15) holds, we get

$$m_{2n} < \infty, \quad \text{for infinitely many } n. \tag{4.19}$$

Now, let us recall that for any non-negative integer j there exists $C > 0$ such that

$$\forall i \leq j,\ \forall x \in \mathbb{R}, \quad |x|^i \leq C(1 + |x|^j).$$

Therefore, if we fix one of the infinitely many n for which (4.19) holds, then we have that for such an n there exists a finite positive constant C such that

$$\forall i \leq 2n,\ \forall \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d), \quad \langle \frac{\chi_R}{k_2}, \eta\rangle^i \leq \left|\langle \frac{\chi_R}{k_2}, \eta\rangle^i\right| \leq C\left(1 + \langle \frac{\chi_R}{k_2}, \eta\rangle^{2n}\right),$$

where R is a positive real number and χ_R is such that

$$\chi_R \in \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ and } \chi_R(\mathbf{r}) := \begin{cases} 1 & \text{if } |\mathbf{r}| \leq R \\ 0 & \text{if } |\mathbf{r}| \geq R+1. \end{cases} \tag{4.20}$$

Integrating both sides, we get for all $i \leq 2n$

$$\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \frac{\chi_R}{k_2}, \eta\rangle^i \mu(d\eta) \leq C\mu(\mathscr{D}'_{proj}(\mathbb{R}^d)) + C\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \frac{\chi_R}{k_2}, \eta\rangle^{2n} \mu(d\eta),$$

that is

$$\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \left(\frac{\chi_R}{k_2}\right)^{\otimes i}, \eta^{\otimes i}\rangle \mu(d\eta) \leq C' + C\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \left(\frac{\chi_R}{k_2}\right)^{\otimes 2n}, \eta^{\otimes 2n}\rangle \mu(d\eta),$$

with $C' := C\mu(\mathscr{D}'_{proj}(\mathbb{R}^d)) < \infty$.
By assumption the sequence m is realized by the measure μ and so the previous inequality becomes

$$\int_{\mathbb{R}^{id}} \prod_{l=1}^{i} \frac{\chi_R(\mathbf{r}_l)}{k_2(\mathbf{r}_l)} m^{(i)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_i) \leq C' + C\int_{\mathbb{R}^{2nd}} \prod_{l=1}^{2n} \frac{\chi_R(\mathbf{r}_l)}{k_2(\mathbf{r}_l)} m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n}).$$

Using the monotone convergence theorem for $R \to \infty$ we have that

$$\int_{\mathbb{R}^{id}} \frac{m^{(i)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_i)}{\prod_{l=1}^{i} k_2(\mathbf{r}_l)} \leq C' + C\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)},$$

i.e.

$$m_i \leq C' + Cm_{2n}.$$

Using (4.19) in the previous relation we get that $m_i < \infty$ for any $i \leq 2n$. But this is true for infinitely many n and so we get that m_i is finite for all $i \in \mathbb{N}$.

□

Proposition 4.2.7.
If a sequence $m = (m^{(n)})_{n=0}^\infty$, with $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ symmetric function of its n variables, satisfies (4.15) and (4.18) for some $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$, for all $\mathbf{r} \in \mathbb{R}^d$, then m is a determining sequence, i.e. (3.16) holds for m.

Proof.
Let $n \in \mathbb{N}$. Then, for any $f_1, \dots, f_{2n} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we get that

$$\begin{aligned}
\left|\left\langle f_1 \otimes \cdots \otimes f_{2n}, m^{(2n)} \right\rangle\right| := & \left| \int_{\mathbb{R}^{2nd}} f_1(\mathbf{r}_1) \cdots f_{2n}(\mathbf{r}_{2n}) m^{(2n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_{2n}) \right| \\
\leq & \int_{\mathbb{R}^{2nd}} \prod_{l=1}^{2n} k_2(\mathbf{r}_l) \left| f_l(\mathbf{r}_l) \right| \frac{m^{(2n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} \\
\leq & \int_{\mathbb{R}^{2nd}} \prod_{l=1}^{2n} C \left\| f_l(\mathbf{r}_l) \right\|_{W_2^{k_1'}(\mathbb{R}^d, k_2'(\mathbf{r})d\mathbf{r})} \frac{m^{(2n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} \\
= & C^{2n} \prod_{l=1}^{2n} \left\| f_l(\mathbf{r}_l) \right\|_{W_2^{k_1'}(\mathbb{R}^d, k_2'(\mathbf{r})d\mathbf{r})} \int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)},
\end{aligned}$$

where k_1' and $k_2'(\mathbf{r})$ are defined as in Proposition 4.1.13 (whose bound is used to get the latter inequality).

Moreover, if we define $\tilde{m}_n := \sqrt{C^{2n} \left(\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} \right)}$ then by (4.18) $\tilde{m}_n$ are finite for all $n \in \mathbb{N}_0$ and (4.15) implies that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\tilde{m}_n}} = \frac{1}{C} \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)}}} = \infty.$$

Since for any $n \in \mathbb{N}$ we have that $\inf\limits_{k \geq n} \sqrt[k]{\tilde{m}_k} \leq \sqrt[n]{\tilde{m}_n}$, then

$$\sum_{n=1}^{\infty} \frac{1}{\inf\limits_{k \geq n} \sqrt[k]{\tilde{m}_k}} = \infty.$$

Thus, by Theorem A.0.19, the class $C\{\tilde{m}_n\}$ is quasi-analytical. Hence, (3.16) holds for $k(m) = (k_1', k_2'(\mathbf{r}))$.

□

Definition 4.2.8.
Given a sequence $m = (m^{(n)})_{n=0}^\infty$, with $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ symmetric function of its n variables, and given a polynomial $P \in \mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'_{proj}(\mathbb{R}^d))$ of the form (4.13), we define the sequence ${}_Pm = \left(({}_Pm)^{(n)}\right)_{n=0}^\infty$ such that for any $f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{nd})$

$$\langle f^{(n)}, ({}_Pm)^{(n)} \rangle := \sum_{j=0}^{N} \langle p^{(j)} \otimes f^{(n)}, m^{(n+j)} \rangle. \tag{4.21}$$

Lemma 4.2.9.
Let $m = (m^{(n)})_{n=0}^{\infty}$ with $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ symmetric function of its n variables. Let P and Q two polynomials in $\mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'_{proj}(\mathbb{R}^d))$. Then,

$$L_m(PQ) = L_{{}_Pm}(Q).$$

Proof.
Two polynomials P and Q of different degree can be represented as

$$P(\eta) := \sum_{k=0}^{N} \langle p^{(k)}, \eta^{\otimes k} \rangle, \quad Q(\eta) := \sum_{i=0}^{N} \langle q^{(i)}, \eta^{\otimes j} \rangle,$$

with $N = \max\{\deg P, \deg Q\}$, by simply adding some coefficients equal to zero in the polynomial of smaller degree. Therefore, the product of P and Q can be written as

$$(PQ)(\eta) = \sum_{k,i=0}^{N} \langle p^{(k)} \otimes q^{(i)}, \eta^{\otimes(k+i)} \rangle.$$

So we have that

$$\begin{aligned}
L_m(PQ) &= L_m\left(\sum_{i=0}^{N} \sum_{k=0}^{N} \langle p^{(k)} \otimes q^{(i)}, \eta^{\otimes(k+i)} \rangle \right) \\
&= \sum_{i=0}^{N} \underbrace{\sum_{k=0}^{N} \langle p^{(k)} \otimes q^{(i)}, m^{(k+i)} \rangle}_{= \langle q^{(i)}, ({}_Pm)^{(i)} \rangle} \\
&= \sum_{i=0}^{N} \langle q^{(i)}, ({}_Pm)^{(i)} \rangle \\
&= L_{{}_Pm}(Q).
\end{aligned}$$

□

Proposition 4.2.10.
Let $P \in \mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'_{proj}(\mathbb{R}^d))$ and let $m = (m^{(n)})_{n=0}^{\infty}$ be such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ symmetric function of its n variables. If m is realized by a non-negative finite measure μ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ then the sequence ${}_Pm$ is realized by the measure $P\mu$ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$.

Proof.
Assume that P is of the form (4.13). We want to prove that for any $f^{(n)} \in$

$\mathcal{C}_c^\infty(\mathbb{R}^{nd})$

$$\langle f^{(n)}, ({}_P m)^{(n)}\rangle = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n}\rangle P(\eta)\mu(d\eta).$$

This is true since the following holds.

$$\begin{aligned}
\langle f^{(n)}, ({}_P m)^{(n)}\rangle &= \sum_{j=0}^{N} \langle p^{(j)} \otimes f^{(n)}, m^{(n+j)}\rangle \\
&= \sum_{j=0}^{N} \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle p^{(j)} \otimes f^{(n)}, \eta^{\otimes(n+j)}\rangle \mu(d\eta) \qquad (4.22)\\
&= \sum_{j=0}^{N} \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n}\rangle \langle p^{(j)}, \eta^{\otimes j}\rangle \mu(d\eta) \\
&= \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n}\rangle \sum_{j=0}^{N} \langle p^{(j)}, \eta^{\otimes j}\rangle \mu(d\eta) \\
&= \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n}\rangle P(\eta)\mu(d\eta).
\end{aligned}$$

Note that in (4.22) we made use of the assumption that m is realized by μ.

□

Note that if the sequence m is realized by a finite non-negative measure μ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$, then the equality in Lemma 4.2.9 can be alternatively proved by using Proposition 4.2.10 in the following way.

First of all, let us observe that for any polynomial $P \in \mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'_{proj}(\mathbb{R}^d))$ of the form (4.13) we have that

$$\begin{aligned}
L_m(P) &= \sum_{j=0}^{N} \langle p^{(j)}, m^{(j)}\rangle \\
&= \sum_{j=0}^{N} \left(\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle p^{(j)}, \eta^{\otimes j}\rangle \, \mu(d\eta)\right) \\
&= \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \left(\sum_{j=0}^{N} \langle p^{(j)}, \eta^{\otimes j}\rangle \right) \mu(d\eta) \\
&= \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} P(\eta)\, \mu(d\eta). \qquad (4.23)
\end{aligned}$$

Hence,

$$L_m(PQ) = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} P(\eta)Q(\eta)\, \mu(d\eta) = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} Q(\eta)P(\eta)\, \mu(d\eta) = L_{{}_P m}(Q),$$

where in the last equality we used Proposition 4.2.10.

Proposition 4.2.11.
Let $m = (m^{(n)})_{n=0}^{\infty}$ be such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ is a symmetric function of its n variables and (4.15) holds for some $k_2(\mathbf{r}) \in \mathcal{C}^{\infty}(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$. Suppose that the associated sequence $(m_n)_{n=0}^{\infty}$ with

$$m_n := \int_{\mathbb{R}^{nd}} \frac{m^{(n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n)}{\prod_{l=1}^{n} k_2(\mathbf{r}_l)}$$

is log-convex (w.l.o.g. suppose also $m_0 = 1$).
Then, the sequence ${}_Pm$ satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\int_{\mathbb{R}^{2nd}} \frac{({}_Pm)^{(2n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)}}} = \infty. \tag{4.24}$$

Proof.
Since in the polynomial P the coefficients $p^{(j)} \in \mathcal{C}_c^{\infty}(\mathbb{R}^{jd})$ and $k_2(\mathbf{r}) \in \mathcal{C}^{\infty}(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$, we get that for all $\mathbf{y} \in \mathbb{R}^{jd}$

$$0 \leq |p^{(j)}(\mathbf{y}_1, \ldots, \mathbf{y}_j)| \prod_{l=1}^{j} k_2(\mathbf{y}_l) \leq \max_{(\mathbf{x}_1,\ldots,\mathbf{x}_j) \in supp(p^{(j)})} |p^{(j)}(\mathbf{x}_1, \ldots, \mathbf{x}_j)| \prod_{l=1}^{j} k_2(\mathbf{x}_l) < \infty.$$

Let $\delta_j := \max_{(\mathbf{x}_1,\ldots,\mathbf{x}_j) \in supp(p^{(j)})} |p^{(j)}(\mathbf{x}_1, \ldots, \mathbf{x}_j)| \prod_{l=1}^{j} k_2(\mathbf{x}_l)$, then we have that

$$\begin{aligned}
& \int_{\mathbb{R}^{2nd}} \frac{({}_Pm)^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} \\
= & \sum_{j=0}^{N} \int_{\mathbb{R}^{(2n+j)d}} \frac{p^{(j)}(\mathbf{r}_{2n+1}, \ldots, \mathbf{r}_{2n+j}) m^{(2n+j)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n+j})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} \\
\leq & \sum_{j=0}^{N} \int_{\mathbb{R}^{(2n+j)d}} \frac{\prod_{l=2n+1}^{2n+j} k_2(\mathbf{r}_l) |p^{(j)}(\mathbf{r}_{2n+1}, \ldots, \mathbf{r}_{2n+j})| m^{(2n+j)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n+j})}{\prod_{l=1}^{2n+j} k_2(\mathbf{r}_l)} \\
\leq & \sum_{j=0}^{N} \delta_j \int_{\mathbb{R}^{(2n+j)d}} \frac{m^{(2n+j)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n+j})}{\prod_{l=1}^{2n+j} k_2(\mathbf{r}_l)} \\
\leq & \sum_{j=0}^{N} \delta_j m_{2n+j} \leq \left(\sum_{j=0}^{N} \delta_j \right) \max_{j=0,\ldots,N} m_{2n+j} = \delta(N) \max\{m_{2n}, m_{2n+N}\},
\end{aligned}$$

where $\delta(N) := \sum_{j=0}^{N} \delta_j$ and in the last equality we used that the sequence m_n is unimodal by Proposition A.0.31. We can distinguish two cases:

- If $\max\{m_{2n}, m_{2n+N}\} = m_{2n}$, then

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\int_{\mathbb{R}^{2nd}} \frac{(_P m)^{(2n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)}}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\delta(N) m_{2n}}}. \tag{4.25}$$

  By assumption, we have that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{m_{2n}}} = \infty$ and so, by Lemma A.0.28, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\delta(N) m_{2n}}} = \infty$ as well. Hence, the left-hand side of (4.25) diverges.

- If $\max\{m_{2n}, m_{2n+N}\} = m_{2n+N}$, then

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\int_{\mathbb{R}^{2nd}} \frac{(_P m)^{(2n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)}}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\delta(N) m_{2n+N}}}. \tag{4.26}$$

  By assumption, we have that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{m_{2n}}} = \infty$ and so $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\delta(N) m_{2n+N}}} = \infty$ by Theorem A.0.30 and Lemma A.0.28. Hence, the left-hand side of (4.26) diverges.

$\square$

Proposition 4.2.12.
Let $m = (m^{(n)})_{n=0}^{\infty}$ be such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is symmetric function of its n variables. If m is realized by a measure μ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ with finite local moments, then the sequence

$$m_n := \int_{\mathbb{R}^{nd}} \frac{m^{(n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_n)}{\prod_{l=1}^{n} k_2(\mathbf{r}_l)}$$

is log-convex.

Proof.
Since $m = (m^{(n)})_{n=0}^{\infty}$ is realized by a measure μ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$, then for any

$\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d)$ and for χ_R defined as in (4.20) we get

$$
\begin{aligned}
m_n^2 &= \left(\int_{\mathbb{R}^{nd}} \frac{m^{(n)}(d\mathbf{r}_1,\dots,d\mathbf{r}_n)}{\prod_{l=1}^n k_2(\mathbf{r}_l)}\right)^2 \\
&= \left(\lim_{R\to\infty}\int_{\mathbb{R}^{nd}} \frac{\prod_{l=1}^n \chi_R(\mathbf{r}_l)}{\prod_{l=1}^n k_2(\mathbf{r}_l)} m^{(n)}(d\mathbf{r}_1,\dots,d\mathbf{r}_n)\right)^2 \\
&= \lim_{R\to\infty}\left(\int_{\mathbb{R}^{nd}} \frac{\prod_{l=1}^n \chi_R(\mathbf{r}_l)}{\prod_{l=1}^n k_2(\mathbf{r}_l)} m^{(n)}(d\mathbf{r}_1,\dots,d\mathbf{r}_n)\right)^2 \\
&= \lim_{R\to\infty}\left(\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle\left(\frac{\chi_R}{k_2}\right)^{\otimes n}, \eta^{\otimes n}\rangle \mu(d\eta)\right)^2 \\
&= \lim_{R\to\infty}\left(\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle\frac{\chi_R}{k_2}, \eta\rangle^n \mu(d\eta)\right)^2 \\
&= \lim_{R\to\infty}\left(\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle\frac{\chi_R}{k_2}, \eta\rangle^{\frac{n-1}{2}}\langle\frac{\chi_R}{k_2}, \eta\rangle^{\frac{n+1}{2}} \mu(d\eta)\right)^2 \\
&\leq \lim_{R\to\infty}\left[\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle\frac{\chi_R}{k_2}, \eta\rangle^{n-1} \mu(d\eta)\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle\frac{\chi_R}{k_2}, \eta\rangle^{n+1} \mu(d\eta)\right] \\
&= \lim_{R\to\infty}\left[\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle\left(\frac{\chi_R}{k_2}\right)^{\otimes n-1}, \eta^{\otimes n-1}\rangle \mu(d\eta)\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle\left(\frac{\chi_R}{k_2}\right)^{\otimes n+1}, \eta^{\otimes n+1}\rangle \mu(d\eta)\right] \\
&= \lim_{R\to\infty}\left[\int_{\mathbb{R}^{(n-1)d}} \frac{\prod_{l=1}^{n-1} \chi_R(\mathbf{r}_l)}{\prod_{l=1}^{n-1} k_2(\mathbf{r}_l)} m^{(n-1)}(d\mathbf{r}_1,\dots,d\mathbf{r}_{n-1})\int_{\mathbb{R}^{(n+1)d}} \frac{\prod_{l=1}^{n+1} \chi_R(\mathbf{r}_l)}{\prod_{l=1}^{n+1} k_2(\mathbf{r}_l)} m^{(n+1)}(d\mathbf{r}_1,\dots,d\mathbf{r}_{n+1})\right] \\
&= \int_{\mathbb{R}^{(n-1)d}} \frac{m^{(n-1)}(d\mathbf{r}_1,\dots,d\mathbf{r}_{n-1})}{\prod_{l=1}^{n-1} k_2(\mathbf{r}_l)}\int_{\mathbb{R}^{(n+1)d}} \frac{m^{(n+1)}(d\mathbf{r}_1,\dots,d\mathbf{r}_{n+1})}{\prod_{l=1}^{n+1} k_2(\mathbf{r}_l)} \\
&= m_{n-1}m_{n+1}.
\end{aligned}
$$

□

Proof. (of Theorem 4.2.4)

Necessity

Assume that m is realized by a non-negative finite measure μ on $\mathcal{S}$. Then, by using (4.23), we get that for any $h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right)$ and for any $i \in Y$

$$L_m(h^2) = \int_{\mathcal{S}} h^2(\eta)\,\mu(d\eta) \quad \text{and} \quad L_m(P_i h^2) = \int_{\mathcal{S}} P_i(\eta)h^2(\eta)\,\mu(d\eta).$$

Hence, the inequalities (4.17) follow from the obvious fact that integrals of non-negative functions w.r.t. a non-negative measure are non-negative. Moreover, (4.18) follows by Lemma 4.2.6.

Sufficiency

Using the fact that a generic polynomial $h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right)$ has the form

$h(\eta) = \sum_{n=0}^{N} \langle h^{(n)}, \eta^{\otimes n} \rangle$ with $h^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{nd})$, the condition

$$L_m(h^2) \geq 0, \quad \forall\, h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right),$$

can be rewritten as

$$\sum_{i,j=0}^{\infty} \langle h^{(i)} \otimes h^{(j)}, m^{(i+j)} \rangle \geq 0, \quad \forall\, h^{(i)} \in \mathcal{C}_c^\infty(\mathbb{R}^{id}).$$

The latter means that the sequence m is positive semidefinite in the sense of Definition 3.2.7. Moreover, (4.15) and (4.18) imply that m is also determining by Proposition 4.2.7.

