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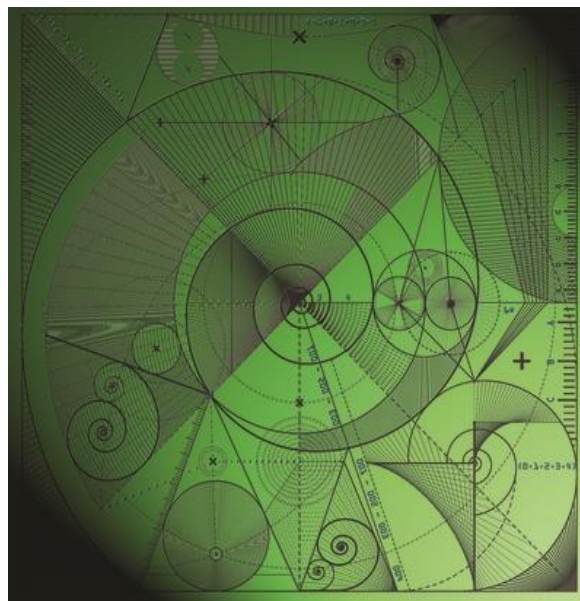
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Vectorial variational principles in L^∞ and
their characterisation through PDE systems

by

Birzhan Ayanbayev and **Nikos Katzourakis**



VECTORIAL VARIATIONAL PRINCIPLES IN L^∞ AND THEIR CHARACTERISATION THROUGH PDE SYSTEMS

BIRZHAN AYANBAYEV AND NIKOS KATZOURAKIS

ABSTRACT. We discuss two distinct minimality principles for general supremal first order functionals for maps and characterise them through solvability of associated second order PDE systems. Specifically, we consider Aronsson's standard notion of absolute minimisers and the concept of ∞ -minimal maps introduced more recently by the second author. We prove that C^1 absolute minimisers characterise a divergence system with parameters probability measures and that C^2 ∞ -minimal maps characterise Aronsson's PDE system. Since in the scalar case these different variational concepts coincide, it follows that the non-divergence Aronsson's equation has an equivalent divergence counterpart.

1. INTRODUCTION

Let $n, N \in \mathbb{N}$ and $H \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n})$ with $\Omega \subseteq \mathbb{R}^n$ an open set. In this paper we consider the supremal functional

$$(1.1) \quad E_\infty(u, \mathcal{O}) := \operatorname{ess\,sup}_{\mathcal{O}} H(\cdot, u, Du), \quad u \in W_{\operatorname{loc}}^{1,\infty}(\Omega; \mathbb{R}^N), \quad \mathcal{O} \Subset \Omega,$$

defined on maps $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$. In (1.1) and subsequently, we see the gradient as a matrix map $Du = (D_i u_\alpha)_{i=1 \dots n}^{\alpha=1 \dots N} : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^{N \times n}$. Variational problems for (1.1) have been pioneered by Aronsson in the 1960s in the scalar case $N = 1$ ([2]-[6]). Nowadays the study of such functionals (and of their associated PDEs describing critical points) form a fairly well-developed area of vivid interest, called Calculus of Variations in L^∞ . For pedagogical general introductions to the theme we refer to [10, 19, 32].

One of the main difficulties in the study of (1.1) which prevents us from utilising the standard machinery of Calculus of Variations for conventional (integral) functionals as e.g. in [24] is that it is non-local, in the sense that a global minimiser u of $E_\infty(\cdot, \Omega)$ in $W_g^{1,\infty}(\Omega; \mathbb{R}^N)$ for some fixed boundary data g may not minimise $E_\infty(\cdot, \mathcal{O})$ in $W_u^{1,\infty}(\mathcal{O}; \mathbb{R}^N)$. Namely, global minimisers are not generally local minimisers, a property which is automatic for integral functionals. The remedy proposed by Aronsson (adapted) to the vector case is to build locality into the minimality notion:

Definition 1. Let $u \in W_{\operatorname{loc}}^{1,\infty}(\Omega; \mathbb{R}^N)$. We say that u is an *absolute minimiser* of (1.1) on Ω if

$$(1.2) \quad \left. \begin{array}{l} \forall \mathcal{O} \Subset \Omega, \\ \forall \phi \in W_0^{1,\infty}(\mathcal{O}; \mathbb{R}^N) \end{array} \right\} \implies E_\infty(u, \mathcal{O}) \leq E_\infty(u + \phi, \mathcal{O}).$$

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In the scalar case of $N = 1$, Aronsson's concept of absolute minimisers turns out to be the appropriate substitute of mere minimisers. Indeed, absolute minimisers possess the desired uniqueness properties subject to boundary conditions and, most importantly, the possibility to characterise them through a necessary (and sufficient) condition of satisfaction of a certain nonlinear nondivergence second order PDE, known as the Aronsson equation ([9, 10, 12, 11, 13, 14, 17, 18, 20, 21, 27, 39, 44]). The latter can be written for functions $u \in C^2(\Omega)$ as

$$(1.3) \quad \mathbb{H}_P(\cdot, u, Du) \cdot D(\mathbb{H}(\cdot, u, Du)) = 0.$$

The Aronsson equation, being degenerate elliptic and non-divergence when formally expanded, is typically studied in the framework of viscosity solutions. In the above, $\mathbb{H}_P, \mathbb{H}_\eta, \mathbb{H}_x$ denotes the derivatives of $\mathbb{H}(x, \eta, P)$ with respect to the respective arguments and “ \cdot ” is the Euclidean inner product.

