

Riemann-Hilbert Problems and their applications in
mathematical physics

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Thesis submitted for the degree of Doctor of Philosophy

April, 2017

Declaration: I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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Abstract

The aim of this thesis is to present the reader with the very effective and rigorous Riemann-Hilbert approach of solving asymptotic problems. We consider a transition problem for a Toeplitz determinant; its symbol depends on an additional parameter t . When $t > 0$, the symbol has one Fisher-Hartwig singularity at an arbitrary point $z_1 \neq 1$ on the unit circle (with associated $\alpha_1, \beta_1 \in \mathbb{C}$ strengths) and as $t \rightarrow 0$, a new Fisher-Hartwig singularity emerges at the point $z_0 = 1$ (with $\alpha_0, \beta_0 \in \mathbb{C}$ strengths). The asymptotics we present for the determinant are uniform for sufficiently small t . The location of the β -parameters leads to the consideration of two cases, both of which are addressed in this thesis. In the first case, when $|\operatorname{Re} \beta_0 - \operatorname{Re} \beta_1| < 1$ we see a transition between two asymptotic regimes, both given by the same result by Ehrhardt, but with different parameters, thus producing different asymptotics. In the second case, when $|\operatorname{Re} \beta_0 - \operatorname{Re} \beta_1| = 1$ the symbol has Fisher-Hartwig representations at $t = 0$, and the asymptotics are given the Tracy-Basor conjecture. These double scaling limits are used to explain transition in the theory of XY spin chains between different regions in the phase diagram across critical lines.

Acknowledgements

Firstly, I would like to express my great appreciation to my supervisor, Jani Virtanen. His unfailing support and assistance through this work was invaluable. I would also like to offer my special thanks to Beatrice Pelloni, who shared supervision over this project with Jani in the early years of my PhD.

I would like to thank all of the faculty and administrative staff at Reading University. I am particularly grateful for the assistance given by Titus Hilberdink, especially in the final days of writing up my thesis. I am also grateful for the emotional support and encouragement provided by Peta-Ann King, and the rest of the lovely administrative staff: Brigitte, Ruth, Marzena and Kris for being a constant source of support and enjoyable chatter!

I would like to express my deep gratitude to Percy Deift for giving me the opportunity to visit Courant Institute and allowing me to study Riemann-Hilbert problems directly under his wing. I also wish to thank Tom Claeys for allowing me to visit him at UCL in Louvain-la-Neuve and the constructive discussions which ensued. I would like to extend my special thanks to Igor Krasovsky for informing me of the problem contained in this thesis and for helping me understand it.

I would like to acknowledge the financial support of the EPSRC in funding my studentship and the University of Reading and the IMA for helping to fund my numerous trips to conferences and visits.

I am particularly grateful for the constant encouragement and support of my fellow PhD students. Anne, who provided me with refreshing distractions and helpful insights; Noeleene and Aamena, for being a huge motivation in their hard working presence; Andrew, who provided our office with much needed homely cosiness and inspirations for healthy eating habits; James, who liked to pop in for a distracting chat; and Hasen for drinking gallons of tea and coffee with me during the good and the bad times alike.

I would like to say a huge thank you to my boyfriend and best friend Jamie for his unwavering support during difficult times, a pair of eyes outside of the box and a computer specialist when I needed it. I would like to thank my brother Michal for his encouragement and constructive conversations. Finally, I would like to thank my parents, and the rest of my family for always believing in me.

Dziękuję! Danke! Thank you!

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Introduction

The aim of this thesis is to acquaint the reader with the greatly effective and rigorous Riemann-Hilbert approach to solving many asymptotic problems. These types of problems have applications in statistical mechanics, most prominent being the spontaneous magnetisation problem for the Ising model, which is discussed throughout Sections 1.3, 1.4 and 1.5, also see [15] for a thorough review by Deift, Its and Krasovsky. For applications to random matrix theory and asymptotic results for various orthogonal polynomials see for example [10], [12], [26], [31], [32].

Chapter 1 reviews established techniques and ideas that are subsequently used to solve the problem in Chapter 2, as well as historical reasoning behind some of the results. In this thesis we consider a transition problem for a Toeplitz determinant; its symbol depends on an additional parameter t and the asymptotics we obtain for the determinant are uniform for sufficiently small t . A considerable role in this problem is played by a Painlevé V function $\sigma(x)$, which is the same function that was considered by Claeys, Its and Krasovsky in [11]. The symbol considered within this thesis has one Fisher-Hartwig singularity at an arbitrary point z_1 on the unit circle for a $t > 0$ (with associated α_1 and β_1 strengths, where $\alpha_1, \beta_1 \in \mathbb{C}$) and as $t \rightarrow 0$, a new Fisher-Hartwig singularity emerges (away from z_1) at the point $z_0 = 1$ (with α_0 and β_0 strengths, $\alpha_0, \beta_0 \in \mathbb{C}$). The location of the β -parameters leads to the consideration of two cases, both of which are addressed in this thesis. The first case provides an expression unifying two asymptotic regimes—the first regime describes the Toeplitz determinant with one Fisher-Hartwig singularity for $0 < t < t_0$, and the second regime arises for a $t = 0$, when the determinant has two singularities and the seminorm $|||\beta||| = |\operatorname{Re} \beta_0 - \operatorname{Re} \beta_1| < 1$, where the complete definition of the seminorm of β -parameters can be found in (1.4.10). These two asymptotic regimes are both given by the same result of Ehrhardt [22] (but with different parameters, thus producing different asymptotics). In the case when $|||\beta||| = |\operatorname{Re} \beta_0 - \operatorname{Re} \beta_1| = 1$ we see a transition between asymptotics of a determinant with one singularity in the symbol and an

asymptotic regime for a determinant whose symbol possesses Fisher-Hartwig representations. For a symbol possessing these representations the asymptotics for the associated Toeplitz determinant are then given as a linear combination of those same results by Ehrhardt; each term corresponding to a new, non-trivial representation of the symbol. This is referred to as the Tracy-Basor conjecture, or the generalised Fisher-Hartwig asymptotics. We use the nonlinear steepest descent method which was introduced by Deift and Zhou in [20] and further developed by Deift, Venakides and Zhou in [19] and by Deift, Kriecherbauer, McLaughlin, Venakides and Zhou in [17] and [18], to solve the Riemann-Hilbert problem for orthogonal polynomials with the weight given by our symbol. We also utilise many results that were described in [11], where the authors presented a transition between a smooth symbol and one possessing one Fisher-Hartwig singularity, and [14] where Deift, Its and Krasovsky considered the case of $m + 1$ number of fixed singularities. To obtain the result for $\|\beta\| = |\operatorname{Re} \beta_0 - \operatorname{Re} \beta_1| = 1$, we used ideas from the proof of the Tracy-Basor conjecture given in [14] to find a relation between the Toeplitz determinant whose symbol possesses representations and a determinant with no representations and asymptotics of related orthogonal polynomials.

This transition problem for a Toeplitz determinant can be applied to statistical mechanics. This application arises from the work of Franchini and Abanov in [25] on the emptiness formation probability for the one-dimensional anisotropic XY spin-1/2 chain in a transverse magnetic field. In Chapter 3 we explore this application in more detail.

Chapter 1

The Essentials

1.1 Toeplitz Determinants

Throughout this thesis, we will often be talking about the analyticity of certain functions. Recall from a first course in complex analysis, that an *analytic function* is a function which has a Taylor series expansion at every point in the region of its analyticity. Such functions are holomorphic—satisfying the Cauchy-Riemann equations—and so, infinitely differentiable. We will denote the unit circle in \mathbb{C} by \mathbb{T} and *Fourier coefficients* of a function $g(z) \in L^1(\mathbb{T})$ by

$$g_k = \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) e^{-ik\theta} d\theta. \quad (1.1.1)$$

Definition 1.1.1. For $1 \leq p \leq \infty$, we define the **Hardy space** $H^p(\mathbb{T})$, **on the circle**, to be the following,

$$H^p(\mathbb{T}) = \{g \in L^p(\mathbb{T}) : g_k = 0, k < 0\}.$$

It is a closed subspace of $L^p(\mathbb{T})$, making it a Banach space.

Definition 1.1.2. The **Hardy space on the open unit disk** \mathbb{D} is defined to be

$$H^p(\mathbb{D}) = \left\{ g \in H(\mathbb{D}) : \sup_{0 < r < 1} \left(\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \right)^{1/p} < \infty \right\}$$

where $H(\mathbb{D})$ denotes the space of analytic functions on \mathbb{D} .

The two spaces are isometrically isomorphic which allows us to write simply H^p below.

Consider a function $g \in L^2(\mathbb{T})$; we can write it uniquely as a Fourier series, $g(z) = \sum_{k=-\infty}^{\infty} g_k e^{ik\theta}$ with respect to the orthonormal basis $(e_k)_{k=-\infty}^{\infty}$ on $L^2(\mathbb{T})$, where $e_k(\theta) = e^{ik\theta}$ for $k \in \mathbb{Z}$.

Definition 1.1.3. The *orthogonal projection* $P : L^2(\mathbb{T}) \rightarrow H^2$ mapping $g(z)$ onto H^2 is given by

$$\sum_{k=-\infty}^{\infty} g_k e^{ik\theta} \longmapsto \sum_{k=0}^{\infty} g_k e^{ik\theta}. \quad (1.1.2)$$

This projection can also be viewed as a singular integral operator, see (1.2.2).

Definition 1.1.4. The operator $T_f : H^2 \rightarrow H^2$, for $f \in L^\infty(\mathbb{T})$, defined as

$$T_f g = P(fg)$$

is called the **Toeplitz operator** with symbol f .

We multiply two functions together, g in H^2 and a prescribed function f in L^∞ . The resulting function fg is in L^2 , and so we project it back onto H^2 . Some authors also define the multiplication operator M_f , to write the Toeplitz operator as $T_f g = P M_f g$. The above definition specifies that the function f is in L^∞ . This is a sufficient and necessary reason for the operator to be bounded. If we take a function with Fisher-Hartwig singularities (1.4.1) for example, the resulting Toeplitz operator is unbounded.

Definition 1.1.5. Let $f(z) \in L^1(\mathbb{T})$. We call

$$T(f) = \begin{pmatrix} f_0 & f_{-1} & f_{-2} & \cdots \\ f_1 & f_0 & f_{-1} & \cdots \\ f_2 & f_1 & f_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (1.1.3)$$

the **Toeplitz matrix** with symbol $f(z)$.

Such matrices are representations of the Toeplitz operator T_φ with respect to the standard basis $\{e^{ik\theta} : k \geq 0\}$ acting on the sequence space $l^2(\mathbb{Z}_+) = \{u = (u_0, u_1, \dots) : \sum_{i=0}^{\infty} |u_i|^2 < \infty\}$.

