School of Mathematical, Physical
and Computational Sciences

University of Reading

Department of Mathematics and Statistics

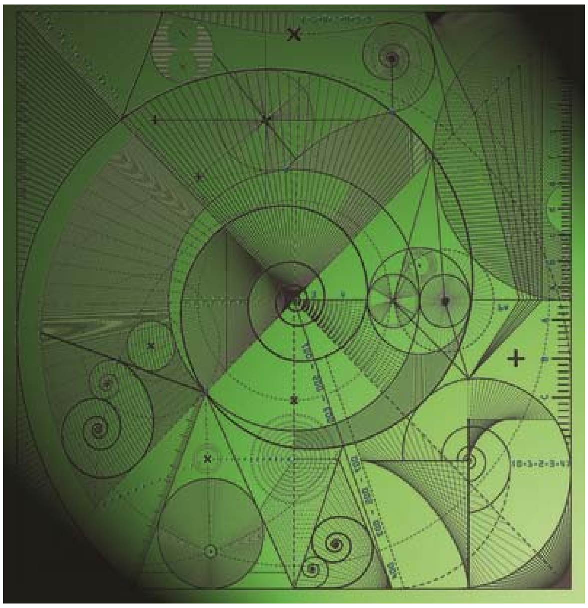
Preprint MPCS-2018-02

16 April 2018

Phase separation of n-dimensional ∞-harmonic mappings

by

Hussien Abugirda



PHASE SEPARATION OF n-DIMENSIONAL ∞-HARMONIC MAPPINGS

HUSSIEN ABUGIRDA

ABSTRACT. Among other interesting results, in a recent paper [15], Katzourakis analysed the phenomenon of separation of the solutions $u:\mathbb{R}^2 \supseteq \Omega \longrightarrow \mathbb{R}^N$, to the ∞-Laplace system

$$\Delta_\infty u \,:=\, \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes I\Big) : \mathrm{D}^2 u \,=\, 0,$$

to phases with qualitatively different behavior in the case of $n = 2 \leq N$. The solutions of the ∞-Laplace system are called the ∞-Harmonic mappings. In this paper we discuss an extension of Katzourakis' result mentioned above to higher dimensions by studying the phase separation of n-dimensional ∞-Harmonic mappings in the case $N \geq n \geq 2$.

1. Introduction

In this paper we study the phase separation of n-dimensional ∞-Harmonic mappings $u:\mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ by which we mean the classical solutions to the ∞-Laplace system

$$\Delta_\infty u \,:=\, \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes I\Big) : \mathrm{D}^2 u \,=\, 0, \quad \text{on } \Omega, \tag{1.1}$$

where n, N are integers such that $N \geq n \geq 2$ and Ω an open subset of $\mathbb{R}^n$. Here, for the map u with components $(u_1, ..., u_N)^\top$ the notation $\mathrm{D}u$ symbolises the gradient matrix

$$\mathrm{D}u(x) = \big(\mathrm{D}_i u_\alpha(x)\big)_{i=1...n}^{\alpha=1...N} \,\in\, \mathbb{R}^{N\times n}\ , \quad \mathrm{D}_i \equiv \partial/\partial x_i, \tag{1.2}$$

and for any $X \in \mathbb{R}^{N\times n}$, $[\![X]\!]^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of linear map $X : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![X]\!]^\perp := \mathrm{Proj}_{\mathrm{R}(X)^\perp}. \tag{1.3}$$

In index form, the system (1.1) reads

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n}\Big(\mathrm{D}_i u_\alpha\, \mathrm{D}_j u_\beta + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp_{\alpha\beta}\, \delta_{ij}\Big)\mathrm{D}^2_{ij} u_\beta \,=\, 0, \quad \alpha = 1, ..., N.$$

Our general notation will be either self-explanatory, or otherwise standard as e.g. in [9, 10, 28]. Throughout this paper we reserve $n, N \in \mathbb{N}$ for the dimensions of Euclidean spaces and $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$.

2010 *Mathematics Subject Classification.* 49N99; 49N60; 35B06; 35B65; 35D99.

Key words and phrases. ∞-Laplace system, ∞-Harmonic mappings, Calculus of Variations in L^∞, Rigidity, Gradient flows.

