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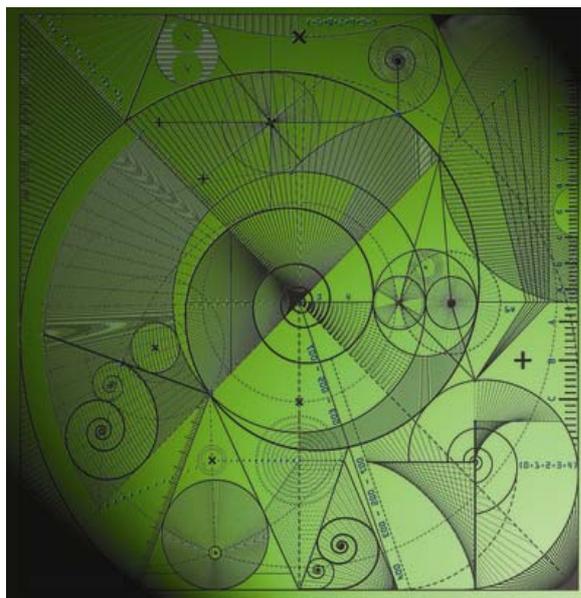
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Aperiodic dynamics in a deterministic model of attitude formation in social groups

by

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Abstract

Homophily and social influence are the fundamental mechanisms that drive the evolution of attitudes, beliefs and behaviour within social groups. Homophily relates the similarity between pairs of individuals' attitudinal states to their frequency of interaction, and hence structural tie strength, while social influence causes the convergence of individuals' states during interaction. Building on these basic elements, we propose a new mathematical modelling framework to describe the evolution of attitudes within a group of interacting agents. Specifically, our model describes sub-conscious attitudes that have an activator-inhibitor relationship. We consider a homogeneous population using a deterministic, continuous-time dynamical system. Surprisingly, the combined effects of homophily and social influence do not necessarily lead to group consensus or global monoculture. We observe that sub-group formation and polarisation-like effects may be transient, the long-time dynamics being quasi-periodic with sensitive dependence to initial conditions. This is due to the interplay between the evolving interaction network and Turing instability associated with the attitudinal state dynamics.

Keywords: Social dynamics, Cultural dissemination, Coevolving networks, Adaptive networks, Social influence, Homophily, Activator-inhibitor
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1. Introduction

Our attitudes and opinions have a reciprocal relationship with those around us: who we know depends on what we have in common, while simultaneously our beliefs influence, and are influenced by, those of our peers. These two mechanisms—homophily and social influence—underpin a wide range of social phenomena, including the diffusion of innovations [1, 2, 3, 4, 5], complex contagions [6, 7, 8], collective action [9, 10, 11], opinion dynamics [12, 13, 14, 15, 16, 17, 18, 19, 20] and the emergence of social norms [21, 22, 23]. Thus homophily and social influence represent the atomistic ingredients for models of social dynamics [24]. Starting with these basic elements, we investigate a new type of modelling framework intended to describe the coevolution of sub-conscious attitudinal states and social tie strengths in a population of interacting agents.

The first ingredient in our modelling framework, homophily, relates the similarity of individuals to their frequency of interaction [25]. Thus homophily is structural, affecting the strength of ties between people and hence the underlying social network. Homophily has been observed over a broad range of sociodemographics: implicit characteristics, such as age, gender and race; acquired characteristics, such as education, religion and occupation; and internal states that govern attitudes and behaviour [26, 25]. Homophily inextricably links state dynamics with the evolution of social tie strength, and consequently a faithful model must be coevolutionary, connecting both the dynamics *of* the social network and the dynamics *on* the social network [27]. Such network models are known as *coevolving* or *adaptive*; see [28] for a review. There has been a recent surge of interest in coevolving networks, particularly models of opinion dynamics [15, 16, 17, 20], which build on simple models of voting behaviour.

The second ingredient is social influence, which affects people's attitudinal state through typically dyadic interactions. It is a fundamental result of social psychology that people tend to modify their behaviour and attitudes in response to the opinions of others [29, 30, 31], sometimes even when this conflicts sharply with what they know to be true [32] or believe to be morally justifiable [33]. Similarly to Flache and Macy [34], we use diffusion to model social influence: agents adjust their state according to a weighted sum of the differences between their state and their neighbours'. The weights, which represent the strength of influence between pairs of agents, are the corresponding elements of the undirected (dynamic) social network, whose

evolution is driven by homophily. Although our model is built on the notions of homophily and social influence described above, we point out that differentiating between the effects of these processes, particularly in observational settings, may be very difficult [35, 36].

