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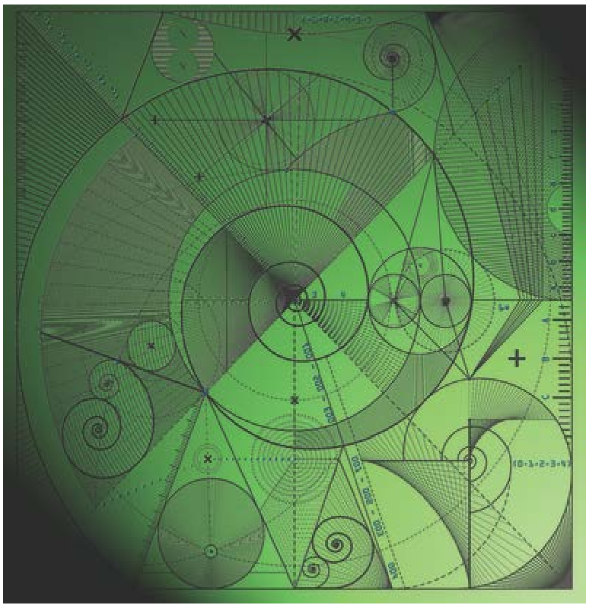

A POINTWISE CHARACTERISATION OF THE PDE SYSTEM OF VECTORIAL CALCULUS OF VARIATIONS IN L^∞

BIRZHAN AYANBAYEV AND NIKOS KATZOURAKIS

Abstract. Let $n, N \in \mathbb{N}$ with $\Omega \subseteq \mathbb{R}^n$ open. Given $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn})$, we consider the functional

$$\mathrm{E}_\infty(u, \mathcal{O}) := \underset{\mathcal{O}}{\mathrm{ess\,sup}}\, \mathrm{H}(\cdot, u, \mathrm{D}u), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N), \quad \mathcal{O} \Subset \Omega. \tag{1}$$

The associated PDE system which plays the role of Euler-Lagrange equations in L^∞ is

$$\begin{cases} \mathrm{H}_P(\cdot, u, \mathrm{D}u)\, \mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big) \,=\, 0, \\ \mathrm{H}(\cdot, u, \mathrm{D}u)\, [\![\mathrm{H}_P(\cdot, u, \mathrm{D}u)]\!]^\perp \Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big) \,=\, 0, \end{cases} \tag{2}$$

where $[\![A]\!]^\perp := \mathrm{Proj}_{R(A)^\perp}$. Herein we establish that generalised solutions to (2) can be characterised as local minimisers of (1) for appropriate classes of affine variations of the energy. Generalised solutions to (2) are understood as $\mathcal{D}$-solutions, a general framework recently introduced by one of the authors.

1. Introduction

Given $n, N \in \mathbb{N}$, let $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn})$ be a given function, where $\Omega \subseteq \mathbb{R}^n$ is an open set. In this paper we are interested in the variational characterisation of the PDE system arising as the analogue of the Euler-Lagrange equations when one considers vectorial minimisation problems for supremal functionals of the form

$$\mathrm{E}_\infty(u, \mathcal{O}) := \underset{x \in \mathcal{O}}{\mathrm{ess\,sup}}\, \mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big), \quad \mathcal{O} \Subset \Omega, \tag{1.1}$$

on maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ in $W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. The associated PDE system is

$$\mathcal{F}_\infty(\cdot, u, \mathrm{D}u, \mathrm{D}^2 u) \,=\, 0 \quad \text{in } \ \Omega, \tag{1.2}$$

where $\mathcal{F}_\infty : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s \longrightarrow \mathbb{R}^N$ is the Borel measurable map

$$\begin{aligned} \mathcal{F}_\infty(x, \eta, P, \mathbf{X}) \,:=\, & \mathrm{H}_P(x, \eta, P)\Big(\mathrm{H}_P(x, \eta, P) : \mathbf{X} + \mathrm{H}_\eta(x, \eta, P)^\top P + H_x(x, \eta, P)\Big) \\ & + \mathrm{H}(x, \eta, P)\, [\![\mathrm{H}_P(x, \eta, P)]\!]^\perp \Big(\mathrm{H}_{PP}(x, \eta, P) : \mathbf{X} + \mathrm{H}_{P\eta}(x, \eta, P) : P \\ & + \mathrm{H}_{Px}(x, \eta, P) : \mathrm{I} - \mathrm{H}_\eta(x, \eta, P)\Big). \end{aligned} \tag{1.3}$$

In the above, $\mathbb{R}^{Nn}$ and $\mathbb{R}^{Nn^2}_s$ denote respectively the space of matrices and the space of symmetric tensors wherein the gradient matrix and the hessian tensor

$$\mathrm{D}u(x) \,=\, \big(\mathrm{D}_i u_\alpha(x)\big)^{\alpha=1,\dots,N}_{i=1,\dots,n}, \qquad \mathrm{D}^2 u(x) \,=\, \big(\mathrm{D}^2_{ij} u_\alpha(x)\big)^{\alpha=1,\dots,N}_{i,j=1,\dots,n}$$

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of (regular) maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ are valued, whilst subscripts of H denotes derivatives with respect to the respective variables (x, η, P). We use the symbolisations $x = (x_1, ..., x_n)^\top$, $u = (u_1, ..., u_N)^\top$, $\mathrm{D}_i \equiv \partial/\partial x_i$, whilst Latin indices $i, j, k, ...$ will run in $\{1, ..., n\}$ and Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{1, ..., N\}$. Further, for any linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$, the notation $[\![A]\!]^\perp$ used above denotes the orthogonal projection onto the orthogonal complement of its range $R(A) \subseteq \mathbb{R}^N$:

$$[\![A]\!]^\perp := \mathrm{Proj}_{R(A)^\perp}. \tag{1.4}$$

Also, "$\mathcal{O} \Subset \Omega$" means that $\mathcal{O}$ is open and $\overline{\mathcal{O}} \subseteq \Omega$. In index form, $\mathcal{F}_\infty$ reads

$$\begin{aligned}\mathcal{F}_\infty(x, \eta, \mathrm{P}, \mathbf{X})_\alpha := \sum_i \mathrm{H}_{P_{\alpha i}}(x, \eta, P)\Bigg(&\sum_{\beta, j} \mathrm{H}_{P_{\beta j}}(x, \eta, P)\mathbf{X}_{\beta i j} + \sum_\beta \mathrm{H}_{\eta_\beta}(x, \eta, P)\, P_{\beta i} \\ &+ \mathrm{H}_{x_i}(x, \eta, P)\Bigg) + \mathrm{H}(x, \eta, P) \sum_\beta [\![\mathrm{H}_P(x, \eta, P)]\!]^\perp_{\alpha\beta} \cdot \\ &\cdot \Bigg(\sum_{i,j} \mathrm{H}_{P_{\alpha i} P_{\beta j}}(x, \eta, P)\, \mathbf{X}_{\beta i j} + \sum_i \mathrm{H}_{P_{\alpha i}\eta_\beta}(x, \eta, P)\, P_{\beta i} \\ &+ \sum_i \mathrm{H}_{P_{\alpha i} x_i}(x, \eta, P) - \mathrm{H}_{\eta_\beta}(x, \eta, P)\Bigg),\end{aligned}$$