One way of seeing this, is to let $f(e^{ik\theta}) = \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}$ as $f \in L^\infty(\mathbb{T})$, then

$$T_f e_n = P \sum_{k=-\infty}^{\infty} f_k e^{ikt} e^{int} = \sum_{p=0}^{\infty} f_{p-n} e^{ipt}, \quad \text{where } p = n + k.$$

Another way is by considering the inner product in L^2 ,

$$\begin{aligned} f_{jk} &= (T_f e_k, e_j) = (P M_f e_k, e_j) = (f e_k, P^* e_j) = (f e_k, e_j) = \int_0^{2\pi} f(e^{i\theta}) e^{ik\theta} e^{-ij\theta} \frac{d\theta}{2\pi} \\ &= \int_{\mathbb{T}} f(e^{i\theta}) e^{(k-j)i\theta} \frac{d\theta}{2\pi} = f_{j-k}. \end{aligned}$$

For great reference on matrix representation of operators (among other things) see [8].

The Toeplitz matrix is a semi-infinite matrix—entries continue for infinity in the lower (in this case) half of the matrix—and so to understand its determinant, we need to make the matrix finite first. We will then look at the asymptotic behaviour of the determinant as we increase the size of the matrix to infinity.

Definition 1.1.6. Given $f_k \in \mathbb{C}$ we denote by $T_n(f)$ the $n \times n$ **finite sections** of the matrix in (1.1.3), i.e.

$$T_n(f) = \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{-(n-1)} \\ f_1 & f_0 & \cdots & f_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1} & f_{n-2} & \cdots & f_0 \end{pmatrix}, \quad (1.1.4)$$

and its **determinant** by $D_n(f) = \det T_n(f)$.

1.2 Riemann-Hilbert Problems

In what follows we take Σ to be a Carleson curve as they are the most general setting for the results below to hold. For the definition of a Carleson curve, see [6, Chapter 1]. Let us give a short example.

Example 1.2.1. [6, Example 1.3] Let $\alpha > 0$, and define

$$\Gamma = \{\tau \in \mathbb{C} : \tau = x + if(x), 0 \leq x \leq 1\}.$$

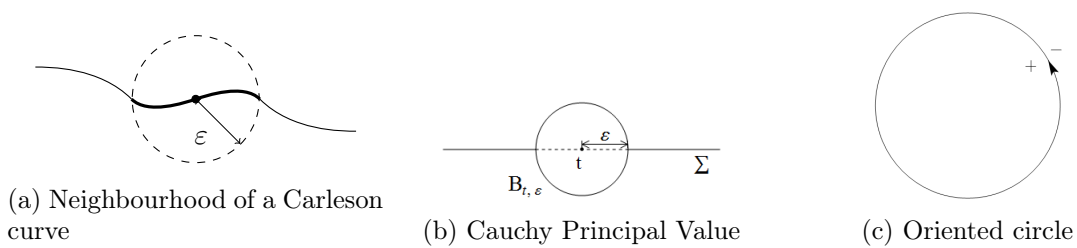


Figure 1.1

Let

$$f(x) = x^\alpha \sin(1/x) \quad \text{for } x \in (0, 1] \quad \text{and } f(0) = 0.$$

The function $f(x)$ is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$. It is also bounded on $[0, 1]$. For $\alpha \geq 2$ its derivative is bounded on $(0, 1)$ and Γ is Carleson. However for $0 < \alpha < 2$, the curve is not Carleson.

We do not go into any detail as to what these curves are as we do not require it, we merely mention this for completeness. Throughout this section we follow [1], [6], [21] and [38].

Definition 1.2.2. *The Cauchy singular integral operator $S : L^p(\Sigma) \rightarrow L^p(\Sigma)$ is given by*

$$(Sf)(t) = \frac{1}{\pi i} \oint_{\Sigma} \frac{f(\tau)}{\tau - t} d\tau.$$

This integral exists in the Cauchy principal value sense, often also denoted by PV , defined as

$$\oint_{\Sigma} \frac{f(\tau)}{\tau - t} d\tau = PV \int_{\Sigma} \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \rightarrow 0} \int_{\Sigma \setminus B_{t,\varepsilon}} \frac{f(\tau)}{\tau - t} d\tau,$$

where we first integrate along Σ with a small ball centred at the singularity t of radius ε (i.e. $B_{t,\varepsilon}$) removed (see Figure 1.1b), and then let ε decrease to 0. If the limit exists, we say that it exists in Cauchy principle value sense.

Given an oriented curve we can define positive and negative regions on the left and right side of the curve to the direction of travel respectively, see Figure 1.1c for example.

Theorem 1.2.3. (Plemelj formulae, [1, Lemma 7.2.1]) *Let Σ be a (Carleson) curve. For every $\varphi \in L^p(\Sigma)$, $1 < p < \infty$, define*

$$\Phi(z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{\varphi(\tau)}{\tau - z} d\tau, \quad z \in \mathbb{C} \setminus \Sigma.$$

The following Plemelj formulae hold,

$$\Phi^\pm(t) = \frac{1}{2} (\pm\varphi(t) + (S\varphi)(t)). \quad (1.2.1)$$

Here, Φ^+ denotes the limit as $z \in \mathbb{C}$ tends to a point $t \in \Sigma$ entirely from the positive side of Σ non-tangentially (within a sector of opening angle less than π). An analogous definition holds for Φ^- (cf. Figure 1.1c for $\Sigma = \mathbb{T}$). The projection (Definition 1.1.3) can be written in integral form as $P = \frac{1}{2}(I + S)$. Denote by $Q = I - P = \frac{1}{2}(I - S)$ its orthogonal complement, then

$$\Phi^+ = P\varphi \quad \text{and} \quad \Phi^- = -Q\varphi. \quad (1.2.2)$$

Further, (1.2.1) is thus equivalent to

$$\Phi^+ - \Phi^- = (P + Q)\varphi = I\varphi \quad (1.2.3)$$

and

$$\Phi^+ + \Phi^- = (P - Q)\varphi = S\varphi. \quad (1.2.4)$$

The Plemelj formulae are a celebrated result which is used time and time again. The equation (1.2.3) above is what is known as an additive Riemann-Hilbert problem, with the jump φ . Using Plemelj's formulae, it is straight-forward to proceed to its solution, it is just the singular integral, $\Phi = \frac{1}{2}S\varphi$. The reason why this is so celebrated is that it provides a way to tackle (2×2) matrix Riemann-Hilbert problems. The idea is to reduce it to a form where it can be solved using this result. The usual point of action in higher dimensional cases (see for example [32] for higher dimensional R-H problem for multiple orthogonal polynomials), is to try and reduce it to a lower dimensional problem. In general, the matrix Riemann-Hilbert problem, which from now on will be referred to as R-H problem, is defined in the following way.

Definition 1.2.4. (Riemann-Hilbert Problem) *Let $\Sigma \subset \mathbb{C}$ be a smooth oriented contour and let there be a map from the contour mapping to the space of $n \times n$ invertible matrices (called the jump), $v : \Sigma \rightarrow GL(n, \mathbb{C})$. Given the pair (v, Σ) , R-H problem is the problem of finding an $n \times n$ matrix valued function $m(z)$ such that*

1. m is analytic in $\mathbb{C} \setminus \Sigma$

$$2. m_+(t) = v(t)m_-(t), t \in \Sigma$$

$$3. m(z) \rightarrow I \text{ as } z \rightarrow \infty.$$

A matrix-valued function is a matrix whose entries are functions. We say a matrix-valued function is analytic if each of the entry functions are analytic in the usual sense. Condition 3 above is not necessary, it is possible to work with an R-H problem that is not normalised at infinity, the first step to tackle it would be just to transform it into a normalised R-H problem. In this thesis, I will be considering 2×2 R-H problem, which have to go through series of various transformations which with every turn make the problem easier/possible to solve. The well-posedness of the R-H problem is equivalent to the question of the existence and uniqueness of a system of Fredholm¹ singular integral equations; see [35] for more details.

1.3 Asymptotic behaviour of Toeplitz determinants

This thesis deals with the asymptotics of truncated Toeplitz determinants as we increase the size of the matrix, $n \rightarrow \infty$. It is thus necessary for us to outline the following notation which will be used throughout. Note that the two definitions below cover complex-valued functions. If $z \in \mathbb{C}$ and we have say, $f(z) = \mathcal{O}(1/z)$ as $z \rightarrow \infty$, we mean $f(|z|) = \mathcal{O}(1/|z|)$ as $|z| \rightarrow \infty$.

Definition 1.3.1. [1, Definition 6.1.1]

1.
 - The notation $f(z) = \mathcal{O}(g(z))$ as $z \rightarrow z_0$ means there exists a finite constant $M > 0$ in the neighbourhood of z_0 such that $|f| \leq M|g|$.
 - The notation $f(z) = \mathcal{O}(g(z))$ as $z \rightarrow \infty$ means there exists a constant $M > 0$, such that $|f| \leq M|g|$ for $z > z_0$, for some sufficiently large z_0 .
2. The notation $o(g(z)) = f(z)$ as $z \rightarrow z_0$ means that

$$\lim_{z \rightarrow z_0} \left| \frac{f(z)}{g(z)} \right| = 0.$$

Here we note that throughout this thesis, the big $\mathcal{O}(\cdot)$ and small $o(\cdot)$ notations are used interchangeably as a matrix with each of the elements with that order or just a scalar, depending on the

¹An operator is Fredholm if it is a bounded linear operator between two Banach spaces with a finite-dimensional kernel and cokernel.

context, i.e. we could write,

$$I + \mathcal{O}(1/n) = \begin{pmatrix} 1 + \mathcal{O}(1/n) & \mathcal{O}(1/n) \\ \mathcal{O}(1/n) & 1 + \mathcal{O}(1/n) \end{pmatrix}. \quad (1.3.1)$$

Definition 1.3.2. [1, Definition 6.1.2]

1. An ordered sequence of functions $\{\delta_j(z)\}, j = 1, 2, \dots$ is called an asymptotic sequence as $z \rightarrow z_0$ if

$$\delta_{j+1}(z) = o(\delta_j(z)), \quad z \rightarrow z_0$$

for each j .

2. Let $I(z)$ be continuous in z . Let $\{\delta_j(z)\}$ be an asymptotic sequence as $z \rightarrow z_0$. The formal series $\sum_{j=1}^N a_j \delta_j(z)$ is called an asymptotic expansion of $I(z)$, as $z \rightarrow z_0$, valid to order $\delta_N(z)$, if

$$I(z) = \sum_{j=1}^m a_j \delta_j(z) + \mathcal{O}(\delta_{m+1}(z)), \quad z \rightarrow z_0, \quad m = 1, 2, \dots, N.$$

If we denote $\eta(z) := \sum_{j=1}^N a_j \delta_j(z)$. This is the same as saying,

$$I(z) \sim \eta(z), \quad z \rightarrow z_0,$$

or

$$\lim_{z \rightarrow z_0} \left| \frac{I(z)}{\eta(z)} \right| = 1.$$

The central result in the study of asymptotic behaviour of Toeplitz determinants is the theorem by Gábor Szegő. In 1915 [41], he proved the following result which was conjectured by George Pólya and provides the leading order asymptotics for a Toeplitz determinant with a smooth symbol.