Social scientists have developed 'agent-based' models that incorporate homophily and social influence in order to examine a variety of social-phenomena, including group stability [37], social differentiation [38] and cultural dissemination [39], where a culture is defined as an attribute that is subject to social influence. In such models, an agent's state is typically described by a vector of discrete cultures and the more similar (according to some metric) two agent's states are, the higher the probability of dyadic interactions between them (homophily) in which one agent replicates certain attributes of the other (social influence). Surprisingly, the feedback between homophily and social influence does not necessarily lead to a global monoculture [39]. In fact, the dissolution of ties between culturally distinct groups, or equivalently the creation of 'structural holes' [40], may lead to *cultural polarisation*—equilibrium states that preserve diversity. However, such multi-cultural states are not necessarily stable when there is 'cultural drift', i.e. small, random perturbations or noise, which inevitably drive the system towards monoculture [41]. There have been a number of attempts to develop models with polarised states that are stable in the presence cultural drift [27, 34], but this is still an open area of research [24].

Two key features differentiate our approach from those described above. Firstly, we specifically focus on *sub-conscious* attitude formation driven by a general class of activator-inhibitor processes. This is motivated by neuropsychological evidence that the activation of emotional responses are associated with the (evolutionarily older) regions of the brain know as the limbic system and our inhibitions are regulated by the (evolutionarily younger) prefrontal cortex [42]. This has led psychologists to develop theories in which various personality traits (such as extraversion, impulsivity, neuroticism and anxiety) form an independent set of dimensions along which different types of behaviour may be excited or regulated [43, 44, 45]. Thus it is natural in our modelling framework for these processes to be communicated independently and in parallel through distinct transmission channels and hence via distinct diffusion coefficients. The consequence of activator-inhibitor attitudinal state dynamics is that we would expect to encounter Turing instability, since the rates at which social influence can change homophilious attributes may differ dramatically.

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Secondly, and in sharp contrast to recent models of cultural dissemination [39, 27, 34] and indeed many other types of behavioural model [24] that are stochastic or probabilistic[1], we consider a *deterministic*, continuous-time dynamical systems formulation. While this does not reflect the mercurial nature seemingly ingrained in human interaction, it allows us to probe the underlying mechanisms driving dynamical phenomena. In fact, our principle observation is that the tension between Turing instability and the coevolution of the social network and attitudinal states gives rise to aperiodic dynamics that are sensitive to initial conditions and surprisingly unpredictable. This begs the question, are the mechanisms that govern our behaviour the cause of its volatility? For parsimony, we also consider systems of homogeneous agents. This allows us to identify parameters that destabilise the global monocultural steady state, giving rise to transient sub-group formation.

This paper is organised as follows: in Section 2, we describe our model in detail, analyse the stability of global monoculture and describe the underlying dynamical mechanisms; in Section 3 we illustrate typical numerical results from both a large population of individuals and a simple example consisting of just two agents; in Section 4 we summarise our work and finally in Section 5 we discuss our results in the context of other models of cultural dynamics and polarisation phenomena.

2. A deterministic model of cultural dynamics

Consider a population of N identical individuals (agents/actors), each described by a set of M real attitude state variables that are continuous functions of time t. Let $\mathbf{x}_i(t) \in \mathbb{R}^M$ denote the ith individual's attitudinal state. In the absence of any influence or communication between agents we assume that each individual's state obeys an autonomous rate equation of the form

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i), \quad i = 1, ..., N, \tag{1}$$

where $\mathbf{f}$ is a given smooth field over $\mathbb{R}^M$, such that $\mathbf{f}(\mathbf{x}^*) = 0$ for some $\mathbf{x}^*$. Thus (1) has a uniform population equilibrium $\mathbf{x}_i = \mathbf{x}^*$, for $i = 1, ..., N$, which we shall assume is locally asymptotically stable. As discussed in the

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[1] A notable exception is the deterministic, discrete-time model of Friedkin and Johnsen [12]; see also [13] and references therein.

introduction, we shall more specifically assume that (1) is drawn from a class of activator-inhibitor systems.

Now suppose that the individuals are connected up by a dynamically evolving weighted network. Let $A(t)$ denote the $N \times N$ weighted adjacency matrix for this network at time t, with the ijth term, $A_{ij}(t)$, representing the instantaneous weight (frequency and/or tie strength) of the mutual influence between individual i and individual j at time t. Throughout we assert that $A(t)$ is symmetric, contains values bounded in [0,1] and has a zero diagonal (no self influence). We extend (1) and adopt a first order model for the coupled system:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + D \sum_{j=1}^{N} A_{ij} \left(\mathbf{x}_j - \mathbf{x}_i \right), \quad i = 1, ..., N. \tag{2}$$

Here D is a real, diagonal and non-negative matrix containing the maximal transmission coefficients (diffusion rates) for the corresponding attitudinal variables between neighbours. Thus some of the attitude variables may be more easily or readily transmitted, and are therefore influenced to a greater extent by (while simultaneously being more influential to) those of neighbours. Note that $\mathbf{x}_i = \mathbf{x}^*$, for $i = 1, ..., N$, is also a uniform population equilibrium of the augmented system.