Summarizing, the sequence m is positive semidefinite and determining. Hence, Theorem 3.2.9 guarantees the existence of a unique non-negative measure μ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ with all finite generalized moment functions realizing m, i.e. for any $f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{nd})$

$$\langle f^{(n)}, m^{(n)} \rangle = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle \mu(d\eta).$$

Moreover, by Proposition 4.2.10, the sequence ${}_{P_i}m$ is realized by the signed measure $P_i\mu$, i.e. for any $f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{nd})$

$$\langle f^{(n)}, ({}_{P_i}m)^{(n)} \rangle = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle P_i(\eta)\mu(d\eta). \tag{4.27}$$

On the other hand, since by Lemma 4.2.9 $L_m(P_i h^2) = L_{{}_{P_i}m}(h^2)$, we have that $L_{{}_{P_i}m}(h^2) = L_m(P_i h^2) \geq 0$ for any $h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right)$, i.e. the sequence ${}_{P_i}m$ is positive semidefinite. By Proposition 4.2.12, the sequence of all $m_n := \int_{\mathbb{R}^{nd}} \frac{m^{(n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{n} k_2(\mathbf{r}_l)}$ is log-convex. The latter and (4.15) imply that the sequence ${}_{P_i}m$ fulfills (4.24) by Proposition 4.2.11. Arguing as before (note that the equivalent of (4.18) for the sequence ${}_{P_i}m$ is true by Lemma 4.2.6 applied to ${}_{P_i}m$), we get that the sequence ${}_{P_i}m$ is also determining.
Hence, by Theorem 3.2.9, the sequence ${}_{P_i}m$ is realized by a unique non-negative measure ν on $\mathscr{D}'_{proj}(\mathbb{R}^d)$, i.e. for any $f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{nd})$

$$\langle f^{(n)}, ({}_{P_i}m)^{(n)} \rangle = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle \nu(d\eta). \tag{4.28}$$

Then by (4.27) and (4.28) we get that for any $f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{nd})$

$$\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle P_i(\eta) \mu(d\eta) = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle \nu(d\eta)$$

or, equivalently,

$$\int_{A_i \cup B_i} \langle f^{(n)}, \eta^{\otimes n} \rangle P_i(\eta) \mu(d\eta) = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle \nu(d\eta),$$

where $A_i := \left\{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : P_i(\eta) \geq 0\right\}$ and $B_i := \left\{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : P_i(\eta) < 0\right\}$.

The latter can be rewritten as

$$\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle 1\!\!1_{A_i}(\eta) P_i(\eta) \mu(d\eta) = \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle f^{(n)}, \eta^{\otimes n} \rangle \Big(\nu(d\eta) - 1\!\!1_{B_i}(\eta) P_i(\eta) \mu(d\eta)\Big),$$

which shows that the two non-negative measures on $\mathscr{D}'_{proj}(\mathbb{R}^d)$

$$1\!\!1_{A_i} P_i d\mu \quad \text{and} \quad d\nu - 1\!\!1_{\mathscr{D}'_{proj}(\mathbb{R}^d) \setminus A_i} P_i d\mu \tag{4.29}$$

have the same moment functions.

Let us call m^+ the sequence of all moment functions of $1\!\!1_{A_i} d\mu$ and let us show that ${}_{P_i}m^+$ is determining.

Since m is realized by μ on $\mathscr{D}'_{proj}(\mathbb{R}^d)$, for any $n \in \mathbb{N}_0$ and for any positive real number R we have that

$$\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \frac{\chi_R}{k_2}, \eta \rangle^n \mu(d\eta) = \int_{\mathbb{R}^{nd}} \prod_{l=1}^{n} \frac{\chi_R(\mathbf{r}_l)}{k_2(\mathbf{r}_l)} m^{(n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n),$$

where $\chi_R \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is the one defined in (4.20). Using the monotone convergence theorem for $R \to \infty$ we have that

$$\int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \frac{1}{k_2}, \eta \rangle^n \mu(d\eta) = \int_{\mathbb{R}^{nd}} \frac{m^{(n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n)}{\prod_{l=1}^{n} k_2(\mathbf{r}_l)}. \tag{4.30}$$

Since $\mathbb{1}_{A_i} d\mu \leq \mu$ and since (4.30) holds, we have that

$$\begin{aligned}\int_{\mathbb{R}^{2nd}} \frac{m^{+(2n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} &= \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \frac{1}{k_2}, \eta\rangle^{2n} \mathbb{1}_{A_i}(\eta)\mu(d\eta)\\ &\leq \int_{\mathscr{D}'_{proj}(\mathbb{R}^d)} \langle \frac{1}{k_2}, \eta\rangle^{2n} \mu(d\eta)\\ &= \int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)}.\end{aligned}$$

From the latter inequality follows that m^+ satisfies the weighted Carleman's condition and, for any $n \in \mathbb{N}_0$, $\int_{\mathbb{R}^{2nd}} \frac{m^{+(2n)}(d\mathbf{r}_1,\ldots,d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} < \infty$. Moreover, since m^+ is realized by the measure $\mathbb{1}_{A_i}\mu$, we have that $(m_n^+)_{n\in\mathbb{N}_0}$ is log-convex by Proposition 4.2.12. By Proposition 4.2.11 also ${}_{P_i}m^+$ satisfies the weighted Carleman's condition. Hence, by Proposition 4.2.7, ${}_{P_i}m^+$ is determining.

As the two non-negative measures in (4.29) both realize the determining sequence ${}_{P_i}m^+$, they coincide since Theorem 3.2.9 also guarantees the uniqueness of the realizing measure. It follows that $P_i d\mu = d\nu$, i.e. the signed measure $P_i d\mu$ is actually a non-negative measure on $\mathscr{D}'_{proj}(\mathbb{R}^d)$ as well as ν. Therefore, we have that

$$\forall\, i \in Y, \quad \mu\left(\mathscr{D}'_{proj}(\mathbb{R}^d) \setminus A_i\right) = 0. \tag{4.31}$$

The set $\mathcal{S} = \bigcap_{i\in Y} A_i \in \sigma(\tau_w^{ind})$, as the intersection of closed sets (see Proposition 4.2.2). Since $\mathcal{S} \subseteq \mathscr{D}'_{proj}(\mathbb{R}^d)$, by Corollary 4.1.8 we also get that $\mathcal{S} \in \sigma(\tau_w^{proj})$. It remains to show that μ is concentrated on $\mathcal{S}$. If Y is countable, then the conclusion immediately follows from (4.31) by using the $\sigma-$subadditivity of μ. In the case when Y is uncountable, the latter argument does not work anymore but we can still get that the measure is concentrated on $\mathcal{S}$ proceeding as follows.
Since $\sigma(\tau_w^{ind})$ restricted to $\mathscr{D}'_{proj}(\mathbb{R}^d)$ coincides with $\sigma(\tau_w^{proj})$ by Corollary 4.1.8, we extend the measure μ to a measure μ' on $\mathscr{D}'_{ind}(\mathbb{R}^d)$ in the following way

$$\mu'(M) := \mu(M \cap \mathscr{D}'_{proj}(\mathbb{R}^d)), \quad \forall\, M \in \sigma(\tau_w^{ind}).$$

As $(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_w^{ind})$ is a Radon space by Proposition 4.1.9, the finite measure μ' is inner regular. This means that for any $M \in \sigma(\tau_w^{ind})$ and for any $\varepsilon > 0$ there exists a compact set K_ε in $\mathscr{D}'_{ind}(\mathbb{R}^d)$ such that

$$K_\varepsilon \subseteq M, \tag{4.32}$$

with

$$\mu'(M) < \mu'(K_\varepsilon) + \varepsilon. \tag{4.33}$$

Let us apply this property to $M = \mathscr{D}'_{ind}(\mathbb{R}^d) \setminus \mathcal{S}$. Using the definition of $\mathcal{S}$, we get

$$M = \mathscr{D}'_{ind}(\mathbb{R}^d) \setminus \mathcal{S} = \mathscr{D}'_{ind}(\mathbb{R}^d) \setminus \left(\bigcap_{i \in Y} A_i \right) = \bigcup_{i \in Y} \left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus A_i \right).$$

Hence, due to (4.32), for any $\varepsilon > 0$ there exists a compact set K_ε in $\mathscr{D}'_{ind}(\mathbb{R}^d)$ which fulfills (4.33), i.e.

$$K_\varepsilon \subseteq \bigcup_{i \in Y} \left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus A_i \right).$$

As the collection of the sets $\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus A_i$ forms an open cover of K_ε, the compactness of K_ε in $\left(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_w^{ind}\right)$ implies that there exists a finite open subcover of K_ε, i.e. there exists a finite subset $J \subset Y$ such that

$$K_\varepsilon \subseteq \bigcup_{i \in J} \left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus A_i \right).$$

Therefore, we have that

$$\begin{aligned}
0 \leq \mu'(K_\varepsilon) &\leq \mu' \left(\bigcup_{i \in J} \left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus A_i \right) \right) \\
&\leq \sum_{i \in J} \mu' \left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus A_i \right) \\
&= \sum_{i \in J} \mu \Big(\left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus A_i \right) \cap \mathscr{D}'_{proj}(\mathbb{R}^d) \Big) \\
&= \sum_{i \in J} \mu \left(\mathscr{D}'_{proj}(\mathbb{R}^d) \setminus A_i \right) \\
&= 0,
\end{aligned}$$

where in the last equality we used (4.31).
By (4.33), we then have that

$$\mu' \left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus \mathcal{S} \right) < \varepsilon + \mu'(K_\varepsilon) = \varepsilon.$$

Since the previous relation holds for any $\varepsilon > 0$, we have that $\mu' \left(\mathscr{D}'_{ind}(\mathbb{R}^d) \setminus \mathcal{S} \right) = 0$, which means that μ' is concentrated on $\mathcal{S}$ and so is its restriction μ.

It remains to show (4.16). Since the measure μ is concentrated on $\mathcal{S}$, (4.30) gives that

$$\int_{\mathcal{S}} \langle \frac{1}{k_2}, \eta \rangle^n \mu(d\eta) = \int_{\mathbb{R}^{nd}} \frac{m^{(n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_n)}{\prod_{l=1}^n k_2(\mathbf{r}_l)} < \infty,$$

where the inequality holds by Lemma 4.2.6.

□

4.3 Applications

In the following we provide some concrete applications of Theorem 4.2.4.

4.3.1 The space of Radon measures $\mathcal{R}(\mathbb{R}^d)$

Theorem 4.3.1.
The set $\mathcal{R}(\mathbb{R}^d)$ of all Radon measures on $\mathbb{R}^d$ is a semi-algebraic subset of $\mathscr{D}'_{proj}(\mathbb{R}^d)$, i.e.

$$\mathcal{R}(\mathbb{R}^d) = \bigcap_{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \left\{ \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : \Phi_\varphi(\eta) \geq 0 \right\}$$

where $\Phi_\varphi(\eta) := \langle \varphi, \eta \rangle$ as in (4.7).

In order to prove Theorem 4.3.1 let us introduce some useful embeddings which involve $\mathcal{R}(\mathbb{R}^d)$ and $\mathscr{D}'_{proj}(\mathbb{R}^d)$. First of all, let us consider the dual pairing in (4.7) as a function of the first variable, i.e. for any Radon measure η

$$\begin{array}{rccl} \langle \cdot, \eta \rangle : & \mathcal{C}_c^\infty(\mathbb{R}^d) & \to & \mathbb{R} \\ & \varphi & \mapsto & \langle \varphi, \eta \rangle = \int_{\mathbb{R}^d} \varphi(\mathbf{r}) \eta(d\mathbf{r}). \end{array} \tag{4.34}$$

Moreover, let us recall that given a space Ω_τ and $C \subset \Omega_\tau$, the dual cone $C^\perp \subset \Omega'_\tau$ of C is defined as follows.

$$C^\perp := \{ F \in \Omega'_\tau : F(\varphi) \geq 0 \,, \ \forall \varphi \in C \} \,.$$

Theorem 4.3.2.
There exists a bijective correspondence between the Radon measures on $\mathbb{R}^d$ and the continuous non-negative linear functionals on the space $\mathscr{D}_{proj}(\mathbb{R}^d)$. Namely,

$$\mathcal{R}(\mathbb{R}^d) \cong \left(\mathscr{D}^+_{proj}(\mathbb{R}^d) \right)^\perp .$$

Proof.
Let $\eta \in \mathcal{R}(\mathbb{R}^d)$. The functional $\langle \cdot, \eta \rangle$ defined as in (4.34) is an element of $\left(\mathscr{D}^+_{proj}(\mathbb{R}^d) \right)^\perp$.

In fact, by Proposition 4.1.14, $\langle \cdot, \eta \rangle$ is an element of $\mathscr{D}'_{proj}(\mathbb{R}^d)$ (i.e. it is continuous w.r.t. the projective limit topology). Moreover, the functional $\langle \cdot, \eta \rangle$ is non-negative because for any $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ we have that $\langle \varphi, \eta \rangle = \int_{\mathbb{R}^d} \varphi(\mathbf{r})\eta(d\mathbf{r}) \geq 0$ since η is a non-negative measure.

Conversely, by a theorem due to L. Schwartz (similar to the Riesz representation theorem, see [11, Theorem 5.3.1], [70, Theorem V]), every non-negative linear functional on $\mathscr{D}_{ind}(\mathbb{R}^d)$ can be represented as integral w.r.t. a Radon measure on $\mathbb{R}^d$. In particular, this theorem holds for every continuous non-negative linear functional on $\mathscr{D}_{proj}(\mathbb{R}^d)$. $\square$

The following is Theorem 4.2.4 stated for $\mathcal{S} = \mathcal{R}(\mathbb{R}^d)$ represented as in Theorem 4.3.1.

Theorem 4.3.3.
Let $m = (m^{(n)})_{n=0}^{\infty}$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. Assume that m fulfills the condition (4.15) for some function $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$. Then m is realized by a unique non-negative finite measure μ on $\mathcal{R}(\mathbb{R}^d)$ satisfying (4.16) if and only if the following inequalities hold.

$$L_m(h^2) \geq 0\,,\ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \tag{4.35}$$

$$L_m(\Phi_\varphi h^2) \geq 0\,,\ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \tag{4.36}$$

$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} < \infty,\ \forall n \in \mathbb{N}_0,$$

where $\Phi_\varphi := \langle \varphi, \eta \rangle$.

Remark 4.3.4.
Using the fact that a generic polynomial $h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right)$ has the form $h(\eta) = \sum_{n=0}^{N} \langle h^{(n)}, \eta^{\otimes n} \rangle$ with $h^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{nd})$, the conditions (4.35) and (4.36) can be rewritten as

$$\sum_{i,j} \langle h^{(i)} \otimes h^{(j)},\, m^{(i+j)} \rangle \geq 0, \quad \forall\, h^{(i)} \in \mathcal{C}_c^\infty(\mathbb{R}^{id}),$$

and

$$\sum_{i,j} \langle h^{(i)} \otimes h^{(j)} \otimes \varphi,\, m^{(i+j+1)} \rangle \geq 0, \quad \forall\, h^{(i)} \in \mathcal{C}_c^\infty(\mathbb{R}^{id}), \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d).$$

Recalling Definition 4.2.8, we can easily see that these conditions respectively

mean that the sequence $(m^{(n)})_{n\in\mathbb{N}_0}$ *and its shifted version* $((_{\Phi_\varphi}m)^{(n)})_{n\in\mathbb{N}_0}$ *are positive semidefinite as in Definition 3.2.7.*
In particular, if for each $n \in \mathbb{N}_0$ *there exists a function* $\alpha^{(n)} \in L^1(\mathbb{R}^{dn}, \lambda)$ *such that* $m^{(n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) = \alpha^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n)d\mathbf{r}_1 \cdots d\mathbf{r}_n$*, then* (4.35) *and* (4.36) *assume the following concrete form*

$$\sum_{i,j} \int_{\mathbb{R}^{d(i+j)}} h^{(i)}(\mathbf{r}_1, \ldots, \mathbf{r}_i)h^{(j)}(\mathbf{r}_{i+1}, \ldots, \mathbf{r}_{i+j})\alpha^{(i+j)}(\mathbf{r}_1, \ldots, \mathbf{r}_{i+j})d\mathbf{r}_1 \cdots d\mathbf{r}_{i+j} \geq 0,$$

$$\sum_{i,j} \int_{\mathbb{R}^{d(i+j+1)}} h^{(i)}(\mathbf{r}_1, \ldots, \mathbf{r}_i)h^{(j)}(\mathbf{r}_{i+1}, \ldots, \mathbf{r}_{i+j})\varphi(\mathbf{y})\alpha^{(i+j+1)}(\mathbf{r}_1, \ldots, \mathbf{r}_{i+j}, \mathbf{y})d\mathbf{r}_1 \cdots d\mathbf{r}_{i+j}d\mathbf{y} \geq 0,$$

for all $h^{(i)} \in \mathcal{C}_c^\infty(\mathbb{R}^{id})$ *and for all* $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$*.*

These conditions respectively mean that $(\alpha^{(n)})_{n\in\mathbb{N}_0}$ *is positive semidefinite and that for* $\lambda-$*almost all* $\mathbf{y} \in \mathbb{R}^d$ *the sequence* $(\alpha^{(n+1)}(\cdot, \mathbf{y}))_{n\in\mathbb{N}_0}$ *is positive semidefinite (in the generalized sense). This reformulation makes clear the analogy with the Stieltjes moment problem where necessary and sufficient conditions for the realizability of a sequence of numbers* $(m_n)_{n\in\mathbb{N}_0}$ *on* $\mathbb{R}^+$ *are that* $(m_n)_{n\in\mathbb{N}_0}$ *and* $(m_{n+1})_{n\in\mathbb{N}_0}$ *are positive semidefinite.*

4.3.2 The space of sub-probability measures $\mathcal{SP}(\mathbb{R}^d)$

Theorem 4.3.5.
The set $\mathcal{SP}(\mathbb{R}^d)$ *of all sub-probabilities on* $\mathbb{R}^d$*, i.e.*

$$\mathcal{SP}(\mathbb{R}^d) := \{\eta \in \mathcal{R}(\mathbb{R}^d) : \eta(\mathbb{R}^d) \leq 1\} \tag{4.37}$$

is a semi-algebraic subset of $\mathscr{D}'_{proj}(\mathbb{R}^d)$*. More precisely, we get that*

$$\mathcal{SP}(\mathbb{R}^d) = \mathcal{R}(\mathbb{R}^d) \cap \bigcap_{\substack{\varphi\in\mathcal{C}_c^{+,\infty}(\mathbb{R}^d)\\ \|\varphi\|_\infty\leq 1}} \left\{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : \Gamma_\varphi(\eta) \geq 0\right\} \tag{4.38}$$

where $\Gamma_\varphi(\eta) := 1 - \langle\varphi, \eta\rangle^2$*.*

Proof.
Step I: $\subseteq$
Let $\eta \in \mathcal{SP}(\mathbb{R}^d)$ as in (4.37), then $\eta \in \mathcal{R}(\mathbb{R}^d)$ and $\eta(\mathbb{R}^d) \leq 1$. The latter relations imply that, for any $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ with $\|\varphi\|_\infty \leq 1$,

$$0 \leq \langle\varphi, \eta\rangle \leq 1$$

and then

$$\langle \varphi, \eta \rangle^2 \leq 1.$$

Step II: $\supseteq$

Let $\eta \in \mathcal{R}(\mathbb{R}^d)$ such that for any $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ with $\|\varphi\|_\infty \leq 1$

$$1 - \langle \varphi, \eta \rangle^2 \geq 0.$$

Therefore,

$$0 \leq \langle \varphi, \eta \rangle \leq 1. \tag{4.39}$$

To prove $\eta \in \mathcal{SP}(\mathbb{R}^d)$, it remains to show that $\eta(\mathbb{R}^d) = \langle 1\!\!1_{\mathbb{R}^d}, \eta \rangle \leq 1$.

