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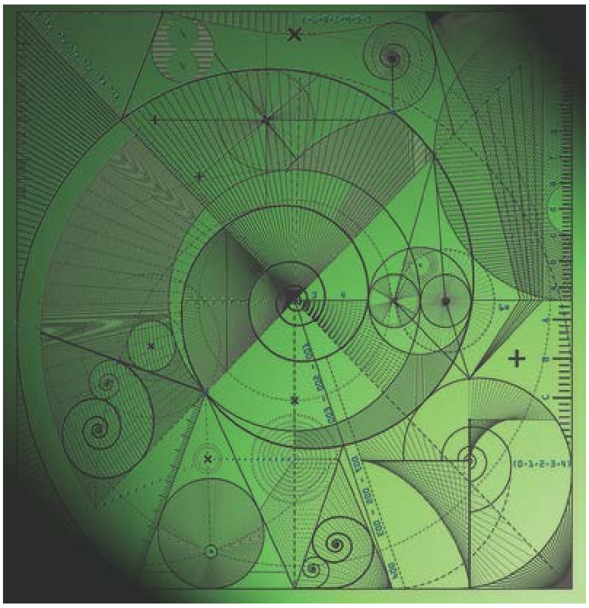

EXISTENCE OF $1D$ VECTORIAL ABSOLUTE MINIMISERS IN L^∞ UNDER MINIMAL ASSUMPTIONS

HUSSIEN ABUGIRDA AND NIKOS KATZOURAKIS

Abstract. We prove the existence of vectorial Absolute Minimisers in the sense of Aronsson to the supremal functional $E_\infty(u,\Omega') = \|\mathscr{L}(\cdot,u,\mathrm{D}u)\|_{L^\infty(\Omega')}$, $\Omega' \Subset \Omega$, applied to $W^{1,\infty}$ maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ with given boundary values. The assumptions on $\mathscr{L}$ are minimal, improving earlier existence results previously established by Barron-Jensen-Wang and by the second author.

1. Introduction

The main goal of this paper is to prove the existence of vectorial Absolute Minimisers with given boundary values to the supremal functional

$$E_\infty(u,\Omega') := \operatorname*{ess\,sup}_{x\in\Omega'} \mathscr{L}\left(x,u(x),\mathrm{D}u(x)\right), \qquad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N),\ \Omega' \Subset \Omega, \tag{1.1}$$

applied to maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$, $N \in \mathbb{N}$, where Ω is an open interval and $\mathscr{L} \in C(\Omega \times \mathbb{R}^N \times \mathbb{R}^N)$ is a non-negative function which we call Lagrangian. By Absolute Minimiser we mean a map $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ such that

$$E_\infty(u,\Omega') \;\leq\; E_\infty(u+\phi,\Omega'), \tag{1.2}$$

for all $\Omega' \Subset \Omega$ and all $\phi \in W^{1,\infty}_0(\Omega',\mathbb{R}^N)$. This is the appropriate minimality notion for supremal functionals of the form (1.1); requiring at the outset minimality on all subdomains is necessary because of the lack of additivity in the domain argument. The study of (1.1) was pioneered by Aronsson in the 1960s [A1]-[A5] who considered the case $N = 1$. Since then, the (higher dimensional) scalar case of $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ has developed massively and there is a vast literature on the topic (see for instance the lecture notes [C, K7]). Of particular interest has been the study of the (single) equation associated to (1.1) which is the equivalent of the Euler-Lagrange equation for supremal functionals and is known as the "Aronsson equation":

$$A_\infty u \;:=\; \mathrm{D}\big(\mathscr{L}(\cdot,u,\mathrm{D}u)\big)\mathscr{L}_P(\cdot,u,\mathrm{D}u) \;=\; 0. \tag{1.3}$$

Herein we are interested in the vectorial case $N \geq 2$ but in one spatial dimension. Unlike the scalar case, the literature for $N \geq 2$ is much more sparse and starts much more recently. Perhaps the first most important contributions were by Barron-Jensen-Wang [BJW1, BJW2] who among other deep results proved the existence of Absolute Minimisers for (1.1) under certain assumptions on $\mathscr{L}$ which we recall later. However, their contributions were at the level of the functional and the appropriate (non-obvious) vectorial analogue of the Aronsson equation was not known at the time. The systematic study of the vectorial case of (1.1) (actually in the general