Theorem 1.3.3. (Szegő's First Theorem) *Let $\varphi(e^{i\theta}) > 0$ be a continuous, positive function on \mathbb{T} and $D_n(\varphi)$ be the associated Toeplitz determinant (see Definition 1.1.6), then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log D_n(\varphi) = (\log \varphi)_0. \quad (1.3.2)$$

Note that we are using the notation from (1.1.1), $(\log \varphi)_k = \frac{1}{2\pi} \int_0^{2\pi} \log \varphi(e^{i\theta}) e^{-ik\theta} d\theta$. This

equation can be rewritten equivalently as follows,

$$D_n(\varphi) = \exp \{n (\log \varphi)_0 + o(n)\}, \quad (1.3.3)$$

as $n \rightarrow \infty$.

The full asymptotics were computed in 1952, again by Szegő [42]. He revisited his calculations from nearly 40 years before, encouraged by the work of Lars Onsager and Bruria Kaufman on the Ising model in the 1940's. The following, is a celebrated result, which finds uses to this day and was generalised many times over by a multitude of mathematicians from diverse backgrounds. It has several proofs from different walks of mathematics—many of them can be found in the OPUC book by Barry Simon [40]; see also [8, 9].

Theorem 1.3.4. (Szegő's Strong Limit Theorem (SSLT)) *Let $\varphi(e^{i\theta})$ be positive, $C^{1+\varepsilon}$ (φ is C^1 with φ' Hölder continuous of some positive order $\varepsilon > 0$) function on \mathbb{T} , then*

$$\lim_{n \rightarrow \infty} \frac{D_n(\varphi)}{e^{n(\log \varphi)_0}} = e^{E(\varphi)}, \quad (1.3.4)$$

where

$$E(\varphi) = \sum_{k=1}^{\infty} k |(\log \varphi)_k|^2. \quad (1.3.5)$$

Theorem 1.3.4 evaluates the error term in (1.3.3), it is given by $o(n) = E(\varphi) + o(1)$. This error term was of the main interest to Onsager and Kaufman at the time.

1.4 Fisher and Hartwig

A number of problems in statistical mechanics have brought the need to consider more complicated symbols for Toeplitz matrices. Instead of continuous or analytic functions, these new symbols were now required to have zeros, integrable singularities and non-zero winding numbers. Michael Fisher and Robert Hartwig have devised a practical way to factorise out these singularities, see [23]. They are referred to as Fisher-Hartwig (F-H) symbols and are defined on the unit circle in the following way,

$$f(z) = e^{V(z)} z^{\sum_{j=0}^m \beta_j} \prod_{j=0}^m |z - z_j|^{2\alpha_j} g_{z_j, \beta_j}(z) z_j^{-\beta_j}, \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi), \quad (1.4.1)$$

for some $m = 0, 1, \dots$, where

$$z_j = e^{i\theta_j}, \quad j = 0, \dots, m, \quad 0 = \theta_0 < \theta_1 < \dots < \theta_m < 2\pi,$$

$$g_{z_j, \beta_j}(z) = \begin{cases} e^{i\pi\beta_j}, & \text{if } 0 \leq \arg z < \theta_j, \\ e^{-i\pi\beta_j}, & \text{if } \theta_j \leq \arg z < 2\pi. \end{cases} \quad (1.4.2)$$

and

$$\operatorname{Re} \alpha_j > -1/2, \quad \beta_j \in \mathbb{C}, \quad j = 0, \dots, m, \quad (1.4.3)$$

and $V(z)$ is analytic in a neighbourhood of the unit circle.

To put this into words, the symbol has $m + 1$ singularities at the pairwise distinct points $z_j = e^{i\theta_j}$, $j = 0, \dots, m$, $\theta_j \in [0, 2\pi)$. The root type singularities exist for a non-zero $\alpha_j \in \mathbb{C}$, and we stipulate that $\operatorname{Re} \alpha_j > -1/2$ to ensure integrability. The jump type singularities are given for non-zero $\beta_j \in \mathbb{C}$. For the purposes of this thesis, the function $V(z)$ is analytic in a neighbourhood of the unit circle, however in [14], the authors have proven results for a more general $V(z)$, see (1.4.14). In this thesis, we take $\arg z \in [0, 2\pi)$, unless stated otherwise.

One of the driving forces behind generalising the SSLT (Theorem 1.3.4) to this new class of symbols, was the hunt for the solution of the spontaneous magnetisation problem for the Ising model. I refer the reader to [15] for a thorough and captivating account of the history of the problem and the path which was taken in generalising SSLT. The Ising model in 2 dimensions concerns the interaction of random spins $\sigma_{i,j} = \pm 1$ at sites $(i, j) \in \mathbb{Z}^2$ at a temperature T . One is interested in determining the magnetisation of the system depending on the temperature, by first considering a finite box² in \mathbb{Z}^2 with size depending on n , and then letting $n \rightarrow \infty$. There exists a critical temperature T_c , also called the Curie point. For instance, given a magnet, there will exist a critical temperature, such that for temperatures $T < T_c$, it will exhibit spontaneous magnetisation, whereas for $T > T_c$ the magnetisation will be zero. The square of the magnetisation can be expressed as the large n limit of a Toeplitz determinant with a particular symbol. More precisely,

$$M_T^2 = \lim_{n \rightarrow \infty} \langle \sigma_{1,1}, \sigma_{1,n+1} \rangle_T,$$

²With suitable boundary conditions. There are several one can consider, cf [15].

where the inner product denotes the correlations between the nearest neighbouring spins. Thanks to the work of Lars Onsager and Bruria Kaufman we know that the correlation function can be written as a determinant of a Toeplitz matrix,

$$\langle \sigma_{1,1}, \sigma_{1,n+1} \rangle_T = D_n(f_T)$$

where the symbol f_T is given by

$$f_T(z) = \left[\frac{(z_1 z_2 z - 1)(z_1 z - z_2)}{(z - z_1 z_2)(z_2 z - z_1)} \right]^{1/2}. \quad (1.4.4)$$

The case $0 < z_2 < z_1 < 1$ corresponds to $T < T_c$, and the net winding of f_T is 0. One can use the SSLT in this case (Theorem 1.3.4—extended to complex symbols, see [15, Section 3]) to get the expression of the magnetisation. In the case when $0 < z_1 < z_2 < 1$ we have that $T > T_c$ and the winding number of f_T is -1 . The SSLT breaks down and we have to consider a determinant of a Toeplitz matrix with a F-H singularity. There are complicated cancellations in the asymptotic formula of the determinant, but one finally achieves 'zero' to be the magnetisation—meaning there is none.

The symbol for $T > T_c$ can be written as

$$f_T(z) = e^{V(z)} z^{-1} g_{1,-1}(z) (1)^{+1}, \quad (1.4.5)$$

which is a F-H symbol with a singularity at 1 with (strength) $\beta_0 = -1$ (compare with (1.4.1), $\alpha_j = 0$, $j = 0, \dots, m$, $\beta_k = 0$, $k = 1, \dots, m$).

Remark 1.4.1. There are other ways of writing Fisher-Hartwig symbols. They are considered in the form (1.4.1) in the works of Claeys, Deift, Its and Krasovsky because it makes it easier in the method they use (via R-H problems). In the works of Basor, Böttcher, Ehrhardt, Fisher, Hartwig, Silberman, Tracy and Widom (among many others), the symbols are written as,

$$f(z) = b(z) \prod_{j=1}^m |z - z_j|^{2\alpha_j} \varphi_{z_j, \beta_j}(z), \quad (z \in \mathbb{T}), \quad (1.4.6)$$

where

$$\varphi_{z_j, \beta_j}(z) := \exp \{ i \beta_j \arg(-z/z_j) \}. \quad (1.4.7)$$

Here we take $\arg z \in (-\pi, \pi]$. One can draw parallels between the two immediately. The pioneers of the Riemann-Hilbert approach changed the argument of z , which particularly affected the function $\varphi(z)$, and factorised out some of the terms. The factors $z_j^{-\beta_j}$ are singled out to simplify comparisons with other literature and the $z^{\sum_{j=0}^{\infty} \beta_j}$ are factored out because of the nature of the work which was carried out in [14] on the Tracy-Basor conjecture (see Section 1.4.2). The only real difference between the two, are the conditions on $V(t)$ and $b(z)$ which is a continuous function. In [16] the authors also relaxed the conditions on $V(z)$.

Notice also that the product in (1.4.1) begins at $j = 0$ and the authors look at $m+1$ singularities. In their papers, the authors fix $\theta_0 = 0$ and so $z_0 = 1$ always. This is without any loss of generality however, as the problems considered loc. cit. are translation invariant.

1.4.1 F-H Conjecture

Many people have worked on producing the asymptotics of the determinant of a Toeplitz matrix associated with the F-H symbol. A thorough historical account can be found in [15, Section 6], where the authors present the mathematical and physical motivation for this result, as well as who produced what results and when. In short summary, it was Andrew Lenard in [33, 34] and Fisher and Hartwig in [23], who propelled the study of Toeplitz determinants with discontinuous symbols and conjectured the results for asymptotics, close together in time. In [45], Harold Widom verified the conjecture made by Lenard which differed only by the parameters $\beta_j = 0, \forall j$ from the conjecture made by Fisher and Hartwig. Subsequently, Estelle Basor in [2] worked on adding β singularities and then together with J. William Helton in [3] analyzed pure F-H singularities (with no analytic part, $V(z) \equiv 0$) using iterative techniques. Albrecht Böttcher and Bernd Silbermann have obtained in [7] an explicit formula for the determinant with such pure F-H symbols,

$$D_n(f) = \frac{G(1 + \alpha + \beta)G(1 + \alpha - \beta)}{G(1 + 2\alpha)} \frac{G(n)G(n + 2\alpha)}{G(n + \alpha + \beta)G(n + \alpha - \beta)}$$

and $D_n(f) = 0$ if $\alpha \pm \beta$ is a negative integer. This formula played an important role in their proof (also in [7]) of the conjecture posed by Fisher and Hartwig in its original form. The function $G(z)$ is the Barnes G -function, which is an entire function defined as

$$G(z + 1) = (2\pi)^{z/2} e^{-z(z+1)/2 - Cz^2/2} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n e^{-z + z^2/(2n)} \right\}, \quad (1.4.8)$$

where $C = 0.577\dots$ is the Euler's constant. It is worth noting the following identity which draws parallels between the Barnes G - and the Gamma $\Gamma(z)$ functions, just as $\Gamma(z+1) = z\Gamma(z)$, we have that $G(z+1) = \Gamma(z)G(z)$. The following asymptotic relation can also be derived from (1.4.8),

$$\frac{G(n)G(n+\gamma+\delta)}{G(n+\gamma)G(n+\delta)} \sim n^{\gamma\delta} \quad \text{as } n \rightarrow \infty \quad (1.4.9)$$

for an arbitrary $\delta, \gamma \in \mathbb{C}$.