Speaking about the system (1.1), we would like to mention that the system (1.1) is called the"∞-Laplacian" and it arises as a sort of Euler-Lagrange PDE of vectorial variational problems in L^∞ for the supremal functional

$$E_\infty(u,\mathcal{O}) := \|H(\mathrm{D}u)\|_{L^\infty(\mathcal{O})}, \quad u \in W^{1,\infty}_{loc}(\Omega,\mathbb{R}^N),\ \mathcal{O} \Subset \Omega, \tag{1.4}$$

when the Hamiltonian (the non-negative function $H \in C^2(\mathbb{R}^{N\times n})$) is chosen to be $H(\mathrm{D}u) = \frac{1}{2}\big|\mathrm{D}u\big|^2$, with $\big|.\big|$ is the Euclidean norm on the space $\mathbb{R}^{N\times n}$. the ∞-Laplacian is a special case of the system

$$\Delta_\infty u \,:=\, \Big(H_P \otimes H_P + H[\![H_P]\!]^\perp H_{PP}\Big)(\mathrm{D}u) : \mathrm{D}^2 u \,=\, 0, \tag{1.5}$$

which was first formally derived by Katzourakis [11] as the limit of the Euler-Lagrange equations of the integral functionals $E_m(u,\Omega) := \int_\Omega (H(\mathrm{D}u))^p$ as $p \longrightarrow \infty$.

Eventhough the theory of weak solutions has witnessed a significant development so far, particularly the new theory of "$\mathcal{D}$-solutions" introduced by Katzourakis [20], which applies to nonlinear PDE systems of any order and allows for merely measurable maps to be rigorously interpreted and studied as solutions of PDE systems fully nonlinear and with discontinuous coefficients, yet the structure of weak solutions are complicated to understand. In this paper, we restrict our attention to classical solutions which might be helpful to imagine and understand the behavior and the structure of the weak solutions.

For the ∞-Laplace system (1.1) the orthogonal projection on the orthogonal complement of the range, $[\![\mathrm{D}u]\!]^\perp$, coincides with the projection on the geometric normal space of the image of the solution.

It is worth noting that ∞-Harmonic maps are affine when $n = 1$ since in this case the system (1.1) simplifies to

$$\Delta_\infty u \,=\, \Big(u' \otimes u'\Big)u'' + |u'|^2\Big(I - \frac{u' \otimes u'}{|u'|^2}\Big)u'' \,=\, |u'|^2 u'', \tag{1.6}$$

and hence no interesting phenomena arise when $n = 1$.

For the case $N = 1$, the system (1.1) reduced to the single ∞-Laplacian PDE

$$\Delta_\infty u \,:=\, \Big(\mathrm{D}u \otimes \mathrm{D}u\Big) : \mathrm{D}^2 u \,=\, 0, \tag{1.7}$$

since the normal coefficient $|\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp$ vanishes identically. This also happen when u is submersion. The single ∞-Laplacian PDE (1.7), and the related scalar L^∞-variational problems, started being studied in the '60s by Aronsson in [4, 5]. Today it is being studied in the context of Viscosity Solutions (see for example Crandall [3], Barron-Evans-Jensen [7] and Katzourakis [16]).

The vectorial case $N \geq 2$ first arose in the early 2010s in the work of Katzourakis [11]. Due to both the mathematical significance as well as the importance for applications particularly in Data Assimilation, the area is developing very rapidly (see [2], [6],[8], [12]-[15], and also [17]-[27]).

In a joint work with Katzourakis and Ayanbayev [1], among other results, we have proved that the image $u(\Omega)$ of a solution $u \in C^2(\Omega,\mathbb{R}^N)$ to the nonlinear system (1.1) satisfying that the rank of its gradient matrix is at most one, $\mathrm{rk}(\mathrm{D}u) \leq 1$ in Ω, is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections). Hence the component $[\![\mathrm{D}u]\!]^\perp \Delta u$ of Δ_∞ forces flatness of the image of solutions.

Interestingly, even when the operator Δ_∞ is applied to C^∞ maps, which may even be solutions, (1.1) may have discontinuous coefficients. This further difficulty of the vectorial case is not present in the scalar case. As an example consider

$$u(x,y) := e^{ix} - e^{iy}, \qquad u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2. \tag{1.8}$$

Katzourakis has showed in [11] that even though (1.8) is a smooth solution of the ∞-Laplacian near the origin, we still have the coefficient $|\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp$ of (1.1) is discontinuous. This is because when the dimension of the image changes, the projection $[\![\mathrm{D}u]\!]^\perp$ "jumps". More precisely, for (1.8) the domain splits to three components according to the $\mathrm{rk}(\mathrm{D}u)$, the "2D phase Ω_2", whereon u is essentially 2D, the"1D phase Ω_1", whereon u is essentially $1D$ (which is empty for (1.8))and the "interface S" where the coefficients of Δ_∞ become discontinuous.