In this paper we are interested in characterising appropriately defined minimisers of (1.1) in the general vectorial case of $N \geq 2$ through solvability of associated PDE systems which generalise the Aronsson equation (1.3). As the wording suggests and we explain below, when $N \geq 2$ Aronsson's notion of Definition 1 is no longer the unique possible L^∞ variational concept. In any case, the extension of Aronsson's equation to the vectorial case reads

$$(1.4) \quad \begin{aligned} & \mathbb{H}_P(\cdot, u, Du) D(\mathbb{H}(\cdot, u, Du)) \\ & + \mathbb{H}(\cdot, u, Du) [\mathbb{H}_P(\cdot, u, Du)]^\perp \left(\text{Div}(\mathbb{H}_P(\cdot, u, Du)) - \mathbb{H}_\eta(\cdot, u, Du) \right) = 0. \end{aligned}$$

In the above, for any linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^N$, $[A]^\perp$ symbolises the orthogonal projection $\text{Proj}_{\mathbb{R}(A)^\perp}$ on the orthogonal complement of its range $\mathbb{R}(A) \subseteq \mathbb{R}^N$. We will refer to the PDE system (1.4) as the “Aronsson system”, in spite of the fact it was actually derived by the second author in [28], wherein the connections between general vectorial variational problems and their associated PDEs were first studied, namely those playing the role of Euler-Lagrange equations in L^∞ . The Aronsson system was derived through the well-known method of L^p -approximations and is being studied quite systematically since its discovery, see e.g. [28]-[31], [34, 37]. The additional normal term which is not present in the scalar case imposes an extra layer of complexity, as it might be discontinuous even for smooth solutions (see [29, 31]).

For simplicity and in order to illustrate the main ideas in a manner which minimises technical complications, *in this paper we restrict our attention exclusively to regular minimisers and solutions*. In general, solutions to (1.4) are nonsmooth and the lack of divergence structure combined with its vectorial nature renders its study beyond the reach of viscosity solutions. To this end, the theory of \mathcal{D} -solutions introduced in [34] and subsequently utilised in several works (see e.g. [1, 22, 34, 35]) offers a viable alternative for the study of general locally Lipschitz solutions to (1.4), and in fact it works far beyond the realm of Calculus of Variations in L^∞ . We therefore leave the generalisation of the results herein to a lower regularity setting for future work.

Additionally to absolute minimisers, for reasons to be explained later, in the paper [30] a special case of the next L^∞ variational concept was introduced (therein for $\mathbb{H}(x, \eta, P) = |P|^2$):

Definition 2. Let $u \in C^1(\Omega; \mathbb{R}^N)$. We say that u is an ∞ -minimal map for (1.1) on Ω if (i) and (ii) below hold true:

(i) u is a *rank-one absolute minimiser*, namely it minimises with respect to essentially scalar variations vanishing on the boundary along fixed unit directions:

$$(1.5) \quad \left. \begin{array}{l} \forall \mathcal{O} \Subset \Omega, \forall \xi \in \mathbb{R}^N \\ \forall \phi \in C_0^1(\overline{\mathcal{O}}; \text{span}[\xi]) \end{array} \right\} \implies E_\infty(u, \mathcal{O}) \leq E_\infty(u + \phi, \mathcal{O}).$$

(ii) u has *∞ -minimal area*, namely it minimises with respect to variations which are normal to the range of the matrix field $H_P(\cdot, u, Du)$ and free on the boundary:

$$(1.6) \quad \left. \begin{array}{l} \forall \mathcal{O} \Subset \Omega, \forall \phi \in C^1(\mathbb{R}^n; \mathbb{R}^N) \\ \text{with } \phi^\top H_P(\cdot, u, Du) = 0 \text{ on } \mathcal{O} \end{array} \right\} \implies E_\infty(u, \mathcal{O}) \leq E_\infty(u + \phi, \mathcal{O}).$$

In the above,

$$C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N) := \{\psi \in C^1(\mathbb{R}^n; \mathbb{R}^N) : \psi = 0 \text{ on } \partial\mathcal{O}\}.$$

Note also that *when $N = 1$ absolute minimisers and ∞ -minimal maps coincide*, at least when $\{H_P = 0\} \subseteq \{H = 0\}$. Further, in the event that $H_P(\cdot, u, Du)$ has discontinuous rank on \mathcal{O} , the only continuous normal vector fields ϕ may be only those vanishing on the set of discontinuities.