The leading order asymptotics of Toeplitz determinants with F-H symbols were conclusively computed by Ehrhardt in his PhD thesis in 1997 (see also [22]). To state the result, we first need to introduce the following seminorm,

$$\|\beta\| = \max_{j,k} |\operatorname{Re} \beta_j - \operatorname{Re} \beta_k|, \quad (1.4.10)$$

where $1 \leq j, k \leq m$ if $\alpha_0 = \beta_0 = 0$, and $0 \leq j, k \leq m$ otherwise. If $m = 0$, set $\|\beta\| = 0$.

The Wiener-Hopf factorisation of the function $e^{V(t)}$ ($V(z)$ which appeared in (1.4.1)) is given by,

$$e^{V(z)} = b_+(z)e^{V_0}b_-(z), \quad b_+(z) = e^{\sum_{k=1}^{\infty} V_k z^k}, \quad b_-(z) = e^{\sum_{k=-\infty}^{-1} V_k z^k} \quad (1.4.11)$$

where V_k are the Fourier coefficients of the function $V(t)$.

Theorem 1.4.2. *Let $f(z)$ be given by (1.4.1), $V(z) \in C^\infty(\mathbb{T})$, $\|\beta\| < 1$, $\operatorname{Re} \alpha_j > -\frac{1}{2}$ and $\alpha_j \pm \beta_j \neq -1, -2, \dots$ for $j, k = 0, 1, \dots, m$. Then as $n \rightarrow \infty$,*

$$\begin{aligned} D_n(f) &= \exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \prod_{j=0}^m b_+(z_j)^{-(\alpha_j - \beta_j)} b_-(z_j)^{-(\alpha_j + \beta_j)} \\ &\times n^{\sum_{j=0}^m (\alpha_j^2 - \beta_j^2)} \prod_{0 \leq j < k \leq m} |z_j - z_k|^{2(\beta_j \beta_k - \alpha_j \alpha_k)} \left(\frac{z_k}{z_j e^{i\pi}} \right)^{\alpha_j \beta_k - \alpha_k \beta_j} \\ &\times \prod_{j=0}^m G_{\alpha_j + \beta_j, \alpha_j - \beta_j} (1 + o(1)). \end{aligned} \quad (1.4.12)$$

Here, the product over $j < k$ is set to 1 if $m = 0$ and

$$G_{\alpha_j + \beta_j, \alpha_j - \beta_j} = \frac{G(1 + \alpha_j + \beta_j)G(1 + \alpha_j - \beta_j)}{G(1 + 2\alpha_j)}. \quad (1.4.13)$$

In [14, 16] the authors prove Theorem 1.4.2 again, but using Riemann-Hilbert problem approach and they also generalise it further to hold for less smooth functions $V(t)$; finite degree of smoothness, more specifically, satisfying condition

$$\sum_{k=-\infty}^{\infty} |k|^s |V_k| < \infty \quad (1.4.14)$$

for s such that

$$s > \frac{1 + \sum_{j=0}^m [(\operatorname{Im} \alpha_j)^2 + (\operatorname{Re} \beta_j)^2]}{1 - \|\beta\|}. \quad (1.4.15)$$

In [14], the authors also prove the result for $\|\beta\| = 1$ if $V(z) \in C^\infty(\mathbb{T})$, i.e. the generalised F-H, or Tracy-Basor conjecture which stems from F-H representations, all of which is described in the next section.

Remark 1.4.3. Note, that for $T > T_c$ in the Ising model, the F-H symbol, given by (1.4.5), has the parameters $\alpha = 0$, $\beta = -1$. We call this a degenerate case of the Theorem 1.4.2. The reason for this degeneracy is a property of the Barnes G -function, which will vanish if the sum or difference of α_j and β_j is a negative integer. The symbol can be written in the following way, $f_T(z) = -z^{-1} \tilde{f}(z)$, with $\tilde{f}(z)$ being a smooth function. One has the following relation for the two Toeplitz determinants (see also Lemma 2.6.1),

$$D_n(f) = \hat{\pi}_n(0) D_n(\tilde{f}),$$

where $\hat{\pi}_n(z)$ are orthogonal polynomials with respect to the weight $\tilde{f}(z)$, see (1.6.1). One can compute the asymptotics of the orthogonal polynomials using for example a R-H problem and the determinant $D_n(\tilde{f})$ is given using SSLT as \tilde{f} is smooth.

1.4.2 F-H representation and the Tracy-Basor Conjecture

Basor and Craig Tracy considered in [4] a symbol with two jump singularities, at $z_0 = 1$ and $z_1 = e^{i\pi} = -1$ with strengths $\beta_0 = 1/2$ and $\beta_1 = -1/2$ respectively, which meant that the seminorm, $\|\beta\| = 1$. Their asymptotics for the determinant of the associated Toeplitz matrix as $n \rightarrow \infty$ were not of the general F-H asymptotic form. However, upon closer inspection they arrived at the conclusion, that the asymptotics they obtained were in fact a linear combination of two F-H asymptotic forms (1.4.12). This gave birth to their conjecture, now Theorem 1.4.6. One asymptotic form corresponded to the original symbol as though $\|\beta\| < 1$, and the second was again F-H

asymptotics but for a symbol with jump singularities at the same sites, $z_0 = 1$ and $z_1 = e^{i\pi} = -1$, but with strengths $\beta_0 = -1/2$ and $\beta_1 = 1/2$. The second symbol only differed from the original one by a constant. This is indeed the case for any F-H symbol. One can translate the β parameters by integer values, as described in the definition below and because of uniformity in the α and β parameters [14, Remark 1.6], one can apply the translations to the asymptotics without obtaining anything new. However, the representations do play a significant role when $\|\beta\| = 1$, as we will find out in what follows.

Definition 1.4.4. *Let $f(z)$ be a F-H symbol, defined as in (1.4.1). If $\beta_j \neq 0$ or $\alpha_j \neq 0$ or both, replace β_j by $\beta_j + n_j =: \hat{\beta}_j$, $n_j \in \mathbb{Z}$, where n_j satisfy the condition $\sum_{j=0}^m n_j = 0$, but are otherwise arbitrary integers. The resulting function $f(z; n_0, \dots, n_m)$ is a **F-H representation** of $f(z)$.*

Note that all F-H representations of $f(z)$ differ only by multiplicative constants, we have that

$$f(z) = \prod_{j=0}^m z_j^{n_j} \times f(z; n_0, \dots, n_m). \quad (1.4.16)$$

Indeed, if we look at $\beta_j \mapsto \beta_j + n_j =: \hat{\beta}_j$,

$$f(z, n_0, \dots, n_m) = e^{V(z)} z^{\sum_{j=0}^m b_j + \sum_{j=0}^m n_j} \prod_{j=0}^m |z - z_j|^{2\alpha_j} z_j^{-\beta_j - n_j} \quad (1.4.17)$$

$$\times \begin{cases} e^{i\pi\beta_j} e^{i\pi n_j} (= (-1)^{n_j} e^{i\pi\beta_j}), & \text{if } 0 \leq \arg z < \theta_j \\ e^{-i\pi\beta_j} e^{-i\pi n_j} (= (-1)^{n_j} e^{-i\pi\beta_j}), & \text{if } \theta_j \leq \arg z < 2\pi, \end{cases} \quad (1.4.18)$$

and because $\sum_{j=0}^m n_j = 0$ and $\prod_{j=0}^m (-1)^{n_j} = (-1)^{\sum_{j=0}^m n_j} = 1$ the representations only differ from $f(z)$ by the product of $z_j^{n_j}$'s.

We are interested in those F-H representations for which

$$\sum_{j=0}^m (\operatorname{Re} \beta_j + n_j)^2 \quad (1.4.19)$$

is minimal. The number of such representations is finite. There exists an algorithm for finding them explicitly which was given in [14, Lemma 1.12] and will be outlined below. We will denote the set of all representations for which (1.4.19) is minimal by \mathcal{M} . We call the F-H representation degenerate if $\alpha + \hat{\beta}_j = \alpha_j + (\beta_j + n_j)$ or $\alpha - \hat{\beta}_j = \alpha_j - (\beta_j + n_j)$ is a negative integer for some j . We call \mathcal{M}

nondegenerate if it contains no degenerate F-H representations. Let us now denote by O_β the set corresponding to all F-H representations of $f(z)$, i.e. the orbit of $\beta = (\beta_0, \beta_1, \dots, \beta_m)$,

$$O_\beta = \left\{ \hat{\beta} : \hat{\beta}_j = \beta_j + n_j, \sum_{j=0}^m n_j = 0 \right\}. \quad (1.4.20)$$

Similarly as before we define the seminorm $\|\|\beta\|\| = \max_{j,k} |\operatorname{Re} \hat{\beta}_j - \operatorname{Re} \hat{\beta}_k|$, (see (1.4.10)). The set \mathcal{M} can be characterised in the following way.

Lemma 1.4.5. *There exist only the following two mutually exclusive possibilities:*

- $\exists \hat{\beta} \in O_\beta$ such that $\|\|\beta\|\| < 1$. Then such $\hat{\beta}$ is unique and it is the unique element of $\mathcal{M} = \{\hat{\beta}\}$.
- $\exists \hat{\beta} \in O_\beta$ such that $\|\|\beta\|\| = 1$. Then there are at least two such $\hat{\beta}$'s and all of them are obtained from each other by a repeated application of the following rule: add 1 to a $\hat{\beta}_j$ with the smallest real part and subtract 1 from a $\hat{\beta}_j$ with the largest. Moreover, $\mathcal{M} = \{\hat{\beta} \in O_\beta : \|\|\beta\|\| = 1\}$.

Proof. Suppose that the seminorm $\|\|\beta\|\| > 1$. Then, following the algorithm we write:

- $\beta_s^{(1)} = \beta_s + 1$, where $\beta_s = \min_j \operatorname{Re} \beta_j$,
- $\beta_t^{(1)} = \beta_t - 1$, where $\beta_t = \max_j \operatorname{Re} \beta_j$,
- $\beta_j^{(1)} = \beta_j$ if $j \neq s, t$.

Obviously, now $\|\|\beta^{(1)}\|\| \leq \|\|\beta\|\|$ and $f(z)$ corresponding to $\beta^{(1)}$ is a F-H representation with $n_s = 1$ and $n_t = -1$. We repeat the process until we arrive at either $\|\|\beta^{(r)}\|\| < 1$ or $\|\|\beta^{(r)}\|\| = 1$, and r is the number of steps we needed to get there. Applying the algorithm again will not change anything in the case when $\|\|\beta^{(r)}\|\| = 1$ and in the case $\|\|\beta^{(r)}\|\| < 1$ the seminorm will oscillate periodically, taking values $\|\|\beta^{(r)}\|\|$ and $2 - \|\|\beta^{(r)}\|\|$. Thus we can conclude that all F-H symbols belong to those two distinct classes for which

1. $\|\|\beta^{(r)}\|\| < 1$, or
2. $\|\|\beta^{(r)}\|\| = 1$.