Let us note that the function $1\!\!1_{\mathbb{R}^d}$ can be approximated pointwise by an increasing sequence of functions $\{\chi_R\}_{R \in \mathbb{R}^+} \subset \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ with $\|\chi_R\|_\infty = 1$ (see (4.20)). Hence, by using the monotone convergence theorem and (4.39), we have that

$$\eta(\mathbb{R}^d) = \langle 1\!\!1_{\mathbb{R}^d}, \eta \rangle = \langle \lim_{R \to \infty} \chi_R, \eta \rangle = \lim_{R \to \infty} \langle \chi_R, \eta \rangle \leq 1.$$

$\square$

Using the representation (4.38), we can explicitly rewrite Theorem 4.2.4 for $\mathcal{S} = \mathcal{SP}(\mathbb{R}^d)$ as follows.

Theorem 4.3.6.

Let $m = (m^{(n)})_{n=0}^\infty$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. Assume that m fulfills the condition (4.15) for some function $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$. Then m is realized by a unique non-negative finite measure μ on $\mathcal{SP}(\mathbb{R}^d)$ satisfying (4.16) if and only if the following inequalities hold.

$$L_m(h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \tag{4.40}$$

$$L_m(\Phi_\varphi h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \tag{4.41}$$

$$L_m(\Gamma_\varphi h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \|\varphi\|_\infty \leq 1, \tag{4.42}$$

$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} < \infty, \quad \forall n \in \mathbb{N}_0, \tag{4.43}$$

where $\Phi_\varphi(\eta) := \langle \varphi, \eta \rangle$ and $\Gamma_\varphi(\eta) := 1 - \langle \varphi, \eta \rangle^2$.

Actually, the result in Theorem 4.3.6 also holds if we drop the assumption that m fulfills (4.15) and (4.43) as they follow from the remaining ones. Indeed, we can prove the following theorem.

Theorem 4.3.7.
Let $m = (m^{(n)})_{n=0}^{\infty}$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. Then m is realized by a unique non-negative finite measure μ on $\mathcal{SP}(\mathbb{R}^d)$ if and only if the (4.40), (4.41), (4.42) *hold.*

Proof.
Sufficiency
Assume that (4.40), (4.41) and (4.42) are fulfilled and let us show that (4.15) and (4.43) hold for the function $k_2 \equiv 1$. In fact, for any $\varphi \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$ and for any $n \in \mathbb{N}$ we can apply (4.42) for $h(\eta) = \langle \varphi, \eta \rangle^{(n-1)}$. Then, we have the following

$$L_m(\langle \varphi, \eta \rangle^{2n}) \leq L_m(\langle \varphi, \eta \rangle^{2(n-1)}).$$

Iterating, we get that

$$L_m(\langle \varphi, \eta \rangle^{2n}) \leq L_m(1).$$

Consequently, for any real positive constant R, if we take in the previous inequality $\varphi = \chi_R$ as in (4.20), then we have that

$$\int_{\mathbb{R}^{2nd}} \prod_{i=1}^{2n} \chi_R(\mathbf{r}_i) m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n}) = L_m(\langle \chi_R, \eta \rangle^{2n}) \leq L_m(1).$$

Therefore, using the monotone convergence theorem as $R \to \infty$

$$\int_{\mathbb{R}^{2nd}} m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n}) \leq L_m(\langle 1, \eta^{\otimes 0} \rangle) = m^{(0)} < \infty.$$

Hence, the conditions (4.43) and (4.15) hold for $k_2 \equiv 1$ and so we can apply Theorem 4.3.6.

□

This proof was inspired by the results of Schmüdgen about the moment problem on a semi-algebraic compact subset of $\mathbb{R}^d$ in [68]. In fact, $\mathcal{SP}(\mathbb{R}^d)$ is a compact subset of $\mathcal{R}(\mathbb{R}^d)$ w.r.t. the vague topology τ_v. The compactness follows from [18, Corollary A2.6.V], using the observation that $\mathcal{SP}(\mathbb{R}^d)$ is closed in $\big(\mathcal{R}(\mathbb{R}^d), \tau_v\big)$ and that $\sup_{\eta \in \mathcal{SP}(\mathbb{R}^d)} \eta(A) < \infty$ for every bounded Borel set A in $\mathbb{R}^d$.
However, Schmüdgen's technique does not apply straightforwardly to the case of realizability on $\mathcal{SP}(\mathbb{R}^d)$ because he treats the case when the semi-algebraic set is defined by finitely many polynomials and not by infinitely many as in our case.

Remark 4.3.8.
The representation (4.38) *is not unique. In fact, it is possible to give other representations of* $\mathcal{SP}(\mathbb{R}^d)$ *as semi-algebraic set using slight modifications in the proof of Theorem 4.3.5. For example, we can write*

$$\mathcal{SP}(\mathbb{R}^d) = \bigcap_{\substack{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d) \\ \|\varphi\|_\infty \leq 1}} \left\{ \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : \langle \varphi, \eta \rangle - \langle \varphi, \eta \rangle^2 \geq 0 \right\},$$

or also

$$\mathcal{SP}(\mathbb{R}^d) = \bigcap_{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \left\{ \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : \langle \varphi, \eta \rangle \geq 0 \right\} \cap \bigcap_{\substack{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d) \\ \|\varphi\|_\infty \leq 1}} \left\{ \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : 1 - \langle \varphi, \eta \rangle \geq 0 \right\}. \tag{4.44}$$

Depending on the choice of the representation, we get different versions of Theorem 4.3.6. Indeed, if

$$\mathcal{SP}(\mathbb{R}^d) = \bigcap_{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \left\{ \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : P_\varphi(\eta) \geq 0 \right\}$$

then necessary and sufficient conditions for the realizability of the sequence $(m^{(n)})_{n \in \mathbb{N}_0}$ *on* $\mathcal{SP}(\mathbb{R}^d)$ *are that the sequence* $(m^{(n)})_{n \in \mathbb{N}_0}$ *and all its shifted versions* $(({}_{P_\varphi}m)^{(n)})_{n \in \mathbb{N}_0}$ *(see Definition 4.2.8) are positive semidefinite in the sense of Definition 3.2.7.*
For instance, using the representation (4.44)*, we get Theorem 4.3.6 with the condition* (4.42) *replaced by*

$$L_m(Q_\varphi h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \ \|\varphi\|_\infty \leq 1, \tag{4.45}$$

where $Q_\varphi(\eta) := 1 - \langle \varphi, \eta \rangle$*. Note that we cannot drop the assumptions* (4.15) *and* (4.43) *with the trick inspired by Schmüdgen and used in the proof of Theorem 4.3.7 because it does not work for the representation* (4.44)*.*
The condition in (4.45) *can be rewritten more explicitly in terms of moment measures as*

$$\sum_{i,j} \langle h^{(i)} \otimes h^{(j)}, m^{(i+j)} \rangle - \sum_{i,j} \langle h^{(i)} \otimes h^{(j)} \otimes \varphi, m^{(i+j+1)} \rangle \geq 0,$$

for all $h^{(i)} \in \mathcal{C}_c^\infty(\mathbb{R}^{id})$ *and for all* $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ *with* $\|\varphi\|_\infty \leq 1$*.*
In particular, if for each $n \in \mathbb{N}_0$ *there exists a function* $\alpha^{(n)} \in L^1(\mathbb{R}^{dn}, \lambda)$ *such that* $m^{(n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) = \alpha^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) d\mathbf{r}_1 \cdots d\mathbf{r}_n$ *then* (4.45) *assumes the following*

concrete form

$$\sum_{i,j}\int_{\mathbb{R}^{d(i+j)}} h^{(i)}(\mathbf{r}_1,\dots,\mathbf{r}_i)h^{(j)}(\mathbf{r}_{i+1},\dots,\mathbf{r}_{i+j})\alpha^{(i+j)}(\mathbf{r}_1,\dots,\mathbf{r}_{i+j})d\mathbf{r}_1\cdots d\mathbf{r}_{i+j}$$

$$-\int_{\mathbb{R}^{d(i+j+1)}} h^{(i)}(\mathbf{r}_1,\dots,\mathbf{r}_i)h^{(j)}(\mathbf{r}_{i+1},\dots,\mathbf{r}_{i+j})\varphi(\mathbf{y})\alpha^{(i+j+1)}(\mathbf{r}_1,\dots,\mathbf{r}_{i+j},\mathbf{y})d\mathbf{r}_1\cdots d\mathbf{r}_{i+j}d\mathbf{y}\geq 0$$

for all $h^{(i)} \in \mathcal{C}_c^{\infty}(\mathbb{R}^{id})$ *and for all* $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ *with* $\|\varphi\|_\infty \leq 1$. *This means that for* $\lambda-$*almost all* $\mathbf{y} \in \mathbb{R}^d$ *the sequence* $(\alpha^{(n)}(\cdot) - \alpha^{(n+1)}(\cdot,\mathbf{y}))_{n\in\mathbb{N}_0}$ *is positive semidefinite. Moreover, as already discussed in Remark 4.3.4, the conditions* (4.40) *and* (4.41) *of Theorem 4.3.6 give that* $(\alpha^{(n)})_{n\in\mathbb{N}_0}$ *is positive semidefinite and for* $\lambda-$*almost all* $\mathbf{y} \in \mathbb{R}^d$ *the sequence* $(\alpha^{(n+1)}(\cdot,\mathbf{y}))_{n\in\mathbb{N}_0}$ *is positive semidefinite.*

This reformulation makes clear the analogy with the Hausdorff moment problem as treated in [19], where $[0,1]$ *is represented like*

$$[0,1] = \{x \in \mathbb{R} : x \geq 0\} \cap \{x \in \mathbb{R} : 1-x \geq 0\}$$

and so necessary and sufficient conditions for the realizability on $[0,1]$ *of a sequence of numbers* $(m_n)_{n\in\mathbb{N}_0}$ *are that* $(m_n)_{n\in\mathbb{N}_0}$, $(m_{n+1})_{n\in\mathbb{N}_0}$ *and* $(m_n - m_{n+1})_{n\in\mathbb{N}_0}$ *are positive semidefinite. Also here, we can get different (but, a posteriori, equivalent) conditions on* $(m_n)_{n\in\mathbb{N}_0}$ *depending on the representation we choose for* $[0,1]$ *(see [10]).*

4.3.3 The space of probability measures $\mathcal{P}(\mathbb{R}^d)$

Using the results in Subsections 4.3.1 and 4.3.2, it is possible to prove the following version of Theorem 4.2.4 for $\mathcal{S} = \mathcal{P}(\mathbb{R}^d)$ the set of all probabilities.

Theorem 4.3.9.
Let $m = (m^{(n)})_{n=0}^{\infty}$ *be a sequence such that* $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ *and* $m^{(n)}$ *is a symmetric functions of its* n *variables. Then,* m *is realized by a unique non-negative finite measure* μ *on* $\mathcal{P}(\mathbb{R}^d)$ *satisfying* (4.16) *if and only if the following inequalities hold.*

$$L_m(h^2) \geq 0\ ,\ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \tag{4.46}$$
$$L_m(\Phi_\varphi h^2) \geq 0\ , \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right),\ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \tag{4.47}$$
$$L_m(\Gamma_\varphi h^2) \geq 0\ , \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right),\ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d),\ \|\varphi\|_\infty \leq 1, \tag{4.48}$$
$$m^{(1)}(\mathbb{R}^d) = m^{(0)}, \tag{4.49}$$

where $\Phi_\varphi(\eta) := \langle \varphi, \eta \rangle$, $\Gamma_\varphi(\eta) := 1 - \langle \varphi, \eta \rangle^2$.

Proof.
Necessity
Let us assume that the sequence m is realized by a non-negative finite measure μ on $\mathcal{P}(\mathbb{R}^d)$. W.l.o.g. let us assume that μ is a probability, i.e. $m^{(0)} = \mu(\mathcal{P}(\mathbb{R}^d)) = 1$. In particular, m is realized on the set $\mathcal{SP}(\mathbb{R}^d) \supset \mathcal{P}(\mathbb{R}^d)$ by the same measure μ. Hence, Theorem 4.2.4 applied for $\mathcal{S} = \mathcal{SP}(\mathbb{R}^d)$, implies the conditions (4.46), (4.47) and (4.48).
It remains to show the condition (4.49). Let us approximate $1\!\!1_{\mathbb{R}^d}$ by the increasing sequence of functions $\{\chi_R\}_{R\in\mathbb{R}^+} \subset \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ introduced in (4.20). By using the monotone convergence theorem and the assumption that m is realized by μ on $\mathcal{P}(\mathbb{R}^d)$, we have that

$$
\begin{aligned}
m^{(1)}(\mathbb{R}^d) &= \langle 1\!\!1_{\mathbb{R}^d}, m^{(1)} \rangle \\
&= \langle \lim_{R\to\infty} \chi_R, m^{(1)} \rangle \\
&= \lim_{R\to\infty} \langle \chi_R, m^{(1)} \rangle \\
&= \lim_{R\to\infty} \int_{\mathcal{P}(\mathbb{R}^d)} \langle \chi_R, \eta \rangle \mu(d\eta) \\
&= \int_{\mathcal{P}(\mathbb{R}^d)} \lim_{R\to\infty} \langle \chi_R, \eta \rangle \mu(d\eta) \\
&= \int_{\mathcal{P}(\mathbb{R}^d)} \langle 1\!\!1_{\mathbb{R}^d}, \eta \rangle \mu(d\eta) \\
&= \int_{\mathcal{P}(\mathbb{R}^d)} \mu(d\eta) = \mu(\mathcal{P}(\mathbb{R}^d)) = 1.
\end{aligned}
$$

Sufficiency
Let us assume that (4.46), (4.47), (4.48) and (4.49) hold. Due to Theorem 4.3.7, the first four conditions imply that there exists a unique finite non-negative measure μ realizing m on $\mathcal{SP}(\mathbb{R}^d)$. W.l.o.g. we can assume μ to be a probability on $\mathcal{SP}(\mathbb{R}^d)$. It remains to prove that actually

$$\mu(\mathcal{P}(\mathbb{R}^d)) = 1.$$

This is equivalent to prove that

$$\mu\Big(\{\eta \in \mathcal{SP}(\mathbb{R}^d) : 1 - \langle 1\!\!1_{\mathbb{R}^d}, \eta \rangle = 0\} \Big) = 1. \tag{4.50}$$

Let us note that the function $1\!\!1_{\mathbb{R}^d}$ can be approximated pointwise by an increasing sequence of functions $\{\chi_R\}_{R\in\mathbb{R}^+} \subset \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ with $\|\chi_R\|_\infty = 1$ (as in (4.20)). Hence, we have that for all $\eta \in \mathcal{SP}(\mathbb{R}^d)$

$$\lim_{R\to\infty}(1 - \langle \chi_R, \eta\rangle) = 1 - \langle 1\!\!1_{\mathbb{R}^d}, \eta\rangle.$$

Moreover, by (4.38) we have that for all $\eta \in \mathcal{SP}(\mathbb{R}^d)$ and for any $R \in \mathbb{R}^+$

$$1 - \langle \chi_R, \eta\rangle \geq 0$$

and so we also get that for all $\eta \in \mathcal{SP}(\mathbb{R}^d)$

$$1 - \langle 1\!\!1_{\mathbb{R}^d}, \eta\rangle \geq 0. \tag{4.51}$$

By the monotone convergence theorem, we get that

$$\lim_{R\to\infty}\int_{\mathcal{SP}(\mathbb{R}^d)}(1 - \langle \chi_R, \eta\rangle)\mu(d\eta) = \int_{\mathcal{SP}(\mathbb{R}^d)}(1 - \langle 1\!\!1_{\mathbb{R}^d}, \eta\rangle)\mu(d\eta). \tag{4.52}$$

On the other hand, we also have that

$$\lim_{R\to\infty}\int_{\mathcal{SP}(\mathbb{R}^d)}(1 - \langle \chi_R, \eta\rangle)\mu(d\eta) = 1 - \lim_{R\to\infty}\int_{\mathbb{R}^d}\chi_R(\mathbf{r})m^{(1)}(d\mathbf{r}) = 1 - m^{(1)}(\mathbb{R}^d) = 0, \tag{4.53}$$

where we used that m is realized by μ on $\mathcal{SP}(\mathbb{R}^d)$ and the assumption (4.49). Thus, by (4.52) and (4.53), we have

$$\int_{\mathcal{SP}(\mathbb{R}^d)}(1 - \langle 1\!\!1_{\mathbb{R}^d}, \eta\rangle)\mu(d\eta) = 0.$$

Since μ is non-negative and, by (4.51), the integrand is also non-negative on $\mathcal{SP}(\mathbb{R}^d)$, the previous relation implies that

$$1 - \langle 1\!\!1_{\mathbb{R}^d}, \eta\rangle = 0, \ \ \mu - \text{a.s.},$$

which is (4.50).

□

As we have already observed in the previous section, when we write Theorem 4.2.4 for $\mathcal{SP}(\mathbb{R}^d)$ we can always choose $k_2 \equiv 1$. Furthermore, note that if m

is realized by a finite non-negative measure μ on $\mathcal{SP}(\mathbb{R}^d)$ then for $k_2 \equiv 1$ we get

$$\begin{aligned}
\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} &= \int_{\mathbb{R}^{2nd}} m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n}) \\
&= \int_{\mathcal{SP}(\mathbb{R}^d)} \eta(\mathbb{R}^d)^{\otimes 2n} \mu(d\eta) \\
&\leq \mu(\mathcal{SP}(\mathbb{R}^d)) < \infty.
\end{aligned}$$

Hence, the conditions (4.18) and (4.15) hold.
The previous consideration is also true when we state Theorem 4.2.4 for the set of all probabilities $\mathcal{P}(\mathbb{R}^d)$.

4.3.4 The set of $L^\infty-$bounded density measures

Theorem 4.3.10.
Let $c \in \mathbb{R}^+$. The set $\mathcal{S}_c$ of all Radon measures with density w.r.t. the Lebesgue measure λ on $\mathbb{R}^d$ which is $L^\infty-$bounded by c, i.e.

$$\mathcal{S}_c := \left\{\eta \in \mathcal{R}(\mathbb{R}^d) : \eta(d\mathbf{r}) = f(\mathbf{r})\lambda(d\mathbf{r}) \text{ with } f \geq 0 \text{ and } \|f\|_{L^\infty} \leq c\right\} \tag{4.54}$$

is a semi-algebraic subset of $\mathscr{D}'_{proj}(\mathbb{R}^d)$.
More precisely, we get that

$$\mathcal{S}_c = \mathcal{R}(\mathbb{R}^d) \cap \bigcap_{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \left\{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : c\langle\varphi, \lambda\rangle - \langle\varphi, \eta\rangle \geq 0\right\}. \tag{4.55}$$

Proof.
Step I: $\subseteq$
Let $\eta \in \mathcal{S}_c$, then by definition (4.54), we have $\eta \in \mathcal{R}(\mathbb{R}^d)$ and $\eta(d\mathbf{r}) = f(\mathbf{r})\lambda(d\mathbf{r})$ for some $f \geq 0$ with $\|f\|_{L^\infty} \leq c$.
Hence, for any $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ we get

$$\begin{aligned}
\langle\varphi, \eta\rangle &= \int_{\mathbb{R}^d} \varphi(\mathbf{r})\eta(d\mathbf{r}) \\
&= \int_{\mathbb{R}^d} \varphi(\mathbf{r})f(\mathbf{r})\lambda(d\mathbf{r}) \\
&\leq \|f\|_{L^\infty} \int_{\mathbb{R}^d} \varphi(\mathbf{r})\lambda(d\mathbf{r}) \\
&\leq c\int_{\mathbb{R}^d} \varphi(\mathbf{r})\lambda(d\mathbf{r}) = c\langle\varphi, \lambda\rangle.
\end{aligned}$$

Step II: $\supseteq$

Let $\eta \in \mathcal{R}(\mathbb{R}^d)$ such that

$$c\langle \varphi, \lambda \rangle - \langle \varphi, \eta \rangle \geq 0, \ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d). \tag{4.56}$$

Since $\mathcal{C}_c^{\infty}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \mu)$ for any signed Radon measure μ, we have that $\mathcal{C}_c^{\infty}(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d, \lambda - \eta)$. Hence, the condition (4.56) holds for all $\varphi \in L^1(\mathbb{R}^d, \lambda - \eta)$ and in particular for $\varphi = 1\!\!1_A$, where $A \in \mathcal{B}(\mathbb{R}^d)$ bounded, we have

$$\eta(A) \leq c\lambda(A), \ \forall A \in \mathcal{B}(\mathbb{R}^d) \text{ bounded.} \tag{4.57}$$

The latter relation implies that if $\lambda(A) = 0$ then $\eta(A) = 0$, i.e. $\eta \ll \lambda$. Consequently, by the Radon-Nikodym theorem, there exists $0 \leq f \in L^1(\mathbb{R}^d, \lambda)$ such that

$$\eta(d\mathbf{r}) = f(\mathbf{r})\lambda(d\mathbf{r}). \tag{4.58}$$

By (4.58) and by (4.56) for $\varphi = 1\!\!1_A$, for any $A \in \mathcal{B}(\mathbb{R}^d)$ bounded, we get that

$$\int_A f(\mathbf{r})\lambda(d\mathbf{r}) = \int_A \eta(d\mathbf{r}) \leq c \int_A \lambda(d\mathbf{r}).$$

Hence, $f(\mathbf{r}) \leq c \ \lambda-$a.e. in each bounded A and therefore $\|f\|_{L^\infty} \leq c$.