In [30] it was proved that C^2 ∞ -minimal maps of full rank (namely immersions or submersions) are ∞ -Harmonic, that is solutions to the so-called ∞ -Laplace system. The latter is a special case of (1.4), corresponding to the choice $H(x, \eta, P) = |P|^2$:

$$(1.7) \quad Du D(|Du|^2) + |Du|^2 [Du]^\perp \Delta u = 0.$$

The fullness of rank was assumed because of the possible discontinuity of the coefficient $[Du]^\perp$, which may well happen even for smooth solutions (for explicit examples see [29]). In this paper we bypass this difficulty by replacing the orthogonal projection $[\cdot]^\perp$ by the projection on the subspace of those normal vectors which have local normal C^1 extensions in a open neighbourhood:

Definition 3. Let $V : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N \times n}$ be a matrix field and note that

$$R(V(x))^\perp = N(V(x)^\top),$$

where for any $x \in \Omega$, $N(V(x)^\top)$ is the nullspace of the transpose $V(x)^\top \in \mathbb{R}^{n \times N}$. We define the orthogonal projection

$$\llbracket V(x) \rrbracket^\perp := \text{Proj}_{\tilde{N}(V(x)^\top)}, \quad \llbracket V(\cdot) \rrbracket^\perp : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N \times N},$$

where $\tilde{N}(V(x)^\top)$ is the reduced nullspace, given by

$$\begin{aligned} \tilde{N}(V(x)^\top) := & \left\{ \xi \in N(V(x)^\top) \mid \exists \varepsilon > 0 \ \& \ \exists \bar{\xi} \in C^1(\mathbb{R}^n; \mathbb{R}^N) : \right. \\ & \left. \bar{\xi}(x) = \xi \ \& \ \bar{\xi}(y) \in N(V(y)^\top), \ \forall y \in \mathbb{B}_\varepsilon(x) \right\}. \end{aligned}$$

It is a trivality to check that $\tilde{N}(V(x)^\top)$ is indeed a vector space and that

$$\llbracket V(x) \rrbracket^\perp [V(x)]^\perp = \llbracket V(x) \rrbracket^\perp,$$

where $[V(x)]^\perp = \text{Proj}_{N(V(x)^\top)}$. Note that the definition could be written in a more concise manner by using the algebraic language of *sheaves and germs*, but we refrained from doing so as there is no real benefit in this simple case.

The first main result in this paper is the next variational characterisation of the Aronsson system (1.4).

Theorem 4 (Variational Structure of Aronsson's system). *Let $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ be a map in $C^2(\Omega; \mathbb{R}^N)$. Then:*

(I) *If u is a rank-one absolute minimiser for (1.1) on Ω (Definition 2(i)), then it solves*

$$(1.8) \quad H_P(\cdot, u, Du) D(H(\cdot, u, Du)) = 0 \quad \text{on } \Omega.$$

The opposite is true if in addition H does not depend on $\eta \in \mathbb{R}^N$ and $H_P(\cdot, Du)$ has full rank on Ω .

(II) *If u has ∞ -minimal area for (1.1) on Ω (Definition 2(ii)), then it solves*

$$(1.9) \quad H(\cdot, u, Du) \llbracket H_P(\cdot, u, Du) \rrbracket^\perp \left(\text{Div}(H_P(\cdot, u, Du)) - H_\eta(\cdot, u, Du) \right) = 0 \quad \text{on } \Omega.$$

The opposite is true if in addition for any $x \in \Omega$, $H(x, \cdot, \cdot)$ is convex on $\mathbb{R}^n \times \mathbb{R}^{N \times n}$.

(III) *If u is ∞ -minimal map for (1.1) on Ω , then it solves the (reduced) Aronsson system*

$$\begin{aligned} A_\infty u &:= H_P(\cdot, u, Du) D(H(\cdot, u, Du)) \\ &+ H(\cdot, u, Du) \llbracket H_P(\cdot, u, Du) \rrbracket^\perp \left(\text{Div}(H_P(\cdot, u, Du)) - H_\eta(\cdot, u, Du) \right) = 0. \end{aligned}$$

The opposite is true if in addition H does not depend on $\eta \in \mathbb{R}^N$, $H_P(\cdot, Du)$ has full rank on Ω and for any $x \in \Omega$ $H(x, \cdot)$ is convex in $\mathbb{R}^{N \times n}$.

The emergence of two distinct sets of variations and a pair of separate PDE systems comprising (1.4) might seem at first glance mysterious. However, it is a manifestation of the fact that the (reduced) Aronsson system in fact consists of two linearly independent differential operators because of the perpendicularity between $\llbracket H_P \rrbracket^\perp$ and H_P ; in fact, one may split $A_\infty u = 0$ to

$$\begin{cases} H_P(\cdot, u, Du) D(H(\cdot, u, Du)) = 0, \\ H(\cdot, u, Du) \llbracket H_P(\cdot, u, Du) \rrbracket^\perp \left(\text{Div}(H_P(\cdot, u, Du)) - H_\eta(\cdot, u, Du) \right) = 0. \end{cases}$$

Theorem 4 makes clear that Aronsson's absolute minimisers do **not** characterise the Aronsson system when $N \geq 2$, at least when the additional natural assumptions hold true. This owes to the fact that, unlike the scalar case, *the Aronsson system admits arbitrarily smooth non-minimising solutions*, even in the model case of the ∞ -Laplacian. For details we refer to [37].