Let $\mathbf{X}(t)$ denote the $M \times N$ matrix with ith column given by $\mathbf{x}_i(t)$, and $\mathbf{F}(\mathbf{X})$ be the $M \times N$ matrix with ith column given by $\mathbf{f}(\mathbf{x}_i(t))$. Then (2) may be written as

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) - D\mathbf{X}\Delta. \tag{3}$$

Here $\Delta(t)$ denotes the weighted graph Laplacian for $A(t)$, given by $\Delta(t) = \mathrm{diag}(\mathbf{k}(t)) - A(t)$, where $\mathbf{k}(t) \in \mathbb{R}^N$ is a vector containing the degrees of the vertices ($k_i(t) = \sum_{j=1}^{N} A_{ij}(t)$). Equation (3) has a rest point at $\mathbf{X} = \mathbf{X}^*$, where the ith column of $\mathbf{X}^*$ is given by $\mathbf{x}^*$ for all $i = 1, ..., N$.

To close the system, consider the evolution equation for $A(t)$ given by

$$\dot{A} = \alpha A \circ (\mathbf{1} - A) \circ (\varepsilon \mathbf{1} - \Phi(\mathbf{X})). \tag{4}$$

Here $\mathbf{1}$ denotes the adjacency matrix of the fully weighted connected graph (with all off-diagonal elements equal to one and all diagonal elements equal to zero); $\circ$ denotes the element-wise 'Hadamard' matrix product; $\alpha > 0$ is a rate parameter; $\varepsilon > 0$ is a homophily scale parameter; and $\Phi : \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$ is a symmetric matrix function that incorporates homophily effects. We assume

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Φ to be of the form $\Phi_{ij} := \phi(|\mathbf{x}_i - \mathbf{x}_j|) \geq 0$, where $|\cdot|$ is an appropriate norm or semi-norm, and the real function ϕ is monotonically increasing with $\phi(0) = 0$. Note that the sign of the differences held in $\varepsilon\mathbf{1} - \Phi(X)$ controls the growth or decay of the corresponding coupling strengths. All matrices in (4) are symmetric, so in practice we need only calculate the super-diagonal terms. For the ijth edge, from (4), we have

$$\dot{A}_{ij} = \alpha A_{ij}(1 - A_{ij})(\varepsilon - \phi(|\mathbf{x}_i - \mathbf{x}_j|)).$$

The nonlinear "logistic growth"-like term implies that the weights remain in [0,1], while we refer to the term $\varepsilon - \phi(|\mathbf{x}_i - \mathbf{x}_j|)$ as the *switch* term.

2.1. Stability analysis

By construction, there are equilibria at $\mathbf{X} = \mathbf{X}^*$ with either $A = 0$ or $A = \mathbf{1}$. To understand their stability, let us assume that $\alpha \to 0$ so that $A(t)$ evolves very slowly. We may then consider the stability of the uniform population, $\mathbf{X}^*$, under the fast dynamic (3) for any fixed network A. Assuming that A is constant, writing $\mathbf{X}(t) = \mathbf{X}^* + \tilde{\mathbf{X}}(t)$ and Linearising (3) about $\mathbf{X}^*$, we obtain

$$\dot{\tilde{\mathbf{X}}} = \mathbf{df}(\mathbf{x}^*)\tilde{\mathbf{X}} - D\tilde{\mathbf{X}}\Delta. \tag{5}$$

Here $\mathbf{df}(\mathbf{x}^*)$ is an $M \times M$ matrix given by the linearisation of $\mathbf{f}(\mathbf{x})$ at $\mathbf{x}^*$. Letting $(\lambda_i, \mathbf{w}_i) \in [0, \infty) \times \mathbb{R}^N$, $i = 1, ..., N$, be the eigen-pairs of Δ, then we may decompose uniquely [46]:

$$\tilde{\mathbf{X}}(t) = \sum_{i=1}^{N} \mathbf{u}_i(t)\mathbf{w}_i^T,$$

where each $\mathbf{u}_i(t) \in \mathbb{R}^M$. The stability analysis of (5) is now trivial since decomposition yields

$$\dot{\mathbf{u}}_i = (\mathbf{df}(\mathbf{x}^*) - D\lambda_i)\mathbf{u}_i.$$

Thus the uniform equilibrium, $\mathbf{X}^*$, is asymptotically stable if and only if all N matrices, $(\mathbf{df}(\mathbf{x}^*) - D\lambda_i)$, are simultaneously stability matrices; and conversely is unstable in the ith mode of the graph Laplacian if $(\mathbf{df}(\mathbf{x}^*) - D\lambda_i)$ has an eigenvalue with positive real part.

Consider the spectrum of $(\mathbf{df}(\mathbf{x}^*) - D\lambda)$ as a function of λ. If λ is small then this is dominated by the stability of the autonomous system, $\mathbf{df}(\mathbf{x}^*)$, which we assumed to be stable. If λ is large then this is again a stability matrix, since D is positive definite. The situation, dependent on some collusion

between choices of D and $\mathbf{df}(\mathbf{x}^*)$, where there is a *window of instability* for an intermediate range of λ, is know as a Turing instability. Turing instabilities occur in a number of mathematical applications and are tied to the use of activator-inhibitor systems (in the state space equations, such as (1) here), where inhibitions diffuse faster than activational variables.