where $\alpha = 1, ..., N$. Note that, although H is C^2, the projection map $[\![\mathrm{H}_P(\cdot, u, \mathrm{D}u)]\!]^\perp$ is *discontinuous* when the rank of $\mathrm{H}_P(\cdot, u, \mathrm{D}u)$ changes. Further, we remark that because of the perpendicularity of H_P and $[\![\mathrm{H}_P]\!]^\perp$, the system can be decoupled into two independent systems which we write in a contracted fashion:

$$\left\{\begin{aligned} \mathrm{H}_P(\cdot, u, \mathrm{D}u)\, \mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big) &= 0, \\ \mathrm{H}(\cdot, u, \mathrm{D}u)\, [\![\mathrm{H}_P(\cdot, u, \mathrm{D}u)]\!]^\perp \Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big) &= 0. \end{aligned}\right.$$

When $\mathrm{H}(x, \eta, P) = |P|^2$ (the Euclidean norm on $\mathbb{R}^{Nn}$ squared), the system (1.2)-(1.4) simplifies to the so-called ∞-Laplacian:

$$\Delta_\infty u := \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes \mathrm{I}\Big) : \mathrm{D}^2 u = 0. \tag{1.5}$$

The scalar case $N = 1$ first arose in the work of G. Aronsson in the 1960s [A1, A2] who initiated the area of Calculus of Variations in the space L^∞. The field is fairly well-developed today and the relevant bibliography is vast. For a pedagogical introduction to the topic accessible to non-experts, we refer to [K8]. We just mention that in the scalar case, generalised solutions to the respective PDE which is commonly referred to as the Aronsson equation and simplifies to

$$\mathrm{H}_P(\cdot, u, \mathrm{D}u) \cdot \Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)^\top \mathrm{D}^2 u + \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\mathrm{D}u + \mathrm{H}_x(x, \eta, P)\Big) = 0$$

are understood in the viscosity sense (see [C, CIL, K8]). The study of the vectorial case $N \geq 2$ started much more recently and the full system (1.2)-(1.4) first appeared in the work [K1] of one of the authors in the early 2010s and it is being studied quite systematically ever since (see [K2]-[K7], [K9]-[K13], as well as the joint works of the second author with Abugirda, Pryer, Croce and Pisante [AK], [CKP], [KP, KP2]).

In this paper we are interested in the characterisation of appropriately defined generalised vectorial solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to (1.2)-(1.4) in terms of the functional (1.1). It is well known even from classical scalar considerations for $N = 1$ that the solutions to (1.2)-(1.4) in general cannot be expected to be smooth. Since the viscosity theory does not appear to work for (1.2)-(1.4) when $N \geq 2$, we will interpret solutions in the so-called $\mathcal{D}$-sense. This is a new notion of solution for fully nonlinear systems of very general applicability recently introduced in [K9]-[K10]. Deferring temporarily the details of this new theory of $\mathcal{D}$-solutions, we would like to stress the next purely vectorial peculiar occurrence: the obvious adaptation of Aronsson's variational notion of Absolute Minimisers to $N \geq 2$, i.e.

$$\mathrm{E}_\infty(u, \mathcal{O}) \,\leq\, \mathrm{E}_\infty(u + \phi, \mathcal{O}), \qquad \mathcal{O} \Subset \Omega,\ \phi \in W^{1,\infty}_0(\mathcal{O}, \mathbb{R}^N)$$

is not yet known *whether it is the appropriate one when* $\min\{n, N\} \geq 2$. Accordingly, in the model case of (1.5) and for C^2 solutions, the relevant notion of so-called ∞-Minimal maps allowing to *characterise variationally* solutions to (1.5) in term of $u \mapsto \|\mathrm{D}u\|_{L^\infty(\cdot)}$ was introduced in [K4] and in a sense its nature stems from the emergence (when $N \geq 2$) of the extra system which has discontinuous coefficients but vanishes identically when $\min\{N, n\} = 1$. These findings are compatible with the early vectorial observations made in [BJW1, BJW2], wherein the appropriate L^∞ quasiconvexity notion in the vectorial case is essentially different from its scalar counterpart and the existence of absolute minimisers was established only for $\min\{N, n\} = 1$. Notwithstanding, in the most recent paper [K13] a new characterisation has been discovered that allows to connect $\mathcal{D}$-solutions of (1.5) to local minimisers of $u \mapsto \|\mathrm{D}u\|_{L^\infty(\cdot)}$ in terms of certain classes of *local affine variations.*

In this paper we characterise general $\mathcal{D}$-solutions to (1.2)-(1.4) in terms of local affine variations of (1.1). Our main result is Theorem 7 which follows and claims that $\mathcal{D}$-solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to (1.2)-(1.4) in $C^1_{\mathrm{pw}}(\Omega, \mathbb{R}^N)$ can be characterised variationally in terms of (1.1). The a priori regularity of piecewise C^1 assumed for our putative solutions is slightly higher than the generic membership in the space $W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, but as a compensation we impose *no convexity* of any kind for the hamiltonian H.

In special case of classical C^2 solutions, our result reduces to the following corollary which shows the geometric nature of our characterisation:

Corollary 1 (C^2 ∞-Harmonic mappings)**.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be open,* $u \in C^2(\Omega, \mathbb{R}^N)$ *and* $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn})$ *a function satisfying*

$$\big\{\mathrm{H}_P(x, \eta, \cdot) = 0\big\} \,\subseteq\, \big\{\mathrm{H}(x, \eta, \cdot) = 0\big\}, \quad (x, \eta) \in \Omega \times \mathbb{R}^N.$$

Then,

$$\mathcal{F}_\infty(\cdot, u, \mathrm{D}u, \mathrm{D}^2u) \,=\, 0 \ \text{ in } \Omega \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \mathrm{E}_\infty(u,\, \mathcal{O}) \,\leq\, \mathrm{E}_\infty(u + A,\, \mathcal{O})\,, \\ \forall\ \mathcal{O} \Subset \Omega,\ \forall\ A \in \big(\mathcal{A}^{\|,\infty}_{\mathcal{O}} \cup \mathcal{A}^{\perp,\infty}_{\mathcal{O}}\big)(u). \end{array} \right.$$