Since Aronsson's absolute minimisers do not characterise the Aronsson system, the natural question arises as to what is their PDE counterpart. The next theorem which is our second main result answers this question:

Theorem 5 (Divergence PDE characterisation of Absolute minimisers). *Let $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ be a map in $C^1(\Omega; \mathbb{R}^N)$. Consider the following statements:*

(I) *u is a vectorial absolute minimiser for (1.1) on Ω .*

(II) *For any $\mathcal{O} \Subset \Omega$ and for all Radon probability measures $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ satisfying*

$$(1.10) \quad \text{supp}(\sigma) \subseteq \text{Argmax}\{H(\cdot, u, Du) : \overline{\mathcal{O}}\},$$

we have

$$(1.11) \quad -\text{div}(H_P(\cdot, u, Du)\sigma) + H_\eta(\cdot, u, Du)\sigma = 0,$$

in the dual space $(C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N))^$.*

(III) For any $\mathcal{O} \Subset \Omega$ and any $\psi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$, we have

$$(1.12) \quad \mathbb{H}_P(\cdot, u, Du) : D\psi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \psi = 0$$

on $\text{Argmax}\{\mathbb{H}(\cdot, u, Du) : \overline{\mathcal{O}}\}$.

Then, we have (I) \implies (II) \implies (III). If in addition for any $x \in \Omega$, $\mathbb{H}(x, \cdot, \cdot)$ is convex, then (III) \implies (I) and all three statements are equivalent.

The result above provides an interesting characterisation of Aronsson’s concept of Absolute minimisers in terms of divergence PDE systems with measures as parameters. The exact distributional meaning of (1.11) is

$$\int_{\overline{\mathcal{O}}} \left(\mathbb{H}_P(\cdot, u, Du) : D\psi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \psi \right) d\sigma = 0$$

for all $\psi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$, where the “:” notation in the PDE symbolises the Euclidean (Frobenius) inner product in $\mathbb{R}^{N \times n}$.

The idea of Theorem 5 is inspired by the paper [25] of Evans and Yu, wherein a particular case of the divergence system is derived (in the special scalar case $N = 1$ for the ∞ -Laplacian and only for $\Omega = \mathcal{O}$), as well as by new developments on higher order Calculus of variations in L^∞ in [36, 38, 40].

Note that, it does not suffice to consider only $\Omega = \mathcal{O}$ as in [25] in order to describe absolute minimisers. For a subdomain $\mathcal{O} \subseteq \Omega$, it may well happen that the only measure σ “charging” the points of $\overline{\mathcal{O}}$ where the energy density $\mathbb{H}(\cdot, u, Du)$ is maximised is the Dirac measure at a single point $x \in \partial\mathcal{O}$. This is for instance the case for the standard “Aronsson solution” of the ∞ -Laplacian on \mathbb{R}^2 , given by $u(x, y) = |x|^{4/3} - |y|^{4/3}$, as well as for any other ∞ -Harmonic function which is nowhere Eikonal (i.e. $|Du|$ is non-constant on all open subsets).

By specialising Theorems 4-5 to the scalar case $N = 1$, we obtain the next set of equivalences (for simplicity stated in the C^2 case only):

Corollary 6 (The scalar case $N = 1$). *Let $u \in C^2(\Omega)$. Suppose that $\mathbb{H}(x, \cdot, \cdot)$ is convex on $\mathbb{R} \times \mathbb{R}^n$ and that $\{\mathbb{H}_P = 0\} \subseteq \{\mathbb{H} = 0\}$. Then, the next statements are equivalent:*

- (I) u is an absolute minimiser for (1.1) on Ω .
- (II) For any $\mathcal{O} \Subset \Omega$ and for any Radon probability measure $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ satisfying (1.10), (u, σ) solves (1.11) in the dual space $(C_0^1(\overline{\mathcal{O}}))^*$.
- (III) For any $\mathcal{O} \Subset \Omega$, u satisfies (1.12) on $\text{Argmax}\{\mathbb{H}(\cdot, u, Du) : \overline{\mathcal{O}}\}$ for any $\psi \in C_0^1(\overline{\mathcal{O}})$.
- (IV) u solves the Aronsson equation (1.3) on Ω .

We conclude this introduction by noting that the two vectorial variational concepts we are considering herein (Definitions 1-2) do not exhaust the plethora variational concepts in L^∞ . In particular, in the paper [43] the concept of *tight maps* was introduced in the case of $\mathbb{H}(x, \eta, P) = \|P\|$ where $\|\cdot\|$ is the operator norm on $\mathbb{R}^{N \times n}$. Additionally, in the papers [1, 35] a concept of special affine variations was considered which also characterises the Aronsson system, in fact in the generality of merely locally Lipschitz \mathcal{D} -solutions. Finally, in the paper [8] new concepts of absolute minimisers for constrained minimisation problems have been proposed, whilst results relevant to variational principles in L^∞ and applications appear in [15, 16, 18, 26, 41, 42].