Now we can see the possible tension between homophily and Turing instability in the attitude dynamics when the timescale of the evolving network, α, is comparable to the changes in agents' states. There are two distinct types of dynamical behaviour at work. In one case, $\Delta(t)$ has presently no eigenvalues within the window of instability and each individual's states $\mathbf{x}_i(t)$ approach the mutual equilibrium, $\mathbf{x}^*$; consequently all switch terms become positive and the edge weights all grow towards unity, i.e. the fully weighted clique. In the alternative case, unstable eigen-modes cause the individual states to diverge from $\mathbf{x}^*$, and subsequently some of the corresponding switch terms become negative, causing those edges to begin losing weight and hence partitioning the network.

The eigenvalues of the Laplacian for the fully weighed clique, $A = \mathbf{1}$, are at zero (simple) and at N (with multiplicity $N - 1$). So the interesting case is where the system parameters are such that $\lambda = N$ lies within the window of instability. Then the steady state $(\mathbf{X}, A) = (\mathbf{X}^*, \mathbf{1})$ is unstable and thus state variable patterns will form, echoing the structure of (one or many of) the corresponding eigen-mode(s). This Turing driven symmetry loss may be exacerbated by the switch terms (depending upon the choice of ε being small enough), and then each sub-network will remain relatively well intra-connected, while becoming less well connected to the other sub-networks. Once relatively isolated, individuals within each of these sub-networks may evolve back towards the global equilibrium at $\mathbf{x}^*$, providing that $A(t)$ is such that the eigenvalues of Δ have by that time left the window of instability. Within such a less weightily connected network, all states will approach $\mathbf{x}^*$, the switch terms will become positive, and then the whole qualitative cycle can begin again.

Thus we expect aperiodic or pseudo-cyclic emergence and diminution of patterns, representing transient variations in attitudes in the form of different *norms* adopted by distinct sub-populations. As we shall see though, the trajectory of any individual may be sensitive and therefore effectively unpredictable, while the dynamics of the global behaviour is qualitatively predictable.

In the next section we introduce a specific case of the more general setting

described here.

3. Examples

We wish to consider activator-inhibitor systems as candidates for the attitudinal dynamics in (1) and hence (3). The simplest such system has $M = 2$, with a single inhibitory variable, $x(t)$, and a single activational variable, $y(t)$. Let $\mathbf{x}_i(t) = (x_i(t), y_i(t))^T$ in (2), and consider the Scnackenberg dynamics defined by the field

$$\mathbf{f}(\mathbf{x}_i) = (p - x_i y_i^2, \; q - y_i + x_i y_i^2)^T, \tag{6}$$

where $p > q \geq 0$ are constants. The equations have the required equilibrium point at

$$\mathbf{x}^* = \left(\frac{p}{(p+q)^2}, \; p + q \right)^T, \tag{7}$$

and in order that $\mathbf{df}$ be a stability matrix, we must have

$$p - q < (p + q)^3.$$

We employ $\phi_{ij} = (x_i - x_j)^2$ as the homophily function and we must have $D = \mathrm{diag}(D_1, D_2)$ in (3).

When $M = 2$, the presence of Turing instability depends on the sign of the determinant of $(\mathbf{df}(\mathbf{x}^*) - D\lambda)$, which is quadratic in λ. For the Schnakenberg dynamics defined above, the roots of this quadratic are given by

$$\lambda_\pm = \frac{(p-q) - \frac{D_2}{D_1}(p+q)^3 \pm \sqrt{\left[(p-q) - \frac{D_2}{D_1}(p+q)^3 \right]^2 - 4\frac{D_2}{D_1}(p+q)^4}}{2 D_2 (p+q)} > 0.$$

It is straightforward to show that if

$$\frac{D_2}{D_1} < \frac{3p + q - 2\sqrt{2p(p+q)}}{(p+q)^3} := \sigma_{\mathrm{c}}, \tag{8}$$

then $\lambda_\pm$ are real positive roots and hence $(\mathbf{df}(\mathbf{x}^*) - D\lambda)$ is a stability matrix if and only if λ lies outside of the interval (λ_-, λ_+), the *window of instability*. Inside there is always one positive and one negative eigenvalue, and the equilibrium $\mathbf{X}^*$ is unstable for any fixed network A. Note that, as is well known, it is the *ratio* of the diffusion constants that determines whether there is a window of instability.
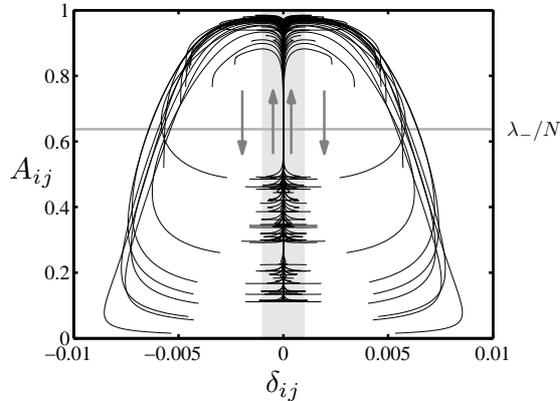
Figure 1: Trajectories of $\delta_{ij} := x_i - x_j$ and A_{ij} for all (i, j) pairs for unstable parameters integrated until $t \approx 440$. Parameter values and initial conditions are described in the main text. In the light grey shaded region, $\delta_{ij} < \varepsilon$ and the direction of trajectories are indicated with arrows. The grey horizontal line indicates the scaled stability threshold (unstable above, stable below).