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case of maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$) together with its associated system of equations begun in the early 2010s by the second author in a series of papers, see [K1]-[K6], [K8]-[K12], [CKP, KP]. The ODE system associated to (1.1) turns out to be

$$\mathcal{F}_\infty\big(\cdot, u, \mathrm{D}u, \mathrm{D}^2u\big) \,=\, 0, \quad \text{on } \Omega, \tag{1.4}$$

where

$$\begin{aligned}
\mathcal{F}_\infty(x,\eta,P,X) \,:=\, &\Big[\mathscr{L}_P(x,\eta,P) \otimes \mathscr{L}_P(x,\eta,P) \\
&+\, \mathscr{L}(x,\eta,P)[\mathscr{L}_P(x,\eta,P)]^\perp \mathscr{L}_{PP}(x,\eta,P)\Big] X \\
&+\, \Big(\mathscr{L}_\eta(x,\eta,P)\cdot P \,+\, \mathscr{L}_x(x,\eta,P)\Big)\mathscr{L}_P(x,\eta,P) \\
&+\, \mathscr{L}(x,\eta,P)\big[\mathscr{L}_P(x,\eta,P)\big]^\perp\Big(\mathscr{L}_{P\eta}(x,\eta,P)P \\
&+\, \mathscr{L}_{Px}(x,\eta,P) \,-\, \mathscr{L}_\eta(x,\eta,P)\Big).
\end{aligned} \tag{1.5}$$

In (1.5) the notation of subscripts denotes derivatives with respect to the respective variables and $\big[\mathscr{L}_P(x,\eta,P)\big]^\perp$ is the orthogonal projection

$$\big[\mathscr{L}_P(x,\eta,P)\big]^\perp \,:=\, I - \,\mathrm{sgn}\big(\mathscr{L}_P(x,\eta,P)\big) \otimes \mathrm{sgn}\big(\mathscr{L}_P(x,\eta,P)\big). \tag{1.6}$$

The system (1.4) reduces to the equation (1.3) when $N = 1$. In the paper [K9] the existence of an absolutely minimising *generalised solution* to (1.4) was proved, together with extra partial regularity and approximation properties. Since (1.4) is a quasilinear non-divergence degenerate system with discontinuous coefficients, a notion of appropriately defined "weak solution" is necessary because in general solutions are non-smooth. To this end, the general new approach of $\mathcal{D}$-solutions which has recently been proposed in [K8] has proven to be the appropriate setting for vectorial Calculus of Variations in L^∞ (see [K8]-[K10]), replacing to some extent viscosity solutions which essentially apply only in the scalar case.

Herein we are concerned with the existence of absolute minimisers to (1.1) without drawing any connections to the differential system (1.4). Instead, we are interested in obtaining existence under the weakest possible assumptions. Accordingly, we establish the following result.

Theorem 1. *Let $\Omega \subseteq \mathbb{R}$ be a bounded open interval and let also*

$$\mathscr{L} \;:\; \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow [0,\infty),$$

be given continuous function with $N \in \mathbb{N}$. We assume that:

1. *For each $(x,\eta) \in \overline{\Omega} \times \mathbb{R}^N$, the function $P \longmapsto \mathscr{L}(x,\eta,P)$ is level-convex, that is for each $t \geq 0$ the sublevel set*
$$\big\{P \in \mathbb{R}^N \,:\, \mathscr{L}(x,\eta,P) \leq t\big\}$$
is a convex set in $\mathbb{R}^N$.
2. *there exist non-negative constants C_1, C_2, C_3, and $0 < q \leq r < +\infty$ and a positive locally bounded function $h : \mathbb{R} \times \mathbb{R}^N \longrightarrow [0,+\infty)$ such that for all $(x,\eta,P) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N$*
$$C_1|P|^q \,-\, C_2 \,\leq\, \mathscr{L}(x,\eta,P) \,\leq\, h(x,\eta)|P|^r \,+\, C_3.$$

Then, for any affine map $b : \mathbb{R} \longrightarrow \mathbb{R}^N$, there exist a vectorial Absolute Minimiser $u^\infty \in W^{1,\infty}_b(\Omega,\mathbb{R}^N)$ of the supremal functional (1.1) (Definition (1.2)).