2. PROOFS AND A MAXIMUM-MINIMUM PRINCIPLE FOR $H(\cdot, u, Du)$

In this section we prove our main results Theorems 4-5. Before delving into that, we establish a result of independent interest, which generalises a corresponding result from [30].

Proposition 7 (Maximum-Minimum Principles). *Suppose Let $u \in C^2(\Omega; \mathbb{R}^N)$ be a solution to (1.8), such that H satisfies*

- (a) $H_P(\cdot, u, Du)$ has full rank on Ω ,
- (b) there exists $c > 0$ such that

$$(\xi^\top H_P(x, \eta, P)) \cdot (\xi^\top P) \geq c |\xi^\top H_P(x, \eta, P)|^2,$$

for all $\xi \in \mathbb{R}^N$ and all $(x, \eta, P) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

Then, for any $\mathcal{O} \Subset \Omega$ we have:

$$(2.1) \quad \sup_{\mathcal{O}} H(\cdot, u, Du) = \max_{\partial \mathcal{O}} H(\cdot, u, Du),$$

$$(2.2) \quad \inf_{\mathcal{O}} H(\cdot, u, Du) = \min_{\partial \mathcal{O}} H(\cdot, u, Du).$$

The proof is based on the usage of the following flow with parameters:

Lemma 8. *Let $u \in C^2(\Omega; \mathbb{R}^N)$. Consider the parametric ODE system*

$$(2.3) \quad \begin{cases} \dot{\gamma}(t) = \xi^\top H_P(\cdot, u, Du)|_{\gamma(t)}, & t \neq 0, \\ \gamma(0) = x, \end{cases}$$

for given $x \in \Omega$ and $\xi \in \mathbb{R}^N$. Then, we have

$$(2.4) \quad \frac{d}{dt} \left(H(\cdot, u, Du)|_{\gamma(t)} \right) = \xi^\top H_P(\cdot, u, Du) D(H(\cdot, u, Du))|_{\gamma(t)},$$

$$(2.5) \quad \frac{d}{dt} \xi^\top u(\gamma(t)) \geq c \left| \xi^\top H_P(\cdot, u, Du)|_{\gamma(t)} \right|^2.$$

Proof of Lemma 8. The identity (2.4) follows by a direct computation and (2.3). For the inequality (2.5), we have

$$\begin{aligned} \frac{d}{dt} \xi^\top u(\gamma(t)) &= \left(\xi^\top Du(\gamma(t)) \right) \cdot \dot{\gamma}(t) \\ &= \left(\xi^\top Du(\gamma(t)) \right) \cdot \left(\xi^\top H_P(\cdot, u, Du)|_{\gamma(t)} \right) \\ &\geq c \left| \xi^\top H_P(\cdot, u, Du)|_{\gamma(t)} \right|^2. \end{aligned}$$

The lemma ensues. □

Proof of Proposition 7. Fix $\mathcal{O} \Subset \Omega$. Without loss of generality, we may suppose \mathcal{O} is connected.

Consider first the case where $\text{rk}(H_P(\cdot, u, Du)) \equiv n \leq N$. Then, the matrix-valued map $H_P(\cdot, u, Du)$ is pointwise left invertible. Therefore, by (1.8) we have

$$(H_P(\cdot, u, Du))^{-1} H_P(\cdot, u, Du) D(H(\cdot, u, Du)) = 0$$

which, by the connectivity of \mathcal{O} , gives

$$H(\cdot, u, Du) \equiv \text{const} \quad \text{on } \mathcal{O}.$$

The latter readily implies the desired conclusion.

Consider now the case where $\text{rk}(\mathbf{H}_P(\cdot, u, Du)) \equiv N \leq n$. Fix $x \in \mathcal{O}$ and a unit vector $\xi \in \mathbb{R}^n$ and consider the parametric ODE system (2.3) of Lemma 8. By the fullness of the rank of $\mathbf{H}_P(\cdot, u, Du)$, we have that

$$|\xi^\top \mathbf{H}_P(\cdot, u, Du)| \geq c_1 > 0 \quad \text{on } \mathcal{O}.$$

We will now show that the trajectory $\gamma(t)$ reaches $\partial\mathcal{O}$ in finite time. To this end, we estimate

$$\begin{aligned} \|Du\|_{L^\infty(\mathcal{O})} \text{diam}(\mathcal{O}) &\geq \|Du\|_{L^\infty(\mathcal{O})} |\gamma(t) - \gamma(0)| \\ &\geq \left| \frac{d}{dt} \Big|_{\hat{t}} \xi^\top u(\gamma(t)) \right| t, \end{aligned}$$

for some $\hat{t} \in (0, t)$, by the mean value theorem. Hence,

$$\begin{aligned} \|Du\|_{L^\infty(\mathcal{O})} \text{diam}(\mathcal{O}) &\geq \left| \frac{d}{dt} \Big|_{\hat{t}} \xi^\top u(\gamma(t)) \right| t \\ &= \left| \xi^\top Du(\gamma(\hat{t})) \cdot \dot{\gamma}(\hat{t}) \right| t \\ &= \left| \xi^\top Du(\gamma(\hat{t})) \cdot \left(\xi^\top \mathbf{H}_P(\cdot, u, Du) \Big|_{\gamma(\hat{t})} \right) \right| t \\ &\geq c_0 \left| \xi^\top \mathbf{H}_P(\cdot, u, Du) \Big|_{\gamma(\hat{t})} \right|^2 t \\ &\geq (c_0 c_1^2) t. \end{aligned}$$