3.1. Group dynamics

We now present simulations of the Schnakenberg dynamics with $N = 10$. Parameter values are $p = 1.25$, $q = 0.1$, $\alpha = 10^4$, $\varepsilon = 10^{-6}$, $D_1 \approx 0.571$ and $D_2 \approx 0.037$. The ratio of the diffusion constants is $D_2/D_1 := \sigma = 0.9\sigma_{\mathrm{c}}$, and to ensure that the window of instability is centred on the fully coupled system we have

$$ D_1 = \frac{(p - q) - \sigma(p + q)^3}{2\sigma N(p + q)}. \tag{9} $$

The initial coupling strengths were chosen uniformly at random between 0.1 and 0.5. The initial values of x and y were chosen at equally spaced intervals on a circle of radius 10^{-3} centred on the uniform equilibrium.

In Figure 1 we illustrate the trajectories of $\delta_{ij} := x_i - x_j$ and the corresponding coupling strengths A_{ij} up to $t \approx 440$. The shaded region corresponds to $\delta_{ij} < \varepsilon$, within which the A_{ij} increase and outside of which they decrease, indicated by the dark grey arrows. The horizontal grey line marks the scaled instability threshold λ_-/N, which is indicative of the boundary of instability, above being unstable and below being stable. Because agents are only weakly coupled initially, their attitudes move towards the steady state $\mathbf{x}^*$, which causes the differences δ_{ij} to decrease. The switch terms subsequently become positive and hence the coupling strengths increase, along

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with the eigenvalues of the Laplacian λ_i. When one or more of the λ_i are within the window of instability, some of the differences δ_{ij} begin to diverge. However, this eventually causes their switch terms to become negative, reducing the corresponding coupling strengths and hence some of the λ_i. This then affects the differences δ_{ij}, which start to decrease, completing the qualitative cycle. As the system evolves beyond $t > 440$, this quasi-cyclic behaviour becomes increasingly erratic.

Although the long term behaviour of any given agent is unpredictable, the behaviour of the mean coupling strength of the system fluctuates around the instability boundary k_-/N. In Figure 2(a), we plot the time series of the mean coupling strength, $\bar{A}(t)$, between $t = 5 \times 10^3$ and 10^4. The dashed line indicates the instability boundary $k_-/N \approx 0.6372$, which is very close to the time-averaged mean coupling strength $\langle A \rangle \approx 0.6343$. Also plotted in Figure 2 are snapshots of the network at six sequential times. To improve the visualisation of the network, the positions of nodes have been rotated by approximately $72°$, since the differences in diffusion rates mean that the unrotated coordinates, (x, y), become contracted in one direction. The shading of the nodes corresponds to their average coupling strength and the shading of the edges correspond to their weight. The sequence of figures illustrate the general scenario: agents' trajectories cycle around the origin with the network repeatedly contracting and expanding as agents become more and less similar in attitude respectively.