$\square$

Using the representation (4.55), we can explicitly rewrite Theorem 4.2.4 for $\mathcal{S} = \mathcal{S}_c$ as follows.

Theorem 4.3.11.

Let $c \in \mathbb{R}^+$. Let $m = (m^{(n)})_{n=0}^{\infty}$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. Assume that m fulfills the condition (4.15) for some function $k_2(\mathbf{r}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$. Then m is realized by a unique non-negative finite measure μ on $\mathcal{S}_c$ satisfying (4.16) if and only if the following inequalities hold.

$$L_m(h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \tag{4.59}$$

$$L_m(\Phi_\varphi h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \tag{4.60}$$

$$L_m(\Gamma_{c,\varphi} h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \tag{4.61}$$

$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} < \infty, \ \forall n \in \mathbb{N}_0, \tag{4.62}$$

where $\Phi_\varphi(\eta) := \langle \varphi, \eta \rangle$ and $\Gamma_{c,\varphi}(\eta) := c\langle \varphi, \lambda \rangle - \langle \varphi, \eta \rangle$.

Remark 4.3.12.

Proceeding exactly as in Remark 4.3.8, we can observe the analogy between realizability problem on $\mathcal{S}_c$ and the moment problem on $[0, c]$.
In fact, if for each $n \in \mathbb{N}_0$ there exists a function $\alpha^{(n)} \in L^1(\mathbb{R}^{dn}, \lambda)$ such that

$$m^{(n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) = \alpha^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n)d\mathbf{r}_1 \cdots d\mathbf{r}_n$$

then (4.59), (4.60) and (4.61) give respectively that $(\alpha^{(n)})_{n\in\mathbb{N}_0}$ is positive semidefinite and that for $\lambda-$almost all $\mathbf{y} \in \mathbb{R}^d$ the sequences $(\alpha^{(n+1)}(\cdot, \mathbf{y}))_{n\in\mathbb{N}_0}$ and $(c\alpha^{(n)}(\cdot) - \alpha^{(n+1)}(\cdot, \mathbf{y}))_{n\in\mathbb{N}_0}$ are positive semidefinite. Similarly, necessary and sufficient conditions for the realizability of a sequence of numbers $(m_n)_{n\in\mathbb{N}_0}$ on $[0, c]$, where

$$[0, c] = \{x \in \mathbb{R} : x \geq 0\} \cap \{x \in \mathbb{R} : c - x \geq 0\},$$

are that $(m_n)_{n\in\mathbb{N}_0}$, $(m_{n+1})_{n\in\mathbb{N}_0}$ and $(c \cdot m_n - m_{n+1})_{n\in\mathbb{N}_0}$ are positive semidefinite (see [19]).

4.3.5 The set of point configurations $\mathcal{N}(\mathbb{R}^d)$

Theorem 4.3.13.

The set of all Radon measures on $\mathbb{R}^d$ taking as values either a non-negative integer or infinity, i.e.

$$\mathcal{N}(\mathbb{R}^d) := \left\{\eta \in \mathcal{R}(\mathbb{R}^d) |\ \eta(B) \in (\mathbb{N}_0 \cup \{+\infty\}), \forall B \in \mathcal{B}(\mathbb{R}^d)\right\},$$

is a semi-algebraic subset of $\mathscr{D}'_{proj}(\mathbb{R}^d)$. More precisely, we get that

$$\mathcal{N}(\mathbb{R}^d) = \bigcap_{k\in\mathbb{N}} \bigcap_{\varphi\in\mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \left\{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) :\ \langle \varphi^{\otimes k}, \eta^{\odot k}\rangle \geq 0\right\}. \tag{4.63}$$

The power $\eta^{\odot k}$ of a generalized function $\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d)$ is called *factorial power* and it is defined as follows. For any $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and for any $n \in \mathbb{N}$

$$\langle f^{\otimes n}, \eta^{\odot n}\rangle := \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k!} \sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\ldots+n_k=n}} \frac{n!}{n_1 \cdot \ldots \cdot n_k} \langle T_{n_1,\ldots,n_k} f^{\otimes n}, \eta^{\otimes k}\rangle \tag{4.64}$$

with

$$T_{n_1,\ldots,n_k} f^{\otimes n}(x_1, \ldots, x_k) := f^{n_1}(x_1) \cdots f^{n_k}(x_k).$$

For example, in the cases $n = 1$ and $n = 2$ the previous definition gives

$$\langle f^{\otimes 1}, \eta^{\odot 1} \rangle = \langle f, \eta \rangle \quad \text{and} \quad \langle f^{\otimes 2}, \eta^{\odot 2} \rangle = \langle f, \eta \rangle^2 - \langle f^2, \eta \rangle.$$

The name "factorial power" comes from the fact that for any $\eta \in \mathcal{R}(\mathbb{R}^d)$ and for any measurable set A

$$\langle \mathbb{1}_A^{\otimes n}, \eta^{\odot n} \rangle = \eta(A)(\eta(A) - 1) \cdots (\eta(A) - n + 1).$$

Note that the definition of factorial power results natural in the setting of point configurations $\mathcal{N}(\mathbb{R}^d)$ (see [42]). In fact, if $\eta \in \mathcal{N}(\mathbb{R}^d)$ there exists $I \subseteq \mathbb{N}$ and $x_i \in \mathbb{R}^d$ $(i \in I)$ such that

$$\eta = \sum_{i \in I} \delta_{x_i}, \tag{4.65}$$

where I is either $\mathbb{N}$ or a finite subset of $\mathbb{N}$. Moreover, if $I = \mathbb{N}$ then the sequence $(x_i)_{i \in I}$ has no accumulation points in $\mathbb{R}^d$ (see [18]). Therefore, the definition (4.64) becomes

$$\langle f^{\otimes n}, \eta^{\odot n} \rangle = \sum_{i_1, \ldots, i_n \in I}{}' f(x_{i_1}) \cdots f(x_{i_n}),$$

where $\sum'$ denotes a sum over distinct indices (for more details see [42]).

Proof. (of Theorem 4.3.13)
Step I: $\subseteq$
Let us assume that $\eta \in \mathcal{N}(\mathbb{R}^d)$. Hence, by (4.65) for any $k \in \mathbb{N}$ and any $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$ we have that

$$\langle \varphi^{\otimes k}, \eta^{\odot k} \rangle = \sum_{i_1, \ldots, i_k \in I}{}' \varphi(x_{i_1}) \cdots \varphi(x_{i_k}) \geq 0.$$

Step II: $\supseteq$
Let $\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d)$ such that for any $k \in \mathbb{N}$ and for any $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$

$$\langle \varphi^{\otimes k}, \eta^{\odot k} \rangle \geq 0. \tag{4.66}$$

In particular, for $k = 1$ we have that $\eta \in \mathcal{R}(\mathbb{R}^d)$.
Moreover, since $\mathcal{C}_c^{\infty}(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d, \eta)$, the condition (4.66) also holds for any $\varphi \in L^1(\mathbb{R}^d, \eta)$ with $\varphi \geq 0$. In particular, for $\varphi = \mathbb{1}_A$ with $A \in \mathcal{B}(\mathbb{R}^d)$ bounded

we have that

$$0 \leq \langle \mathbb{1}_A^{\otimes k}, \eta^{\odot k} \rangle = \eta(A)(\eta(A)-1)\cdots(\eta(A)-k+1), \quad \forall k \in \mathbb{N}. \tag{4.67}$$

Hence, for any $A \in \mathcal{B}(\mathbb{R}^d)$ we get that $\eta(A) \in \mathbb{N}_0 \cup \{+\infty\}$.

$\square$

Using the representation (4.63), we can explicitly rewrite Theorem 4.2.4 for $\mathcal{S} = \mathcal{N}(\mathbb{R}^d)$ as follows.

Theorem 4.3.14.
Let $m = (m^{(n)})_{n=0}^{\infty}$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. Assume that m fulfills the condition (4.15) for some function $k_2(\mathbf{r}) \in \mathcal{C}^{\infty}(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$. Then m is realized by a unique non-negative finite measure μ on $\mathcal{N}(\mathbb{R}^d)$ satisfying (4.16) if and only if the following inequalities hold.

$$\begin{aligned}
&L_m(h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \\
&L_m(\Phi_{\varphi,k} h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \ \forall\, k \in \mathbb{N}, \\
&\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \ldots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} < \infty, \ \forall n \in \mathbb{N}_0,
\end{aligned} \tag{4.68}$$

where $\Phi_{\varphi,k}(\eta) := \langle \varphi^{\otimes k}, \eta^{\odot k} \rangle$.

Note that the condition (4.68) involves infinitely many polynomials of arbitrarily large degree. However, we can show another version of the previous theorem which only involves polynomials of at most second degree and which gives a realizing measure on the space $\mathcal{N}_s(\mathbb{R}^d)$ of all simple configurations, i.e.

$$\mathcal{N}_s(\mathbb{R}^d) := \{\eta \in \mathcal{N}(\mathbb{R}^d) | \, \forall\, \mathbf{x} \in \mathbb{R}^d, \, \eta(\{\mathbf{x}\}) \in \{0,1\}\}.$$

Theorem 4.3.15.
Let $m = (m^{(n)})_{n=0}^{\infty}$ be a sequence such that $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ and $m^{(n)}$ is a symmetric function of its n variables. Assume that m fulfills the condition (4.15) for some function $k_2(\mathbf{r}) \in \mathcal{C}^{\infty}(\mathbb{R}^d)$ with $k_2(\mathbf{r}) \geq 1$ for all $\mathbf{r} \in \mathbb{R}^d$. Then m is realized by a unique non-negative finite measure μ on $\mathcal{N}_s(\mathbb{R}^d)$ satisfying (4.16) if and only if

the following inequalities hold.

$$L_m(h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \tag{4.69}$$
$$L_m(\Phi_{\varphi,1}h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \tag{4.70}$$
$$L_m(\Phi_{\varphi,2}h^2) \geq 0, \ \forall h \in \mathscr{P}_{\mathcal{C}_c^\infty}\left(\mathscr{D}'_{proj}(\mathbb{R}^d)\right), \ \forall \varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d), \tag{4.71}$$
$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(d\mathbf{r}_1, \dots, d\mathbf{r}_{2n})}{\prod_{l=1}^{2n} k_2(\mathbf{r}_l)} < \infty, \ \forall n \in \mathbb{N}_0, \tag{4.72}$$
$$m^{(2)}(diag(\Lambda \times \Lambda)) = m^{(1)}(\Lambda), \ \ \forall \Lambda \in \mathcal{B}(\mathbb{R}^d) \text{ compact}, \tag{4.73}$$

where $diag(\Lambda \times \Lambda) := \{(\mathbf{x}, \mathbf{x}) \,|\, \mathbf{x} \in \Lambda\}$, $\Phi_{\varphi,1}(\eta) = \langle \varphi, \eta \rangle$ *and* $\Phi_{\varphi,2}(\eta) = \langle \varphi^{\otimes 2}, \eta^{\odot 2} \rangle$.

Remark 4.3.16.

Note that by Theorem 4.2.4, the conditions (4.69), (4.70), (4.71), (4.72) *are necessary and sufficient for the existence of a unique non-negative finite measure* μ *realizing the sequence* m *on the set*

$$\mathcal{S} := \bigcap_{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : \langle \varphi, \eta \rangle \geq 0\} \cap \bigcap_{\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)} \{\eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : \langle \varphi^{\otimes 2}, \eta^{\odot 2} \rangle \geq 0\}.$$

Let us note that $\mathcal{N}_s(\mathbb{R}^d) \subset \mathcal{N}(\mathbb{R}^d) \subset \mathcal{S}$.

Proof.

Sufficiency

Let us assume that (4.69), (4.70), (4.71), (4.72) and (4.73) hold. W.l.o.g. we can suppose that the measure μ given by Remark 4.3.16 is a probability on $\mathcal{S}$. Hence, it remains to show that μ is actually a probability on $\mathcal{N}_s(\mathbb{R}^d)$.

Let $\eta \in \mathcal{S}$. Then, for any $\varphi \in \mathcal{C}_c^{+,\infty}(\mathbb{R}^d)$

$$\langle \varphi, \eta \rangle \geq 0 \quad \text{and} \quad \langle \varphi^{\otimes 2}, \eta^{\odot 2} \rangle \geq 0.$$

Since $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d, \eta)$, the previous condition also holds for any $\varphi \in L^1(\mathbb{R}^d, \eta)$ with $\varphi \geq 0$. In particular, it holds for $\varphi = 1\!\!1_A$ where $A \in \mathcal{B}(\mathbb{R}^d)$ bounded, i.e.

$$\begin{cases} \eta(A) \geq 0 \\ \eta(A)(\eta(A) - 1) \geq 0. \end{cases}$$

The latter relations imply that $\eta(A) \in \{0\} \cup [1, +\infty]$ and so that there exist $I \subseteq \mathbb{N}$, $\mathbf{x}_i \in \mathbb{R}^d$ and real numbers $a_i \geq 1$ $(i \in I)$ such that

$$\eta = \sum_{i \in I} a_i \delta_{\mathbf{x}_i}. \tag{4.74}$$

On the other hand, using (4.73) and the fact that m is realized by μ on $\mathcal{S}$ we get, via approximation arguments, that for any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ compact

$$0 = m^{(2)}(diag(\Lambda \times \Lambda)) - m^{(1)}(\Lambda) = \int_{\mathcal{S}} \Big(\langle 1\!\!1_{diag(\Lambda\times\Lambda)}, \eta^{\otimes 2}\rangle - \langle 1\!\!1_{\Lambda}, \eta\rangle \Big) \mu(d\eta),$$

and so

$$\langle 1\!\!1_{diag(\Lambda\times\Lambda)}, \eta^{\otimes 2}\rangle - \langle 1\!\!1_{\Lambda}, \eta\rangle = 0, \quad \mu - \text{a.e.}.$$

By (4.74) the latter becomes

$$0 = \sum_{\substack{i,j\in I\\ \mathbf{x}_i=\mathbf{x}_j\in\Lambda}} a_i a_j - \sum_{\substack{i\in I\\ \mathbf{x}_i\in\Lambda}} a_i = \sum_{\substack{i\in I\\ \mathbf{x}_i\in\Lambda}} a_i \left(\sum_{\substack{j\in I\\ \mathbf{x}_j=\mathbf{x}_i}} a_j - 1 \right).$$

Since $a_i \geq 1$ for all $i \in I$, we necessarily have that

$$\sum_{\substack{j\in I\\ \mathbf{x}_j=\mathbf{x}_i}} a_j - 1 = 0.$$

Then for all $i \in I$

$$a_i - 1 + \sum_{\substack{j\neq i\in I\\ \mathbf{x}_j=\mathbf{x}_i}} a_j = 0. \tag{4.75}$$

Since $a_i - 1$ and $\sum_{\substack{j\neq i\in I\\ \mathbf{x}_j=\mathbf{x}_i}} a_j$ are non-negative numbers, (4.75) implies that

$$\forall i \in I,\ a_i = 1 \quad \text{and} \quad \forall j, i \in I \text{ with } j \neq i,\ \mathbf{x}_j \neq \mathbf{x}_i.$$

Hence, we got that for $\mu-$almost all $\eta \in \mathcal{S}$

$$\eta = \sum_{i\in I} \delta_{\mathbf{x}_i} \quad \text{and} \quad \eta(\{\mathbf{x}\}) \in \{0, 1\}$$

which means that for $\mu-$almost all $\eta \in \mathcal{S}$ we have $\eta \in \mathcal{N}_s(\mathbb{R}^d)$, i.e. $\mu(\mathcal{N}_s(\mathbb{R}^d)) = 1$.

Necessity

Let us assume that the sequence m is realized by a non-negative finite measure μ on $\mathcal{N}_s(\mathbb{R}^d)$. W.l.o.g. we can suppose $\mu(\mathcal{N}_s(\mathbb{R}^d)) = 1$. Hence, we can extend μ to the whole $\mathcal{S}$ by setting $\mu(\eta) = 0$ for all $\eta \in \mathcal{S} \setminus \mathcal{N}_s(\mathbb{R}^d)$. In this way we have that m is realized by μ also on $\mathcal{S}$ and $\mu(\mathcal{S}) = 1$.

By Remark 4.3.16, it only remains to show the condition (4.73).

Recall that for any $\eta \in \mathcal{N}_s(\mathbb{R}^d)$ we have that there exist $I \subseteq \mathbb{N}$ and $\mathbf{x}_i \in \mathbb{R}^d$ such that

$$\eta = \sum_{i \in I} \delta_{\mathbf{x}_i} \quad \text{and} \quad \eta(\{\mathbf{x}\}) \in \{0, 1\}.$$

Therefore, for any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ compact

$$\langle \mathbb{1}_{diag(\Lambda\times\Lambda)}, \eta^{\otimes 2} \rangle - \langle \mathbb{1}_{\Lambda}, \eta \rangle = \sum_{i,j \in I} \mathbb{1}_{diag(\Lambda\times\Lambda)}(\mathbf{x}_i, \mathbf{x}_j) - \sum_{i \in I} \mathbb{1}_{\Lambda}(\mathbf{x}_i) = 0. \tag{4.76}$$

On the other hand, using the fact that m is realized by μ on $\mathcal{N}_s(\mathbb{R}^d)$ we get, via approximation arguments, that for any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ measurable and compact

$$m^{(2)}(diag(\Lambda \times \Lambda)) - m^{(1)}(\Lambda) = \int_{\mathcal{S}} \Big(\langle \mathbb{1}_{diag(\Lambda\times\Lambda)}, \eta^{\otimes 2} \rangle - \langle \mathbb{1}_{\Lambda}, \eta \rangle \Big) \mu(d\eta). \tag{4.77}$$

In conclusion, (4.76) and (4.77) give (4.73).

□

Conclusions and open problems

The main objective of this work was to give necessary and sufficient conditions for a sequence of Radon measures to be the sequence of moment functions of a finite measure concentrated on a pre-given basic semi-algebraic subset of the space of generalized functions on $\mathbb{R}^d$. Getting conditions of semidefinite type was possible by using classical results about the moment problem on nuclear spaces and techniques built to solve the moment problem on basic semi-algebraic subsets of $\mathbb{R}^d$. We demonstrated the usefulness of these results in some concrete situations. The necessary and sufficient conditions depend on the polynomials used in the representation of the semi-algebraic set under consideration. Furthermore, we reviewed and clarified the role of the Carleman condition and the uniqueness result in the context of the moment theory. In the case of the classical moment problem we were able to slightly extend a result of Lasserre.