Here E_∞ *is given by* (1.1)*,* $\mathcal{F}_\infty$ *is given by* (1.2)-(1.4) *and* $\mathcal{A}^{\|,\infty}_{\mathcal{O}}(u), \mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$ *are sets of affine maps given by*

$$\mathcal{A}^{\|,\infty}_{\mathcal{O}}(u) \,=\, \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} \mathrm{D}^2 A \equiv 0,\ A(x) = 0 \ \&\ \text{exist } \xi \in \mathbb{R}^N \\ \&\ x \in \mathcal{O}(u) \text{ s. th. } A \text{ is parallel to} \\ \text{the tangent of } \xi\mathrm{H}(\cdot, u, \mathrm{D}u) \text{ at } x \end{array} \right\},$$

$$\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u) \,=\, \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} \mathrm{D}^2 A \,\equiv\, 0 \text{ and there exists } x \in \mathcal{O}(u) \\ \text{s. th. } A \text{ is normal to } \mathrm{H}_P(\cdot, u, \mathrm{D}u) \text{ at } x \\ \&\ A^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) \text{ is divergenceless at } x \end{array} \right\}$$

and

$$\mathcal{O}(u) \,:=\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\big\}.$$

We conclude the introduction with some rudimentary facts about generalised solutions which are required for the results in the present paper. The notion of $\mathcal{D}$-solutions is based on the probabilistic interpretation of limits of difference quotients by using *Young* measures. Unlike standard PDE approaches which utilise Young measures valued in Euclidean spaces (see e.g. [E, P, FL, CFV, FG, V, KR]), $\mathcal{D}$-solutions are based on Young measures valued in the 1-point compactification $\overline{\mathbb{R}}^{Nn^2}_s := \mathbb{R}^{Nn^2}_s \cup \{\infty\}$ (which is isometric to the sphere of the same dimension). The motivation of the notion in the special case of $W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ to 2nd order fully nonlinear systems is the following: let $u \in W^{2,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a strong solution to

$$\mathcal{F}\Big(x,u(x),\mathrm{D}u(x),\mathrm{D}^2u(x)\Big) \,=\, 0, \qquad \text{a.e. } x\in\Omega, \tag{1.6}$$

where $\mathcal{F}:\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s\longrightarrow\mathbb{R}^N$ is a Borel measurable map which is continuous with respect to the hessian variable. Let $D^{1,h}$ be the usual difference quotient operator, i.e. $\mathrm{D}^{1,h} := \big(\mathrm{D}^{1,h}_1,...,\mathrm{D}^{1,h}_n\big)$ and $\mathrm{D}^{1,h}_i v := \frac{1}{h}\big[v(\cdot+he^i)-v\big]$, $h\neq 0$. By the well known equivalence between weak and strong derivatives, (1.6) gives

$$\mathcal{F}\Big(x,u(x),\mathrm{D}u(x),\lim_{h\to 0}\mathrm{D}^{1,h}u(x)\Big) \,=\, 0, \qquad \text{a.e. } x\in\Omega. \tag{1.7}$$

Further, by the assumed continuity of $\mathbf{X}\mapsto\mathcal{F}(x,\eta,P,\mathbf{X})$, this is equivalent to

$$\lim_{h\to 0}\mathcal{F}\Big(x,u(x),\mathrm{D}u(x),\mathrm{D}^{1,h}u(x)\Big) \,=\, 0, \qquad \text{a.e. } x\in\Omega. \tag{1.8}$$

The crucial observation is that *the limit in* (1.8) *may exist even if that of* (1.7) *does not*, whilst (1.8) makes sense for merely once weakly differentiable maps. In order to represent the limit in a convenient fashion, we need to view the hessian D^2u and the difference quotients of the gradient $\mathrm{D}^{1,h}\mathrm{D}u$ as probability-valued mappings from Ω to $\mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)$, given by the respective Dirac masses: $x\longmapsto\delta_{\mathrm{D}^2u(x)}$ and $x\longmapsto\delta_{\mathrm{D}^{1,h}\mathrm{D}u(x)}$. The exact definition is as follows:

Definition 2 (Diffuse Hessians)**.** Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be in $W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. The **diffuse hessians** $\mathcal{D}^2u$ of u are the subsequential weak* limits of the difference quotients of the gradient in the set of sphere-valued Young measures along infinitesimal sequences $(h_\nu)_{\nu=1}^\infty$:

$$\delta_{\mathrm{D}^{1,h_{\nu_k}}\mathrm{D}u} \xrightarrow{\;*\;} \mathcal{D}^2u \quad \text{ in } \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big), \ \text{ as } k\to\infty.$$

The set $\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$ is defined as

$$\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big) := \Big\{\nu:\Omega\to\mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)\ \Big|\ [\nu(\cdot)](\mathcal{U})\in L^\infty(\Omega) \text{ for any open } \mathcal{U}\subseteq\overline{\mathbb{R}}^{Nn^2}_s\Big\}.$$

Some elementary properties of the set of Young measures $\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$ utilised herein are recalled in the next section. The main fact that we need to record here is that it is *sequentially weakly* compact* with respect to a certain dual topology and hence *every map as above possesses diffuse* 2*nd derivatives.*

Definition 3 ($\mathcal{D}$-solutions to 2nd order systems)**.** Let $\Omega\subseteq\mathbb{R}^n$ be an open set and $\mathcal{F}:\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s\longrightarrow\mathbb{R}^N$ a Borel measurable map which is continuous with respect to the last argument. Consider the PDE system

$$\mathcal{F}\big(\cdot,u,\mathrm{D}u,\mathrm{D}^2u\big) \,=\, 0 \quad \text{on } \Omega. \tag{1.9}$$

We say that the locally Lipschitz continuous map $u : u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is a **$\mathcal{D}$-solution of** (1.9) when for any diffuse hessian $\mathcal{D}^2 u$ of u, we have

$$\sup_{\mathbf{X}_x \in \, \mathrm{supp}_*(\mathcal{D}^2 u(x))} \big|\mathcal{F}\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big)\big| \, = \, 0, \qquad \text{a.e. } x \in \Omega. \tag{1.10}$$

Here "supp_*" symbolises the *reduced support* of a probability measure excluding infinity, namely $\mathrm{supp}_*(\vartheta) := \mathrm{supp}(\vartheta) \setminus \{\infty\}$ when $\vartheta \in \mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)$.

We note that $\mathcal{D}$-solutions are readily compatible with strong/classical solutions: indeed, by Remark 4iii) that follows, if u happens to be twice weakly differentiable then we have $\mathcal{D}^2 u(x) = \delta_{\mathrm{D}^2 u(x)}$ for a.e. $x \in \Omega$ and the notion reduces to

$$\sup_{\mathbf{X}_x \in \, \mathrm{supp}(\delta_{\mathrm{D}^2 u(x)})} \big|\mathcal{F}\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big)\big| \, = \, 0, \qquad \text{a.e. } x \in \Omega,$$

thus recovering strong/classical solutions.