We now illustrate how the quasi-equilibrium end state changes as the window of instability is moved. We fix all parameters as above, but consider a range of values of D_1 whilst keeping the ratio $D_2/D_1 = 0.9\sigma_c$ held fixed. This has the effect of shifting the window of instability from above $\langle A \rangle = 1$ to below as D_1 increases. We integrate until $t = 1.5 \times 10^5$ and then compute the mean coupling strength $\langle A \rangle$ for $t \geq 10^4$. We compute 50 realisations for each set of parameters, the results of which are plotted in Figure 3. The shaded region corresponds to the (scaled) region of instability, the dashed grey line is where $\langle A \rangle = 1$, the black line is the median of the fifty realisations and the dots are the values from each of the realisations. At low values of D_1, where the $A = \mathbf{1}$ equilibrium first becomes unstable, the mean coupling strength fits tightly to the lower edge of the instability boundary at λ_-/N. When the $A = \mathbf{1}$ equilibrium restabilises (at $D_1 \approx 0.78$), the long time behaviour of the mean coupling strength changes, moving away from the λ_-/N boundary. In the region between $D_1 \approx 1.5$ and $D_1 \approx 2.3$, some realisations return to the fully coupled equilibrium $A = \mathbf{1}$, but not all. We would expect that
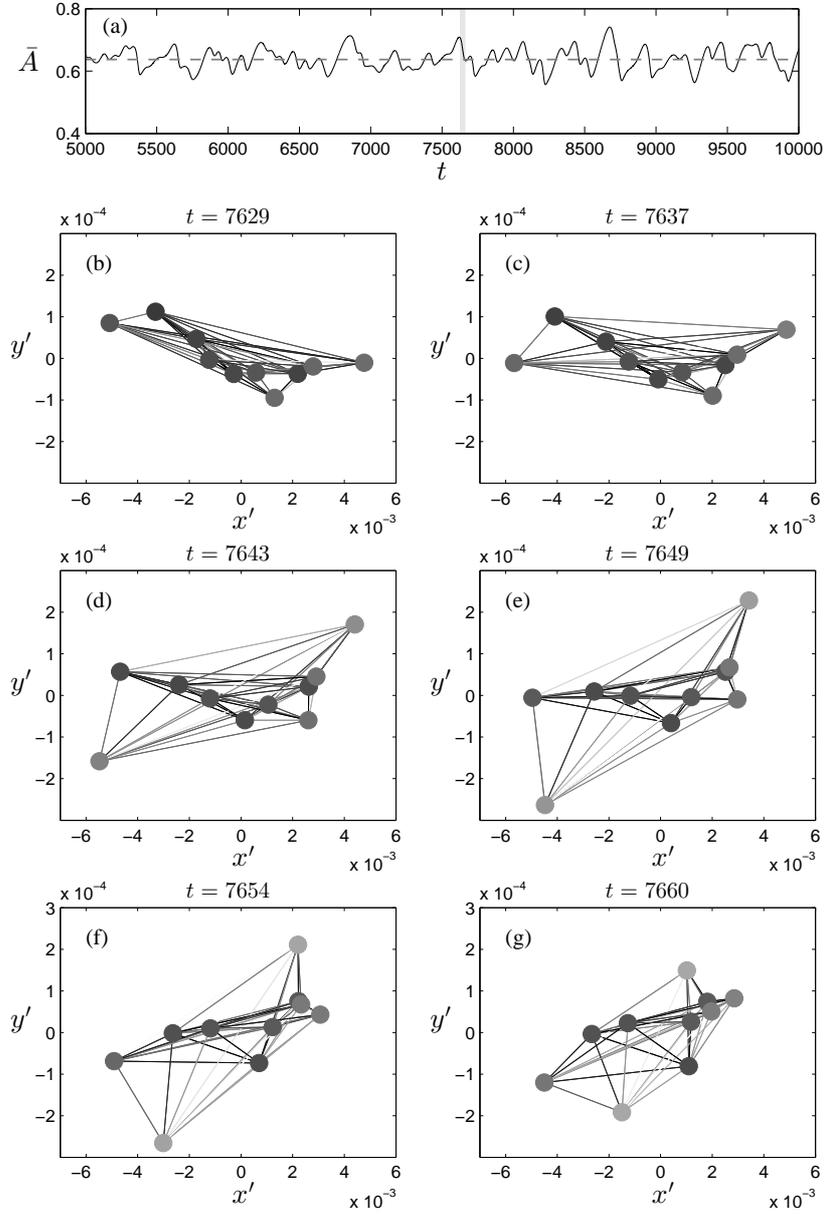
Figure 2: Panel (a): mean coupling strength, $\bar{A}(t)$, time series. Panels (b)–(g): network snapshots at sequential time intervals. Node positions are plotted in the rotated coordinates (x', y'), shading illustrates coupling strength for edges and mean coupling strength for nodes.
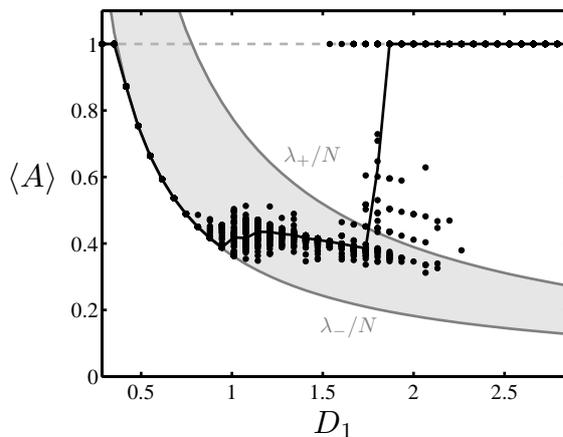
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Figure 3: Mean coupling strength at large times for different values of D_1. Shading indicates the window of instability, the dashed grey line indicates where $A = \mathbf{1}$. The black line is the median of 50 realisations and the black markers are the mean coupling strengths for each of the realisations.

simulating for longer would result in more realisations reaching the fully coupled equilibrium, although it is possible that its basin of attraction does not include every initial condition in the set that we are sampling from.