This proves the desired claim. Further, since u solves (1.8), by (2.4) of Lemma 8 it follows that $\mathbf{H}(\cdot, u, Du)$ is constant along the trajectory. Thus, if $x \in \mathcal{O}$ is chosen as a point realising either the maximum or the minimum in \mathcal{O} , then by moving along the trajectory, we reach a point $y \in \partial\mathcal{O}$ such that $\mathbf{H}(\cdot, u, Du) \Big|_x = \mathbf{H}(\cdot, u, Du) \Big|_y$. This establishes both the maximum and minimum principle. The proposition ensues. \square

The central ingredient in the proofs of Theorems 4-5 is the next consequence of Danskin's theorem [24]: for any $\mathcal{O} \Subset \Omega$ and any $u, \phi \in C^1(\Omega; \mathbb{R}^N)$, we have the identity

$$(2.6) \quad \frac{d}{dt} \Big|_{t=0} \mathbf{E}_\infty(u + t\phi, \mathcal{O}) = \max_{\mathcal{O}(u)} \left(\mathbf{H}_P(\cdot, u, Du) : D\phi + \mathbf{H}_\eta(\cdot, u, Du) \cdot \phi \right),$$

where

$$(2.7) \quad \mathcal{O}(u) := \text{Argmax}\{\mathbf{H}(\cdot, u, Du) : \overline{\mathcal{O}}\}.$$

Indeed, by Danskin's result and the chain rule we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \mathbf{E}_\infty(u + t\phi, \mathcal{O}) &= \frac{d}{dt} \Big|_{t=0} \left(\max_{\overline{\mathcal{O}}} \mathbf{H}(\cdot, u + t\phi, Du + tD\phi) \right) \\ &= \max_{\mathcal{O}(u)} \left(\frac{d}{dt} \Big|_{t=0} \mathbf{H}(\cdot, u + t\phi, Du + tD\phi) \right) \\ &= \max_{\mathcal{O}(u)} \left(\mathbf{H}_P(\cdot, u, Du) : D\phi + \mathbf{H}_\eta(\cdot, u, Du) \cdot \phi \right). \end{aligned}$$

Now we establish Theorem 4.

Proof of Theorem 4. (I) Suppose first that u is a rank-one absolutely minimiser on Ω . The aim is to show that (1.8) is satisfied on Ω . This conclusion in fact follows

by the results in [28], but below we provide a new shorter proof. To this end, fix $x \in \Omega$ and $\rho \in (0, \text{dist}(x, \partial\Omega))$ and let $\mathcal{O} := \mathbb{B}_\rho(x)$. We fix also $\xi \in \mathbb{R}^N$ and choose

$$\phi(y) := \xi(|y - x|^2 - \rho^2).$$

Then, $\phi \in C_0^1(\bar{\mathbb{B}}_\rho(x); \text{span}[\xi])$ and by invoking (2.6)-(2.7), we obtain

$$(2.8) \quad \xi^\top \left(2\mathbb{H}_P(\cdot, u, Du)|_{x_\rho}(x_\rho - x) + \mathbb{H}_\eta(\cdot, u, Du)|_{x_\rho}(|x_\rho - x|^2 - \rho^2) \right) = 0$$

at any x_ρ which lies in the argmax set

$$(\mathbb{B}_\rho(x))(u) = \text{Argmax}\{\mathbb{H}(\cdot, u, Du) : \bar{\mathbb{B}}_\rho(x)\}.$$

If x_ρ lies in the interior of $\mathbb{B}_\rho(x)$, then it is an interior maximum and therefore

$$D(\mathbb{H}(\cdot, u, Du))|_{x_\rho} = 0.$$

This means that (1.8) is satisfied at x_ρ . If x_ρ lies on the boundary of $\mathbb{B}_\rho(x)$, then this means that

$$\forall y \in \bar{\mathbb{B}}_\rho(x), \text{ we have } \mathbb{H}(\cdot, u, Du)|_y \leq \mathbb{H}(\cdot, u, Du)|_{x_\rho}.$$