In [12] Katzourakis constructed additional examples, which are more intricate than (1.8), namely smooth 2D ∞-Harmonic maps whose interfaces have triple junctions and corners and are given by the explicit formula

$$u(x,y) := \int_y^x e^{iK(t)} dt. \tag{1.9}$$

Indeed, for $K \in C^1(\mathbb{R},\mathbb{R})$ with $\|K\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}$, (1.9) defines C^2 ∞-Harmonic map whose phases are as shown in Figures 1(a), 1(b) below, when K qualitatively behaves as shown in the Figures 2(a), 2(b) respectively [1]. Also, on the phase Ω_1 the ∞-Harmonic map (1.9) is given by a scalar ∞-Harmonic function times a constant vector, and on the phase Ω_2 it is a solution of the vectorial Eikonal equation.

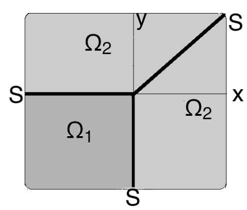


Figure 1(a).

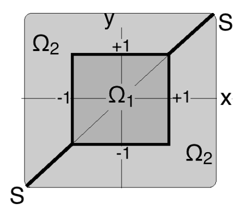


Figure 1(b).

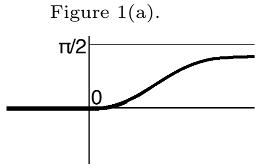


Figure 2(a).

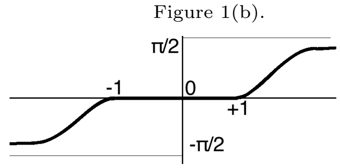


Figure 2(b).

One of the interesting results in [15] was that this phase separation is a general phenomena for smooth 2D ∞-Harmonic maps. Therein the author proves that on each phase the dimension of the tangent space is constant and these phases are separated by interfaces whereon $[\![\mathrm{D}u]\!]^\perp$ becomes discontinuous. Accordingly the author established the next result:

Theorem 1.1 (Structure of 2D ∞-Harmonic maps, cf. [15]).
Let $u : \mathbb{R}^2 \supseteq \Omega \longrightarrow \mathbb{R}^N$ be an ∞-Harmonic map in $C^2\big(\Omega, \mathbb{R}^N\big)$, that is a solution to (1.1). Let also $N \geq 2$. Then, there exist disjoint open sets Ω_1, $\Omega_2 \subseteq \Omega$, and a closed nowhere dense set S such that $\Omega = \Omega_1 \bigcup S \bigcup \Omega_2$ and:

[1] The figures 1(a), 1(b), 2(a) and 2(b) are courtesy of N. Katzourakis.

(i) On Ω_2 we have $\mathrm{rk(Du)} = 2$ *, and the map* $u: \Omega_2 \longrightarrow \mathbb{R}^N$ *is an immersion and solution of the Eikonal equation:*

$$|\mathrm{D}u|^2 = C^2 > 0. \tag{1.10}$$

The constant C may vary on different connected components of Ω_2.

(ii) On Ω_1 we have $\mathrm{rk(Du)} = 1$ *and the map $u: \Omega_1 \longrightarrow \mathbb{R}^N$ is given by an essentially scalar ∞-Harmonic function $f: \Omega_1 \longrightarrow \mathbb{R}$:*

$$u = a + \xi f, \;\; \Delta_\infty f = 0, \;\; a \in \mathbb{R}^N, \;\; \xi \in \mathbb{S}^{N-1}. \tag{1.11}$$

The vectors a, ξ may vary on different connected components of Ω_1.

(iii) On S, $|\mathrm{Du}|^2$ *is constant and also* $\mathrm{rk(Du)} = 1$. *Moreover if $S = \partial\Omega_1 \cap \partial\Omega_2$ (that is if both the* 1D *and* 2D *phases coexist) then $u: S \longrightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:*

$$u = a + \xi f, \;\; |\mathrm{D}f|^2 = C^2 > 0, \;\; a \in \mathbb{R}^N, \;\; \xi \in \mathbb{S}^{N-1}. \tag{1.12}$$

The main result of this paper is to generalise these results to higher dimension $N \geq n \geq 2$. The principle result in this paper in the following extension of theorem 1.1:

Theorem 1.2 (Phase separation of n-dimensional ∞-Harmonic mappings)**.**

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $u: \Omega \longrightarrow \mathbb{R}^N$, $N \geq n \geq 2$, be an ∞-Harmonic map in $C^2\big(\Omega, \mathbb{R}^N\big)$, that is a solution to the ∞-Laplace system (1.1). Then, there exist disjoint open sets $\big(\Omega_r\big)_{r=1}^{n} \subseteq \Omega$, and a closed nowhere dense set S such that $\Omega = S \bigcup \Big(\bigcup_{i=1}^{n} \Omega_i \Big)$ such that:

(i) On Ω_n we have $\mathrm{rk(Du)} \equiv n$ *and the map $u: \Omega_n \longrightarrow \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:*

$$|\mathrm{D}u|^2 = C^2 > 0. \tag{1.13}$$

The constant C may vary on different connected components of Ω_n.