2. Young Measures and Auxiliary Results

Young Measures. Let $\Omega \subseteq \mathbb{R}^n$ be open. The set of Young measures $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$ defined above forms a subset of the unit sphere of a certain L^∞ space of measure-valued maps and this provides its useful properties, including sequential weak* compactness. Consider the separable space $L^1\big(\Omega, C^0\big(\overline{\mathbb{R}}^{Nn^2}_s\big)\big)$ of Bochner integrable maps. The elements of this space are those Carathéodory functions $\Phi : \Omega \times \overline{\mathbb{R}}^{Nn^2}_s \longrightarrow \mathbb{R}$ which satisfy $\|\Phi\|_{L^1(\Omega, C^0(\overline{\mathbb{R}}^{Nn^2}_s))} := \int_\Omega \big\|\Phi(x,\cdot)\big\|_{C^0(\overline{\mathbb{R}}^{Nn^2}_s)} dx < \infty$. We refer e.g. to [FL, Ed, V] and to [K9]-[K13] for background material on these spaces. The dual space of this space is $L^\infty_{w^*}\big(\Omega, \mathcal{M}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)\big)$. This dual Banach space consists of Radon measure-valued maps $\Omega \ni x \longmapsto \nu(x) \in \mathcal{M}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)$ which are weakly* measurable, in the sense that for any open set $\mathcal{U} \subseteq \overline{\mathbb{R}}^{Nn^2}_s$, the function $x \longmapsto [\nu(x)](\mathcal{U})$ is in $L^\infty(\Omega)$. The norm of the space is given by $\|\nu\|_{L^\infty_{w^*}(\Omega, \mathcal{M}(\overline{\mathbb{R}}^{Nn^2}_s))} := \mathrm{ess\,sup}_{x\in\Omega} \|\nu(x)\|$, where "$\|\cdot\|$" denotes the total variation. It thus follows that

$$\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big) \, = \, \Big\{\nu \, \in \, L^\infty_{w^*}\big(\Omega, \mathcal{M}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)\big) \, : \, \nu(x) \in \mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big), \text{ for a.e. } x \in \Omega\Big\}.$$

Remark 4 (Properties of Young Measures)**.** We note the following facts about the set $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$ (proofs can be found e.g. in [FG]):

i) It is convex and sequentially compact in the weak* topology induced from $L^\infty_{w^*}$.

ii) The set of measurable maps $V : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{Nn^2}_s$ can be identified with a subset of it via the embedding $V \longmapsto \delta_V$, $\delta_V(x) := \delta_{V(x)}$.

iii) Let $V^i, V^\infty : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{Nn^2}_s$ be measurable maps, $i \in \mathbb{N}$. Then, up the passage to subsequences, the following equivalence holds true as $i \to \infty$: we have $V^i \longrightarrow V^\infty$ a.e. on Ω if and only if $\delta_{V^i} \overset{*}{\rightharpoonup} \delta_{V^\infty}$ in $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$.

Two Auxiliary Lemmas. We now identify two simple technical results which are needed for our main result.

Lemma 5. *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $u \in C^1(\Omega, \mathbb{R}^N)$ and $\mathrm{H} \in C^2(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nn})$. Fix $\mathcal{O} \Subset \Omega$ and an affine map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$. We set*

$$\mathcal{O}(u) \, := \, \Big\{x \in \overline{\mathcal{O}} \, : \, \mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big) \, = \, \mathrm{E}_\infty(u, \mathcal{O})\Big\}.$$

a) If we have $\mathrm{E}_\infty(u,\,\mathcal{O}) \le \mathrm{E}_\infty(u+tA,\,\mathcal{O})$ *for all* $t>0$*, it follows that*

$$\max_{z\in\overline{\mathcal{O}}}\Big\{\mathrm{H}_P\big(z,u(z),\mathrm{D}u(z)\big):\mathrm{D}A(z)\;+\;\mathrm{H}_\eta\big(z,u(z),\mathrm{D}u(z)\big)\cdot A(z)\Big\}\;\ge\;0.$$

In the above "$:$*" and "*$\cdot$*" denote the inner products in* $\mathbb{R}^{Nn}$ *and* $\mathbb{R}^N$ *respectively.*

b) Let $x\in\mathcal{O}$ *and* $0<\varepsilon<\mathrm{dist}(x,\partial\mathcal{O})$*. The set*

$$\mathcal{O}_\varepsilon(x)\;:=\;\Big\{y\in\mathcal{O}\;\Big|\;\mathrm{H}(y,u(y),\mathrm{D}u(y))<\mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big)\Big\}\cap\mathbb{B}_\varepsilon(x)$$

is open and compactly contained in $\mathcal{O}$ *whilst* $x\in\big(\mathcal{O}_\varepsilon(x)\big)(u)$*, that is*

$$\mathrm{E}_\infty\big(u,\,\mathcal{O}_\varepsilon(x)\big)\;=\;\mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big).$$

Proof of Lemma 5. a) Since $\mathrm{E}_\infty(u,\,\mathcal{O}) \le \mathrm{E}_\infty(u+tA,\,\mathcal{O})$, by Taylor-expanding H we have

$$\begin{aligned}
0\;\le\;&\max_{\overline{\mathcal{O}}}\mathrm{H}(\cdot,u,\mathrm{D}u)\;-\;\max_{\overline{\mathcal{O}}}\mathrm{H}\big(\cdot,u+tA,\mathrm{D}u\,+t\,\mathrm{D}A\big)\\
=\;&\max_{\overline{\mathcal{O}}}\mathrm{H}(\cdot,u,\mathrm{D}u)\;-\;\max_{\overline{\mathcal{O}}}\Big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\;+\;t\,\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot A\\
&\quad+\;t\,\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}A\;+\;O\big(t^2|A|^2+\,t^2|\mathrm{D}A|^2\big)\Big\}\\
\le\;&t\,\max_{\overline{\mathcal{O}}}\Big\{\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot A\;+\;\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}A\Big\}\;+\;O(t^2).
\end{aligned}$$

Consequently, by letting $t\to0$, we discover the desired inequality. Item b) is a direct consequence of the definitions. ☐

Next, we have the following simple consequence of Danskin's theorem [D]:

Lemma 6. *Given an open set* $\Omega\subseteq\mathbb{R}^n$*, consider maps* $u\in C^1(\Omega,\mathbb{R}^N)$ *and* $\mathrm{H}\in C^2(\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^{Nn})$*, an affine map* $A:\mathbb{R}^n\longrightarrow\mathbb{R}^N$ *and* $\mathcal{O}\Subset\Omega$*. We define*

$$r(\lambda)\;:=\;\mathrm{E}_\infty(u+\lambda A,\,\mathcal{O})\;-\;\mathrm{E}_\infty(u,\,\mathcal{O}),\qquad\lambda\ge0.$$