The above can be rewritten as

$$\bar{\mathbb{B}}_\rho(x) \subseteq \mathcal{H}(x_\rho) := \left\{ \mathbb{H}(\cdot, u, Du) \leq \mathbb{H}(\cdot, u, Du)|_{x_\rho} \right\},$$

and note also that $x_\rho \in \partial\mathbb{B}_\rho(x) \cap \partial\mathcal{H}(x_\rho)$. Hence, the sublevel set $\mathcal{H}(x_\rho)$ satisfied an interior sphere condition at x_ρ . If $D(\mathbb{H}(\cdot, u, Du))|_{x_\rho} = 0$ then (1.8) is again satisfied at x_ρ . If on the other hand

$$D(\mathbb{H}(\cdot, u, Du))|_{x_\rho} \neq 0$$

then $\partial\mathcal{H}(x_\rho)$ is a C^1 manifold near x_ρ and the gradient above is the normal vector at the point x_ρ . Due to the interior sphere condition, this implies that this is also the normal vector to the sphere $\partial\mathbb{B}_\rho(x)$ at x_ρ . Thus, there exists $\lambda \neq 0$ such that

$$(2.9) \quad x_\rho - x = \lambda D(\mathbb{H}(\cdot, u, Du))|_{x_\rho}.$$

By inserting (2.9) into (2.8) and noting that $|x_\rho - x| = \rho$, we infer that

$$2\lambda \xi^\top \left(\mathbb{H}_P(\cdot, u, Du) D(\mathbb{H}(\cdot, u, Du)) \right)|_{x_\rho} = 0.$$

By dividing by 2λ and letting $\rho \rightarrow 0$, we deduce that (1.8) is satisfied at the arbitrary $x \in \Omega$.

Conversely, suppose that u satisfies (1.8) on Ω , together with the additional assumptions of the statement.

Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C_0^1(\bar{\mathcal{O}}; \text{span}[\xi])$. Without loss of generality, we may suppose \mathcal{O} is connected. Since $\phi = (\xi^\top \phi)\xi$, for convenience we set $g := \xi^\top \phi$ and then we may write $\phi = g\xi$ with $g \in C_0^1(\bar{\mathcal{O}})$. Then, the matrix-valued map $\mathbb{H}_P(\cdot, Du)$ is pointwise left invertible. Therefore, by (1.8) we have

$$(\mathbb{H}_P(\cdot, Du))^{-1} \mathbb{H}_P(\cdot, Du) D(\mathbb{H}(\cdot, Du)) = 0$$

which, by the connectivity of \mathcal{O} , gives

$$\mathbb{H}(\cdot, Du) \equiv \text{const on } \mathcal{O}.$$

Since $g \in C^1(\mathbb{R}^n)$ with $g = 0$ on $\partial\mathcal{O}$, there exists at least one interior critical point $\bar{x} \in \mathcal{O}$ such that $Dg(\bar{x}) = 0$. By the previous, we have

$$\begin{aligned} E_\infty(u, \mathcal{O}) &= H(\bar{x}, Du(\bar{x})) \\ &= H\left(\bar{x}, Du(\bar{x}) + \xi \otimes Dg(\bar{x})\right) \\ &= H\left(\bar{x}, Du(\bar{x}) + D\phi(\bar{x})\right) \\ &\leq \sup_{x \in \mathcal{O}} H\left(x, Du(x) + D\phi(x)\right) \\ &= E_\infty(u + \phi, \mathcal{O}). \end{aligned}$$

The conclusion ensues.

(II) Suppose that u has ∞ -minimal area. Fix $x \in \Omega$ and $\rho \in (0, \text{dist}(x, \partial\Omega))$. Fix

$$\xi \in \tilde{N}\left(\mathbf{H}_P(\cdot, u, Du)^\top|_x\right),$$

which by Definition 3 is the reduced nullspace of $\mathbf{H}_P(\cdot, u, Du)^\top$ at x . This implies that there exists a C^1 extension $\bar{\xi} \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\bar{\xi}(x) = \xi$ and $(\bar{\xi})^\top \mathbf{H}_P(\cdot, u, Du) = 0$ on the closed ball $\bar{\mathbb{B}}_\varepsilon(x)$ for some $\varepsilon \in (0, \rho)$. By differentiating the relation $(\bar{\xi})^\top \mathbf{H}_P(\cdot, u, Du) = 0$ and taking its trace, we obtain the identity

$$(2.10) \quad \bar{\xi} \cdot \text{div}(\mathbf{H}_P(\cdot, u, Du)) + D\bar{\xi} : \mathbf{H}_P(\cdot, u, Du) = 0,$$

on $\bar{\mathbb{B}}_\varepsilon(x)$. Since u has ∞ -minimal area and $\bar{\xi}$ is an admissible normal variation, by (2.6) it follows that

$$(2.11) \quad \bar{\xi} \cdot \mathbf{H}_\eta(\cdot, u, Du) + D\bar{\xi} : \mathbf{H}_P(\cdot, u, Du) = 0$$

on $(\mathbb{B}_\varepsilon(x))(u)$, where

$$(\mathbb{B}_\varepsilon(x))(u) = \text{Argmax}\{H(\cdot, u, Du) : \bar{\mathbb{B}}_\varepsilon(x)\}.$$

By (2.10)-(2.11), we infer that

$$\bar{\xi} \cdot \left(\text{div}(\mathbf{H}_P(\cdot, u, Du)) - \mathbf{H}_\eta(\cdot, u, Du)\right) = 0,$$

on $(\mathbb{B}_\varepsilon(x))(u)$. By letting $\varepsilon \rightarrow 0$, we deduce that

$$\xi \cdot \left(\text{div}(\mathbf{H}_P(\cdot, u, Du)) - \mathbf{H}_\eta(\cdot, u, Du)\right)\Big|_x = 0,$$

for any $\xi \in \tilde{N}(\mathbf{H}_P(\cdot, u, Du)^\top|_x)$. Hence, u satisfies (1.9) at $x \in \Omega$.