Theorem 1 generalises two respective results in the both the papers [BJW1] and [K9]. On the one hand, *in* [BJW1] *Theorem 1 was established under the extra assumption* $C_2 = C_3 = 0$ *which forces* $\mathscr{L}(x,\eta,0) = 0$, *for all* $(x,\eta) \in \mathbb{R} \times \mathbb{R}^N$. Unfortunately this requirement is incompatible with important applications of (1.1) to problems of L^∞-modelling of variational Data Assimilation (*4DVar*) arising in the Earth Sciences and especially in Meteorology (see [B, BS, K9]). An explicit model of $\mathscr{L}$ is given by

$$\mathscr{L}(x,\eta,P) := \big|k(x) - K(\eta)\big|^2 + \big|P - \mathscr{V}(x,\eta)\big|^2, \tag{1.7}$$

and describes the "error" in the following sense: consider the problem of finding the solution u to the following ODE coupled by a pointwise constraint:

$$\mathrm{D}u(t) = \mathscr{V}\big(t, u(t)\big) \quad \& \quad K(u(t)) = k(t), \quad t \in \Omega.$$

Here $\mathscr{V} : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a time-dependent vector field describing the law of motion of a body moving along the orbit described by $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ (e.g. Newtonian forces, Galerkin approximation of the Euler equations, etc), $k : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^M$ is some partial "measurements" in continuous time along the orbit and $K : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is a submersion which corresponds to some component of the orbit that is observed. We interpret the problem as that u should satisfy the law of motion and also be compatible with the measurements along the orbit. Then minimisation of (1.1) with $\mathscr{L}$ as given by (1.7) leads to a uniformly optimal approximate solution without "spikes" of large deviation of the prediction from the actual orbit.

On the other hand, in the paper [K9] Theorem 1 was proved under assumptions allowing to model Data Assimilation but strong convexity, smoothness and structural assumptions were imposed, allowing to obtain stronger results accordingly. In particular, the Lagrangian was assumed to be radial in P, which means it can be written in the form

$$\mathscr{L}(x,\eta,P) := \mathscr{H}\Big(x,\eta,\frac{1}{2}\big|P - \mathscr{V}(x,\eta)\big|^2\Big).$$

In this paper we relax the hypotheses of both the aforementioned results.

2. Proof of the main result

In this section we establish Theorem 1. In the proof we will utilise the following lemma essentially proved in [BJW1] which we recall right below for the convenience of the reader.

Lemma 2. $\big($*cf.* [BJW1]$\big)$ *In the setting of theorem 1 and under the same hypotheses, for a fixed affine map* $b : \mathbb{R} \longrightarrow \mathbb{R}^N$, *set*

$$C_m := \inf\Big\{E_m(u,\Omega)^{\frac{1}{m}} \,:\, u \in W^{1,qm}_b(\Omega,\mathbb{R}^N)\Big\},$$
$$C_\infty := \inf\Big\{E_\infty(u,\Omega) \,:\, u \in W^{1,\infty}_b(\Omega,\mathbb{R}^N)\Big\}.$$

where E_∞ *is as in* (1.1) *and*

$$E_m(u,\Omega) := \int_\Omega \mathscr{L}\big(x,u(x),\mathrm{D}u(x)\big)^m\,dx. \tag{2.1}$$

Then, there exist $u^\infty \in W^{1,\infty}_b(\Omega,\mathbb{R}^N)$ *which is a (mere) minimiser of* (1.1) *over* $W^{1,\infty}_b(\Omega,\mathbb{R}^N)$ *and a sequence of approximate minimisers* $\{u^m\}_{m=1}^\infty$ *of* (2.1) *in the* *spaces* $W^{1,qm}_b(\Omega,\mathbb{R}^N)$ *such that, for any* $s\geq 1$, $u^m \longrightarrow u^\infty$ *weakly as* $m\to\infty$ *in* $W^{1,s}(\Omega,\mathbb{R}^N)$ *along a subsequence. Moreover,*

$$E_\infty(u^\infty,\Omega)\,=\,C_\infty\,=\,\lim_{m\to\infty}C_m. \tag{2.2}$$

By approximate minimiser we mean that u^m *satisfies*

$$\big|E_m(u^m,\Omega)-C_m\big|\,<\,2^{-m^2}. \tag{2.3}$$

Finally, for any $A\subseteq\Omega$ *measurable of positive measure the following lower semicontinuity inequality holds*

$$E_\infty(u^\infty,A)\,\leq\,\liminf_{m\to\infty}E_m(u^m,A)^{\frac{1}{m}}. \tag{2.4}$$

Proof of Theorem 1. Our goal now is to prove that the candidate u^∞ of Lemma 2 above is actually an Absolute Minimiser of (1.1), which means we need to prove u^∞ satisfies (1.2).