Let also $\mathcal{O}(u)$ *be as in Lemma 5. Then,* r *is convex,* $r(0)=0$ *and also it satisfies*

$$\underline{\mathrm{D}}r(0^+)\;\ge\;\max_{\mathcal{O}(u)}\Big\{\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}A\;+\;\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot A\Big\},$$

where $\underline{\mathrm{D}}r(0^+):=\liminf\limits_{\lambda\to0^+}\frac{r(\lambda)-r(0)}{\lambda}$ *is the lower right Dini derivative of* r *at zero.*

Proof of Lemma 6. The result is deducible from Danskin's theorem (see [D]) but we prove it directly since the 1-sided version above is not given explicitly in the paper. By setting

$$R(\lambda,y)\;:=\mathrm{H}\Big(y\,,\,u(y)+\lambda A(y)\,,\,Du(y)+\,\lambda DA(y)\Big)$$

we have $r(\lambda)=\max_{y\in\overline{\mathcal{O}}}R(\lambda,y)-\max_{y\in\overline{\mathcal{O}}}R(0,y)$, whilst for any $\lambda\ge0$ the maximum $\max_{y\in\overline{\mathcal{O}}}R(\lambda,y)$ is realised at (at least one) point $y^\lambda\in\overline{\mathcal{O}}$. Hence

$$\begin{aligned}
\frac1\lambda\big(r(\lambda)-r(0)\big)\;&=\;\frac1\lambda\Big[\max_{y\in\overline{\mathcal{O}}}R(\lambda,y)\;-\;\max_{y\in\overline{\mathcal{O}}}R(0,y)\Big]\\
&=\;\frac1\lambda\Big[R(\lambda,y^\lambda)\;-\;R(0,y^0)\Big]\\
&=\;\frac1\lambda\Big[\big(R(\lambda,y^\lambda)-R(\lambda,y^0)\big)\;+\;\big(R(\lambda,y^0)-R(0,y^0)\big)\Big]
\end{aligned}$$

and hence

$$\frac{1}{\lambda}\big(r(\lambda)-r(0)\big)\;\geq\;\frac{1}{\lambda}\big(R(\lambda,y^0)\;-\;R(0,y^0)\big),$$

where $y^0\in\overline{\mathcal{O}}$ is any point such that $R(0,y^0)\;=\;\max_{\overline{\mathcal{O}}}R(0,\cdot)$. Hence, we have

$$\begin{aligned}
\underline{\mathrm{D}}r(0^+) =\; & \liminf_{\lambda\to0^+}\frac{1}{\lambda}\big(r(\lambda)-r(0)\big)\\
\geq\; & \max_{y^0\in\overline{\mathcal{O}}}\left\{\liminf_{\lambda\to0^+}\frac{1}{\lambda}\Big(R(\lambda,y^0)-R(0,y^0)\Big)\right\}\\
=\; & \max_{y\in\mathcal{O}(u)}\left\{\liminf_{\lambda\to0^+}\frac{1}{\lambda}\Big(R(\lambda,y)-R(0,y)\Big)\right\}\\
=\; & \max_{\mathcal{O}(u)}\left\{\liminf_{\lambda\to0^+}\frac{1}{\lambda}\Big(\mathrm{H}\big(\,\cdot\,,u+\lambda A,\mathrm{D}u+\lambda\mathrm{D}A\big)-\mathrm{H}\big(\cdot,u,\mathrm{D}u\big)\Big)\right\}\\
=\; & \max_{\mathcal{O}(u)}\left\{\liminf_{\lambda\to0^+}\frac{1}{\lambda}\Big(\mathrm{H}(\cdot,u,\mathrm{D}u)\;+\;\lambda\,\mathrm{H}_\eta\big(\cdot,u,\mathrm{D}u\big)\cdot A\;+\;\lambda\,\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}A\right.\\
& \left.+\;O\big(|\lambda\mathrm{D}A|^2+|\lambda A|^2\big)\;-\;\mathrm{H}(\cdot,u,\mathrm{D}u)\Big)\right\}
\end{aligned}$$

and the desired inequality has been established. $\square$

Let us record the next simple inequality which follows from the definitions of lower right Dini derivative and convexity:

$$r(\lambda)\;-\;r(0)\;\geq\;\underline{\mathrm{D}}r(0^+)\,\lambda, \tag{2.1}$$

for all $\lambda\geq0$.

3. Main result and Proofs

Now we proceed to the main theme of the paper, the variational characterisation of $\mathcal{D}$-solutions to the PDE system (1.2) in terms of appropriate variations of the energy functional (1.1). We recall that the Borel mapping $\mathcal{F}_\infty:\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s\longrightarrow\mathbb{R}^N$ is given by (1.3)-(1.4) and $\Omega\subseteq\mathbb{R}^n$ is a fixed open set.

Notational simplifications and perpendicularity considerations. We begin by rewriting $\mathcal{F}_\infty(\cdot,u,\mathrm{D}u,\mathrm{D}^2u)\;=\;0$ in a more malleable fashion. We define the maps

$$\mathcal{F}^\perp_\infty(x,\eta,P,\mathbf{X})\;:=\;\mathrm{H}_{PP}(x,\eta,P):\mathbf{X}\;+\;\mathrm{H}_{P\eta}(x,\eta,P):P\;+\;\mathrm{H}_{Px}(x,\eta,P):\mathrm{I}\;, \tag{3.1}$$

$$\mathcal{F}^\parallel_\infty(x,\eta,P,\mathbf{X})\;:=\;\mathrm{H}_{P}(x,\eta,P):\mathbf{X}\;+\;\mathrm{H}_{\eta}(x,\eta,P)^\top P\;+\;H_x(x,\eta,P) \tag{3.2}$$

and these are abbreviations of

$$\begin{aligned}
\mathcal{F}^\perp_\infty(x,\eta,P,\mathbf{X})_\alpha\;=\;& \sum_{\beta,i,j}\mathrm{H}_{P_{\alpha i}P_{\beta j}}(x,\eta,P)\,\mathbf{X}_{\beta ij}\;+\;\sum_{\beta,i}\mathrm{H}_{P_{\alpha i}\eta_\beta}(x,\eta,P)\,P_{\beta i}\\
&+\;\sum_{i}\mathrm{H}_{P_{\alpha i}x_i}(x,\eta,P)\;,\\
\mathcal{F}^\parallel_\infty(x,\eta,P,\mathbf{X})_i\;=\;& \sum_{\beta,j}\mathrm{H}_{P_{\beta j}}(x,\eta,P)\mathbf{X}_{\beta ij}\;+\;\sum_{\beta}\mathrm{H}_{\eta_\beta}(x,\eta,P)\,P_{\beta i}\;+\;H_{x_i}(x,\eta,P).
\end{aligned}$$