3.2. Dyad dynamics

To probe the mechanism driving the aperiodic dynamics illustrated in Section 3.1, we consider a simpler dynamical setting consisting of just two agents. This reduces the coupling strength evolution (4) to a single equation, and hence five equations in total,

$$\dot{x}_1 = p - x_1 y_1^2 - D_1 a(x_1 - x_2), \tag{10}$$

$$\dot{x}_2 = p - x_2 y_2^2 + D_1 a(x_1 - x_2), \tag{11}$$

$$\dot{y}_1 = q - y_1 + x_1 y_1^2 - D_2 a(y_1 - y_2), \tag{12}$$

$$\dot{y}_2 = q - y_2 + x_2 y_2^2 + D_2 a(y_1 - y_2), \tag{13}$$

$$\dot{a} = \alpha a(1 - a) \left[\varepsilon - (x_1 - x_2)^2 \right]. \tag{14}$$

In Figure 4, we plot the trajectories for each of the two agents (black and grey lines) in (x, y) space, and in (x, y, a) space in the upper-right inset. The parameter values are $p = 1.25$, $q = 0.1$, $\alpha = 10^4$, $\varepsilon = 10^{-6}$, $D_1 \approx 2.857$ and $D_2 \approx 0.184$. Again, the diffusion constants have the ratio $D_2/D_1 = 0.9\sigma_{\mathrm{c}}$ and the window of instability is centred on the fully coupled system
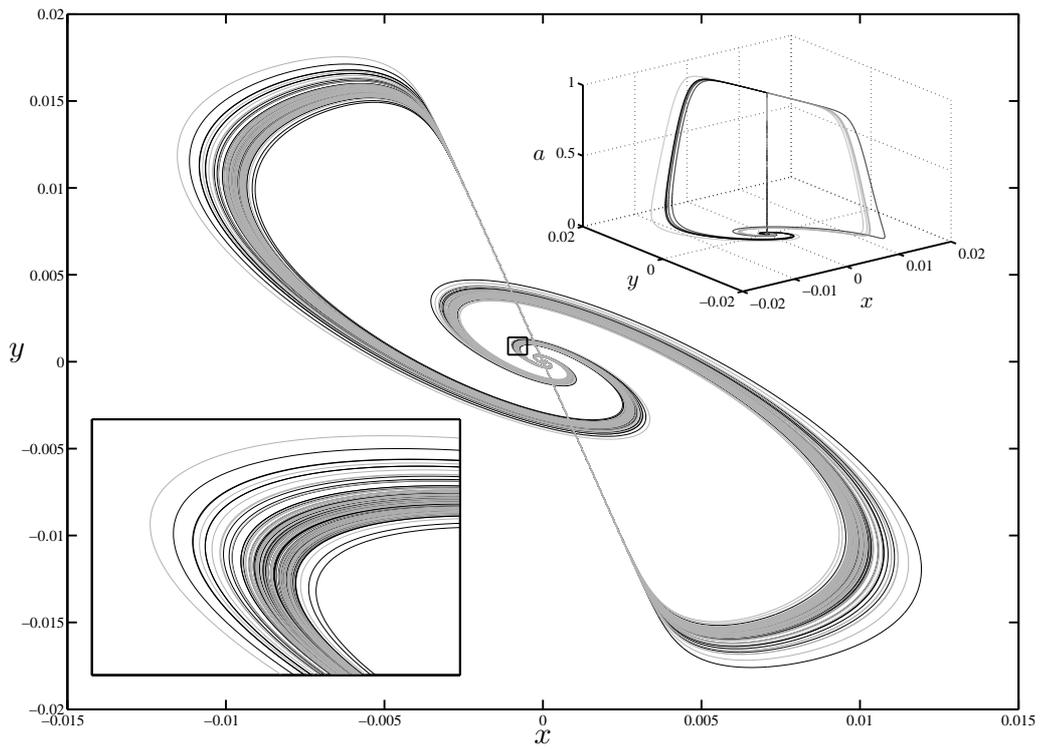
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Figure 4: Main: Trajectories of dyadic system in (x, y) space with unstable parameters. Upper-right inset: Trajectories in (x, y, a) space. Lower-left inset: zoom of boxed region in the main plot. Parameters described in the main text.

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via (9). The initial conditions are chosen near to the uniform equilibrium $\mathbf{x}^* = (x^*, y^*)^T$, specifically $x_1(0) = x^* + 1.5 \times 10^{-4}$, $x_2(0) = x^* - 1.5 \times 10^{-4}$, $y_1(0) = y^* - 1 \times 10^{-6}$, $y_2(0) = y^* + 1 \times 10^{-6}$; the initial coupling strength is $a(0) = 0.1$. This system is numerically stiff—on each cycle, trajectories get very close to the equilibrium $\mathbf{x}^*$ and the invariant planes $a = 0$ and $a = 1$— thus very low error tolerances are necessary in order to accurately resolve the trajectories.

The mechanisms driving the near cyclic behaviour illustrated in Figure 4 are qualitatively similar to those described in Section 2.1, but the present case is much simpler since the coupling constant, a, is a scalar. If we consider a as a parameter in the attitudinal dynamics (10–13), then a Turing instability occurs as a pitchfork bifurcation at some $a = a_*$, where $0 < a_* < 1$. The equilibria at $(\mathbf{x}, a) = (\mathbf{x}^*, 0)$ and $(\mathbf{x}^*, 1)$ are both saddle-foci, where the unstable manifolds are respectively parallel to the a-axis and entirely within the attitudinal state space, $\mathbf{x}$. Near to the $(\mathbf{x}^*, 1)$ equilibrium, a given trajectory tracks the unstable manifold of $(\mathbf{x}^*, 1)$ in one of two opposing directions, the choice of which is sensitively dependent on its earlier position when $a \approx a_*$. The combination of this sensitivity together with the spiral dynamics around the unstable manifold of $(\mathbf{x}^*, 0)$, leads to an orbit switching sides unpredictably on each near-pass of $(\mathbf{x}^*, 1)$ (c.f. the top-right inset of Figure 4). The mechanism by which this chaotic behaviour arises is not standard (e.g. via a Shilnikov bifurcation or homoclinic explosion) and warrants its own study, which we address in an article currently in preparation.