(ii) On Ω_r we have $\mathrm{rk(Du)} \equiv r$, *where r is integer in $\{2, 3, 4, ..., (n-1)\}$, and* $|\mathrm{Du}(\gamma(t))|$ *is constant along trajectories of the parametric gradient flow of $u(\gamma(t, x, \xi))$*

$$\begin{cases} \dot{\gamma}(t, x, \xi) = \xi^\top \mathrm{D}u\big(\gamma(t, x, \xi)\big), & t \in (-\varepsilon, 0) \bigcup (0, \varepsilon), \\ \gamma(0, x, \xi) = x, \end{cases} \tag{1.14}$$

where $\xi \in \mathbb{S}^{N-1}$, and $\xi \notin N\Big(\mathrm{D}u\big(\gamma(t, x, \xi)\big)^\top\Big)$.

(iii) On Ω_1 we have $\mathrm{rk(Du)} \leq 1$ *and the map $u: \Omega_1 \longrightarrow \mathbb{R}^N$ is given by an essentially scalar ∞-Harmonic function $f: \Omega_1 \longrightarrow \mathbb{R}$:*

$$u = a + \xi f, \;\; \Delta_\infty f = 0, \;\; a \in \mathbb{R}^N, \;\; \xi \in \mathbb{S}^{N-1}. \tag{1.15}$$

The vectors a, ξ may vary on different connected components of Ω_1.

(iv) On S, when $S \supseteq \partial\Omega_p \cap \partial\Omega_q = \phi$ for all p and q such that $2 \leq p < q \leq n-1$, then we have that $|\mathrm{Du}|^2$ *is constant and also* $\mathrm{rk(Du)} \equiv 1$. *Moreover on*

$$\partial\Omega_1 \cap \partial\Omega_n \subseteq S,$$

(when both 1D *and* nD *phases coexist), we have that $u: S \longrightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:*

$$u = a + \xi f, \;\; |\mathrm{D}f|^2 = C^2 > 0, \;\; a \in \mathbb{R}^N, \;\; \xi \in \mathbb{S}^{N-1}. \tag{1.16}$$

On the other hand, if there exist some r and q such that $2 \leq r < q \leq n-1$, then on $S \supseteq \partial\Omega_r \cap \partial\Omega_q \neq \phi$ (when both rD *and* qD *phases coexist), we have that* $\mathrm{rk}(\mathrm{D}u) \equiv r$ *and we have same result as in (ii) above.*

2. Preliminaries

For the convenience of the reader, in this section we recall without proof a theorem of rigidity of rank-one maps, proved in [15], which will be used in the proof of the main result of this paper in section 3. We also recall the proposition of Gradient flows for tangentially ∞-Harmonic maps which introduced in [11] and its improved modification lemma in [14].

Theorem 2.1 (Rigidity of Rank-One maps, cf. [15])**.**
Suppose $\Omega \subseteq \mathbb{R}^n$ is open and contractible and $u : \Omega \longrightarrow \mathbb{R}^N$ is in $C^2(\Omega, \mathbb{R}^N)$. Then the following are equivalent:

(i) u is a Rank-One map, that is $rk(\mathrm{D}u) \leq 1$ on Ω or equivalently there exist maps $\xi : \Omega \longrightarrow \mathbb{R}^N$ and $w : \Omega \longrightarrow \mathbb{R}^n$ with $w \in C^1(\Omega, \mathbb{R}^n)$ and $\xi \in C^1\big(\Omega \setminus \{w = 0\}, \mathbb{R}^N\big)$ such that $\mathrm{D}u = \xi \otimes w$.

(ii) There exist $f \in C^2(\Omega, \mathbb{R})$, a partition $\{B_i\}_{i\in\mathbb{N}}$ og Ω to Borel sets where each B_i equals a connected open set with a boundary portion and Lipschitz curves $\{\mathcal{V}^i\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{loc}(\Omega)^N$ such that on each B_i u equals the composition of $\mathcal{V}^i$ with f:

$$u \,=\, \mathcal{V}^i \circ f \quad , \qquad on\, B_i \subseteq \Omega. \tag{2.1}$$