Note that $\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X})\in\mathbb{R}^N$, whilst $\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X})\in\mathbb{R}^n$. By utilising (3.1)-(3.2), we can now express (1.3) as

$$\begin{aligned}\mathcal{F}_{\infty}(x,\eta,P,\mathbf{X}) \,:=\, &\mathrm{H}_P(x,\eta,P)\,\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X}) + \mathrm{H}(x,\eta,P)\,\cdot\\ &\cdot\, [\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}\Big(\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X}) \,-\, \mathrm{H}_\eta(x,\eta,P)\Big).\end{aligned}$$

Further, recall that in view of (1.4), $[\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}$ is the projection on the orthogonal complement of $R(\mathrm{H}_P(x,\eta,P))$. Hence, by the orthogonality of $[\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}\cdot$ $\cdot\big(\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X})-\mathrm{H}_\eta(x,\eta,P)\big)$ and $\mathrm{H}_P(x,\eta,P)\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X})$, we have

$$\mathcal{F}_{\infty}(x,\eta,P,\mathbf{X}) \,=\, 0, \quad \text{for some } (x,\eta,P,\mathbf{X})\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s,$$

if and only if

$$\left\{\begin{aligned} \mathrm{H}_P(x,\eta,P)\,\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X}) \,&=\, 0,\\ \mathrm{H}(x,\eta,P)\,[\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}\Big(\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X})-\mathrm{H}_\eta(x,\eta,P)\Big) \,&=\, 0.\end{aligned}\right.$$

Finally, we note that for the sake of clarity we state and prove our characterisation below only in the case of C^1 solutions, but due to its pointwise nature, the result holds true for piecewise C^1 solutions with obvious adaptations which we refrain from providing. Therefore, our main result is as follows:

Theorem 7 (Variational characterisation of the PDE system arising in L^∞)**.** *Let $\Omega\subseteq\mathbb{R}^n$ be open, $u\in C^1(\Omega,\mathbb{R}^N)$ and $\mathrm{H}\in C^2(\Omega\times\mathbb{R}^n\times\mathbb{R}^{Nn})$ a function satisfying*

$$\big\{\mathrm{H}_P(x,\eta,\cdot)=0\big\}\,\subseteq\,\big\{\mathrm{H}(x,\eta,\cdot)=0\big\},\quad (x,\eta)\in\Omega\times\mathbb{R}^N.$$

Then:

(A) *We have*

$$\mathcal{F}_{\infty}(\cdot,u,\mathrm{D}u,\mathrm{D}^2u) \,=\, 0 \quad \text{in } \Omega,$$

in the $\mathcal{D}$-sense, if and only if

$$\mathrm{E}_{\infty}(u,\,\mathcal{O}) \,\leq\, \mathrm{E}_{\infty}(u+A,\,\mathcal{O}),\quad \forall\;\mathcal{O}\Subset\Omega,\;\forall\;A\in\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u)\bigcup\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u).$$

In the above, the sets $\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u),\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$ consist, for any $\mathcal{O}\Subset\Omega$, by affine mappings as follows:

$$\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u) := \left\{A:\mathbb{R}^n\to\mathbb{R}^N \;\middle|\; \begin{array}{l}\mathrm{D}^2A\equiv 0,\; A(x)=0\,\&\text{ exist }\xi\in\mathbb{R}^N,\,x\in\mathcal{O}(u),\\ \mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)\;\&\;\boldsymbol{X}_x\in\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)\\ \text{s. th. }:\;\mathrm{D}A\,\equiv\,\xi\otimes\mathcal{F}^{\parallel}_{\infty}\big(x,u(x),\mathrm{D}u(x),\boldsymbol{X}_x\big)\end{array}\right\}\bigcup\mathbb{R}^N$$

and

$$\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u) := \left\{A:\mathbb{R}^n\to\mathbb{R}^N \;\middle|\; \begin{array}{l}\mathrm{D}^2A\,\equiv\,0\;\&\text{ there exist }x\in\mathcal{O}(u),\,\mathcal{D}^2u\\ \in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)\;\&\;\boldsymbol{X}_x\in\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)\\ \text{s. th. }:\;A(x)\in R\big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)^{\perp}\\ \&\;\mathrm{D}A\in\mathscr{L}\big(x,A(x),\boldsymbol{X}_x\big)\end{array}\right\}\bigcup\mathbb{R}^N$$

where $\mathscr{L}\big(x,\eta,\boldsymbol{X}\big)$ is an affine space of $N\times n$ matrices, defined as

$$\mathscr{L}\big(x,\eta,\boldsymbol{X}\big) := \left\{\begin{array}{ll}\Big\{Q\in\mathbb{R}^{Nn}\;\Big|\;\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big):Q & \\ \qquad =\,-\eta\cdot\mathcal{F}^{\perp}_{\infty}\big(x,u(x),\mathrm{D}u(x),\boldsymbol{X}_x\big)\Big\}, & \text{if } \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\neq 0,\\ \{0\}, & \text{if } \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)=0,\end{array}\right.$$

for any $(x,\eta,\boldsymbol{X}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn^2}_s$.

(B) *In view of the mutual perpendicularity of the two components of* $\mathcal{F}_\infty$ *(see* (3.1)-(3.2)*),* (A) *is a consequence of the following particular results:*

$$\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathcal{F}^{\|}_\infty(\cdot,u,\mathrm{D}u,\mathrm{D}^2u) \,=\, 0 \quad \text{in } \Omega,$$

in the $\mathcal{D}$*-sense, if and only if*

$$\mathrm{E}_\infty(u,\,\mathcal{O}) \,\leq\, \mathrm{E}_\infty(u+A,\,\mathcal{O}), \quad \forall\, \mathcal{O} \Subset \Omega,\ \forall\, A \in \mathcal{A}^{\|,\infty}_{\mathcal{O}}(u)$$

and also

$$\mathrm{H}(\cdot,u,\mathrm{D}u)\,\big[\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big]^\perp\Big(\mathcal{F}^\perp_\infty(\cdot,u,\mathrm{D}u,\mathrm{D}^2u) - \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big) \,=\, 0 \quad \text{in } \Omega,$$

in the $\mathcal{D}$*-sense, if and only if*

$$\mathrm{E}_\infty(u,\,\mathcal{O}) \,\leq\, \mathrm{E}_\infty(u+A,\,\mathcal{O}), \quad \forall\, \mathcal{O} \Subset \Omega,\ \forall\, A \in \mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u).$$

We note that in the special case of C^2 solutions, Corollary 1 describes the way that classical solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to (1.2)-(1.4) are characterised.