The method we utilise follows similar lines to those of [K9]. Let us fix $\Omega'\Subset\Omega$. Since Ω' is a countable disjoint union of open intervals, there is no loss of generality in assuming Ω' is itself an open interval. By a simple rescaling argument, it suffices to assume that $\Omega'=(0,1)\subseteq\mathbb{R}$. For, let $\phi\in W^{1,\infty}_0((0,1),\mathbb{R}^N)$ be an arbitrary variation and set $\psi^\infty:=u^\infty+\phi$. In order to conclude, it suffices to establish

$$E_\infty\big(u^\infty,(0,1)\big)\,\leq\,E_\infty\big(\psi^\infty,(0,1)\big).$$

Obviously, $u^\infty(0)=\psi^\infty(0)$ and $u^\infty(1)=\psi^\infty(1)$. We define the energy comparison function $\psi^{m,\delta}$, for any fixed $0<\delta<1/3$ as

$$\psi^{m,\delta}(x)\,:=\,\left\{\begin{array}{ll}\left(\dfrac{\delta-x}{\delta}\right)u^m(0)+\left(\dfrac{x}{\delta}\right)\psi^\infty(\delta), & x\in(0,\delta],\\ \psi^\infty(x), & x\in(\delta,1-\delta),\\ \left(\dfrac{1-x}{\delta}\right)\psi^\infty(1-\delta)+\left(\dfrac{x-(1-\delta)}{\delta}\right)u^m(1), & x\in[1-\delta,1),\end{array}\right.$$

where $m\in\mathbb{N}\cup\{\infty\}$. Then, $\psi^{m,\delta}-u^m\in W^{1,\infty}_0\big((0,1),\mathbb{R}^N\big)$ and

$$\mathrm{D}\psi^{m,\delta}(x)\,=\,\left\{\begin{array}{ll}\dfrac{\psi^\infty(\delta)-u^m(0)}{\delta}, & \text{on }(0,\delta),\\ \mathrm{D}\psi^\infty, & \text{on }(\delta,1-\delta),\\ \dfrac{\psi^\infty(1-\delta)-u^m(1)}{-\delta}, & \text{on }(1-\delta,1).\end{array}\right.$$

Now, note that

$$\psi^{m,\delta}\longrightarrow\psi^{\infty,\delta}\text{ in }W^{1,\infty}\big((0,1),\mathbb{R}^N\big),\text{ as }m\to\infty, \tag{2.5}$$

because $\psi^{m,\delta}\longrightarrow\psi^{\infty,\delta}$ in $L^\infty\big((0,1),\mathbb{R}^N\big)$ and for a.e. $x\in(0,1)$ we have

$$\begin{aligned}\Big|\mathrm{D}\psi^{m,\delta}(x)-\mathrm{D}\psi^{\infty,\delta}(x)\Big|\,&=\,\chi_{(0,\delta)}\frac{|u^\infty(0)-u^m(0)|}{\delta}\,+\,\chi_{(1-\delta,1)}\frac{|u^\infty(1)-u^m(1)|}{\delta}\\ &\leq\,\left(\frac{1}{\delta}+\frac{1}{\delta}\right)\|u^m-u^\infty\|_{L^\infty(\Omega)}\\ &=\,o(1),\end{aligned}$$

as $m \to \infty$ along a subsequence. Now, recalling that $\psi^{m,\delta} = u^m$ at the endpoints $\{0,1\}$, and since u^m is an approximate minimiser of (2.1) over $W^{1,m}_b(\Omega,\mathbb{R}^N)$ for each $m \in \mathbb{N}$, by utilising minimality, the additivity of the integral and Hölder inequality, we get

$$E_m\big(u^m,(0,1)\big) \,\le\, E_m\big(\psi^{m,\delta},(0,1)\big) \,+\, 2^{-m^2}$$

and hence

$$\begin{aligned} E_m\big(u^m,(0,1)\big)^{\frac{1}{m}} \,&\le\, E_m\big(\psi^{m,\delta},(0,1)\big)^{\frac{1}{m}} \,+\, 2^{-m} \\ &\le\, E_\infty\big(\psi^{m,\delta},(0,1)\big) \,+\, 2^{-m}. \end{aligned} \tag{2.6}$$