Moreover, $|\dot{\mathcal{V}}^i| \equiv 1$ on $f(B_i)$, $\dot{\mathcal{V}}^i \equiv 0$ on $\mathbb{R} \setminus f(B_i)$ and there exist $\ddot{\mathcal{V}}^i$ on $f(B_i)$, interpreted as 1-sided on $\partial f(B_i)$, if any. Also,

$$\mathrm{D}u \,=\, (\mathcal{V}^i \circ f) \otimes \mathrm{D}f \quad , \qquad on\, B_i \subseteq \Omega, \tag{2.2}$$

and the image $u(\Omega)$ is an 1-rectifiable subset of $\mathbb{R}^N$:

$$u(\Omega) \,=\, \bigcup_{i=1}^{\infty} \mathcal{V}^i(f(B_i)) \subseteq \mathbb{R}^N. \tag{2.3}$$

Proposition 2.2 (Gradient flows for tangentially ∞-Harmonic maps, cf. [11])**.**
Let $u \in C^2\big(\mathbb{R}^n, \mathbb{R}^N\big)$. Then, $\mathrm{D}u\,\mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big) = 0$ on $\Omega \Subset \mathbb{R}^n$ if and only if the flow map $\gamma : \mathbb{R} \times \Xi \longrightarrow \Omega$ with $\Xi := \{(x, \xi) \mid \xi^\top \mathrm{D}u(x) \neq 0\} \subseteq \Omega \times \mathbb{S}^{N-1}$ of

$$\begin{cases} \dot{\gamma}(t, x, \xi) = \xi^\top \mathrm{D}u\big(\gamma(t, x, \xi)\big), \\ \gamma(0, x, \xi) = x, \end{cases} \tag{2.4}$$

satisfies along trajectories

$$\begin{cases} \big|\mathrm{D}u\big(\gamma(t, x, \xi)\big)\big| = \big|\mathrm{D}u\big(\gamma(x)\big)\big|, \quad t \in \mathbb{R} \\ t \longmapsto \xi^\top u\big(\gamma(t, x, \xi)\big) \quad \textit{is increasing.} \end{cases} \tag{2.5}$$

The following lemma is improved modification of proposition 2.2

Lemma 2.3 (cf. [14])**.**
Let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ *be in* $u \in C^2\big(\Omega, \mathbb{R}^N\big)$. *Consider the gradient flow*

$$\begin{cases} \dot{\gamma}(t,x,\xi) = \Big(\frac{|\mathrm{D}u|^2}{|\xi^\top \mathrm{D}u|^2}\xi^\top \mathrm{D}u\Big)\big(\gamma(t,x,\xi)\big), \quad t \neq 0 \\ \gamma(0,x,\xi) = x, \end{cases} \tag{2.6}$$

for $x \in \Omega$, $\xi \in \mathbb{S}^{N-1} \setminus [\![\mathrm{D}u]\!]^\perp$. *Then, we have the differential identities*

$$\frac{d}{dt}\big(\frac{1}{2}|\mathrm{D}u\big(\gamma(t,x,\xi)\big)|^2\big) = \Big(\frac{|\mathrm{D}u|^2}{|\xi^\top \mathrm{D}u|^2}\xi^\top \mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u\Big)\big(\gamma(t,x,\xi)\big), \tag{2.7}$$

$$\frac{d}{dt}\big(\xi^\top \mathrm{D}u\big(\gamma(t,x,\xi)\big)\big) = |\mathrm{D}u\big(\gamma(t,x,\xi)\big)|^2, \tag{2.8}$$

which imply $\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u = 0$ *on* Ω *if and only if* $|\mathrm{D}u\big(\gamma(t,x,\xi)\big)|$ *is constant along trajectories* γ *and* $t \longmapsto \xi^\top u\big(\gamma(t,x,\xi)\big)$ *is affine.*

3. Proof of the main result

In this section we present the proof of the main result of this paper, theorem 1.2.

Proof of Theorem 1.2.
Let $u \in C^2\big(\Omega, \mathbb{R}^N\big)$ be a solution to the ∞-Laplace system (1.1). Note that the PDE system can be decoupled to the following systems

$$\mathrm{D}u\, \mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big) \;=\; 0, \tag{3.1}$$

$$|\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp \Delta u \;=\; 0. \tag{3.2}$$

Set $\Omega_1 := \ \mathrm{int}\{\mathrm{rk}(\mathrm{D}u) \ \leq \ 1\}$, $\Omega_r := \ \mathrm{int}\{\mathrm{rk}(\mathrm{D}u) \ \equiv \ r\}$ and $\Omega_n := \{\mathrm{rk}(\mathrm{D}u) \ \equiv \ n\}$. Then:

On Ω_n we have $\mathrm{rk}(\mathrm{D}u) = \dim(\Omega_n \subseteq \mathbb{R}^n) = n$. Since $N \geq n$ and hence the map $u : \Omega_n \longrightarrow \mathbb{R}^N$ is an immersion (because its derivative has constant rank equal to the dimension of the domain, the arguments in the case of $\mathrm{rk}(\mathrm{D}u) \equiv n$ follows the same lines as in [15, theorem 1.1] but we provide them for the sake of completeness). This means that $\mathrm{D}u$ is injective. Thus, $\mathrm{D}u(x)$ possesses a left inverse $(\mathrm{D}u(x))^{-1}$ for all $x \in \Omega_n$. Therefore, the system (3.1) implies

$$(\mathrm{D}u)^{-1}\mathrm{D}u\, \mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big) \;=\; 0, \tag{3.3}$$

and hence $\mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big) \;=\; 0$ on Ω_n, or equivalently

$$|\mathrm{D}u|^2 \;=\; C^2, \tag{3.4}$$

on each connected component of Ω_n. Moreover, (3.4) holds on the common boundary of Ω_n with any other component of the partition.

On Ω_r we have $\mathrm{rk}(\mathrm{D}u) \equiv r$, where r is integer in $\{2, 3, 4, ..., (n-1)\}$. Consider the gradient flow (2.6). Giving that (3.1) holds, then by the proposition of Gradient flows for tangentially ∞-Harmonic maps and its improved modification lemma which we recalled in the preliminaries, we must have that $|\mathrm{D}u\big(\gamma(t,x,\xi)\big)|$ is constant along trajectories γ and $t \longmapsto \xi^\top u\big(\gamma(t,x,\xi)\big)$ is affine. Moreover, if there exist some r and q such that $2 \leq r < q \leq n-1$, and $\partial\Omega_r \cap \partial\Omega_q \neq \phi$. Then a similar thing happen on $\partial\Omega_r \cap \partial\Omega_q \subseteq S$ (when both rD and qD phases coexist), because in this case we also have that $\mathrm{rk}(\mathrm{D}u) \equiv r$ and we have same result as above.

The proof of the remaining claims of the theorem is very similar to [15, theorem 1.1], which we give below for the sake of completeness:

On $\Omega_1 := \ \mathrm{int}\{\mathrm{rk}(\mathrm{D}u) \le 1\}$ we have $\mathrm{rk}(\mathrm{D}u) \le 1$. Hence there exist vector fields $\xi : \mathbb{R}^n \supseteq \Omega_1 \longrightarrow \mathbb{R}^N$ and $w : \mathbb{R}^n \supseteq \Omega_1 \longrightarrow \mathbb{R}^n$ such that $\mathrm{D}u = \xi \otimes w$. Suppose first that Ω_1 is contractible. Then, by the Rigidity Theorem 2.1, there exist a function $f \in C^2(\Omega_1, \mathbb{R})$, a partition of Ω_1 to Borel sets $\{B_i\}_{i\in\mathbb{N}}$ and Lipschitz curves $\{\mathcal{V}^i\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{\mathrm{loc}}(\Omega)^N$ with $|\dot{\mathcal{V}}^i| \equiv 1$ on $f(B_i)$, $|\dot{\mathcal{V}}^i| \equiv 0$ on $\mathbb{R}\setminus f(B_i)$ twice differentiable on $f(B_i)$, such that $u = \mathcal{V}^i \circ f$ on each $B_i \subseteq \Omega$ and hence $\mathrm{D}u = (\mathcal{V}^i \circ f) \otimes \mathrm{D}f$. By (3.1) , we obtain

$$\Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) \otimes \Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) : \\ : \Big[\big(\ddot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f \otimes \mathrm{D}f + \big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}^2 f\Big] = 0, \tag{3.5}$$

on $B_i \subseteq \Omega_1$. Since $|\dot{\mathcal{V}}^i| \equiv 1$ on $f(B_i)$, we have that $\ddot{\mathcal{V}}^i$ is normal to $\dot{\mathcal{V}}^i$ and hence

$$\Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) \otimes \Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) : \Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}^2 f\Big) = 0, \tag{3.6}$$

on $B_i \subseteq \Omega_1$. Hence, by using again that $|\dot{\mathcal{V}}^i|^2 \equiv 1$ on $f(B_i)$ we get

$$\Big(\mathrm{D}f \otimes \mathrm{D}f : \mathrm{D}^2 f\Big)\big(\dot{\mathcal{V}}^i \circ f\big) = 0, \tag{3.7}$$

on $B_i \subseteq \Omega_1$. Thus, $\Delta_\infty f = 0$ on B_i. By (3.2) and again since $|\dot{\mathcal{V}}^i|^2 \equiv 1$ on $f(B_i)$, we have $[\![\mathrm{D}u]\!]^\perp = [\![\dot{\mathcal{V}}^i \circ f]\!]^\perp$ and hence

$$|\mathrm{D}f|^2 [\![\dot{\mathcal{V}}^i \circ f]\!]^\perp \mathrm{Div}\Big(\big(\dot{\mathcal{V}}^i \circ f\big) \otimes \mathrm{D}f\Big) = 0, \tag{3.8}$$