Conversely, suppose that u solves (1.9) on Ω . Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\phi^\top \mathbf{H}_P(\cdot, u, Du) = 0$ on \mathcal{O} . Note further that by the continuity up to the boundary of all functions involved, the latter identity in fact holds on $\bar{\mathcal{O}}$. By the satisfaction of (1.9) and Definition 3, it follows that

$$\phi \cdot \left(\text{div}(\mathbf{H}_P(\cdot, u, Du)) - \mathbf{H}_\eta(\cdot, u, Du)\right) = 0,$$

on $\bar{\mathcal{O}} \subseteq \Omega$. By differentiating $\phi^\top \mathbf{H}_P(\cdot, u, Du) = 0$, we obtain

$$\phi \cdot \text{div}(\mathbf{H}_P(\cdot, u, Du)) + D\phi : \mathbf{H}_P(\cdot, u, Du) = 0,$$

on $\bar{\mathcal{O}}$. By the above two identities, we deduce

$$\phi \cdot \mathbf{H}_\eta(\cdot, u, Du) + D\phi : \mathbf{H}_P(\cdot, u, Du) = 0$$

on $\overline{\mathcal{O}}$. Since $\mathcal{O}(u) \subseteq \overline{\mathcal{O}}$, the identity (2.6) yields u is a critical point. Since by assumption $H(x, \cdot, \cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N \times n}$, it follows that $E_\infty(\cdot, \mathcal{O})$ is convex on $W^{1,\infty}(\mathcal{O}; \mathbb{R}^N)$. Hence, the critical point u is in fact a minimum point for this class of variations. This establishes our claim.

(III) This is an immediate corollary of items (I) and (II). \square

Now we conclude by establishing Theorem 5.

Proof of Theorem 5. (I) \implies (II): If u is an absolute minimiser, then by (2.6)-(2.7) we have

$$\max_{\mathcal{O}(u)} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) = 0,$$

for any $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. By replacing ϕ with $-\phi$, the above yields the pointwise equality

$$H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi = 0$$

on $\mathcal{O}(u)$. Then, for any Radon probability measure $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ with $\text{supp}(\sigma) \subseteq \mathcal{O}(u)$, we have

$$\int_{\overline{\mathcal{O}}} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) d\sigma = 0$$

for all $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. Hence, we have shown that

$$-\text{div}(H_P(\cdot, u, Du)\sigma) + H_\eta(\cdot, u, Du)\sigma = 0,$$

in the dual space $(C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N))^*$.

(II) \implies (III): By assumption we have

$$\int_{\overline{\mathcal{O}}} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) d\sigma = 0$$

for all $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$ and all Radon probability measures $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ with $\text{supp}(\sigma) \subseteq \mathcal{O}(u)$. Fix any $\bar{x} \in \mathcal{O}(u)$. By choosing the Dirac measure $\bar{\sigma} \in \mathcal{P}(\overline{\mathcal{O}})$ given by

$$\bar{\sigma} := \delta_{\bar{x}}$$

which evidently satisfies $\text{supp}(\bar{\sigma}) = \{\bar{x}\} \subseteq \mathcal{O}(u)$, we obtain

$$\begin{aligned} & \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) \Big|_{\bar{x}} \\ &= \int_{\overline{\mathcal{O}}} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) d\bar{\sigma} \\ &= 0, \end{aligned}$$

for any $\bar{x} \in \mathcal{O}(u)$. The conclusion ensues.

(III) \implies (I): By assumption we have

$$H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \equiv 0$$

on $\mathcal{O}(u)$. By (2.6)-(2.7), this implies

$$\frac{d}{dt} \Big|_{t=0} E_\infty(u + t\phi, \mathcal{O}) = \max_{\mathcal{O}(u)} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) = 0.$$

Hence, since by assumption $H(x, \cdot, \cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N \times n}$, it follows that $E_\infty(\cdot, \mathcal{O})$ is convex on $W^{1,\infty}(\mathcal{O}; \mathbb{R}^N)$. Hence, the critical point u is in fact a minimum point and this completes the proof. \square

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B.A.: DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF READING, WHITEKNIGHTS,
 PO BOX 220, READING RG6 6AX, UNITED KINGDOM
E-mail address: b.ayanbayev@pgr.reading.ac.uk

N.K.: (CORRESPONDING AUTHOR) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY
 OF READING, WHITEKNIGHTS, PO BOX 220, READING RG6 6AX, UNITED KINGDOM
E-mail address: n.katzourakis@reading.ac.uk