On the other hand, we have

$$\begin{aligned} E_\infty\big(\psi^{m,\delta},(0,1)\big) \,=\, \max\Big\{ & E_\infty\big(\psi^{m,\delta},(0,\delta)\big), \\ & E_\infty\big(\psi^{m,\delta},(\delta,1-\delta)\big), \\ & E_\infty\big(\psi^{m,\delta},(1-\delta,1)\big)\Big\} \end{aligned}$$

and since $\psi^{m,\delta} = \psi^\infty$ on $(\delta,1-\delta)$, we have

$$\begin{aligned} E_\infty\big(\psi^{m,\delta},(0,1)\big) \,\le \max\Bigg\{ & E_\infty\big(\psi^{m,\delta},(0,\delta)\big),\, E_\infty\big(\psi^{\infty},(0,1)\big), \\ & E_\infty\big(\psi^{m,\delta},(1-\delta,1)\big)\Bigg\}. \end{aligned} \tag{2.7}$$

Combining (2.5)-(2.7) and (2.4), we get

$$\begin{aligned} E_\infty\big(u^\infty,(0,1)\big) \,&\le\, \liminf_{m\to\infty}\Big(\max\Big\{E_\infty\big(\psi^{m,\delta},(0,\delta)\big),\, E_\infty\big(\psi^{\infty},(0,1)\big), \\ &\qquad\qquad E_\infty\big(\psi^{m,\delta},(1-\delta,1)\big)\Big\}\Big) \\ &\le\, \max\Big\{E_\infty\big(\psi^{\infty},(0,1)\big),\, E_\infty\big(\psi^{\infty,\delta},(0,\delta)\big), \\ &\qquad E_\infty\big(\psi^{\infty,\delta},(1-\delta,1)\big)\Big\}. \end{aligned} \tag{2.8}$$

Let us now denote the difference quotient of a function $v:\mathbb{R}\longrightarrow\mathbb{R}^N$ as $\mathrm{D}^{1,t}v(x) := \frac{1}{t}[v(x+t)-v(x)]$. Then, we may write

$$\begin{aligned} \mathrm{D}\psi^{\infty,\delta}(x) \,&=\, \mathrm{D}^{1,\delta}\psi^\infty(0), \qquad x\in(0,\delta), \\ \mathrm{D}\psi^{\infty,\delta}(x) \,&=\, \mathrm{D}^{1,-\delta}\psi^\infty(1), \quad\ x\in(1-\delta,1), \end{aligned}$$

Note now that

$$\left\{ \begin{aligned} E_\infty\big(\psi^{\infty,\delta},(0,\delta)\big) \,&=\, \max_{0\le x\le\delta} \mathscr{L}\Big(x,\psi^{\infty,\delta}(x),\mathrm{D}^{1,\delta}\psi^\infty(0)\Big), \\ E_\infty\big(\psi^{\infty,\delta},(1-\delta,1)\big) \,&=\, \max_{1-\delta\le x\le 1} \mathscr{L}\Big(x,\psi^{\infty,\delta}(x),\mathrm{D}^{1,-\delta}\psi^\infty(1)\Big). \end{aligned} \right. \tag{2.9}$$

In view of (2.8)-(2.9), it is suffices to prove that there exist an infinitesimal sequence $(\delta_i)_{i=1}^\infty$ such that

$$\begin{aligned} E_\infty\big(\psi^\infty,(0,1)\big) \,\ge\, \max\Bigg\{ & \limsup_{i\to\infty}\max_{[0,\delta_i]}\ \mathscr{L}\Big(\cdot,\psi^{\infty,\delta_i},\mathrm{D}^{1,\delta_i}\psi^\infty(0)\Big), \\ & \limsup_{i\to\infty}\max_{[1-\delta_i,1]}\ \mathscr{L}\Big(\cdot,\psi^{\infty,\delta_i},\mathrm{D}^{1,-\delta_i}\psi^\infty(1)\Big)\Bigg\}. \end{aligned} \tag{2.10}$$

The rest of the proof is devoted to establishing (2.10). Let us begin by recording for later use that

$$\left\{ \begin{array}{ll} \max\limits_{0\leq x\leq\delta}\Big|\psi^{\infty,\delta}(x)-\psi^{\infty}(0)\Big| \longrightarrow 0, & \text{as } \delta\to 0, \\[2ex] \max\limits_{1-\delta\leq x\leq 1}\Big|\psi^{\infty,\delta}(x)-\psi^{\infty}(1)\Big| \longrightarrow 0, & \text{as } \delta\to 0. \end{array} \right. \tag{2.11}$$