This work opens up further developments. A first natural generalization of Theorem 4.2.4 is studied in [35] in which the sequence $m = (m^{(n)})_{n\in\mathbb{N}_0}$ of putative moment functions is made of generalized functions which are not necessarily Radon measures. The essential parts of the proofs are the same as in this thesis. Nevertheless, we decided to present here the results under the conditions that the putative moment functions are Radon measures because this allowed us to give slight different proofs which are more natural and less abstract in this case. For each concrete case, our theorem leads to the question of finding appropriate polynomials for the representation of the semi-algebraic set to get conditions as easy as possible. Different applications given by different physical problems will require a careful investigation of the related polynomials.

Moreover, in view of [4, 6, 48, 49], it would be interesting to discover whether analogous results can be obtained if the sequence m is made of *correlation functions* which are more natural in the context of point configuration type spaces. More precisely, given a sequence of symmetric Radon measure $\rho = (\rho^{(n)})_{n=0}^{\infty}$, we would look for the existence of a generalized process μ on Ω' such that for any

$n \in \mathbb{N}_0$ and for any $f^{(n)} \in \Omega^{\otimes n}$ we have

$$\langle f^{(n)}, \rho^{(n)} \rangle = \int_{\Omega'} \langle f^{(n)}, \eta^{\odot n} \rangle \mu(d\eta),$$

assumed the integral is finite. The difference is that $\eta^{\odot n}$ is a modified tensor power of $\eta^{\otimes}$ (see (4.64)).

The case when the sequence m (or ρ) is truncated up to a certain order N, i.e. $m = (m^{(n)})_{n=0}^{N}$ with $N \in \mathbb{N}_0$, still remains unsolved. The truncated realizability problem is of substantial importance because in a lot of practical applications one has to deal with a limited number of data given by the limitation of observations in experiments, which gives statistical reliable information only about few moments. The main difficulty for such a line of research is that even in the finite dimensional case the truncated moment problem is far less developed than the full one.

It would be interesting to investigate if it is possible to extend the results of R. E. Curto, L. A. Fialkow, M. G. Kreĭn and A. A. Nudel'man (see [17, 40]) for the truncated moment problem to the infinite dimensional case.

Another open path in this theory is to improve the determining condition in concrete cases.

Appendix A

Quasi-analiticity

Let us recall the basic definitions and state the results used throughout this thesis concerning the theory of quasi-analiticity.

Definition A.0.17 (The class $C\{M_n\}$).
Given a sequence of positive real numbers $(M_n)_{n=0}^{\infty}$, we define the class $C\{M_n\}$ as the set of all functions $f \in \mathcal{C}^{\infty}(\mathbb{R})$ such that for any $n \in \mathbb{N}_0$

$$\|D^n f\|_{\infty} \leq \beta_f B_f^n M_n,$$

where $D^n f$ is the $n-th$ derivative of f, $\|D^n f\|_{\infty} := \sup_{x\in\mathbb{R}} |D^n f(x)|$, and β_f, B_f are positive constants only depending on f.

Definition A.0.18 (Quasi-analytical class).
A class $C\{M_n\}$ is said to be quasi-analytical if the conditions

$$f \in C\{M_n\},\ (D^n f)(0) = 0, \quad \forall\, n \in \mathbb{N}_0,$$

imply that $f(x) = 0$ for all $x \in \mathbb{R}$.

The main result in the theory of quasi-analiticity is the following (see [34, Theorem 1.3.8] and [16] for a detailed proof).

Theorem A.0.19 (The Denjoy-Carleman Theorem).
Let $(M_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. Then, the following conditions are equivalent.

1. *$C\{M_n\}$ is quasi-analytical.*
2. *$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty$, where $\beta_n := \inf_{k\geq n} \sqrt[k]{M_k}$.*

The previous theorem, as well as some of the following propositions, can be proved more easily if one assumes that the sequence of positive numbers $(M_n)_{n=0}^{\infty}$ satisfies the assumptions

- $M_0 = 1$,
- $M_n^2 \leq M_{n-1}M_{n+1}$, $n \in \mathbb{N}$ (log-convexity).

Remark A.0.20.
If $\mu \in \mathcal{M}^*(\mathbb{R})$, then the sequence $M = (M_n)_{n=0}^{\infty}$, with $M_n = \int |x|^n \mu(dx)$, is log-convex. In fact, by Cauchy-Schwarz's inequality, we have that for any $n \in \mathbb{N}$

$$\begin{aligned} M_n^2 = \left(\int |x|^n \mu(dx)\right)^2 &= \left(\int |x|^{\frac{n-1}{2}} |x|^{\frac{n+1}{2}} \mu(dx)\right)^2 \\ &\leq \left(\int |x|^{n-1}\mu(dx)\right)\left(\int |x|^{n+1}\mu(dx)\right) \\ &= M_{n-1}M_{n+1}. \end{aligned}$$

By writing $2n = \frac{2n-2}{2} + \frac{2n+2}{2}$ (with $n \in \mathbb{N}$) we get, as before, that $M_{2n}^2 \leq M_{2n-2}M_{2n+2}$ or, in other words, $M_{2n}^2 \leq M_{2(n-1)}M_{2(n+1)}$. The latter means that the sequence of the even *moments of μ, namely the sequence $m = (m_n)_{n=0}^{\infty}$ with $m_n = M_{2n}$, is log-convex.*

Note that, if $\mu \in \mathcal{M}^*(\mathbb{R})$ is a probability, $m_0 = M_0 = 1$.

Let us state the Denjoy-Carleman theorem under the assumption of log-convexity (see [14] and [67, Theorem 19.11] for the proof of the theorem in this case).

Theorem A.0.21 (Denjoy-Carleman's Theorem for log-convex sequences)**.**
Let $(M_n)_{n=0}^{\infty}$ be a log-convex sequence of positive numbers with $M_0 = 1$. Then, the following conditions are equivalent.

1. *$C\{M_n\}$ is quasi-analytical.*
2. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty$.
3. $\sum\limits_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty$.

Remark A.0.22.
The assumption of log-convexity involves no loss of generality regarding the quasi-analytic classes. *In fact, one can prove that for any sequence $(M_n)_{n=0}^{\infty}$ there always exists a log-convex sequence $(\tilde{M}_n)_{n=0}^{\infty}$, with $\tilde{M}_0 = 1$, such that the classes $C\{M_n\}$ and $C\{\tilde{M}_n\}$ coincide. More precisely, the sequence $(\tilde{M}_n)_{n=0}^{\infty}$ is the convex*

regularization of $(\frac{M_n}{M_0})_{n=0}^{\infty}$ by means of the logarithm (for more details see [50, Chapter VI, Theorem 6.5.III] and [25]).

Lemma A.0.23.
A function f is convex if and only if for all $y \leq z \leq x$ the following holds.

$$(x-y)f(z) \leq (z-y)f(x) + (x-z)f(y). \tag{A.1}$$

Proof.
Let us recall that, by definition, a function f is convex if and only if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{A.2}$$

for any $\lambda \in [0,1]$.
Let us consider $z := \lambda x + (1-\lambda)y$ with $\lambda \in [0,1]$. Then we have $\lambda = \frac{z-y}{x-y}$ and, by substituting the latter in (A.2), we get (A.1).

□

Lemma A.0.24.
For a sequence of positive real numbers $(M_n)_{n=0}^{\infty}$ the following properties are equivalent.

(a) $M_n^2 \leq M_{n-1}M_{n+1}$ *for any* $n \geq 1$.

(b) $\left(\frac{M_n}{M_{n-1}}\right)_{n=1}^{\infty}$ *is monotone increasing.*

(c) $(\ln M_n)_{n=1}^{\infty}$ *is convex.*

Proof.
(a) and (b) are obviously equivalent by dividing or multiplying by M_n.
If (c) holds then, by Lemma A.0.23, we have

$$2\ln M_n \leq \ln M_{n+1} + \ln M_{n-1},$$

which is equivalent to (a).
Let us assume (b), then we need to check the convexity of $(\ln M_n)_{n=1}^{\infty}$.
For any positive integers n, m, k such that $n \leq k \leq m$ the following inequality holds.

$$\frac{1}{k-n}\sum_{j=n+1}^{k} \ln\left(\frac{M_j}{M_{j-1}}\right) \leq \frac{1}{m-k}\sum_{j=k+1}^{m} \ln\left(\frac{M_j}{M_{j-1}}\right), \tag{A.3}$$

where we used the assumption (b) and the fact that the denominators of the pre-factors are equal to the number of summands in both sums.

The inequality (A.3) is equivalent to

$$\frac{1}{k-n}\sum_{j=n+1}^{k}\big(\ln M_j - \ln M_{j-1}\big) \leq \frac{1}{m-k}\sum_{j=k+1}^{m}\big(\ln M_j - \ln M_{j-1}\big),$$

which can be rewritten as

$$(m-k)(\ln M_k - \ln M_n) \leq (k-n)(\ln M_m - \ln M_k).$$

The latter becomes

$$(m-n)\ln M_k \leq (k-n)\ln M_m + (m-k)\ln(M_n).$$

Hence, by Lemma A.0.23, the condition (c) holds.

□

Lemma A.0.25.
If the sequence $(M_n)_{n=0}^{\infty}$ of positive real numbers, with $M_0 = 1$, is log-convex then $(\sqrt[n]{M_n})_{n=1}^{\infty}$ is monotone increasing.

Proof.
From (b) it follows that

$$M_n = \frac{M_n}{M_0} = \prod_{j=1}^{n}\frac{M_j}{M_{j-1}} \leq \left(\frac{M_n}{M_{n-1}}\right)^n,$$

i.e.

$$M_{n-1}^{n} \leq M_n^{n-1},$$

or equivalently

$$M_{n-1}^{1/n-1} \leq M_n^{1/n}.$$

□

Lemma A.0.26.
Assume that $(M_n)_{n=0}^{\infty}$, with $M_0 = 1$, is a log-convex sequence of positive real numbers. Then, for any $j \in \mathbb{N}$,

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{M_n}} = \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[jn]{M_{jn}}} = \infty.$$

Proof.

If the series $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[jn]{M_{jn}}}$ diverges for some $j \in \mathbb{N}$, then also $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}}$ does since the latter contains more summands, i.e.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[jn]{M_{jn}}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}}.$$

On the other hand, fixed $j \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} &= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt[jn]{M_{jn}}} + \frac{1}{\sqrt[jn+1]{M_{jn+1}}} + \ldots + \frac{1}{\sqrt[jn+(j-1)]{M_{jn+j-1}}} \right) + \sum_{n=1}^{j-1} \frac{1}{\sqrt[n]{M_n}} \\
&\leq j \sum_{n=1}^{\infty} \frac{1}{\sqrt[jn]{M_{jn}}} + \sum_{n=1}^{j-1} \frac{1}{\sqrt[n]{M_n}},
\end{aligned}$$

where in the last inequality we made use of Lemma A.0.25. Hence, if $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}}$ diverges then $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[jn]{M_{jn}}}$ diverges as well. □

Lemma A.0.27.

Let $(M_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. Then, for any $k \in \mathbb{N}_0$,

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{M_{n+k-1}}{M_{n+k}} = \infty.$$

Proof.

These two series differ only by a finite number of positive summands. In fact, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} &= \sum_{n=1}^{k} \frac{M_{n-1}}{M_n} + \sum_{n=k+1}^{\infty} \frac{M_{n-1}}{M_n} \\
&= \sum_{n=1}^{k} \frac{M_{n-1}}{M_n} + \sum_{n=1}^{\infty} \frac{M_{n+k-1}}{M_{n+k}}.
\end{aligned}$$

□

Lemma A.0.28.
Assume that $(M_n)_{n=0}^{\infty}$ is a sequence of positive real numbers. Then, for any positive constant δ,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\delta\, M_n}} = \infty.$$

Proof.
In the case $\delta = 1$ the theorem trivially holds.

Assume that $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty$ and let us define $t(n) := \delta^{\frac{1}{n}}$.

- If $0 < \delta < 1$ then $t(n)$ is increasing and so for any $n \in \mathbb{N}$ we have $\delta^{\frac{1}{n}} M_n^{\frac{1}{n}} \leq \left(\lim\limits_{n\to\infty} \delta^{\frac{1}{n}}\right) M_n^{\frac{1}{n}}$. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\delta\, M_n}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty.$$

- If $\delta > 1$ then $t(n)$ is decreasing and so for any $n \in \mathbb{N}$ we have $\delta^{\frac{1}{n}}\, M_n^{\frac{1}{n}} \leq \delta\, M_n^{\frac{1}{n}}$. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\delta\, M_n}} \geq \frac{1}{\delta} \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty.$$

Assume that for any $\delta > 0$ we have $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{\delta\, M_n}} = \infty$. Since $t(n) \to 1$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, we have that $t(n) \geq \frac{1}{2}$ and then

$$\frac{1}{\sqrt[n]{\delta\, M_n}} \leq \frac{2}{\sqrt[n]{M_n}}.$$

□

Lemma A.0.29.
Let $(M_n)_{n=0}^{\infty}$, with $M_0 = 1$, be a log-convex sequence of positive real numbers. Then, for any $k \in \mathbb{N}_0$,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_{n+k}}} = \infty.$$

Proof.
By Theorem A.0.21 and Lemma A.0.27 we have that for any $k \in \mathbb{N}_0$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{M_{n+k-1}}{M_{n+k}} = \infty. \tag{A.4}$$

For any $n \in \mathbb{N}_0$, let us define

$$B_n := \frac{M_{n+k}}{M_k}.$$

Then, by the log-convexity of $(M_n)_{n=0}^{\infty}$, we get that $(B_n)_{n=0}^{\infty}$ is log-convex as well. In fact, for any $n \in \mathbb{N}$,

$$B_n^2 = \frac{M_{n+k}^2}{M_k^2} \leq \frac{M_{n+k-1}}{M_k} \frac{M_{n+k+1}}{M_k} = B_{n-1}B_{n+1}.$$

Moreover, $B_0 = 1$. Hence, (A.4) becomes

$$\sum_{n=1}^{\infty} \frac{B_{n-1}}{B_n} = \infty.$$

By Theorem A.0.21, the latter is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{B_n}} = \infty,$$

which is indeed

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\frac{M_{n+k}}{M_k}}} = \infty. \tag{A.5}$$

The conclusion follows by Lemma A.0.28 applied to (A.5) for $\delta = \frac{1}{M_k}$.

$\square$

Theorem A.0.30.
Let $(M_n)_{n=0}^{\infty}$*, with* $M_0 = 1$*, be a log-convex sequence of positive real numbers. If*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{M_{2n}}} = \infty$$

then for any $h \in \mathbb{N}_0$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{M_{2n+h}}} = \infty.$$

Proof.
Let us distinguish two cases.

1. Case h even
Let us consider $h = 2k$ for some $k \in \mathbb{N}_0$ and let us define the following sequence

$$A_n := \sqrt{M_{2n}}, \quad n \in \mathbb{N}_0,$$

which is log-convex. In fact, by assumption, we get that

$$\begin{aligned} M_{2n}^2 &\leq M_{2n-1}M_{2n+1} \\ &\leq \sqrt{M_{2n-2}}\sqrt{M_{2n}}\sqrt{M_{2n}}\sqrt{M_{2n+2}} \\ &= \sqrt{M_{2n-2}}M_{2n}\sqrt{M_{2n+2}}, \end{aligned}$$

which becomes

$$M_{2n} \leq \sqrt{M_{2n-2}}\sqrt{M_{2n+2}}.$$

Hence, the latter implies

$$A_n^2 = M_{2n} \leq \sqrt{M_{2n-2}}\sqrt{M_{2n+2}} = A_{n-1}A_{n+1}.$$

Note that $A_0 = \sqrt{M_0} = 1$.
With this relabeling, $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[2n]{M_{2n}}} = \infty$ becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{A_n}} = \infty$. Applying Lemma A.0.29 to the sequence $(A_n)_{n=0}^{\infty}$ we get that for any $k \in \mathbb{N}_0$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{A_{n+k}}} = \infty,$$

i.e.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{M_{2n+2k}}} = \infty.$$

2. Case h odd

To get our conclusion also in the case $h = 2k-1$ for some $k \in \mathbb{N}$, we need to consider separately two subcases.

2.1. Suppose $(M_n)_{n=0}^{\infty}$ is bounded, i.e. there exists a finite positive constant c such that $M_n \leq c$ for any $n \in \mathbb{N}_0$. In particular, we have that

$$\frac{1}{\sqrt[2n]{M_{2n+h}}} \geq \frac{1}{\sqrt[2n]{c}}, \tag{A.6}$$

for any $n \in \mathbb{N}$.

Since $\frac{1}{\sqrt[2n]{c}} \to 1$ as $n \to \infty$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{c}}$ diverges and the conclusion follows by (A.6).

2.2. Suppose $(M_n)_{n=0}^{\infty}$ diverges. Then there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we have that $M_n \geq 1$. Hence, for any $n \geq N$, we get that

$$\frac{1}{\sqrt[2n-1]{M_{2n-1}}} \leq \frac{1}{\sqrt[2n]{M_{2n-1}}}. \tag{A.7}$$

Moreover, by Lemma A.0.25, we have that for any $n \in \mathbb{N}$

$$\frac{1}{\sqrt[2n]{M_{2n}}} \leq \frac{1}{\sqrt[2n-1]{M_{2n-1}}}. \tag{A.8}$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{M_{2n}}} = \infty$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n-1]{M_{2n-1}}} = \infty$ by (A.8), which implies by (A.7) that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{M_{2n-1}}} = \infty. \tag{A.9}$$

Let us consider now the following sequence defined as

$$\begin{cases} B_0 := 1 \\ B_n := \frac{\sqrt{M_3}}{M_1}\sqrt{M_{2n-1}}, \quad n \in \mathbb{N}. \end{cases}$$

The sequence $(B_n)_{n=0}^{\infty}$ is log-convex. In fact,

- $B_1^2 = \frac{M_3}{M_1^2}(\sqrt{M_1})^2 = \frac{M_3}{M_1} = \frac{\sqrt{M_3}}{M_1}\sqrt{M_3} = B_2 = B_0 B_2$
- By assumption, we get that for any $n \geq 2$

$$\begin{aligned} M_{2n-1}^2 &\leq M_{2n-2}M_{2n} \\ &\leq \sqrt{M_{2n-3}}\sqrt{M_{2n-1}}\sqrt{M_{2n-1}}\sqrt{M_{2n+1}} \\ &= \sqrt{M_{2n-3}}\sqrt{M_{2n+1}}M_{2n-1}, \end{aligned}$$

i.e.

$$M_{2n-1} \leq \sqrt{M_{2n-3}}\sqrt{M_{2n+1}}.$$

Hence, the latter implies

$$\begin{aligned} B_n^2 = \frac{M_3}{M_1^2} M_{2n-1} &\leq \frac{M_3}{M_1^2}\sqrt{M_{2n-3}}\sqrt{M_{2n+1}} \\ &= \frac{\sqrt{M_3}}{M_1}\sqrt{M_{2n-3}}\frac{\sqrt{M_3}}{M_1}\sqrt{M_{2n+1}} \\ &= B_{n-1}B_{n+1}. \end{aligned}$$

With this relabeling, (A.9) becomes

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\frac{M_1}{\sqrt{M_3}} B_n}} = \infty,$$

which, by Lemma A.0.28, is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{B_n}} = \infty.$$

Applying Lemma A.0.29 to the sequence $(B_n)_{n=0}^{\infty}$, we get that for any $k \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{B_{n+k}}} = \infty,$$

and so we have

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\frac{\sqrt{M_3}}{M_1}\sqrt{M_{2n+(2k-1)}}}} = \infty.$$

By Lemma A.0.28, the latter implies the conclusion.

□

Let us recall a simple but useful property.

Proposition A.0.31.
If a finite sequence of positive real numbers $(M_n)_{n=0}^{N}$ is log-convex then it is unimodal, i.e. there exists $\bar{n} \in \{0, 1, \ldots, N\}$ such that

$$\begin{cases} M_n \geq M_{n+1}, & \forall n \leq \bar{n} \\ M_n \leq M_{n+1}, & \forall n > \bar{n}. \end{cases}$$

The following result is a generalization of Theorem A.0.21.