Remark 8 (About pointwise properties of C^1 $\mathcal{D}$-solutions)**.** Let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a $\mathcal{D}$-solution to (1.2)-(1.4) in $C^1(\Omega,\mathbb{R}^N)$. By Definition 3, this means that for any $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$,

$$\mathcal{F}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big) \,=\, 0, \quad \text{a.e. } x \in \Omega \text{ and all } \mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2u(x)\big).$$

By Definition 2, every diffuse hessian of a putative solution is defined a.e. on Ω as a weakly* measurable probability valued map $\mathbb{R}^n \supseteq \Omega \longrightarrow \mathscr{P}\big(\mathbb{R}^{Nn^2}_s \cup \{\infty\}\big)$. Let $\Omega \ni x \longmapsto \mathbf{O}_x \in \mathbb{R}^{Nn^2}_s$ be any selection of elements of the zero level sets

$$\Big\{\mathbf{X} \in \mathbb{R}^{Nn^2}_s \ :\ \mathcal{F}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}\big) \,=\, 0\Big\}.$$

By modifying each diffuse hessian on a Lebesgue nullset and choosing the representative which is redefined $\mathcal{D}^2u(x) = \delta_{\mathbf{O}_x}$ for a negligible set of x's, we may assume that $\mathcal{D}^2u(x)$ exists for all $x \in \Omega$. Further, given that $\mathrm{D}u(x)$ exists for all $x \in \Omega$, by perhaps a further re-definition on a Lebesgue nullset, it follows that u is $\mathcal{D}$-solution to (1.2)-(1.4) if and only if for (any such representative of) any diffuse hessian

$$\mathcal{F}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big) \,=\, 0, \quad \text{for } \textit{all } x \in \Omega \text{ and } \mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2u(x)\big).$$

Note that at points $x \in \Omega$ for which $\mathcal{D}^2u(x) = \delta_{\{\infty\}}$ and hence $\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big) = \emptyset$, the solution criterion is understood as being trivially satisfied.

Proof of Theorem 7. It suffices to establish only (B), since (A) is a consequence of it. Suppose that for any $\mathcal{O} \Subset \Omega$ and any $A \in \mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$ we have $E_\infty(u,\mathcal{O}) \leq \mathrm{E}_\infty(u+A,\mathcal{O})$. Fix a diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, a point $x \in \overline{\mathcal{O}}$ such that $\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big) \neq \emptyset$ and an $\mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)$. In view of (3.1), if $\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big) = 0$, then, by our assumption on the level sets of H, we have $\mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big) = 0$ as well and as a consequence we readily obtain

$$\begin{aligned}&\mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big)\,\big[\!\big[\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\big]\!\big]^\perp\cdot\\ &\cdot\Big(\mathcal{F}^\perp_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big) - \mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\Big) = 0\end{aligned} \tag{3.3}$$

is clearly satisfied at x. If $\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\neq 0$, then we select any direction normal to the range of $\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\in\mathbb{R}^{Nn}$, that is

$$n_x \,\in R\big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\big)^{\perp}\subseteq\mathbb{R}^N$$

which means $n_x^\top\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\,=\,0$. Of course it may happen that the linear map $\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big):\mathbb{R}^n\longrightarrow\mathbb{R}^{Nn}$ is surjective and then only the trivial $n_x=0$ exists. In such an event, the equality (3.3) above is satisfied at x because $[\![\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)]\!]^{\perp}=0$. Hence, we may assume $n_x\neq 0$. Further, fix any matrix N_x in the affine space $\mathscr{L}(x,n_x,\mathbf{X}_x)\subseteq\mathbb{R}^{Nn}$. By the definition of $\mathscr{L}(x,n_x,\mathbf{X}_x)$, we have

$$\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big):N_x\;=\;-n_x\cdot\mathcal{F}^{\perp}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big).$$

Consider the affine map defined by

$$A(z)\;:=\;n_x\;+\;N_x(z-x),\quad z\in\mathbb{R}^n.$$

We remark that $tA\in\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$ for any $t\in\mathbb{R}$. Indeed, this is a consequence of our choices and the next homogeneity property of the space $\mathscr{L}(x,\eta,\mathbf{X})$:

$$\mathscr{L}(x,t\eta,\mathbf{X})\;=\;t\,\mathscr{L}(x,\eta,\mathbf{X}),\quad t\in\mathbb{R}.$$

Let $\varepsilon>0$ be small and let also $\mathcal{O}_\varepsilon(x)$ be as in Lemma 5b). We therefore have

$$\mathrm{E}_\infty\big(u,\mathcal{O}_\varepsilon(x)\big)\;\leq\;\mathrm{E}_\infty\big(u+tA,\mathcal{O}_\varepsilon(x)\big).$$

By applying Lemma 5a), we have

$$\begin{aligned}0\;\leq\;&\max_{z\in\overline{\mathcal{O}_\varepsilon(x)}}\Big\{\mathrm{H}_P\big(z,u(z),\mathrm{D}u(z)\big):\mathrm{D}A(z)\;+\;\mathrm{H}_\eta\big(z,u(z),\mathrm{D}u(z)\big)\cdot A(z)\Big\}\\ \xrightarrow{\varepsilon\to 0}\;&\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big):N_x\;+\;\mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\cdot n_x\\ =\;&-n_x\cdot\Big(\mathcal{F}^{\perp}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)-\mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\Big).\end{aligned}$$

As a result, we have

$$n_x\cdot\Big(\mathcal{F}^{\perp}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)-\mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\Big)\leq 0$$

for any direction $n_x\perp R\big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\big)$ and by the arbitrariness of n_x, we deduce that

$$[\![\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)]\!]^{\perp}\Big(\mathcal{F}^{\perp}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\;-\;\mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\Big)\;=\;0,$$

for any $\mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, $x\in\Omega$ and $\mathbf{X}_x\in\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)$, as desired.

For the tangential component of the system we argue similarly. Suppose that for any $\mathcal{O}\Subset\Omega$ and any $A\in\mathcal{A}^{\|,\infty}_{\mathcal{O}}(u)$ we have $\mathrm{E}_\infty(u,\mathcal{O})\leq\mathrm{E}_\infty(u+A,\mathcal{O})$. Fix $x\in\overline{\mathcal{O}}$, a diffuse hessian $\mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$ such that $\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)\neq\emptyset$, a point $\mathbf{X}_x\in\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)$ and $\xi\in\mathbb{R}^N$. Recalling (3.2), we define the affine map

$$A(z)\;:=\;\xi\otimes\mathcal{F}^{\|}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\cdot(z-x),\quad z\in\mathbb{R}^n.$$

Fix $\varepsilon>0$ small and let $\mathcal{O}_\varepsilon(x)$ be as in Lemma 5b). Then, $tA\in\mathcal{A}^{\|,\infty}_{\mathcal{O}_\varepsilon(x)}(u)$ for any $t\in\mathbb{R}$. Thus,

$$\mathrm{E}_\infty\big(u,\mathcal{O}_\varepsilon(x)\big)\;\leq\;\mathrm{E}_\infty\big(u+tA,\mathcal{O}_\varepsilon(x)\big)$$

and by applying Lemma 5a), we have

$$0 \le \max_{z\in\mathcal{O}_\varepsilon(x)} \Big\{ \mathrm{H}_P\big(z,u(z),\mathrm{D}u(z)\big) : \mathrm{D}A(z) + \mathrm{H}_\eta\big(z,u(z),\mathrm{D}u(z)\big)\cdot A(z)\Big\}$$
$$\xrightarrow{\varepsilon\to 0} \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big) : \Big(\xi\otimes \mathcal{F}^{\parallel}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big)$$

and hence

$$\xi\cdot\Big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\,\mathcal{F}^{\parallel}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big) \ge 0,$$

for any $\xi\in\mathbb{R}^N$. By the arbitrariness of ξ we deduce that

$$\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\,\mathcal{F}^{\parallel}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big) = 0$$

for any $\mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, $x\in\Omega$ and $\mathbf{X}_x\in\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)$, as desired.