For symbols of the first class, \mathcal{M} has only one element $\mathcal{M} = \{\beta^{(r)}\}$. Indeed, writing $b_j = \operatorname{Re} \beta_j$, if $-1/2 < b_j^{(r)} - q < 1/2$ for some $q \in \mathbb{R}$ and all j , then for any $(k_j)_{j=0}^m$ such that $\sum_{j=0}^m k_j = 0$ and not

all $k_j = 0$, we have

$$\sum_{j=0}^m (b_j^{(r)} + k_j)^2 = \sum_{j=0}^m (b_j^{(r)})^2 + 2 \sum_{j=0}^m (b_j^{(r)} - q)k_j + \sum_{j=0}^m k_j^2 \quad (\text{as } \sum_{j=0}^m k_j = 0) \quad (1.4.21)$$

$$> \sum_{j=0}^m (b_j^{(r)})^2 + \sum_{j=0}^m (k_j^2 - |k_j|) \quad (\text{note } -1 < 2(b_j^{(r)} - q) < 1 \forall j) \quad (1.4.22)$$

$$\geq \sum_{j=0}^m (b_j^{(r)})^2 \quad (\text{note } k_j \in \mathbb{Z})$$

where the first inequality is strict because we said that at least $k_j \neq 0$.

Now, for the symbols of the second class, we can find a q such that $-1/2 \leq b_j^{(r)} - q \leq 1/2$ for all j . We have the same above but with the strict inequality replaced by \geq because now we have $-1 \leq 2(b_j^{(r)} - q) \leq 1 \forall j$ in (1.4.22). Hence, there are several F-H representations in \mathcal{M} in the second case and they correspond to the equalities in (1.4.21) - which is when (1.4.19) is minimal. Adding 1 to one of $\beta_s^{(r)}$ with $b_s^{(r)} = \min_j b_j^{(r)} = q - 1/2$ and subtracting 1 from $\beta_t^{(r)}$ with $b_t^{(r)} = \max_j b_j^{(r)} = q + 1/2$ provides a way to find them all. \square

Having established the way of finding all non-trivial and nondegenerate F-H representations in the proof above, we arrive at the Tracy-Basor conjecture which was proven by Deift, Its and Krasovsky in [14]. All of the representations of the symbol $f(z)$ with $\|\beta\| = 1$ —which correspond to all permutations of the β -parameters which lie on the boundary of the strip $-1/2 + q < \operatorname{Re} z < 1/2 + q$, for some $q \in \mathbb{R}, z \in \mathbb{C}$ —give contribution to the final asymptotics. The detailed proof is omitted from this thesis. The proof of our result in Section 2.6 relies heavily on the ideas that were considered in the proof of the Theorem 1.4.6 below.

Theorem 1.4.6. (Generalised F-H Conjecture, Tracy-Basor Conjecture) *Let $f(z)$ be given by (1.4.1), $\operatorname{Re} \alpha_j > -1/2$, $\beta_j \in \mathbb{C}$, $j = 0, \dots, m$. Let \mathcal{M} be nondegenerate. Then, as $n \rightarrow \infty$,*

$$D_n(f) = \sum \left(\prod_{j=0}^m z_j^{n_j} \right)^n \mathcal{R}(f(z; n_0, \dots, n_m))(1 + o(1)), \quad (1.4.23)$$

where the sum is over all F-H representations in \mathcal{M} . Each $\mathcal{R}(f(z; n_0, \dots, n_m))$ stands for the right-hand side of the formula (1.4.12), without the error term, corresponding to $f(z; n_0, \dots, n_m)$.

1.5 Transition between Szegő and F-H

Double-scaling limits play an enormous role in mathematical physics. Recall the short description of the Ising model from Section 1.4. It is clear that we are looking for a limit as the size of the box grows to infinity. This will tell us the magnetisation (or lack thereof) for the original problem on the infinite lattice \mathbb{Z}^2 . But if you recall further, the problem depends on some critical temperature T_c . It was of great interest for physicists to understand the transition between the two states and what happens close to the critical temperature. From a mathematical point of view, we are interested in the change in asymptotic regimes. This was considered in the paper by Claeys, Its and Krasovsky in [11], and we will give a short summary of it in this section.

1.5.1 The Symbol

We introduce a new parameter $t \geq 0$ and we keep the same analytic (in an annulus containing the unit circle) function $V(z)$ as before. We define a new symbol $a : \mathbb{T} \rightarrow \mathbb{C}$ with $t \in \mathbb{R}^+$ as,

$$a(z; t) = (z - e^t)^{\alpha+\beta} (z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)}. \quad (1.5.1)$$

For $t > 0$, $a(z; t)$ is analytic in $\mathbb{C} \setminus ([0, e^{-t}] \cup [e^t, \infty])$ and $\text{ind } a = 0$. This is to some extent (with $\alpha = 0$ and $\beta = -1$, and another considerations, cf. Remark 1.4.3) analogous to the symbol (1.4.4) with $T < T_c$ for the Ising model. The limit as $t \rightarrow 0$ should be thought of as analogous to $T \rightarrow T_c$.

Indeed the function $a(z; t)$ is analytic in the specified region, consider $a(z; t)$ in the following form, with $\arg z \in [0, 2\pi)$,

$$\begin{aligned} a(z; t) = \exp\{ & (\alpha + \beta) \log |z - e^t| + (\alpha + \beta)i \arg(z - e^t) + (\alpha - \beta) \log |z - e^{-t}| \\ & + (\alpha - \beta)i \arg(z - e^{-t}) + (-\alpha + \beta) \log |z| + (-\alpha + \beta)i \arg(z) - i\pi(\alpha + \beta) + V(z)\} \end{aligned}$$

From this expression, it is clear that $a(z; t)$ is analytic if the radius of $z = re^{i\theta}$, $r \in (e^{-t}, e^t)$. For the ease of the next consideration, let us restrict $z \in \mathbb{T}$. If we picture the translated circles as in Figure 1.2, we can see that there is a possible jump in argument where the circles intersect the real line. Let's consider the problem point $z = 1$ from either direction. Now, denoting by $1 + 0$ the limit

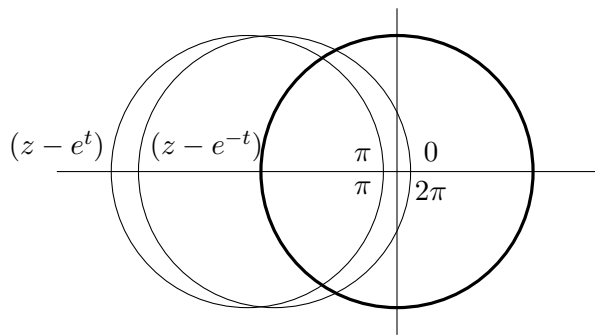


Figure 1.2: Translated circles $(z - e^t)$, $(z - e^{-t})$ and the unit circle $z \in \mathbb{T}$. The $0, \pi, 2\pi$ are the arguments of the complex numbers as $z \rightarrow 0$ along the respective arc.

as $\epsilon > 0$, $\epsilon \rightarrow 0$ in $e^{i(\theta+\epsilon)}$ and $1 - 0$ in $e^{i(\theta-\epsilon)}$. We have the following,

$$\begin{aligned} a(1+0, t) &= \exp\{(\alpha + \beta) \log |z - e^t| + (\alpha + \beta)i\pi + (\alpha - \beta) \log |z - e^{-t}| \\ &\quad + (\alpha - \beta)i0 + (-\alpha + \beta) \log |z| + (-\alpha + \beta)i0 - i\pi(\alpha + \beta) + V(z)\} \\ &= \exp\{(\alpha + \beta) \log |z - e^t| + (\alpha - \beta) \log |z - e^{-t}| + (-\alpha + \beta) \log |z| + V(z)\} \end{aligned}$$

and

$$\begin{aligned} a(1-0, t) &= \exp\{(\alpha + \beta) \log |z - e^t| + (\alpha + \beta)i\pi + (\alpha - \beta) \log |z - e^{-t}| \\ &\quad + (\alpha - \beta)i2\pi + (-\alpha + \beta) \log |z| + (-\alpha + \beta)i2\pi - i\pi(\alpha + \beta) + V(z)\} \\ &= \exp\{(\alpha + \beta) \log |z - e^t| + (\alpha - \beta) \log |z - e^{-t}| + (-\alpha + \beta) \log |z| + V(z)\}. \end{aligned}$$

So $a(1-0, t) = a(1+0, t)$ and thus the function does not have a jump at $z = 1$.

For $t = 0$, $a(z; t)$ has a singularity at $z = 1$, we have that

$$a(e^{i\theta}, t = 0) = |z - 1|^{2\alpha} z^\beta e^{-i\pi\beta} e^{V(z)} = (2 - 2\cos\theta)^\alpha e^{i\beta(\theta-\pi)} e^{V(e^{i\theta})}, \quad 0 < \theta < 2\pi.$$

where $(2 - 2\cos\theta)^\alpha$ corresponds to a root-type singularity if $\alpha \neq 0$ and $e^{i\beta(\theta-\pi)}$ corresponds to jump-type singularity if $\beta \neq 0$. Again, this can be thought of the second case (with $\alpha = 0$ and $\beta = -1$) in the Ising model, when $T > T_c$, see (1.4.5), with the extra considerations raised in Remark 1.4.3.

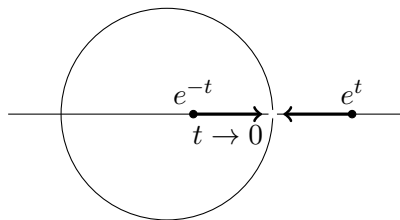


Figure 1.3: Transition from Szegő to F-H

1.5.2 Asymptotic behaviour of the Toeplitz determinant

We now want to consider the asymptotics of the Toeplitz determinant with the symbol $a(z; t)$ given above in (1.5.1), as $n \rightarrow \infty$. We will refer to this determinant as $D_n(t)$. We first note the Fourier coefficients of $\log a(z; t)$, $k \in \mathbb{N}$,

$$\begin{aligned} (\log a)_{-k} &= V_{-k} - (\alpha - \beta) \frac{e^{-tk}}{k}, \\ (\log a)_0 &= t(\alpha + \beta) + V_0, \\ (\log a)_k &= V_k - (\alpha + \beta) \frac{e^{-tk}}{k}. \end{aligned} \tag{1.5.2}$$

There are two cases for the asymptotics of $D_n(t)$:

1. For $t > 0$, we have the Szegő asymptotics, given by SSLT, Theorem 1.3.4. As $n \rightarrow \infty$,

$$\log D_n(t) = nV_0 + nt(\alpha + \beta) + \sum_{k=1}^{\infty} k \left[V_k - (\alpha + \beta) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha - \beta) \frac{e^{-tk}}{k} \right] + o(1). \tag{1.5.3}$$

2. For $t = 0$, we have the Fisher-Hartwig asymptotics, given by Theorem 1.4.2. As $n \rightarrow \infty$,

$$\begin{aligned} \log D_n &= nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} - (\alpha - \beta) \sum_{k=1}^{\infty} V_k - (\alpha + \beta) \sum_{k=1}^{\infty} V_{-k} \\ &+ (\alpha^2 - \beta^2) \log n + \log G_{\alpha+\beta, \alpha-\beta} + o(1), \end{aligned} \tag{1.5.4}$$

if $\alpha \pm \beta \neq -1, -2, \dots$ and $G_{\alpha+\beta, \alpha-\beta}$ is defined in (1.4.13).