Theorem A.0.32 (De Jeu, [38]).
Let $(M_{(1,m)})_{m\in\mathbb{N}_0}$ and $(M_{(2,m)})_{m\in\mathbb{N}_0}$, with $M_{(1,0)} = M_{(2,0)} = 1$, be two log-convex sequences of positive real numbers such that

$$\sum_{m=1}^{\infty} (M_{(1,m)})^{-\frac{1}{m}} = \infty \quad \text{and} \quad \sum_{m=1}^{\infty} (M_{(2,m)})^{-\frac{1}{m}} = \infty.$$

Let $f \in \mathcal{C}^\infty(\mathbb{R}^2)$ and let A and r be non-negative constants such that

$$\left| \frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} (f)(a,b) \right| \leq A r^{\alpha_1+\alpha_2} M_{(1,\alpha_1)} M_{(2,\alpha_2)}$$

for all $\alpha_1, \alpha_2 \in \mathbb{N}_0$ and all $(a,b) \in \mathbb{R}^2$.
Then, if

$$\frac{\partial^{\alpha_2}}{\partial b^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} (f)(0,0) = 0, \quad \forall \alpha_1, \alpha_2 \in \mathbb{N}_0,$$

f is identically equal to zero on $\mathbb{R}^2$.

Proof.
For $\alpha_1 \in \mathbb{N}_0$ we define the function ϕ_{α_1} by

$$\phi_{\alpha_1}(b) := \frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}} f(0,b), \quad b \in \mathbb{R}.$$

Then, for $\alpha_2 \in \mathbb{N}_0$, all the α_2-th derivatives (w.r.t. b) of ϕ_{α_1} vanish at $0 \in \mathbb{R}$ by assumption.
Moreover, since

$$\left| \frac{d^{\alpha_2}}{db^{\alpha_2}} \phi_{\alpha_1}(b) \right| \leq \underbrace{A\, r^{\alpha_1}\, M_{(1,\alpha_1)}}_{:=A'} r^{\alpha_2} M_{(2,\alpha_2)}, \quad b \in \mathbb{R},$$

where A' is a positive constant, by the Denjoy-Carleman Theorem A.0.21 we have that the class $C\{M_{(2,\alpha_2)}\}$ is quasi-analytical, i.e. ϕ_{α_1} is identically zero on $\mathbb{R}$, for an arbitrary $\alpha_1 \in \mathbb{N}_0$.
For each $b \in \mathbb{R}$ define $\psi_b(a) := f(a,b)$ with $a \in \mathbb{R}$.
Since

$$\frac{d^{\alpha_1}}{da^{\alpha_1}} \psi_b(0) = \phi_{\alpha_1}(b)$$

and $\phi_{\alpha_1} \equiv 0$ on $\mathbb{R}$, we have that all the derivatives of ψ_b vanish at $0 \in \mathbb{R}$ too, for arbitrary $b \in \mathbb{R}$.

Additionally, we have that

$$\left|\frac{\partial^{\alpha_1}}{\partial a^{\alpha_1}}\psi_b(a)\right| \leq \underbrace{A\,M_{(2,0)}}_{=:A''}\,r^{\alpha_1}\,M_{(1,\alpha_1)}$$

for all $a, b \in \mathbb{R}$ (A'' is a positive constant).
By the Denjoy-Carleman theorem, the class $C\{M_{(1,\alpha_1)}\}$ is quasi-analytical, i.e. we have that ψ_b is identically zero on $\mathbb{R}$. Hence, $f \equiv 0$ on $\mathbb{R}^2$.

$\square$

Appendix B

Spectral theory

In this chapter we review some basic definitions and fundamental results of the general spectral theory.
Although the notation is standard we recall here the most used objects. We consider a Hilbert space $\mathcal{H}$ with its inner product given by the sesquilinear form $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. We denote by $\mathscr{L}(\mathcal{H})$ the class of all linear and bounded operators $T : \mathcal{H} \to \mathcal{H}$.

The Hellinger-Toeplitz theorem says that an everywhere-defined operator T which satisfies $\langle Tv, w\rangle = \langle v, Tw\rangle$, for all $v, w \in \mathcal{H}$, is necessarily a bounded operator. This suggests that an unbounded symmetric operator T can be only defined on a subset of the Hilbert space $\mathcal{H}$. Then for an unbounded operator T we denote by $\mathcal{D}(T)$ its domain, namely a linear subspace of $\mathcal{H}$ which we will always suppose to be dense in $\mathcal{H}$.
Let $\mathcal{D}(T^*)$ be the set of all $w \in \mathcal{H}$ for which there exists a $z \in \mathcal{H}$ such that

$$\langle Tv, w\rangle = \langle v, z\rangle, \quad \forall v \in \mathcal{D}(T). \tag{B.1}$$

For each $w \in \mathcal{D}(T^*)$, we define $T^*w = z$. The operator T^* is called the *adjoint* of T. For z to be uniquely determined by (B.1) we need the fact that $\mathcal{D}(T)$ is dense in $\mathcal{H}$.

Let us recall that an unbounded operator T is called

- *symmetric* if and only if $\langle Tv, w\rangle = \langle v, Tw\rangle, \quad \forall v, w \in \mathcal{D}(T)$,
- *self-adjoint* if T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$,
- *unitary* if T is invertible and $T^{-1} = T^*$.

There are several, but equivalent, formulations of the spectral theorem. The most well-known version broadly says that a self-adjoint operator can be identified with a multiplication operator. We will analyze these theorems in the case of bounded and unbounded self-adjoint operators. Ultimately, we will show what happens when we deal with a tuple of operators.

B.1 Multiplication operator

Let us first show some preliminary facts.
Let (X, μ) be a finite measure space (i.e. $\mu(X) < +\infty$). By $L^\infty(X, \mu)$ we denote the space of all measurable functions (*complex*-valued) which are essentially bounded, i.e. bounded up to μ-null sets. A norm on this space is given by

$$||g||_{L^\infty} := \inf\{C \geq 0 : |g(x)| \leq C \text{ for } \mu\text{-almost every } x \in X\},$$

for $g \in L^\infty(X, \mu)$.

For any $g \in L^\infty(X, \mu)$, the linear map

$$\begin{aligned} M_g : \quad L^2(X, \mu) &\to L^2(X, \mu) \\ \varphi \qquad &\mapsto M_g\varphi := g\varphi \end{aligned} \tag{B.2}$$

is continuous. In fact, for any $\varphi \in L^2(X, \mu)$ we have

$$||M_g\varphi||^2_{L^2(X,\mu)} = \int_X |g(x)\varphi(x)|^2 \, \mu(dx) \leq ||g||^2_{L^\infty} ||\varphi||^2_{L^2(X,\mu)},$$

so M_g is well defined and continuous with norm $||M_g||_{\mathscr{L}(L^2(X,\mu))} \leq \; ||g||_{L^\infty}$.
Moreover, for any $\varphi_1, \varphi_2 \in L^2(X, \mu)$

$$\langle M_g(\varphi_1), \varphi_2\rangle_{L^2(X,\mu)} = \int_X g(x)\varphi_1(x)\overline{\varphi_2(x)}\mu(dx) = \langle \varphi_1, M_{\bar{g}}(\varphi_2)\rangle_{L^2(X,\mu)}.$$

Therefore, $M_g^* = M_{\bar{g}}$ and M_g is self-adjoint if and only if g is *real*-valued.

B.2 Spectral theorem for unbounded self-adjoint operators

The following is the multiplication operator form of the spectral theorem for unbounded self-adjoint operators.

Theorem B.2.1 (Multiplication operator form, [63] Vol. I, p. 260)**.**
Let $\mathcal{H}$ be a separable Hilbert space and let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint

operator. Then there exist a measure space (X, μ) with μ finite, a real-valued function $g : X \to \mathbb{R}$ which is finite a.e. on X, and a unitary map $U : L^2(X, \mu) \to \mathcal{H}$ such that

- $U^{-1}(\mathcal{D}(T)) = \mathcal{D}(M_g) = \{\varphi \in L^2(X, \mu) \,|\, g\varphi \in L^2(X, \mu)\}$,
- $U^{-1}TU = M_g$ *on* $\mathcal{D}(M_g)$, *i.e.* $U(g\varphi) = T(U\varphi)$ *for* $\varphi \in \mathcal{D}(M_g)$,

where M_g is the operator of multiplication by g.

In other words, the following diagram

$$\begin{tikzcd}
\mathcal{D}(M_g) \arrow[r, "U"] \arrow[d, "M_g"] & \mathcal{D}(T) \arrow[d, "T"] \\
L^2(X, \mu) \arrow[r, "U"] & \mathcal{H}
\end{tikzcd}$$

commutes and the operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ can be identified with the operator $M_g : \mathcal{D}(M_g) \subset L^2(X, \mu) \to L^2(X, \mu)$.

Corollary B.2.2.
Let $\mathcal{H}$ be a separable Hilbert space and T an unbounded self-adjoint operator with domain $\mathcal{D}(T)$ in $\mathcal{H}$. Let v be such that $v \in \mathcal{D}(T)$, $Tv \in \mathcal{D}(T)$, $\ldots$, $T^{\alpha-1}v \in \mathcal{D}(T)$ for any $\alpha \in \mathbb{N}$, then there exists a finite measure μ_v on $\mathbb{R}$ such that

$$\int |r|^\alpha \mu_v(dr) < \infty \quad \text{and} \quad \langle v, T^j v\rangle = \int r^j \mu_v(dr), \tag{B.3}$$

for all $0 \le j \le \alpha$.

Proof.
Let us prove (B.3) for $\alpha = 1$. First, we represent T as a multiplication operator M_g. Let $v \in \mathcal{D}(T)$. Then,

$$\begin{aligned}
\langle v, Tv\rangle &= \langle v, UM_gU^{-1}v\rangle \\
&= \langle U^{-1}v, M_gU^{-1}v\rangle \\
&= \int_X g(x) \left|\left(U^{-1}v\right)(x)\right|^2 \mu(dx) \\
&= \int_{\mathbb{R}} r\, g_* \left(\left|U^{-1}v\right|^2 \mu\right)(dr),
\end{aligned}$$

where $g_*(|U^{-1}v|^2\mu)$ is the image meaure of $|U^{-1}v|^2\mu$ under g (see Definition C.0.8). So we have found

$$\mu_v := g_*\left(\left|U^{-1}v\right|^2\mu\right). \tag{B.4}$$

The measure μ_v is finite because μ is finite and the following holds

$$\mu_v(\mathbb{R}) = \int_{\mathbb{R}} g_*\left(\left|U^{-1}v\right|^2\mu\right)(dr) = \int_X \left|\left(U^{-1}v\right)(x)\right|^2\mu(dx) = \langle U^{-1}v, U^{-1}v\rangle = \langle v, v\rangle.$$

Note that $\langle v, Tv\rangle < +\infty$ because $\langle U^{-1}v, M_gU^{-1}v\rangle < \infty$ being $U^{-1}v \in \mathcal{D}(M_g)$. Moreover, always with the assumption $v \in \mathcal{D}(T)$ we have that

$$\begin{aligned}
\langle Tv, Tv\rangle &= \langle U^{-1}TUU^{-1}v, U^{-1}TUU^{-1}v\rangle \\
&= \langle M_gU^{-1}v, M_gU^{-1}v\rangle \\
&= \int_X (g(x))^2\left|\left(U^{-1}v\right)(x)\right|^2\mu(dx) \\
&= \int_{\mathbb{R}} r^2\, g_*\left(\left|U^{-1}v\right|^2\mu\right)(dr) \\
&= \int_{\mathbb{R}} r^2\,\mu_v(dr).
\end{aligned}$$

Similarly, since $U^{-1}v \in \mathcal{D}(M_g)$, we have $\int_{\mathbb{R}} r^2\mu_v(dr) < \infty$. Since $|r| \le 1 + r^2$ we also deduce that $\int_{\mathbb{R}} |r|\mu_v(dr) < \infty$.

Requiring that $v \in \mathcal{D}(T)$ and $Tv \in \mathcal{D}(T)$, we have $\langle v, T^2v\rangle = \langle Tv, Tv\rangle$ and so (B.3) holds for $\alpha = 2$. The cases $\alpha \ge 3$ similarly follow.

□

B.3 Spectral theorem for bounded self-adjoint operators

Theorem B.3.1 (Multiplication operator form, [63] Vol. I, p. 227)**.**
Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathscr{L}(\mathcal{H})$ a bounded self-adjoint operator. Then there exists a finite measure space (X, μ), a unitary operator $U : L^2(X, \mu) \to \mathcal{H}$ and a bounded real function g on X, such that $U \circ M_g = T \circ U$ where M_g is the operator of multiplication by g.
Equivalently, for any $\varphi \in L^2(X, \mu)$

$$(U^{-1}TU)\varphi(x) = g(x)\varphi(x), \qquad \text{for } \mu\text{-almost every } x \in X.$$

In other words, the following diagram

$$
\begin{array}{ccc}
L^2(X,\mu) & \xrightarrow{U} & \mathcal{H} \\
\Big\downarrow M_g & & \Big\downarrow T \\
L^2(X,\mu) & \xrightarrow{U} & \mathcal{H}
\end{array}
$$

commutes and the operator $T : \mathcal{H} \to \mathcal{H}$ can be identified with the operator $M_g : L^2(X,\mu) \to L^2(X,\mu)$.

Corollary B.3.2.
Let $\mathcal{H}$ be a separable Hilbert space, let $T \in \mathscr{L}(\mathcal{H})$ be a self-adjoint operator and let $v \in \mathcal{H}$ be a fixed vector. There exists a finite non-negative Radon measure μ_v (depending on v) on the compact set $\sigma(T) \subset \mathbb{R}$ such that

$$
\langle v, T^\alpha v \rangle = \int_{\sigma(T)} r^\alpha \mu_v(dr) < +\infty
$$

for any $\alpha \in \mathbb{N}_0$.

The proof of the latter corollary is similar to the one of Corollary B.2.2. Moreover, for any $v \in \mathcal{H}$ we have that

$$
\operatorname{supp} g_*(|U^{-1}v|^2\mu) \subseteq \sigma(T),
$$

where $g_*(|U^{-1}v|^2\mu)$ is the image meaure of $|U^{-1}v|^2\mu$ under g (for more details see [63, Section VII.2]).

B.4 The case of several operators

We can develop a multiplication operator form spectral theorem for a d-tuple of self-adjoint operators. Recall that if S and T are two bounded self-adjoint operators acting on the same Hilbert space $\mathcal{H}$ we say that they commute if $ST = TS$.

Theorem B.4.1.
If $\boldsymbol{T} = (T_1, \ldots, T_d)$ is a d-tuple of commuting, bounded, self-adjoint operators on $\mathcal{H}$ separable, then there exists a finite measure space (X,μ), a unitary map

$W : L^2(X,\mu) \to \mathcal{H}$ *and real-valued* $g_j \in L^\infty(X,\mu)$ *such that*

$$W^{-1}T_jW\varphi(x) = g_j(x)\varphi(x), \quad \varphi \in L^2(X,\mu),\ j = 1,\dots,d.$$

Corollary B.4.2.
Let $\boldsymbol{T} = (T_1,\dots,T_d)$ *be a d-tuple of pairwise commuting bounded self-adjoint operators acting on the same separable Hilbert space* $\mathcal{H}$ *and let* $v \in \mathcal{H}$. *Then there exists a unique non-negative measure* μ_v *(depending on* v*) on* $\mathbb{R}^d$ *such that*

$$\langle v, \boldsymbol{T}^\alpha v\rangle = \int_{\sigma(\boldsymbol{T})} \mathbf{r}^\alpha \mu_v(d\mathbf{r}) < \infty$$

for all $\alpha \in \mathbb{N}_0^d$.

Note that the support of μ_v is compact since it is contained in the compact joint spectrum $\sigma(\mathbf{T}) \subseteq B_{||T_1||}(0) \times \cdots \times B_{||T_d||}(0) \subset \mathbb{R}^d$ (for more details see [69, p. 104]).

Proof. (of Corollary B.4.2)
Let $v \in \mathcal{H}$, let $n \in \mathbb{N}$ and $i_1,\dots,i_n \in \{1,\dots,d\}$. Then,

$$\begin{aligned}
\langle v, T_{i_1}\cdots T_{i_n}v\rangle &= \langle v, WM_{g_{i_1}}W^{-1}\cdots WM_{g_{i_n}}W^{-1}v\rangle \\
&= \langle W^{-1}v, M_{g_{i_1}}\cdots M_{g_{i_n}}W^{-1}v\rangle \\
&= \langle W^{-1}v, M_{g_{i_1}\cdots g_{i_n}}W^{-1}v\rangle \\
&= \int_X g_{i_1}(x)\cdots g_{i_n}(x)\left|\left(W^{-1}v\right)(x)\right|^2\mu(dx) \\
&= \int_X f_n\left(h(x)\right)\left|\left(W^{-1}v\right)(x)\right|^2\mu(dx) \\
&= \int_{\mathbb{R}^d} f_n(\mathbf{r})\, h_*\left(\left|W^{-1}v\right|^2\mu\right)(d\mathbf{r}),
\end{aligned}$$

where

$$\begin{aligned}
h :\ & X && \to \mathbb{R}^d \\
& x && \mapsto \left(g_1(x),\dots,g_d(x)\right),
\end{aligned}$$

and

$$\begin{aligned}
f_n :\ & \mathbb{R}^d && \to \mathbb{R} \\
& (r_1,\dots,r_d) && \mapsto r_{i_1}\cdots r_{i_n}\,.
\end{aligned}$$

□

If S and T are instead unbounded operators, defined as maps from their domain to their range, the previous definition of commutativity may not make sense on any vector in $\mathcal{H}$. For example, we might have the case of $Ran(S) \cap D(T) = \{0\}$. In such a case TS does not have any meaning.
Let us introduce a characterization (which we take as definition) of strong commutativity for unbounded and self-adjoint operators. This makes use of the exponential operator e^{isS} defined via spectral theorem in the functional calculus form[1](see [63, Vol. I, Section VIII.3]). Note that the collection $\left(e^{isS}\right)_{s\in\mathbb{R}}$ is also called the *unitary group* associated to S.

Definition B.4.3 ([63] Vol. I, p. 271, [69] p. 132)**.**
Let S and T be self-adjoint operators on a Hilbert space $\mathcal{H}$.
Then, S and T strongly commute if and only if

$$e^{isS}e^{itT} = e^{itT}e^{isS}$$

for all $s, t \in \mathbb{R}$.

The following is the most general spectral theorem we are going to give so far. Its proof can be obtained similarly to the proof of Theorem B.2.1.

Theorem B.4.4.
If $(T_1, \mathcal{D}(T_1)), \ldots, (T_d, \mathcal{D}(T_d))$ are strongly commuting unbounded self-adjoint operators on $\mathcal{H}$ separable, then there exist a measure space (X, μ), a unitary map $W : L^2(X, \mu) \to \mathcal{H}$, and real-valued $g_j \in L^\infty(X, \mu)$ such that for any $j = 1, \ldots, d$,

- $W^{-1}\left(\mathcal{D}(T_j)\right) = \mathcal{D}(M_{g_j})$,
- $W^{-1}T_jW\varphi(x) = g_j(x)\varphi(x), \quad \varphi \in \mathcal{D}(M_{g_j})$,

where M_{g_j} is the operator of multiplication by g_j.

From the latter theorem one can obtain Theorem B.4.1.
Similarly to the proofs of Corollary B.4.2 and Corollary B.2.2 we derive the following.

Corollary B.4.5.
Let $(T_1, \ldots, T_d)$ be a d-tuple of pairwise strongly commuting self-adjoint operators acting on the same Hilbert space $\mathcal{H}$. Let $v \in \mathcal{H}$ be such that $\forall n \in \mathbb{N}_0$,

[1]If S is bounded we can directly define the exponential by power series converging in norm, i.e. $e^{isS} = \sum_{n=0}^{\infty} \frac{(is)^n S^n}{n!}$.