4. Conclusions

We have proposed a new modelling framework to describe the evolution of sub-conscious attitudes within social groups. We based this framework on the fundamental mechanisms of homophily and social influence, but it differs from previous approaches in two respects. Firstly, we have focused on sub-conscious attitudes, where it is natural to consider dynamics described by a class of activator-inhibitor processes. Secondly, we have formulated a deterministic system, enabling us to highlight (via mathematical analysis and simulation) the mechanisms driving dynamical phenomena. Specifically, we have illustrated that the tension between Turing instability and the evolving network topology gives rise to behaviour that at the system level is qualitatively predictable — sub-group formation and dissolution — yet at the level of individual agent journeys is entirely unpredictable. We point out that a

stochastic model based on similar principles to those described in this paper is presented in [47], where qualitatively similar dynamical phenomena are also observed. Thus we might conclude that even if stochasticity is entirely absent, the mechanisms that govern human behaviour seem to give rise to unpredictable dynamics.

5. Discussion

While we have differentiated our modelling framework from other models of attitudinal dynamics [39, 27, 34], we now discuss our findings in the context of more general cultural models and cultural polarisation.

Current interest in cultural models largely stems from the work of Axelrod [39], who demonstrated that local convergence could lead to cultural polarisation. This topic has particular resonance in our digital society: will global connectivity accelerate a descent into monoculture, or can diversity persist? Models such as Axelrod's provide us with an optimistic outlook, suggesting that even the most basic mechanisms that model social influence and homophily can lead to cultural diversity. But by no means is there presently a completely satisfactory understanding of this phenomena. The polarised states of the Axelrod model are fragile; even low rates of random perturbations to cultural traits can reinstate global monoculture [41]. Thus additional dynamical rules have been investigated in this context. The variant of the Axelrod model proposed by Centola et al. [27] allows agents to disassociate themselves from neighbours that have no similar traits and select a new neighbour at random. Similarly, a number of adaptive network models of opinion dynamics have also found absorbing polarised states, in which groups with differing opinions are completely disconnected [16, 17, 20]. Such polarised states seem artificial and we conjecture that a form of cultural drift, characterised by random rewiring of a small number of edges, would destabilise these states.

This touches on another key issue: in the absence of noise, the mechanisms employed by cultural dissemination models typically *reduce* diversity. It is not surprising then that these types of models can be perturbed in such a way that the eventual result is monoculture. An approach that allows diversity to increase has been suggested by Flache and Macy [34]. They model social influence via diffusion, whereby an agent adjusts their cultural state, described by a vector of continuous real variables, according to a weighted sum of the differences between their state and their neighbours'. The weights

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are dynamic and their evolution is driven by homophily. In some sense, the corresponding elements of our model are like a continuous-time version of the Flache and Macy model. However, Flache and Macy consider the weights embedded on a clustered network and, more importantly, allow their weights to be negative, representing *xenophobia*. It is this feature that allows diversity to both decrease and increase via diffusion and convergence respectively. The effects of cultural drift on polarised states in the Flache and Macy model have not been investigated in detail, but perturbations can cause agents to switch groups [34] and so we would expect that sustained noise would erode smaller groups. Monoculture is also a stable fixed point of their model.

The key element that differentiates our model is that agents' cultural states have an activator-inhibitor dynamic that is independent of other agents. The presence of diffusion allows for Turing instability and hence means that diversity can increase. Moreover, we can identify regions in which global monoculture is *unstable*. For fixed or slowly evolving networks, instability gives rise to stable 'Turing patterns' [46], which could be interpreted as culturally polarised states. However, one would expect inter-group connections to be weaker than intra-group connections within polarised states. But if homophily dissolves such inter-group ties then the patterned or polarised states can no longer be stable, since it is precisely the differences in culture that balance individuals' attitudinal dynamics with diffusion. If non-trivial stable equilibria were to exist in our model, they would involve a delicate balance of cultural differences within the switching terms. However, we have seen no evidence of this occurring in numerical simulations. Thus in its present form, sub-group formation and polarisation are transient phenomena in our model.

It is possible however that extensions to our model could produce stable polarised states. For example, introducing agent heterogeneity, in the form of distinct uncoupled equilibria, offers some promise. Agents could then adopt a state close to their uncoupled equilibrium, allowing distinct groups to form, but Turing instability could still destabilise the monocultural equilibrium. Alternatively, the network evolution equations could include higher order effects such as edge snapping [48] or triangulation. These ideas will be investigated further in follow-up work and we hope that our model may provide a new paradigm from which to explore cultural polarisation phenomena.

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