Figure 1.3 illustrates the situation as $t \rightarrow 0$. We can see the Fisher-Hartwig singularity forming on the unit circle as the points e^{-t} and e^t move towards each other and eventually coalesce. Note that if you let $t \rightarrow 0$ in (1.5.3) you will not obtain (1.5.4).

The purpose of the double-scaling limit is to observe the consequences of taking two limits simultaneously, as both $t \rightarrow 0$ and $n \rightarrow \infty$. Claeys, Its and Krasovsky provide the asymptotic expansion for the Toeplitz determinant with the symbol (1.5.1), which holds uniformly as $n \rightarrow \infty$ for $0 \leq t \leq t_0$ for sufficiently small t_0 . The expression involves, among other things, $\sigma(x)$ which is the solution of a particular Painlevé V equation; for further details on this second order ODE and other Painlevé equations, see [24].

Theorem 1.5.1. [11, Theorem 1.1] *Let $\alpha \in \mathbb{R}$, $\alpha > -1/2$, $\beta \in i\mathbb{R}$. Let f be defined by (1.5.1), and consider the associated Toeplitz determinant denoted by $D_n(t)$ and defined in Definition 1.1.6. The following asymptotic expansion holds as $n \rightarrow \infty$ with the error term $o(1)$ uniform for $0 \leq t \leq t_0$, where t_0 is sufficiently small:*

$$\begin{aligned} \log D_n(t) = nV_0 + (\alpha + \beta)nt + \sum_{k=1}^{\infty} k \left[V_k - (\alpha + \beta) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha - \beta) \frac{e^{-tk}}{k} \right] \\ + \log G_{\alpha+\beta, \alpha-\beta} + \Omega(2nt) + o(1), \end{aligned} \quad (1.5.5)$$

where $G(z)$ is the Barnes G -function, see (1.4.8), (1.4.13), and

$$\Omega(2nt) = \int_0^{2nt} \frac{\sigma(x) - \alpha^2 + \beta^2}{x} dx + (\alpha^2 - \beta^2) \log 2nt. \quad (1.5.6)$$

The function $\sigma(x)$ (see (1.5.9) below) is real analytic on $(0, +\infty)$ and has the following asymptotics for $x > 0$:

$$\sigma(x) = \begin{cases} \alpha^2 - \beta^2 + \frac{\alpha^2 - \beta^2}{2\alpha} \{x - x^{1+2\alpha} C(\alpha, \beta)\} (1 + \mathcal{O}(x)), & x \rightarrow 0, 2\alpha \notin \mathbb{Z}, \\ \alpha^2 - \beta^2 + \mathcal{O}(x) + \mathcal{O}(x^{1+2\alpha}) + \mathcal{O}(x^{1+2\alpha} \log x), & x \rightarrow 0, 2\alpha \in \mathbb{Z}, \\ x^{-1+2\alpha} e^{-x} \frac{1}{\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)} (1 + \mathcal{O}(1/x)), & x \rightarrow +\infty, \end{cases} \quad (1.5.7)$$

where

$$C(\alpha, \beta) = \frac{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)\Gamma(1 - 2\alpha)}{\Gamma(1 - \alpha + \beta)\Gamma(1 - \alpha - \beta)\Gamma(+2\alpha)^2} \frac{1}{1 + 2\alpha} \quad (1.5.8)$$

and $\Gamma(z)$ is the Euler's Γ -function.

The function $\sigma(x)$ is a particular solution to the following second order ODE:

$$\begin{aligned} \left(x \frac{d^2\sigma}{dx^2}\right)^2 &= \left(\sigma - x \frac{d\sigma}{dx} + 2 \left(\frac{d\sigma}{dx}\right)^2 + 2\alpha \frac{d\sigma}{dx}\right)^2 \\ &\quad - 4 \left(\frac{d\sigma}{dx}\right)^2 \left(\frac{d\sigma}{dx} + \alpha + \beta\right) \left(\frac{d\sigma}{dx} + \alpha - \beta\right), \end{aligned} \quad (1.5.9)$$

the σ -form (Jimbo-Miwa-Okamoto [28, 29]) of the Painlevé V equation

$$u_{xx} = \left(\frac{1}{2u} + \frac{1}{u-1}\right) u_x^2 - \frac{1}{x} u_x + \frac{(u-1)^2}{x^2} \left(Au + \frac{B}{u}\right) + \frac{Cu}{x} + D \frac{u(u+1)}{u-1}, \quad (1.5.10)$$

with parameters A, B, C, D given by

$$A = \frac{1}{2}(\alpha - \beta)^2, \quad B = -\frac{1}{2}(\alpha - \beta)^2, \quad C = 1 + 2\beta, \quad D = -\frac{1}{2}.$$

The function $\sigma = \sigma(x; \alpha, \beta)$ is defined for $x \in \mathbb{C}$ with a cut from 0 to infinity and is analytic in the cut plane apart from possible poles. The asymptotics in (1.5.7) imply that there are no poles for x positive and sufficiently large and thus there is a finite number of poles on $(0, +\infty)$.

The authors in [11] show in fact that for $\alpha > -1/2$, $\alpha \in \mathbb{R}$, $i\beta \in \mathbb{R}$ there are no poles on $(0, +\infty)$ and thus took this path of integration in (1.5.6). For arbitrary β and $\operatorname{Re} \alpha > -1/2$ they take a path of integration avoiding the poles $\{x_1, \dots, x_l\}$, giving the following statement:

Theorem 1.5.2. [11, Theorem 1.4] *Let $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re} \alpha > -1/2$, $\alpha \pm \beta \neq -1, -2, \dots$, and let s_δ denote a sector $-\pi/2 + \delta < \arg x < \pi/2 - \delta$, $0 < \delta < \pi/2$. Let $a(z; t)$ be defined by (1.5.1), and consider the related Toeplitz determinants $D_n(t)$. There exists a finite set $\{x_1, \dots, x_l\} \in s_\delta$ (with $l = l(\alpha, \beta, \delta)$ and $x_j = x_j(\alpha, \beta) \neq 0$) such that the expansion (1.5.5) holds uniformly for $t \in s_\delta$, $|t| < t_0$ (with t_0 sufficiently small) as long as $2nt$ remains bounded away from the set $\{x_1, \dots, x_l\}$. The function Ω is defined in (1.5.6), where the path of integration is chosen in s_δ , connecting 0 with $2nt$ and not containing any of the points $\{x_1, \dots, x_l\}$. Moreover, $\sigma(x)$ solves the ODE (1.5.9) and has the asymptotics in the sector s_δ given by (1.5.7).*

1.5.3 Recovering Szegő and F-H asymptotics

From the expression in (1.5.5) we can recover back the asymptotics (1.5.4) and (1.5.3) by considering different values of t . To obtain the F-H asymptotics, we let $t \rightarrow 0$ and let n be fixed. We then look at the function $\Omega(2nt)$ (see (1.5.6)) and the asymptotics for $\sigma(x)$ (see (1.5.7)),

$$\int_0^{2nt} \left\{ \begin{array}{ll} \frac{\alpha^2 - \beta^2}{2\alpha} \{1 - x^{2\alpha} C(\alpha, \beta)\} (1 + \mathcal{O}(x)) & x \rightarrow 0, 2\alpha \notin \mathbb{Z} \\ \mathcal{O}(1) + \mathcal{O}(x^{2\alpha}) + \mathcal{O}(x^{2\alpha} \log x) & x \rightarrow 0, 2\alpha \in \mathbb{Z} \end{array} \right\} dx. \quad (1.5.11)$$

Considering them separately,

$$\begin{aligned} & \int_0^{2nt} \frac{\alpha^2 - \beta^2}{2\alpha} \{1 - x^{2\alpha} C(\alpha, \beta)\} (1 + \mathcal{O}(x)) dx \\ &= \int_0^{2nt} \frac{\alpha^2 - \beta^2}{2\alpha} (1 - x^{2\alpha} C(\alpha, \beta)) dx + \int_0^{2nt} \frac{\alpha^2 - \beta^2}{2\alpha} (\mathcal{O}(x) + \mathcal{O}(x^{2\alpha})) dx \\ &\leq \int_0^{2nt} (Const_1 - Const_2 x^{2\alpha}) dx + \int_0^{2nt} (Const_3 x + Const_4 x^{2\alpha+1}) dx \\ &\rightarrow 0 \text{ as } t \rightarrow 0, \end{aligned}$$

as $x^{2\alpha}$, $x^{2\alpha+1}$ are integrable for $\text{Re } \alpha > -1/2$. And for $2\alpha \in \mathbb{Z}$,

$$\begin{aligned} & \int_0^{2nt} \mathcal{O}(1) + \mathcal{O}(x^{2\alpha}) + \mathcal{O}(x^{2\alpha} \log x) dx \\ &\leq \int_0^{2nt} (Const_1 + Const_2 x^{2\alpha} + (Const_3 x^{2\alpha} \log x)) dx \\ &= o(1) + \frac{Const_3}{2\alpha + 1} (x^{2\alpha+1} (\log x - 1)) \Big|_0^{2nt} \\ &= o(1) + Const_4 x^{2\alpha+1} \log x \Big|_0^{2nt} \\ &= o(1) \text{ as } \lim_{x \rightarrow 0} x^{2\alpha+1} \log x = 0 \text{ for } 2\alpha > -1 \text{ by L'Hopitals rule.} \end{aligned}$$

Therefore,

$$\Omega(2nt) = (\alpha^2 - \beta^2) \log(2nt) + o(1). \quad (1.5.12)$$

Substituting into (1.5.5) gives,

$$\log D_n(t) = nV_0 + (\alpha + \beta)nt + \sum_{k=1}^{\infty} k \left[V_k - (\alpha + \beta) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha - \beta) \frac{e^{-tk}}{k} \right]$$

$$\begin{aligned}
& + \log \frac{G(1 + \alpha + \beta)G(1 + \alpha - \beta)}{G(1 + 2\alpha)} + (\alpha^2 - \beta^2) \log(2nt) + o(1) \\
& = nV_0 + (\alpha + \beta)nt + \sum_{k=1}^{\infty} kV_kV_{-k} - (\alpha + \beta) \sum_{k=1}^{\infty} e^{-tk}V_{-k} - (\alpha - \beta) \sum_{k=1}^{\infty} e^{-tk}V_k \\
& \quad + (\alpha + \beta)(\alpha - \beta) \sum_{k=1}^{\infty} \frac{e^{-2tk}}{k} + \log G_{\alpha+\beta, \alpha-\beta} + (\alpha^2 - \beta^2) \log(2nt) + o(1)
\end{aligned}$$

and using the following identity for $t > 0$,

$$\sum_{k=1}^{\infty} \frac{e^{-2kt}}{k} = -\log(1 - e^{-2t}),$$

we obtain,

$$\begin{aligned}
\log D_n(t) & = nV_0 + (\alpha + \beta)nt + \sum_{k=1}^{\infty} kV_kV_{-k} - (\alpha + \beta) \sum_{k=1}^{\infty} e^{-tk}V_{-k} - (\alpha - \beta) \sum_{k=1}^{\infty} e^{-tk}V_k \\
& \quad + (\alpha^2 - \beta^2) \log \left(\frac{2nt}{1 - e^{-2t}} \right) + \log G_{\alpha+\beta, \alpha-\beta} + o(1).
\end{aligned}$$

As $t \rightarrow 0$ we have $\log \left(\frac{2nt}{1 - e^{-2t}} \right) \rightarrow \log(n)$ (L'Hopital) and with each $e^{-tk} \rightarrow 1$, we finally get the F-H asymptotics as in (1.5.4).