$\forall\, i_1, \ldots, i_{n+1} \in \{1, \ldots, d\}$ we have $T_{i_n} \cdots T_{i_1} v \in \mathcal{D}(T_{i_{n+1}})$ (for $n = 0$ we set $T_{i_n} \cdots T_{i_1} v = v$). Then there exists a unique non-negative spectral measure μ_v (depending on v) supported on $\mathbb{R}^d$ such that for any $n \in \mathbb{N}_0$

$$\int |r_{i_n} \cdots r_{i_1}| \mu_v(d\mathbf{r}) < \infty \quad \textit{and} \quad \langle v, T_{i_n} \cdots T_{i_1} v \rangle = \int r_{i_1} \cdots r_{i_n} \mu_v(d\mathbf{r}). \tag{B.5}$$

B.5 Results of independent interest

Proposition B.5.1 ([69] p. 145)**.**
Let T be an unbounded self-adjoint operator on its domain $\mathcal{D}(T)$ dense in the Hilbert space $\mathcal{H}$. Then $v \in \mathcal{H}$ belongs to $\mathcal{D}^\infty(T)$ (see Definition 1.5.5) if and only if the function $f(t) := e^{itT} v$ is a $\mathcal{C}^\infty$-map of $\mathbb{R}$ into $\mathcal{H}$.

Proof.
($\Leftarrow$) Since f is a $\mathcal{C}^\infty$-map of $\mathbb{R}$ into $\mathcal{H}$, the n-th derivative of f,

$$f^{(n)}(t) = i^n T^n e^{itT} v,$$

is continuous for all $n \in \mathbb{N}_0$. Then necessarily $v \in \mathcal{D}(T^n)$ for all $n \in \mathbb{N}_0$ and therefore $v \in \mathcal{D}^\infty(T)$.
($\Rightarrow$) We want to prove that for any $n \in \mathbb{N}_0$

$$\lim_{\epsilon \to 0} \left\| \frac{f^{(n)}(t+\epsilon) - f^{(n)}(t)}{\epsilon} - f^{(n+1)}(t) \right\| = 0,$$

where $f^{(n)}(t)$ is the n-th derivative of f w.r.t. the variable t.

In our case, fixed $n \in \mathbb{N}_0$, we have

$$\begin{aligned} f^{(n)}(t) &= i^n T^n e^{itT} v, \\ f^{(n+1)}(t) &= i^{n+1} T^{n+1} e^{itT} v, \\ f^{(n)}(t+\epsilon) &= i^n T^n e^{i(t+\epsilon)T} v. \end{aligned}$$

Hence,

$$\left\|\frac{f^{(n)}(t+\epsilon)-f^{(n)}(t)-\epsilon f^{(n+1)}(t)}{\epsilon}\right\|^2=\left\|\frac{T^n e^{itT}\left(e^{i\epsilon T}-I-i\epsilon T\right)v}{\epsilon}\right\|^2=$$

$$\begin{aligned}
&= \langle\frac{T^n e^{itT}\left(e^{i\epsilon T}-I-i\epsilon T\right)v}{\epsilon},\frac{T^n e^{itT}\left(e^{i\epsilon T}-I-i\epsilon T\right)v}{\epsilon}\rangle\\
&= \langle v,\frac{T^{2n} e^{i2tT}\left(e^{i\epsilon T}-I-i\epsilon T\right)^2 v}{\epsilon^2}\rangle\\
&= \int x^{2n}e^{i2tx}\left(\frac{e^{i\epsilon x}-1-i\epsilon x}{\epsilon}\right)^2\mu_v(dx),
\end{aligned}$$

where in the last equality we made use of the functional calculus form of the spectral theorem. Applying the dominated convergence theorem and de L'Hôpital's formula we get our conclusion, i.e.

$$\begin{aligned}
\lim_{\epsilon\to 0}\left\|\frac{f^{(n)}(t+\epsilon)-f^{(n)}(t)-\epsilon f^{(n+1)}(t)}{\epsilon}\right\|^2 &= \int x^{2n}e^{i2tx}\underbrace{\lim_{\epsilon\to 0}\left(\frac{e^{i\epsilon x}-1-i\epsilon x}{\epsilon}\right)^2}_{=0}\mu_v(dx)\\
&= 0.
\end{aligned}$$

Let us remember that we can apply the dominated convergence theorem if

$$\forall x\in\mathbb{R},\ \left|x^{2n}e^{i2tx}\left(\frac{e^{i\epsilon x}-1-i\epsilon x}{\epsilon}\right)^2\right|<g(x)\quad\text{with}\quad\int g(x)\mu_v(dx)<\infty.$$

This is true because, since

$$e^{i\epsilon x}-1=\int_0^1 i\epsilon x\, e^{is\epsilon x}ds=i\epsilon x+\int_0^1 i\epsilon x\,\left(e^{is\epsilon x}-1\right)ds,$$

we have that

$$\left|e^{i\epsilon x}-1-i\epsilon x\right|=\left|\int_0^1 i\epsilon x\,\left(e^{is\epsilon x}-1\right)ds\right|\le\epsilon|x|2$$

and so

$$\left|x^{2n}e^{i2tx}\left(\frac{e^{i\epsilon x}-1-i\epsilon x}{\epsilon}\right)^2\right|\le 4\,x^{2n+2}.$$

Thus, if we take $g(x):=x^{2n+2}$ we have, by (B.3), that

$$\int g(x)\mu_v(dx)=\int x^{2n+2}\mu_v(dx)=\int|x|^{2n+2}\mu_v(dx),$$

which is finite for all $n \in \mathbb{N}$ because $v \in \mathcal{D}^\infty(T)$. $\square$

Appendix C

Miscellanea

Let us report here some standard results which we used throughout this thesis.

Proposition C.0.2 (Heine-Borel property).
A subset A of $\mathbb{R}^d$ ($d \in \mathbb{N}$) is bounded and closed in $\mathbb{R}^d$ if and only if A is compact.

Corollary C.0.3.
Let K be a closed[1] subset of $\mathbb{R}^d$. Then a subset A of K is bounded and closed in K if and only if A is compact.

Theorem C.0.4 (Riesz, [67, 2]).
Let K be a locally compact topological space with a countable base. For any bounded linear functional L on $\mathcal{C}^b(K)$, there exists a unique Borel regular measure (see Definition C.3.1) μ on K such that

$$L(f) = \int_K f(x)\mu(dx)$$

for all f in $\mathcal{C}^b(K)$. Moreover, $\mathcal{C}^b(X) \subset L^1(\mu)$ and since $1 \in \mathcal{C}^b(X)$ the measure is finite.

Theorem C.0.5 (Riesz-Markov, [63, 2]).
Let X be a compact space. For any non-negative linear functional L on $\mathcal{C}(X)$, there exists a unique non-negative measure μ on X such that

$$L(f) = \int_X f(x)\mu(dx)$$

[1]The assumption of K closed is important. In fact, if we consider K open in $\mathbb{R}$, then we can always find a subset of K which is bounded and closed in K but not compact. For istance, if $K = (0, 1)$ and $A = (0, \frac{1}{2}]$ we have that A is bounded and closed in K but not compact.

for all f in $\mathcal{C}(X)$.
Moreover, $\mathcal{C}(X) \subset L^1(\mu)$ and since $1 \in \mathcal{C}(X)$ the measure is finite.

Let V be a vector space.
V is a *vector lattice* if for every $v \in V$ we also have that $|v| \in V$.
We say that a subspace V_0 of V *dominates* V if for every $v \in V$ there exist $v_1^0, v_2^0 \in V_0$ such that $v_1^0 \leq v \leq v_2^0$.

Theorem C.0.6 (Riesz-Krein, [49, 1])**.**
If V is a vector lattice and V_0 is a subspace which dominates V, then any non-negative linear functional on V_0 has at least one non-negative linear extension on V.

Daniell's integration theory gives the following theorem.

Theorem C.0.7 (Daniell, [65, 57])**.**
Let V be a vector space of functions on K which is a vector lattice. Let L be a non-negative linear functional on V for which the following condition holds.

(Dec) If $(v_n)_{n\in\mathbb{N}_0}$ is a sequence of functions in V monotonically decreasing to zero, then $\lim_{n\to\infty} L(v_n) = 0$.

Then there exists a unique *non-negative measure μ on $(K, \sigma(V))$, where $\sigma(V)$ is the σ-algebra generated by V, such that*

$$L(v) = \int_K v(\kappa)\mu(d\kappa)$$

for any $v \in V$.
Moreover, $V \subset L^1(\mu)$ and if $1 \in V$ the measure is finite.

Definition C.0.8 (Image measure, [28])**.**
Let Ω_1 and Ω_2 be two measurable spaces. Given a measure μ on Ω_1, we define the image measure of μ under a measurable function $F : \Omega_1 \to \Omega_2$ as the unique measure μ_F on Ω_2 such that

$$\int_{\Omega_1} f(F(\omega_1))\mu(d\omega_1) = \int_{\Omega_2} f(\omega_2)\mu_F(d\omega_2)$$

for any measurable $f : \Omega_2 \mapsto \mathbb{R}$ such that $f \circ F$ is integrable.

C.1 General properties of locally convex spaces

In this section we collect some classical results about locally convex spaces. The main reference is [63, Section V, vol. I].

Definition C.1.1.
A seminorm *on a vector space X is a map $\rho : X \to [0, +\infty)$ such that*

1. $\rho(x + y) \leq \rho(x) + \rho(y), \quad \forall x, y \in X.$
2. $\rho(\alpha x) = |\alpha|\rho(x), \quad \forall \alpha \in \mathbb{R}\,(\textit{or}\ \mathbb{C}), \forall x \in X.$

Definition C.1.2.
A locally convex space *is a vector space X (over $\mathbb{R}$ or $\mathbb{C}$) with a family $(\rho_\alpha)_{\alpha\in A}$ of seminorms (A is an index set).*

Definition C.1.3.
A net $(x_\beta)_{\beta\in B}$ in a locally convex space X is called Cauchy net *if and only if, for all $\varepsilon > 0$, and for each seminorm ρ_α there is a $\beta_0 \in B$ such that if $\beta, \gamma > \beta_0$, then we have $\rho_\alpha(x_\beta - x_\gamma) < \varepsilon$.*
X is called complete *if every Cauchy net converges, i.e. there exists $x \in X$ such that $\rho_\alpha(x_\beta - x) \to 0$ for all $\alpha \in A$.*

Theorem C.1.4.
A locally convex space X is metrizable if and only if the topology on X is generated by some countable family of seminorms.

Definition C.1.5.
A complete metrizable locally convex space is called a Fréchet space.

Definition C.1.6.
A subset O of a vector space X is called convex *if for any $x, y \in X$ and $t \in [0, 1]$ we have that $tx + (1 - t)y \in O$. The set O is called* balanced *if for any $x \in O$ and $\lambda \in \mathbb{R}$ with $|\lambda| = 1$, we have $\lambda x \in O$. The set O is called* absorbing *if for every $x \in X$ we have $tx \in O$ for some $t > 0$.*

Proposition C.1.7.
If $\rho_{\alpha_1}, \dots, \rho_{\alpha_n}$ are seminorms on a vector space X, then the set

$$\{x \in X : \ |\rho_{\alpha_1}(x)| < \varepsilon, \dots, |\rho_{\alpha_n}(x)| < \varepsilon\}$$

is balanced, convex, absorbing set.

Theorem C.1.8.
Let X be a complete real (or complex) vector space. Let X_n be a family of subspaces with $X_n \subseteq X_{n+1}$ such that $X = \bigcup_{n=1}^{\infty} X_n$. Suppose that each X_n has a locally convex topology so that the restriction of the topology of X_{n+1} to X_n is the given topology on X_n. Let $\mathscr{U}$ be the collection of balanced, absorbing, convex sets O in X for which $O \cap X_n$ is open in X_n for each n. Then,

1. *$\mathscr{U}$ is a neighborhood base about 0 for a locally convex topology.*
2. *The topology generated by $\mathscr{U}$ is the strongest locally convex topology on X so that the injections $X_n \to X$ are continuous.*
3. *The restriction of the topology on X to each X_n is the given topology on X_n.*
4. *If each X_n is complete, so is X.*

Definition C.1.9.
The locally convex space X constructed in Theorem C.1.8 is called the strict inductive limit *of the spaces X_n.*

Theorem C.1.10.
Let X be the strict inductive limit of the locally convex spaces $\{X_n\}_{n=1}^{\infty}$. Then a linear map T from X to a locally convex space Y is continuous if and only if the restriction of T to each X_n is continuous.

C.2 Topological and measurable structures on $\mathcal{R}(\mathbb{R}^d)$

Let us consider $\mathscr{D}'_{proj}(\mathbb{R}^d)$ equipped with the weak topology τ_w^{proj} defined in Section 4.1.2 and recall that a neighbourhood base for τ_w^{proj} about $\nu \in \mathscr{D}'_{proj}(\mathbb{R}^d)$ is given by the following family (see [5, Vol. I, Chapter I, p. 16])

$$\mathscr{B}_{\tau_w}(\nu) := \left\{ \bigcap_{j=1}^{n} O_{\psi_j;\varepsilon}(\nu) : n \in \mathbb{N}, \psi_1, \dots, \psi_n \in \mathcal{C}_c^\infty(\mathbb{R}^d), 0 < \varepsilon \in \mathbb{R} \right\}$$

where $O_{\psi;\varepsilon}(\nu) := \left\{ \eta \in \mathscr{D}'_{proj}(\mathbb{R}^d) : |\langle \psi, \eta - \nu \rangle| < \varepsilon \right\}$. Note that a neighborhood about any vector ν is taken to be a translated neighborhood of zero introduced in Section 3.2.

Let us consider now the space $\mathcal{R}(\mathbb{R}^d)$. By Proposition 4.1.14 we have that $\mathcal{R}(\mathbb{R}^d)$ is a subset of $\mathscr{D}'_{proj}(\mathbb{R}^d)$, so we can equip $\mathcal{R}(\mathbb{R}^d)$ with the relative topology

$\tilde{\tau}_w^{proj}$ induced by τ_w^{proj} defined on $\mathscr{D}'_{proj}(\mathbb{R}^d)$, i.e.

$$\tilde{\tau}_w^{proj} := \left\{ U \cap \mathcal{R}(\mathbb{R}^d) : U \in \tau_w^{proj} \right\}.$$

On the other hand, we can equip it with the vague topology τ_v, i.e. the smallest topology such that the mappings

$$\begin{aligned} \Phi_f : \quad \mathcal{R}(\mathbb{R}^d) & \to \mathbb{R} \\ \eta \quad & \mapsto \langle f, \eta \rangle = \int_{\mathbb{R}^d} f(\mathbf{r}) \eta(d\mathbf{r}) \end{aligned}$$

are continuous for all $f \in \mathcal{C}_c(\mathbb{R}^d)$ (see [2, p. 192]). Note that Φ_f is exactly the functional defined in (4.34) as a function of the second variable.
The space $\left(\mathcal{R}(\mathbb{R}^d), \tau_v\right)$ is Polish (see [18]).

Let us give now a neighbourhood base of each of the two topologies introduced on $\mathcal{R}(\mathbb{R}^d)$. According to the definitions of $\tilde{\tau}_w^{proj}$ and τ_v on $\mathcal{R}(\mathbb{R}^d)$, whenever $U_{\psi;\varepsilon}(\nu) := \left\{ \eta \in \mathcal{R}(\mathbb{R}^d) : |\langle \psi, \eta - \nu \rangle| < \varepsilon \right\}$ with $\varepsilon > 0$, we have that

- A neighbourhood base for $\tilde{\tau}_w^{proj}$ about $\nu \in \mathcal{R}(\mathbb{R}^d)$ is given by

$$\mathscr{B}_{\tilde{\tau}_w^{proj}}(\nu) := \left\{ \bigcap_{j=1}^{n} U_{\psi_j;\varepsilon}(\nu) : n \in \mathbb{N},\ \psi_1, \ldots, \psi_n \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ 0 < \varepsilon \in \mathbb{R} \right\}.$$

- A neighbourhood base for τ_v about $\nu \in \mathcal{R}(\mathbb{R}^d)$ is given by

$$\mathscr{B}_{\tau_v}(\nu) := \left\{ \bigcap_{j=1}^{n} U_{\varphi_j;\varepsilon}(\nu) : n \in \mathbb{N},\ \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c(\mathbb{R}^d),\ 0 < \varepsilon \in \mathbb{R} \right\}.$$

(Note that for any $\nu \in \mathcal{R}(\mathbb{R}^d)$ and any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ we clearly have that $U_{\psi;\varepsilon}(\nu) = O_{\psi;\varepsilon}(\nu) \cap \mathcal{R}(\mathbb{R}^d)$.)

Proposition C.2.1.
The topology $\tilde{\tau}_w^{proj}$ and the vague topology τ_v are equivalent on $\mathcal{R}(\mathbb{R}^d)$.

Proof.
Step I: $\tau_v \subseteq \tilde{\tau}_w^{proj}$
For any function $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$ there exists a positive real number R_φ such that $supp(\varphi) \subset B_{R_\varphi}$, where B_{R_φ} is the open ball in $\mathbb{R}^d$ of radius R_φ and centered at the origin. Let us denote simply by χ_φ the function $\chi_{R_\varphi} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ given in

(4.20). Fixed a finite positive integer N and $\nu \in \mathcal{R}(\mathbb{R}^d)$, if we consider the set

$$U_{\chi_\varphi;N}(\nu) := \left\{\eta \in \mathcal{R}(\mathbb{R}^d) : |\langle \chi_\varphi, \eta - \nu\rangle| < N\right\},$$

then the families

$$\mathscr{B}^*_{\tilde{\tau}_w^{proj}}(\nu) := \left\{W \cap U_{\chi_\varphi;N}(\nu) : W \in \mathscr{B}_{\tilde{\tau}_w^{proj}}(\nu),\ \varphi \in \mathcal{C}_c(\mathbb{R}^d)\right\}$$

$$\mathscr{B}^*_{\tau_v}(\nu) := \left\{V \cap U_{\chi_\varphi;N}(\nu) : V \in \mathscr{B}_{\tau_v}(\nu),\ \varphi \in \mathcal{C}_c(\mathbb{R}^d)\right\}$$

are neighbourhood bases about ν for $\tilde{\tau}_w^{proj}$ and τ_v, respectively.
In fact, for any $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$, the set $U_{\chi_\varphi;N}(\nu)$ is an open neighbourhood of ν w.r.t. both $\tilde{\tau}_w^{proj}$ and τ_v. Moreover, since $\mathscr{B}_{\tilde{\tau}_w^{proj}}(\nu)$ and $\mathscr{B}_{\tau_v}(\nu)$ are neighbourhood bases about ν then for any other neighbourhood $N(\nu)$ of ν there exist $W \in \mathscr{B}_{\tilde{\tau}_w^{proj}}(\nu)$ and $V \in \mathscr{B}_{\tau_v}(\nu)$ such that $\nu \in W \subseteq N(\nu)$ and $\nu \in V \subseteq N(\nu)$. This implies that, for any $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$, the sets $W^* := U_{\chi_\varphi;N}(\nu) \cap W \in \mathscr{B}^*_{\tilde{\tau}_w}(\nu)$ and $V^* := U_{\chi_\varphi;N}(\nu) \cap V \in \mathscr{B}^*_{\tau_v}(\nu)$ are such that $\nu \in W^* \subseteq N(\nu)$ and $\nu \in V^* \subseteq N(\nu)$, which exactly means that $\mathscr{B}^*_{\tilde{\tau}_w}(\nu)$ and $\mathscr{B}^*_{\tau_v}(\nu)$ are neighbourhood bases about ν for $\tilde{\tau}_w^{proj}$ and τ_v, respectively.