Conversely, we fix $\mathcal{O}\Subset\Omega$, $x\in\mathcal{O}(u)$, $\mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, $\mathbf{X}_x\in\mathrm{supp}_*(\mathcal{D}^2u(x))$ and $\xi\in\mathbb{R}^N$ corresponding to an $A\in\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u)$. Let r be the function of Lemma 6. By applying Lemma 6 to the above setting, we have

$$\begin{aligned}\underline{\mathrm{D}}r(0^+) &\ge \max_{y\in\mathcal{O}(u)}\Big\{\mathrm{H}_P(y,u(y),\mathrm{D}u(y)) : \mathrm{D}A(y) + \mathrm{H}_\eta(y,u(y),\mathrm{D}u(y))\cdot A(y)\Big\}\\ &\ge \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big) : \mathrm{D}A(x) + \mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\cdot A(x)\\ &= \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big) : \Big(\xi\otimes\mathcal{F}^{\parallel}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big)\\ &= \xi\cdot\Big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\mathcal{F}^{\parallel}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big)\end{aligned}$$

and hence $\underline{\mathrm{D}}r(0^+)\ge 0$ because u is a $\mathcal{D}$-solution. Due to the fact that $r(0)=0$ and r is convex, by inequality (2.1) we have $r(t)\ge 0$ for all $t\ge 0$. Therefore,

$$\mathrm{E}_\infty(u,\mathcal{O}) \le \mathrm{E}_\infty(u+A,\mathcal{O}),\quad \forall\,\mathcal{O}\Subset\Omega,\ \forall\,A\in\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u).$$

The case of $A\in\mathcal{A}^{\perp,\infty}_{\mathcal{O}}$ is completely analogous. Fix $\mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, $\mathcal{O}\Subset\Omega$, $x\in\mathcal{O}(u)$, $\mathbf{X}_x\in\mathrm{supp}_*(\mathcal{D}^2u(x))$ and an A with $A(x)\perp R(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)$ and $\mathrm{D}A\in\mathscr{L}\big(x,A(x),\mathbf{X}_x\big)$. By applying Lemma 6 again, we have

$$\begin{aligned}\underline{\mathrm{D}}r(0^+) &\ge \max_{y\in\mathcal{O}(u)}\Big\{\mathrm{H}_P(y,u(y),\mathrm{D}u(y)) : \mathrm{D}A(y) + \mathrm{H}_\eta(y,u(y),\mathrm{D}u(y))\cdot A(y)\Big\}\\ &\ge \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big) : \mathrm{D}A(x) + \mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\cdot A(x).\end{aligned}$$

If $\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\neq 0$, then by the definition of $\mathscr{L}\big(x,A(x),\mathbf{X}_x\big)$ we have

$$\begin{aligned}\underline{\mathrm{D}}r(0^+) &\ge \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big) : \mathrm{D}A(x) + \mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\cdot A(x)\\ &= -A(x)\cdot\Big(\mathcal{F}^{\perp}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big) - \mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\Big)\\ &= -A(x)^\top[\![\mathrm{H}_P(x,u(x),\mathrm{D}u(x))]\!]^\perp\Big(\mathcal{F}^{\perp}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\\ &\qquad - \mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\Big)\end{aligned}$$

and hence $\underline{\mathrm{D}}r(0^+)\ge 0$ because u is a $\mathcal{D}$-solution on Ω. If $\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)=0$, then again $\underline{\mathrm{D}}r(0^+)\ge 0$ because $A(x)=0$. In either cases, by inequality (2.1) we obtain $r(t)\ge 0$ for all $t\ge 0$ and hence

$$\mathrm{E}_\infty(u,\mathcal{O}) \le \mathrm{E}_\infty(u+A,\mathcal{O}),\quad \forall\,\mathcal{O}\Subset\Omega,\ \forall\,A\in\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u).$$

The theorem has been established. $\square$

Proof of Corollary 1. If $u \in C^2(\Omega, \mathbb{R}^N)$, then by Lemma 4 any diffuse hessian of u satisfies $\mathcal{D}^2 u(x) = \delta_{\mathrm{D}^2 u(x)}$ for a.e. $x \in \Omega$. By Remark 8, we may assume this happens for all $x \in \Omega$. Therefore, the reduced support of $\mathcal{D}^2 u(x)$ is the singleton set $\{\delta_{\mathrm{D}^2 u(x)}\}$. Hence, for $\mathcal{A}^{\|,\infty}_{\mathcal{O}}(u)$, we have that any possible affine map A satisfies $\mathrm{D}A \equiv \mathrm{D}\big(\xi \mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big)\big)$ and $A(x) = 0$. In the case of $\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$, we have that any possible affine map A satisfies

$$A(x)^\top \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \,=\, 0\ , \quad \mathrm{D}A \in \mathscr{L}\big(x, A(x), \mathrm{D}^2 u(x)\big),$$

which gives

$$\begin{aligned} \mathrm{D}A(x) : \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \,&=\, -A(x) \cdot \Big(\mathrm{H}_{PP}\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{D}^2 u(x) \,+ \\ &\quad +\, \mathrm{H}_{P\eta}\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{D}u(x) \,+\, \mathrm{H}_{Px}\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{I}\Big) \\ &=\, -A(x) \cdot \mathrm{Div}\big(\mathrm{H}_P\big(\cdot, u, \mathrm{D}u\big)\big)(x). \end{aligned}$$

As a consequence, the divergence $\mathrm{Div}\big(A^\top \mathrm{H}_P\big(\cdot, u, \mathrm{D}u\big)\big)(x)$ vanishes because

$$\mathrm{D}A(x) : \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \,+\, A(x) \cdot \mathrm{Div}\big(\mathrm{H}_P\big(\cdot, u, \mathrm{D}u\big)\big)(x) \,=\, 0.$$

The corollary has been established. $\square$

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