The expansion (1.5.5) is also consistent with the Szegő asymptotics—comparing the $\mathcal{O}(n)$ terms with (1.5.3) for a fixed t . Additionally, comparing $\mathcal{O}(1)$ terms gives an identity involving the Painlevé function $\sigma(x)$ via (1.5.6),

$$\Omega(+\infty) = -\log G_{\alpha+\beta, \alpha-\beta} = -\log \frac{G(1 + \alpha + \beta)G(1 + \alpha - \beta)}{G(1 + 2\alpha)}. \quad (1.5.13)$$

1.6 Orthogonal Polynomials

In this section we present the links between Toeplitz determinants $D_n(f)$, Riemann-Hilbert problems (R-H problems) and orthogonal polynomials (OPs). We will look at the R-H problem for polynomials orthogonal on the unit circle (oriented in the positive direction) with respect to the complex weight $f(z)$. Suppose $D_n(f) \neq 0$, $n = n_0, n_0 + 1, \dots$, for some sufficiently large n_0 . Then the polynomials $\phi_n(z) = \chi_n z^n + \dots$, $\hat{\phi}_n(z) = \chi_n z^n + \dots$ of degree n , $n = n_0, n_0 + 1, \dots$, exist, satisfying the

orthogonality conditions

$$\int_{\mathbb{T}} \phi_n(z) z^{-j} f(z) \frac{dz}{2\pi iz} = \chi_n^{-1} \delta_{jn}, \quad \int_{\mathbb{T}} \hat{\phi}_n(z^{-1}) z^j f(z) \frac{dz}{2\pi iz} = \chi_n^{-1} \delta_{jn}, \quad (1.6.1)$$

which are equivalent to,

$$\int_{\mathbb{T}} \phi_n(z) \hat{\phi}_j(z^{-1}) f(z) \frac{dz}{2\pi iz} = \delta_{jn}, \quad j = 0, 1, \dots, n. \quad (1.6.2)$$

We say that these polynomials are orthogonal on the unit circle (sometimes abbreviated to OPUC) with respect to the complex weight $f(z)$ —this means they are orthogonal with respect to the measure $d\mu(z) = \frac{f(z)dz}{2\pi iz}$. We will be using two representations of the standard measure on the unit circle, $\int_0^{2\pi} \frac{d\theta}{2\pi} = \int_{\mathbb{T}} \frac{dz}{2\pi iz}$, interchangeably, determined by convenience. We also denote $D_n(f) = D_n$ below to ease notation. The polynomials are given by the following expressions

$$\phi_n(z) = \frac{1}{\sqrt{D_n D_{n+1}}} \begin{vmatrix} f_{0,0} & f_{0,1} & \cdots & f_{0,n} \\ f_{1,0} & f_{1,1} & \cdots & f_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1,0} & f_{n-1,1} & \cdots & f_{n-1,n} \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad (1.6.3)$$

and,

$$\hat{\phi}_n(z^{-1}) = \frac{1}{\sqrt{D_n D_{n+1}}} \begin{vmatrix} f_{0,0} & f_{0,1} & \cdots & f_{0,n-1} & 1 \\ f_{1,0} & f_{1,1} & \cdots & f_{1,n-1} & z^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n,0} & f_{n,1} & \cdots & f_{n,n-1} & z^{-n} \end{vmatrix}, \quad (1.6.4)$$

where we have momentarily redefined the Fourier coefficients to be,

$$f_{m,n} = f_{m-n} = \int_{\mathbb{T}} f(z) z^{-(m-n)} \frac{dz}{2\pi iz}. \quad (1.6.5)$$

The simplest way to see these, is to think about evaluating the determinants along the bottom row

and far right column respectively. This immediately gives the leading coefficient for both to be

$$\chi_n = \sqrt{\frac{D_n}{D_{n+1}}}. \quad (1.6.6)$$

Before we proceed, we will recall some basic theory of orthogonal polynomials, in a slightly more general setting. It is a fact, see for example [21], that we can construct orthogonal polynomials $p_n(z) = k_n z^n + \dots$, $n \geq 0$, given any Borel measure $d\mu(z)$ on some contour $\Sigma \subset \mathbb{C}$ (see [6]), with bounded or unbounded support. For measures with unbounded support, we require the moments of the measure to be finite, i.e.

$$\int_{\Sigma} |z|^m d\mu(z) < \infty, \quad m = 0, 1, 2, \dots \quad (1.6.7)$$

The polynomials are obtained via the Gram-Schmidt procedure being applied to $1, z, z^2, \dots$ in $L^2(d\mu)$. Two elements of $L^2(\mathbb{T})$ —in our case polynomials p_n and q_n —are orthonormal if they satisfy

$$\int_{\Sigma} p_n(z) \overline{q_m(z)} d\mu(z) = \delta_{mn}, \quad m, n \geq 0. \quad (1.6.8)$$

The leading coefficient k_n can be chosen to always be positive—which makes the polynomials unique. This is what we stipulate in this thesis also. In addition, we will make use of what is known as monic polynomials associated with the measure $d\mu$, i.e.

$$\pi_n(z) = k_n^{-1} p_n(z) = z^n + \dots, \quad n \geq 0 \quad (1.6.9)$$

1.6.1 The link between Orthogonal Polynomials and Toeplitz determinants

Given that the OPs in (1.6.1), (1.6.2) exist, we will show that they are necessarily given by (1.6.3) and (1.6.4). And vice versa, if the polynomials are of the form (1.6.3) and (1.6.4), they are necessarily orthogonal with respect to the weight that is the symbol on the Toeplitz matrix in the given expressions, i.e. (1.6.1), (1.6.2) hold. Below we only consider $\phi_k(z)$, the results for $\hat{\phi}_k(z)$ are obtained analogously.

Proposition 1.6.1. *The orthogonal polynomials satisfying (1.6.1), (1.6.2) exist if and only if they are given by (1.6.3) and (1.6.4).*

Proof. We start with OPs satisfying the orthogonality conditions (1.6.1), (1.6.2). Consider $k = 0$,

$$\phi_0(z) = \hat{\phi}_0(z) = \chi_0,$$

$$\int_{\mathbb{T}} \phi_0(z) \hat{\phi}_0(z^{-1}) f(z) \frac{dz}{2\pi iz} = 1$$

$$\chi_0^2 \int_{\mathbb{T}} f(z) \frac{dz}{2\pi iz} = 1$$

$$\chi_0^2 = \frac{1}{f_{00}} = \frac{1}{D_1}$$

Thus the first coefficient χ_0 is a Toeplitz determinant. Now considering $k = 1$, we have simultaneous equations,

$$\int_{\mathbb{T}} \phi_1(z) z^{-1} f(z) \frac{dz}{2\pi iz} = \chi_1^{-1}, \quad \int_{\mathbb{T}} \phi_1(z) f(z) \frac{dz}{2\pi iz} = 0,$$

$$\chi_1^2 f_{00} + \chi_1 \phi_1(0) f_{10} = 1, \quad \chi_1 f_{01} + \phi_1(0) f_{00} = 0.$$

Solving them gives,

$$\chi_1^2 = \frac{f_{00}}{f_{00}^2 - f_{01} f_{10}} = \sqrt{\frac{D_1}{D_2}}, \quad \phi_1(0) = \frac{-f_{01}}{\sqrt{D_1 D_2}}.$$

Which means,

$$\phi_1(z) = \frac{D_1}{\sqrt{D_1 D_2}} z + \frac{-f_{01}}{\sqrt{D_1 D_2}} = \frac{1}{\sqrt{D_1 D_2}} \begin{vmatrix} f_{00} & f_{01} \\ 1 & z \end{vmatrix}$$

Using recurrence relations, which can be obtained from the orthogonality conditions directly, it follows that the rest of the OPs $\phi_k(z)$ are explicitly given using Toeplitz determinants, for any k .

We can carry on like this to obtain (1.6.3).

Conversely, assume $\phi_k(z)$ is given by (1.6.3). Then by definition,

$$\int_{\mathbb{T}} \phi_k(z) z^{-j} f(z) \frac{dz}{2\pi iz} = \frac{1}{\sqrt{D_k D_{k+1}}} \int_{\mathbb{T}} \begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0k} \\ f_{10} & f_{11} & \cdots & f_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k-1\ 0} & f_{k-1\ 1} & \cdots & f_{k-1\ k} \\ 1 & z & \cdots & z^k \end{vmatrix} z^{-j} f(z) \frac{dz}{2\pi iz}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0k} \\ f_{10} & f_{11} & \cdots & f_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k-1\ 0} & f_{k-1\ 1} & \cdots & f_{k-1\ k} \\ \int_{\mathbb{T}} 1 \cdot z^{-j} f(z) \frac{dz}{2\pi iz} & \int_{\mathbb{T}} z^{1-j} f(z) \frac{dz}{2\pi iz} & \cdots & \int_{\mathbb{T}} z^{k-j} f(z) \frac{dz}{2\pi iz} \end{vmatrix} \\
&= \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0k} \\ f_{10} & f_{11} & \cdots & f_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k-1\ 0} & f_{k-1\ 1} & \cdots & f_{k-1\ k} \\ f_{j\ 0} & f_{j\ 1} & \cdots & f_{j\ k} \end{vmatrix} \\
&= \begin{cases} 0, & \text{if } j \leq k-1, \\ \frac{D_{k+1}}{\sqrt{D_k D_{k+1}}} = \sqrt{\frac{D_{k+1}}{D_k}} = \chi_k^{-1}, & \text{if } j = k. \end{cases}
\end{aligned}$$

Thus proving (1.6.1). □

1.6.2 Riemann-Hilbert Problem (R-H Problem)

Consider the following 2×2 matrix valued function $Y^{(n)}(z) \equiv Y(z), n \geq n_0$:

$$Y(z) = \begin{pmatrix} \chi_n^{-1} \phi_n(z) & \chi_n^{-1} \int_{\mathbb{T}} \frac{\phi_n(\xi) f(\xi) d\xi}{\xi-z} \frac{1}{2\pi i \xi^n} \\ -\chi_{n-1} z^{n-1} \hat{\phi}_{n-1}(z^{-1}) & -\chi_{n-1} \int_{\mathbb{T}} \frac{\hat{\phi}_{n-1}(\xi^{-1}) f(\xi) d\xi}{\xi-z} \frac{1}{2\pi i \xi} \end{pmatrix}, \quad (1.6.10)$$

with $f(z)$ which possesses one F-H singularity at z_1 with the strengths α_1, β_1 (see (1.4.1)). The matrix-valued function above is the unique solution to the following Riemann-Hilbert problem:

R-H problem for Y (OPs with weight $f(z)$)

(Y1) $Y : \mathbb{C} \setminus \mathbb{T} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(Y2) Let $z \in \mathbb{T} \setminus \{z_1\}$. Y has continuous boundary values $Y_+(z)$ as z approaches the unit circle from the inside, and $Y_-(z)$ from the outside, related by the jump condition

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{T}. \quad (1.6.11)$$

(Y3) $Y(z)$ has a the following asymptotic behaviour at ininity:

$$Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as } z \rightarrow \infty. \quad (1.6.12)$$

(Y4) As $z \rightarrow z_1$, $z \in \mathbb{C} \setminus \mathbb{T}$,

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z - z_1|^{2\alpha_1}) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z - z_1|^{2\alpha_1}) \end{pmatrix}, \quad \text{if } \alpha_1 \neq 0, \quad (1.6.13)$$

and

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(\log |z - z_1|) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(\log |z - z_1|) \end{pmatrix}, \quad \text{if } \alpha_1 = 0, \beta_1 \neq 0. \quad (1.6.14)$$

We will now show the connection between the R-H problem above and the OPs (1.6.1).