By Hausdorff criterion (see [80, Theorem 4.8, p. 35]), $\tau_v \subseteq \tilde{\tau}_w^{proj}$ if and only if

$$\forall \nu \in \mathcal{R}(\mathbb{R}^d), \forall V^* \in \mathscr{B}^*_{\tau_v}(\nu) \quad \exists W^* \in \mathscr{B}^*_{\tilde{\tau}_w^{proj}}(\nu)\ s.t.\ W^* \subseteq V^*. \tag{C.1}$$

Actually, it is possible to prove the following stronger property

$$\begin{aligned}&\forall\, \nu \in \mathcal{R}(\mathbb{R}^d),\ \forall\, \varphi \in \mathcal{C}_c(\mathbb{R}^d),\ \forall\, \varepsilon > 0 \quad \exists\, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \exists\, \varepsilon^* > 0\\ &s.t. \quad U_{\psi;\varepsilon^*}(\nu) \cap U_{\chi_\varphi;N}(\nu) \subseteq U_{\varphi;\varepsilon}(\nu) \cap U_{\chi_\varphi;N}(\nu).\end{aligned} \tag{C.2}$$

Let us recall that for any $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$ there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{C}_c^\infty(B_{R_\varphi})$ such that $\|\varphi - \psi_n\|_\infty \to 0$ as $n \to \infty$, i.e. for any $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$ we have $\|\varphi - \psi_n\|_\infty < \varepsilon$ (see [22, p. 47]).

Hence, for any $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$ we can choose $\psi := \psi_n \in \mathcal{C}_c^\infty(B_{R_\varphi})$ for a sufficiently large $n \in \mathbb{N}$ such that

$$\|\varphi - \psi\|_\infty < \frac{\varepsilon}{2N}. \tag{C.3}$$

In addition, let us choose

$$\varepsilon^* := \frac{\varepsilon}{2}. \tag{C.4}$$

Then, for any $\eta \in U_{\psi;\varepsilon^*}(\nu) \cap U_{\varphi;N}(\nu)$, we have that

$$
\begin{aligned}
|\langle \varphi, \eta - \nu \rangle| &\leq |\langle \varphi - \psi, \eta - \nu \rangle| + |\langle \psi, \eta - \nu \rangle| \\
&\leq \left| \int_{\mathbb{R}^d} (\varphi - \psi) d(\eta - \nu) \right| + \varepsilon^* \\
&= \left| \int_{\mathbb{R}^d} \mathbb{1}_{supp(\varphi - \psi)} (\varphi - \psi) d(\eta - \nu) \right| + \varepsilon^* \\
&\leq \|\varphi - \psi\|_\infty \left| \int_{\mathbb{R}^d} \mathbb{1}_{supp(\varphi - \psi)} d(\eta - \nu) \right| + \varepsilon^* \\
&\leq \|\varphi - \psi\|_\infty \left| \int_{\mathbb{R}^d} \mathbb{1}_{B_{R_\varphi}} d(\eta - \nu) \right| + \varepsilon^* \\
&\leq \|\varphi - \psi\|_\infty \left| \int_{\mathbb{R}^d} \chi_\varphi d(\eta - \nu) \right| + \varepsilon^* \\
&< \frac{\varepsilon}{2N} \cdot N + \varepsilon^*, = \varepsilon
\end{aligned}
$$

where in the last inequality we used the relations (C.3) and (C.4).

To complete the proof it remains to prove that (C.2) implies (C.1).
Fixed $\nu \in \mathcal{R}(\mathbb{R}^d)$, let us consider $V^* \in \mathscr{B}^*_{\tau_v}(\nu)$. By definition we have $V^* = \bigcap_{j=1}^{n} U_{\varphi_j;\varepsilon}(\nu) \cap U_{\chi_\varphi;N}(\nu)$, for some $\varphi_j \in \mathcal{C}_c(\mathbb{R}^d)$, some $n \in \mathbb{N}$, some $0 < \varepsilon \in \mathbb{R}$ and some $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$. By (C.2) we have that there exist $\psi_j \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ for $j = 1, \ldots, n$ and $0 < \varepsilon^* \in \mathbb{R}$ such that

$$U_{\psi_j;\varepsilon^*}(\nu) \cap U_{\chi_\varphi;N}(\nu) \subseteq U_{\varphi_j;\varepsilon}(\nu) \cap U_{\chi_\varphi;N}(\nu).$$

Hence, if we define $W^* := \bigcap_{j=1}^{n} U_{\psi_j;\varepsilon^*}(\nu) \cap U_{\chi_\varphi;N}(\nu)$ then we have that $W^* \subseteq V^*$ and $W^* \in \mathscr{B}^*_{\tilde{\tau}_w^{proj}}(\nu)$, i.e. (C.1) holds.
Step II: $\tilde{\tau}_w^{proj} \subseteq \tau_v$
By Hausdorff criterion, $\tilde{\tau}_w^{proj} \subseteq \tau_v$ if and only if

$$\forall \nu \in \mathcal{R}(\mathbb{R}^d), \forall W \in \mathscr{B}_{\tilde{\tau}_w^{proj}}(\nu) \quad \exists V \in \mathscr{B}_{\tau_v}(\nu) \ s.t. \ V \subseteq W. \tag{C.5}$$

The property (C.5) trivially holds because any $W \in \mathscr{B}_{\tilde{\tau}_w^{proj}}(\nu)$ is also an element of $\mathscr{B}_{\tau_v}(\nu)$ since $\mathcal{C}_c^\infty(\mathbb{R}^d) \subseteq \mathcal{C}_c(\mathbb{R}^d)$.

□

Corollary C.2.2.
The $\sigma-$algebra generated by $\tilde{\tau}_w^{proj}$ coincides with the one generated by τ_v on $\mathcal{R}(\mathbb{R}^d)$, i.e. $\sigma(\tilde{\tau}_w^{proj}) \equiv \sigma(\tau_v)$.

Proposition C.2.3.
The $\sigma-$*algebra* $\sigma(\tau_w^{proj}) \cap \mathcal{R}(\mathbb{R}^d)$ *coincides with* $\sigma(\tilde{\tau}_w^{proj})$.

Proof.
Step I: $\sigma(\tilde{\tau}_w^{proj}) \subseteq \sigma(\tau_w^{proj}) \cap \mathcal{R}(\mathbb{R}^d)$
The $\sigma-$algebra generated by $\tilde{\tau}_w^{proj}$ is the smallest $\sigma-$algebra containing the topology $\tilde{\tau}_w^{proj}$, i.e. the smallest $\sigma-$algebra such that the sets $O \cap \mathcal{R}(\mathbb{R}^d)$ are measurable for any $O \in \tau_w^{proj}$.
Hence, it remains to show that, for any $O \in \tau_w^{proj}$, the sets $O \cap \mathcal{R}(\mathbb{R}^d)$ are measurable w.r.t. the $\sigma-$algebra generated by τ_w^{proj} restricted to $\mathcal{R}(\mathbb{R}^d)$.
This is true because a set $O \in \tau_w^{proj}$ is trivially measurable w.r.t. the σ-algebra generated by τ_w^{proj} and therefore, $O \cap \mathcal{R}(\mathbb{R}^d)$ belongs to the σ-algebra generated by τ_w^{proj} restricted to $\mathcal{R}(\mathbb{R}^d)$.

Step II: $\sigma(\tau_w^{proj}) \cap \mathcal{R}(\mathbb{R}^d) \subseteq \sigma(\tilde{\tau}_w^{proj})$
The $\sigma-$algebra generated by τ_w^{proj} restricted to $\mathcal{R}(\mathbb{R}^d)$ is the smallest $\sigma-$algebra which makes the inclusion map $i : \mathcal{R}(\mathbb{R}^d) \hookrightarrow \mathscr{D}'_{proj}(\mathbb{R}^d)$ measurable.
Hence, it remains to show that the inclusion map i is measurable w.r.t. the $\sigma-$algebra generated by $\tilde{\tau}_w^{proj}$.
This is true because the inclusion map i results to be continuous w.r.t. $\tilde{\tau}_w^{proj}$ and therefore i is also measurable w.r.t. the $\sigma-$algebra generated by $\tilde{\tau}_w^{proj}$.

□

Corollary C.2.4.
The $\sigma-$*algebra* $\sigma(\tau_w^{proj}) \cap \mathcal{R}(\mathbb{R}^d)$ *coincides with* $\sigma(\tau_v)$.

All these considerations still hold if we replace $\big(\mathscr{D}'_{proj}(\mathbb{R}^d), \tau_w^{proj}\big)$ with the space $\big(\mathscr{D}'_{ind}(\mathbb{R}^d), \tau_w^{ind}\big)$, where τ_w^{ind} is the weak topology on $\mathscr{D}'_{ind}(\mathbb{R}^d)$ defined in Section 4.1.2. Hence, $\sigma(\tau_v)$ coincides with the trace $\sigma-$algebra on $\mathscr{D}'_{ind}(\mathbb{R}^d)$, i.e. $\sigma(\tau_v) \equiv \sigma(\tau_w^{proj}) \cap \mathcal{R}(\mathbb{R}^d)$. In particular, every measurable set in $\big(\mathcal{R}(\mathbb{R}^d), \sigma(\tau_v)\big)$ is also measurable in $\big(\mathscr{D}'_{ind}(\mathbb{R}^d), \sigma(\tau_w^{ind})\big)$.

C.3 Notes on Radon measures on topological spaces

These notes are taken from [71] and [2]. In [2], a Borel measure is not only a measure defined on the Borel $\sigma-$algebra on a space X but also finite on all compact subsets of X. However, we will consider the latter as an extra requirement. In the following we are going to consider always non-negative measures.

Preliminaries

Let us recall some basic definitions.

Definition C.3.1.
Let (X, τ) be a Hausdorff topological space and let $\mathcal{B}(X)$ be the Borel $\sigma-$algebra on X. Let μ be a Borel measure *on X, i.e. a measure defined on $\mathcal{B}(X)$.*
The measure μ is said to be

- Locally finite *if every point of X has an open neighborhood of finite $\mu-$measure, i.e.*
$$\forall\, x \in X,\ \exists\, U_x \in \tau \ \textit{ with } \ x \in U_x \ \textit{ s.t. } \ \mu(U_x) < \infty.$$

- Inner regular *if for every $B \in \mathcal{B}(X)$ we have that*
$$\mu(B) = \sup\{\mu(K) :\ K \subset B,\ K \textit{ compact}\},$$
or equivalently if
$$\forall B \in \mathcal{B}(X), \forall \varepsilon > 0,\ \exists\, K_\varepsilon \subset B \ \textit{ compact s.t. } \ \mu(B) \leq \mu(K_\varepsilon) + \varepsilon.$$

- Outer regular *if for every $B \in \mathcal{B}(X)$ we have that*
$$\mu(B) = \inf\{\mu(O) :\ B \subset O,\ O \textit{ open}\},$$
or equivalently if
$$\forall B \in \mathcal{B}(X), \forall \varepsilon > 0,\ \exists\, B \subset O_\varepsilon \ \textit{ open s.t. } \ \mu(B) \geq \mu(O_\varepsilon) - \varepsilon.$$

- Regular *if it is both inner regular and outer regular, or equivalently if $\forall B \in \mathcal{B}(X)$, $\forall \varepsilon > 0$, $\exists\, K_\varepsilon$ compact and O_ε open such that*
$$K_\varepsilon \subset B \subset O_\varepsilon \ \textit{ and } \ \mu(O_\varepsilon \setminus K_\varepsilon) \leq \varepsilon.$$

Definition C.3.2 (Radon measure)**.**
Let X be a Hausdorff topological space. A Borel measure μ is called Radon measure *if*

- *μ is locally finite*
- *μ is inner regular.*

Proposition C.3.3 ([2] p. 154).
Let (X, τ) be a Hausdorff topological space and μ a Borel measure on X. Then,

$$(\mu \text{ locally finite}) \Rightarrow (\mu \text{ finite on compact subset of } X).$$

Proof.
Let $K \subset X$ compact. Since μ is locally finite, each point $x \in K$ has an open neighborhood U_x with $\mu(U_x) < \infty$, then $K \subseteq \bigcup_{x \in K} U_x$. By compactness, there exist finitely many of these neighborhoods, say the ones corresponding to $x_1, \ldots, x_n$, covering K. Then,

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^{n} U_{x_i}\right) \leq \sum_{i=1}^{n} \mu(U_{x_i}) < +\infty.$$

□

Concerning the opposite direction, we have the following result.

Lemma C.3.4 ([2] Lemma 25.4, p. 155).
Let (X, τ) be a Hausdorff topological space in which every point has a countable neighborhood basis and let μ be a Borel measure on X. Then,

$$(\mu \text{ locally finite}) \Leftarrow (\mu \text{ finite on compact subset of } X \text{ and inner regular}).$$

Hence, μ is a Radon measure.

Polish spaces

Definition C.3.5 (Polish space, [71] p. 92, [2] p. 157).
Let (X, τ) be a Hausdorff topological space.
(X, τ) is Polish if X is separable and τ can be defined by a metric d_X on X for which X is complete. Equivalently, (X, τ) is Polish if τ has a countable basis and τ can be defined by a metric d_X on X for which X is complete.

Recall that X separable means that there exists a countable dense subset of X and that a countable basis for τ is a countable system of open sets such that every open set O in τ is the union of those from the system which are subsets of O. The equivalence of the two definitions stated above is due to the fact that for a metrizable space, the existence of a countable basis for its topology is equivalent to the existence of a countable dense subset.

The class of Polish spaces is closed under countable products and topological sums, countable intersections and projective limit of countable subfamilies. An open (or closed) subset of a Polish space is Polish. However, finite unions of Polish spaces are not necessarily Polish.
Let us also recall that a Frechét space is Polish if and only if it is separable.

Radon measure on Polish spaces

Lemma C.3.6 ([2] Lemma 26.2, p. 158)**.**
Let (X, τ) be a Hausdorff topological space which is Polish. Then,

$$(\mu \textit{ finite Borel measure on } X) \Rightarrow (\mu \textit{ regular}).$$

In particular, μ is a Radon measure.

Theorem C.3.7 ([2] Theorem 26.3, p. 160)**.**
Let (X, τ) be a Hausdorff topological space which is Polish. Then,

$$(\mu \textit{ locally finite Borel measure on } X) \Rightarrow (\mu \textit{ is a } \sigma\textit{−finite Radon measure on } X).$$

Corollary C.3.8.
Let (X, τ) be a Hausdorff topological space which is Polish. Then,

$$(\mu \textit{ is Radon measure on } X) \Leftrightarrow (\mu \textit{ is a locally finite Borel measure on } X).$$

Corollary C.3.9 ([2] Corollary 26.4, p. 161)**.**
Let (X, τ) be a Hausdorff topological space which is Polish. Then,

$$(\mu \textit{ Radon measure on } X) \Rightarrow (\mu \textit{ outer regular}).$$

The previous corollary can be restated as follows: "every locally finite Borel measure on a Polish space X is regular".

Lusin and Suslin spaces

Definition C.3.10 (Lusin space, [71] p. 94)**.**
Let (X, τ) be a Hausdorff topological space.
(X, τ) is said to be Lusin if there exists a topology τ' on X stronger than τ (i.e. $\tau \subseteq \tau'$) such that (X, τ') is Polish.

Equivalently, (X, τ) is said to be Lusin if there exists a Polish space (Y, σ) and a continuous bijective map from Y to X.

The class of Lusin spaces is closed under countable product and topological sums, disjoint countable unions, countable intersections, complements, countable projective limits. Every open (or closed or Borel) subset of a Lusin space is Lusin.

Definition C.3.11 (Suslin space, [71] p. 96)**.**
Let (X, τ) be a Hausdorff topological space.
(X, τ) is said to be Suslin if there exists a Polish space (Y, σ) and a continuous surjective map from Y to X.

All stability properties given for Lusin spaces also hold for Suslin spaces. This class is further closed under quotient, continuous image, countable inductive limit. However, the class of Suslin spaces is not closed under complementation. Moreover, every Suslin space is separable.

Let us recall the following important property.

Proposition C.3.12 ([71] Corollary 2, p. 101)**.**
If τ_1 and τ_2 are two comparable topologies on a Suslin space X, then the Borel $\sigma-$algebra generated by τ_1 and τ_2 coincide.

Radon spaces

Definition C.3.13 (Radon space, [71] p. 117)**.**
Let (X, τ) be a Hausdorff topological space.
(X, τ) is called a Radon space if every finite Borel measure on X is inner regular (and so a Radon measure).

Note that if (X, τ) is Polish, then by Lemma C.3.6 we have that

Proposition C.3.14 (Radon space, [71] p. 117)**.**
Let (X, τ) be a Hausdorff topological space which is Polish.
(X, τ) is Radon space if and only if every finite Borel measure on X is regular.

The class of Radon spaces is closed under countable topological sums, countable unions, countable intersections, complements, continuous images, injective images.
It is possible to show that

$$(X \text{ Polish}) \Rightarrow (X \text{ Lusin}) \Rightarrow (X \text{ Suslin}) \Rightarrow (X \text{ Radon}).$$

Locally compact spaces

Definition C.3.15 (Locally compact space, [2] p. 166)**.**
Let (X, τ) be a Hausdorff topological space.
(X, τ) is locally compact if each of its point has at least one compact neighborhood.

Theorem C.3.16 ([71] Theorem 6, p. 111)**.**
Let (X, τ) be a Hausdorff topological space which is locally compact. Then, the following are equivalent.

- *X is Polish.*
- *X is Lusin.*
- *X is Suslin.*
- *X has a countable basis for its topology.*

Definition C.3.17 ($\sigma-$compact, [2] p. 181)**.**
A locally compact space (X, τ) is $\sigma-$compact when it can be covered by a sequence of compact subsets.

Proposition C.3.18 ([2] p. 182)**.**
Every locally compact space which has a countable basis for its topology is also $\sigma-$compact.

Radon measure on locally compact spaces

First of all let us note that if (X, τ) is locally compact then the opposite direction of Proposition C.3.3 holds. Namely, we have the following.

Proposition C.3.19.
Let (X, τ) be a locally compact Hausdorff topological space and μ a Borel measure on X. Then,

$$(\mu \text{ locally finite }) \Leftrightarrow (\mu \text{ finite on compact subset of } X).$$

Proof.
Let us show the missing direction.
Since X is locally compact, for any $x \in X$ there exists a compact neighborhood C_x of x. Since μ is assumed to be finite on compact sets then $\mu(C_x) < \infty$. Consequently, μ is locally finite.

□

The latter proposition, together with Theorem C.3.16 and Corollary C.3.8, allows us to give the following characterization.

Proposition C.3.20.
Let (X, τ) be a locally compact Hausdorff topological space which has a countable basis. Then,

$$(\mu \text{ is a Radon measure on } X) \Leftrightarrow (\mu \text{ is finite on compact subset of } X).$$

It is possible to prove the following.

Proposition C.3.21 ([2] Corollary 29.7)**.**
Let (X, τ) be a locally compact Hausdorff topological space which is $\sigma-$compact. Then,

$$(\mu \text{ Radon measure on } X) \Rightarrow (\mu \text{ outer regular}).$$

This result holds also for a locally compact Hausdorff topological space which has a countable basis. This can be obtained either by using Proposition C.3.18 and then Proposition C.3.21 or by using Theorem C.3.16 and then Corollary C.3.9.

Proposition C.3.22 ([2] Corollary 29.11)**.**
Let (X, τ) be a locally compact Hausdorff topological space. Then,

$$(\mu \text{ finite Radon measure on } X) \Rightarrow (\mu \text{ outer regular}).$$

Theorem C.3.23 ([2] Theorem 29.12)**.**
Let (X, τ) be locally compact Hausdorff topological space which has a countable basis. Then,

$$(\mu \text{ locally finite Borel measure on } X) \Rightarrow (\mu \text{ regular}).$$

Hence, μ is a Radon measure.

The previous theorem can be deduced by using Theorem C.3.16 and then Corollary C.3.8 and Corollary C.3.9.

We can easily conclude that if X is locally compact and has a countable basis, then

$$(\mu \text{ is a locally finite}) \Leftrightarrow (\mu \text{ is Radon measure}).$$

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