on $B_i \subseteq \Omega_1$. Hence,

$$|\mathrm{D}f|^2 [\![\dot{\mathcal{V}}^i \circ f]\!]^\perp \Big(\big(\ddot{\mathcal{V}}^i \circ f\big)|\mathrm{D}f|^2 + \big(\dot{\mathcal{V}}^i \circ f\big)\Delta f\Big) = 0, \tag{3.9}$$

on B_i, which by using once again $|\dot{\mathcal{V}}^i|^2 \equiv 1$ gives

$$|\mathrm{D}f|^4 \big(\ddot{\mathcal{V}}^i \circ f\big) = 0, \tag{3.10}$$

on B_i. Since $\Delta_\infty f = 0$ on B_i and $\Omega_1 = \cup_1^\infty B_i$, f is ∞-Harmonic on Ω_1. Thus, by Aronsson's theorem in , either $|\mathrm{D}f|>0$ or $|\mathrm{D}f| \equiv 0$ on Ω_1.

If the first alternative holds, then by (3.10) we have $\ddot{\mathcal{V}}^i \equiv 0$ on $f(B_i)$ for all i and hence, $\mathcal{V}^i$ is affine on $f(B_i)$, that is $\mathcal{V}^i = t\xi^i + a^i$ for some $|\xi^i| = 1, a^i \in \mathbb{R}^N$. Thus, since $u = \mathcal{V}^i \circ f$ and $u \in C^2\big(\Omega_1, \mathbb{R}^N\big)$, all ξ^i and all a^i coincide and consequently $u = \xi f + a$ for $\xi \in \mathbb{S}^{N-1}, a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1, \mathbb{R})$.

If the second alternative holds, then f is constant on Ω_1 and hence, by the representation $u = \mathcal{V}^i \circ f$, u is piecewise constant on each B_i. Since $u \in C^2\big(\Omega_1, \mathbb{R}^N\big)$ and $\Omega_1 = \cup_i^\infty B_i$, necessarily u is constant on Ω_1. But then $|\mathrm{D}u|_{\Omega_2}| = |\mathrm{D}f|_{\mathscr{S}}| = 0$ and necessarily $\Omega_2 = \phi$. Hence, $|\mathrm{D}u| \equiv 0$ on Ω, that is u is affine on each of the connected components of Ω.

If $\Omega_1$is not contractible, cover it with balls $\{\mathbb{B}_m\}_{m\in\mathbb{N}}$ and apply the previous argument. Hence, on each $\mathbb{B}_m$, we have $u = \xi^m f^m + a^m, \xi^m \in \mathbb{S}^{N-1}, a^m \in \mathbb{R}^N$ and $f^m \in C^2(\mathbb{B}_m, \mathbb{R})$ with $\Delta_\infty f^m = 0$ on $\mathbb{B}_m$ and hence either $|\mathrm{D}f^m| > 0$ or $|\mathrm{D}f^m| \equiv 0$. Since $C^2\big(\Omega_1, \mathbb{R}^N\big)$, on the other overlaps of the balls the different expressions of

u must coincide and hence, we obtain $u = \xi f + a$ for $\xi \in \mathbb{S}^{N-1}, a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1, \mathbb{R})$ where ξ and a may vary on different connected components of Ω_1. The theorem follows.

□

Acknowledgement. The author is indebted to N. Katzourakis and would like to thank him for his encouragement and support towards him and for giving the permission to use his own figures, as well as for all the comments and suggestions which improved the content of this paper.

References

[1] H. Abugirda, B. Ayanbayev, and N. Katzourakis, *Rigidity and flatness of the image of certain classes of mappings having tangential Laplacian.* ArXiv preprint, https://arxiv.org/pdf/1704.04492.pdf

[2] H. Abugirda, N. Katzourakis , *Existence of 1D Vectorial Absolute Minimisers in L^∞ under Minimal Assumptions*, Proceeding of the AMS 145, 2567 - 2575 (2017).

[3] M. G. Crandall, *A visit with the ∞-Laplacian*, in *Calculus of Variations and Non-Linear Partial Differential Equations*, Springer Lecture notes in Mathematics 1927, CIME, Cetraro Italy 2005.

[4] G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Arkiv für Mat. 6 (1967), 551 - 561.

[5] G. Aronsson, *On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$*, Arkiv für Mat. 7 (1968), 395 - 425.

[6] B. Ayanbayev, N. Katzourakis, *A Pointwise Characterisation of the PDE system of vectorial Calculus of variations in L^∞*, Proc. Royal Soc. Edinburgh A, Nov. 2016.