Fix a generic $u\in W^{1,\infty}(\Omega,\mathbb{R}^N)$, $x\in[0,1]$ and $0<\varepsilon<1/3$ and define

$$A_\varepsilon(x) \,:=\, [x-\varepsilon,x+\varepsilon]\cap[0,1].$$

We claim that there exist an increasing modulus of continuity $\omega\in C(0,\infty)$ with $\omega(0^+)=0$ such that

$$E_\infty\big(u,A_\varepsilon(x)\big) \,\geq\, \underset{y\in A_\varepsilon(x)}{\mathrm{ess\,sup}}\,\mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big) \,-\, \omega(\varepsilon). \tag{2.12}$$

Indeed for a.e. $y\in A_\varepsilon(x)$ we have $|x-y|\leq\varepsilon$ and by the continuity of $\mathscr{L}$ and the essential boundedness of the derivative Du, there exist ω such that

$$\Big|\mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big)-\mathscr{L}\Big(y,u(y),\mathrm{D}u(y)\Big)\Big| \,\leq\, \omega(\varepsilon)$$

for a.e. $y\in A_\varepsilon(x)$, leading directly to (2.12). Now, we show that

$$\sup_{A_\varepsilon(x)}\left\{\limsup_{t\to 0}\,\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big)\right\} \,\leq\, \underset{A_\varepsilon(x)}{\mathrm{ess\,sup}}\,\mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big). \tag{2.13}$$

Indeed, for any Lipschitz function u, we have

$$\mathrm{D}^{1,t}u(y) \,=\, \frac{u(y+t)-u(y)}{t} \,=\, \int_0^1 Du(y+\lambda t)\,d\lambda, \tag{2.14}$$

when $y,y+t\in A_\varepsilon(x)$, $t\neq 0$. Further, for any $x\in\Omega$ the function $\mathscr{L}(x,u(x),\cdot)$ is level-convex and the Lebesgue measure on $[0,1]$ is a probability measure, thus Jensen's inequality for level-convex functions (see e.g. [BJW1, BJW2]) yields

$$\begin{aligned}\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big) \,&=\, \mathscr{L}\left(x,u(x),\int_0^1 Du(y+\lambda t)\,d\lambda\right) \\ &\leq\, \underset{0\leq\lambda\leq 1}{\mathrm{ess\,sup}}\,\mathscr{L}\Big(x,u(x),\mathrm{D}u(y+\lambda t)\Big),\end{aligned}$$

when $y\in A_\varepsilon(x)$, $0<x<1$. Consequently,

$$\begin{aligned}&\sup_{A_\varepsilon(x)}\left\{\limsup_{t\to 0}\,\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big)\right\} \\ &\qquad\leq\, \sup_{A_\varepsilon(x)}\left\{\limsup_{t\to 0}\left[\underset{0\leq\lambda\leq 1}{\mathrm{ess\,sup}}\,\mathscr{L}\Big(x,u(x),\mathrm{D}u(y+\lambda t)\Big)\right]\right\} \\ &\qquad\leq\, \sup_{A_\varepsilon(x)}\left\{\lim_{s\to 0}\left[\underset{y-s\leq z\leq y+s}{\mathrm{ess\,sup}}\,\mathscr{L}\Big(x,u(x),\mathrm{D}u(z)\Big)\right]\right\} \\ &\qquad=\, \underset{A_\varepsilon(x)}{\mathrm{ess\,sup}}\,\mathscr{L}\Big(x,u(x),\mathrm{D}u(y)\Big),\end{aligned}$$

as desired. Above we have used the following known property of the L^∞ norm (see e.g. [C])

$$\|f\|_{L^\infty(\Omega)} \,=\, \sup_{x\in\Omega}\left(\lim_{\varepsilon\to 0}\left\{\underset{(x-\varepsilon,x+\varepsilon)}{\mathrm{ess\,sup}}\,|f|\right\}\right).$$

Note now that by (2.12) we have

$$E_\infty\big(u,(0,1)\big) \,\geq\, E_\infty\big(u,A_\varepsilon(x)\big) \,\geq\, \underset{A_\varepsilon(x)}{\mathrm{ess\,sup}}\, \mathscr{L}\left(x,u(x),\mathrm{D}u(y)\right) \,-\, \omega(\varepsilon)$$

which combined with (2.13) leads to

$$\begin{aligned} E_\infty\big(u,(0,1)\big) \,&\geq\, \sup_{A_\varepsilon(x)} \left(\limsup_{t\to 0}\, \mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(y)\Big)\right) \,-\, \omega(\varepsilon) \\ &\geq\, \limsup_{t\to 0}\, \left(\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(x)\Big)\right) \,-\, \omega(\varepsilon). \end{aligned}$$

By passing to the limit as $\varepsilon\to 0$ we get

$$E_\infty\big(u,(0,1)\big) \,\geq\, \limsup_{t\to 0}\, \left(\mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t}u(x)\Big)\right), \tag{2.15}$$

for any fixed $u\in W^{1,\infty}(\Omega,\mathbb{R}^N)$ and $x\in[0,1]$. Now, since

$$\Big|\mathrm{D}^{1,t}u(x)\Big| \,\leq\, \|\mathrm{D}u\|_{L^\infty(\Omega)}, \qquad x\in(0,1),\ t\neq 0,$$

for any *finite* set of points $x\in(0,1)$, there is a common infinitesimal sequence $(t_i(x))_{i=1}^\infty$ such that

$$\text{the limit vectors } \lim_{i\to\infty}\mathrm{D}^{1,t_i(x)}u(x) \text{ exists in } \mathbb{R}^N. \tag{2.16}$$

Utilising the continuity of $\mathscr{L}$ together with (2.15)-(2.16) we obtain

$$\begin{aligned} E_\infty\big(u,(0,1)\big) \,&\geq\, \limsup_{i\to\infty}\, \mathscr{L}\Big(x,u(x),\mathrm{D}^{1,t_i(x)}u(x)\Big) \\ &=\, \mathscr{L}\left(x,u(x),\lim_{i\to\infty}\mathrm{D}^{1,t_i(x)}u(x)\right). \end{aligned} \tag{2.17}$$

Now we apply (2.17) to $u=\psi^\infty$ and $x\in\{0,1\}$ to deduce the existence of a sequence $(\delta_i)_{i=1}^\infty$ along which

$$\text{the limit vectors } \lim_{i\to\infty}\mathrm{D}^{1,\delta_i}\psi^\infty(0),\ \lim_{i\to\infty}\mathrm{D}^{1,-\delta_i}\psi^\infty(1) \quad \text{exist in } \mathbb{R}^N \tag{2.18}$$

and also

$$\begin{aligned} E_\infty\big(\psi^\infty,(0,1)\big) \,\geq\, \max\bigg\{&\mathscr{L}\left(0,\psi^\infty(0),\lim_{i\to\infty}\mathrm{D}^{1,\delta_i}\psi^\infty(0)\right), \\ &\mathscr{L}\left(1,\psi^\infty(1),\lim_{i\to\infty}\mathrm{D}^{1,-\delta_i}\psi^\infty(1)\right)\bigg\}. \end{aligned} \tag{2.19}$$

By recalling (2.9), (2.11) and (2.18), for $\delta=\delta_i$ we obtain

$$\begin{aligned} \lim_{i\to\infty} E_\infty\big(\psi^{\infty,\delta_i},(0,\delta_i)\big) \,&=\, \lim_{i\to\infty}\max_{[0,\delta_i]}\mathscr{L}\Big(\cdot,\psi^{\infty,\delta_i},\mathrm{D}^{1,\delta_i}\psi^\infty(0)\Big) \\ &=\, \mathscr{L}\left(0,\psi^\infty(0),\lim_{i\to\infty}\mathrm{D}^{1,\delta_i}\psi^\infty(0)\right), \end{aligned} \tag{2.20}$$

and also

$$\begin{aligned} \lim_{i\to\infty} E_\infty\big(\psi^{\infty,\delta_i},(1-\delta_i,1)\big) \,&=\, \lim_{i\to\infty}\max_{[1-\delta_i,1]}\mathscr{L}\Big(\cdot,\psi^{\infty,\delta_i},\mathrm{D}^{1,-\delta_i}\psi^\infty(1)\Big) \\ &=\, \mathscr{L}\left(1,\psi^\infty(1),\lim_{i\to\infty}\mathrm{D}^{1,-\delta_i}\psi^\infty(1)\right). \end{aligned} \tag{2.21}$$

By putting together (2.19)-(2.21), (2.10) ensues and we conclude the proof. □

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