[7] E. N. Barron, L. C. Evans, R. Jensen, *The Infinity Laplacian, Aronsson's Equation and their Generalizations*, Transactions of the AMS, Vol. 360, Nr 1, Jan 2008.

[8] G. Croce, N. Katzourakis, G. Pisante, *$\mathcal{D}$-solutions to the system of vectorial Calculus of Variations in L^∞ via the Baire Category method for the singular values*, Discrete and Continuous Dynamical Systems, Volume 37, Issue 12, 6165-6181 (2017).

[9] B. Dacorogna, *Direct Methods in the Calculus of Variations*, 2nd Edition, Volume 78, Applied Mathematical Sciences, Springer, 2007.

[10] L.C. Evans, *Partial Differential Equations*, AMS, Graduate Studies in Mathematics Volume 19.1, Second Edition, 2010.

[11] N. Katzourakis, *L^∞ Variational Problems for maps and the Aronsson PDE System*, J. Differential Equations, Volume 253, Issue 7, 1 October 2012, 2123 - 2139.

[12] N. Katzourakis, *Explicit 2D ∞-harmonic maps whose interfaces have junctions and corners*, Comptes Rendus Acad. Sci. Paris, Ser.I, 351 (2013) 677–680.

[13] N. Katzourakis, *Nonuniqueness in vector-valued Calculus of Variations in L^∞ and some linear elliptic systems.* Communications on Pure and Applied Analysis,Volume 14, Issue 1, (2013) 313–327.

[14] N. Katzourakis, *∞-Minimal Submanifolds.* Proceedings of the American Mathematical Society, Volume 142, Issue 8, (2014) 2797–2811.

[15] N. Katzourakis, *On the Structure of ∞-Harmonic Maps*, Communications in PDE, Volume 39, Issue 11 (2014), 2091 - 2124.

[16] N. Katzourakis, *An Introduction to viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in L^∞*, Springer International Publishing, 2015.

[17] N. Katzourakis, *Optimal ∞-Quasiconformal Immersions*, ESAIM: Control, Optimisation and Calculus of Variations, Volume 21, Issue 2, (2015) 561- 582.

[18] N. Katzourakis, *A new characterisation of ∞-harmonic and p-harmonic maps via affine variations in L^∞*, Electronic J. Differential Equations, Vol. 2017 (2017), No. 29, 1 - 19.

[19] N. Katzourakis, *Absolutely minimising generalised solutions to the equations of vectorial Calculus of Variations in L^∞*, Calculus of Variations and PDE 56 (1), 1 - 25 (2017) (DOI: 10.1007/s00526-016-1099-z).

[20] N. Katzourakis, *Generalised solutions for fully nonlinear PDE systems and existence-uniqueness theorems*, Journal of Differential Equations 23 (2017), 641-686, DOI: 10.1016/j.jde.2017.02.048.

[21] N. Katzourakis, *Solutions of vectorial Hamilton-Jacobi equations are rank-one Absolute Minimisers in L^∞*, Advances in Nonlinear Analysis, June 2017.

[22] N. Katzourakis, J. Manfredi, *Remarks on the Validity of the Maximum Principle for the ∞-Laplacian*, Le Matematiche, Vol. LXXI (2016) Fasc. I, 63 - 74, DOI: 10.4418/2016.71.1.5.

[23] N. Katzourakis, R. Moser, *Existence, uniqueness and structure of second order absolute minimisers*, ArXiv preprint, http://arxiv.org/pdf/1701.03348.pdf.

[24] N. Katzourakis, E. Parini, *The eigenvalue problem for the ∞-Bilaplacian*, Nonlinear Differential Equations and Applications NoDEA, Volume 24, Issue 68, (2017).

[25] N. Katzourakis, T. Pryer, *On the numerical approximation of ∞-Biharmonic and p-Biharmonic functions*, ArXiv preprint, https://arxiv.org/pdf/1701.07415.pdf.

[26] N. Katzourakis, T. Pryer, *On the numerical approximation of ∞-Harmonic mappings*, Nonlinear Differential Equations & Applications 23 (6), 1-23 (2016).

[27] N. Katzourakis, T. Pryer, *2nd order L^∞ variational problems and the ∞-Polylaplacian*, Advances in Calculus of Variations, in press.

[28] N. Katzourakis, E. Varvaruca *An Illustrative Introduction to Modern Analysis*, Chapman and Hall/CRC, 2017.

Department of Mathematics , College of Science, University of Basra, Basra, Iraq AND Department of Mathematics and Statistics, University of Reading, Whiteknights,, PO Box 220, Reading RG6 6AX, UK

E-mail address: h.a.h.abugirda@pgr.reading.ac.uk