Proposition 1.6.2. *If $Y(z)$ solves the R-H problem (Y1)-(Y4) above, then $\phi_n(z)$ and $\hat{\phi}_n(z)$ in (1.6.10) are polynomials satisfying (1.6.1).*

Proof. We write the solution to the R-H problem, $Y(z)$ in the following way, to consider each entry of the matrix separately, and we also note each of their asymptotic behaviour as $z \rightarrow \infty$ by condition (Y3) above,

$$Y(z) = \begin{pmatrix} Y_{11}(z) & Y_{12}(z) \\ Y_{21}(z) & Y_{22}(z) \end{pmatrix} = \begin{pmatrix} z^n + \mathcal{O}(z^{n-1}) & \mathcal{O}(z^{-n-1}) \\ \mathcal{O}(z^{n-1}) & z^{-n} + \mathcal{O}(z^{-n-1}) \end{pmatrix}. \quad (1.6.15)$$

We suppose $n \geq 1$ and consider the first row of the jump condition (Y2),

$$(Y_{11}(z) \quad Y_{12}(z))_+ = (Y_{11}(z) \quad Y_{12}(z))_- \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}.$$

Matching the entries of the (now) vector, we have the following,

$$(1) \quad (Y_{11})_+(z) = (Y_{11})_-(z),$$

$$(2) \quad (Y_{12})_+(z) = (Y_{11})_-(z)z^{-n}f(z) + (Y_{12})_-(z).$$

In point (1), the limits from both sides coincide and so the function $Y_{11}(z)$ is analytic in the complex plane. From (1.6.15) we have that $Y_{11}(z) = z^n + \mathcal{O}(z^{n-1})$ is a polynomial of degree n , let us call it

$\pi_n(z) := Y_{11}(z)$. Point (2) gives us,

$$(Y_{12})_+(z) = (Y_{12})_-(z) + \pi_n(z)z^{-n}f(z),$$

which is an additive R-H problem (1.2.3). By the Plemelj formulae (1.2.1), the solution is given by,

$$(Y_{12})(z) = \int_{\mathbb{T}} \frac{\pi_n(s) f(s) ds}{s - z} \frac{1}{s^n 2\pi i}.$$

Expanding $\frac{1}{s-z}$ for z at infinity, and noting $(Y_{12})(z) = \mathcal{O}(z^{-n-1})$, we now have,

$$(Y_{12})(z) = - \int_{\mathbb{T}} \pi_n(s) s^{-n} \left(\frac{s}{z} + \frac{s^2}{z^2} + \dots + \frac{s^n}{z^n} + \dots \right) \frac{f(s) ds}{2\pi i s} = \mathcal{O}(z^{-n-1}),$$

which gives,

$$\int_{\mathbb{T}} \pi_n(s) \frac{s^j f(s) ds}{s^n 2\pi i s} = 0, \quad \text{for } j = 1, \dots, n$$

$$\int_{\mathbb{T}} \pi_n(s) s^{-k} \frac{f(s) ds}{2\pi i s} = 0, \quad \text{for } k = 0, \dots, n-1, \text{ where } k = n - j.$$

But we have that $\pi_n(z) = z^n + a_{n,n-1}z^{n-1} + \dots + a_{n,0}$, so $\pi_n(z)$ is the n 'th monic orthogonal polynomial w.r.t. the weight $f(z)$ (see (1.6.9)), which means we necessarily have $\pi_n(z) = \chi_n^{-1}\phi_n(z)$, as these polynomials are unique.

We now look at the second column,

$$(Y_{21}(z) \quad Y_{22}(z))_+ = (Y_{21}(z) \quad Y_{22}(z))_- \begin{pmatrix} 1 & z^{-k}f(z) \\ 0 & 1 \end{pmatrix},$$

which gives us the following:

$$(1) \quad (Y_{21})_+(z) = (Y_{21})_-(z),$$

$$(2) \quad (Y_{22})_+(z) = (Y_{21})_-(z)z^{-k}f(z) + (Y_{22})_-(z).$$

Again, from point (1) we deduce that $Y_{21}(z)$ is analytic in the complex plane and from the asymptotic condition (Y4) and also (1.6.15), we conclude it's a polynomial of degree $n-1$. From point (2) we

again have by the Plemelj formulae (1.2.1),

$$Y_{22}(z) = \int_{\mathbb{T}} \frac{Y_{21}^{(n-1)}(s) f(s) ds}{(s-z) 2\pi i s^n}.$$

Similarly to before,

$$Y_{22}(z) = - \int_{\mathbb{T}} Y_{21}^{(n-1)}(s) s^{-n} \left(\frac{s}{z} + \frac{s^2}{z^2} + \dots + \frac{s^n}{z^n} + \dots \right) \frac{f(s) ds}{2\pi i s} = z^{-n} + \mathcal{O}(z^{-(n+1)}),$$

which implies two things,

$$\int_{\mathbb{T}} Y_{21}^{(n-1)}(s) s^{-n} s^j \frac{f(s) ds}{2\pi i s} = 0 \quad \text{for } j = 1, \dots, n-1,$$

and

$$- \int_{\mathbb{T}} Y_{21}^{(n-1)}(s) \frac{f(s) ds}{2\pi i s} = 1, \quad (j = n).$$

We can rewrite both in the following way,

$$\int_{\mathbb{T}} s^{-(n-1)} Y_{21}^{(n-1)}(s) s^j \frac{f(s) ds}{2\pi i s} = -\delta_{j, n-1} \quad \text{for } j = 0, \dots, n-1. \quad (1.6.16)$$

Thus $z^{-(n-1)} Y_{21}^{(n-1)}(z)$ is a polynomial orthogonal with respect to the measure $\frac{f(s) ds}{2\pi i s}$, and we have that $Y_{21}^{(n-1)}(z) = -\chi_{n-1} \hat{\phi}_{n-1}(z^{-1}) z^{n-1}$. Now, using this in $Y_{22}(z)$ and what we found for the first row gives (1.6.10). \square

Chapter 2

Emergence of an additional Fisher-Hartwig singularity

The problem this thesis aims to present is to find uniform asymptotics for a Toeplitz determinant with a varying symbol. We begin with a fixed singularity in an arbitrary position on the unit circle (away from $z = 1$). The symbol will depend on an additional parameter $t > 0$, and as t tends to 0 we will see a new F-H singularity emerging at the point $z = 1$. This situation is visualised in Figure 2.1. Because our symbol has two β -singularities, there are two separate cases we will have to consider, see Lemma 1.4.5. The first case $\|\beta\| < 1$ will be dealt with via a Riemann-Hilbert Problem (R-H problem—see (Y1)-(Y4) in Section 1.6.2) for polynomials orthogonal with respect to our symbol $f(z;t)$, which is defined in Section 2.2. We will then relate the determinant of the associated Toeplitz matrix and the R-H problem using a differential identity (see Section 2.3). In the second case, when $\|\beta\| = 1$ we have to take into account F-H representations of the symbol at $t = 0$, i.e. $f(z;0)$ (see Section 1.4.2). For this we will use the ideas in [14]. Specifically we will

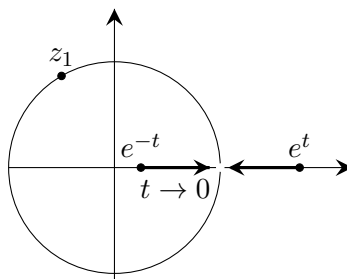


Figure 2.1: Transition between 1 F-H and 2 F-H

use [14, Lemma 2.4] (or Lemma 2.6.1 in this thesis) to obtain an expression for the determinant with the symbol where $\|\beta\| = 1$ in terms of the determinant with seminorm $\|\beta\| < 1$ and some specific asymptotics of associated orthogonal polynomials.

2.1 Summary of results

Theorem 2.1.1. *Let $\alpha_0 \in \mathbb{R}$, $\alpha_1 \in \mathbb{C}$, $\alpha_0, \operatorname{Re} \alpha_1 > -1/2$, $\beta_0 \in i\mathbb{R}$, $\beta_1 \in \mathbb{C}$ such that $\|\beta\| < 1$, where the seminorm is defined in (1.4.10). Let f be defined by (2.2.1), and consider the associated Toeplitz determinant denoted by $D_n(t)$. The following asymptotic expansion holds as $n \rightarrow \infty$ with the error term $o(1)$ uniform for $0 \leq t \leq t_0$, where t_0 is sufficiently small,*

$$\begin{aligned} D_n(t) &= \exp \{nV_0 + nt(\alpha_0 + \beta_0)\} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \right\} \\ &\quad \times \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} \\ &\quad \times \exp \left\{ (\alpha_1 + \beta_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} \\ &\quad \times n^{(\alpha_1^2 - \beta_1^2)} G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} \tilde{\Omega}(2nt) (1 + o(1)), \end{aligned} \quad (2.1.1)$$

where $G_{\alpha_j + \beta_j, \alpha_j - \beta_j}$ is the product of Barnes G -functions, see (1.4.8), (1.4.13), and

$$\tilde{\Omega}(2nt) := \exp \{ \Omega(2nt) \} = \exp \left\{ \int_0^{2nt} \frac{\sigma(x) - \alpha_0^2 + \beta_0^2}{x} dx + (\alpha_0^2 - \beta_0^2) \log 2nt \right\} \quad (2.1.2)$$

The function $\sigma(x)$ (see (1.5.9)) is real analytic on $(0, +\infty)$ whose asymptotic behaviour for $x > 0$ is given by (1.5.7)

This theorem is proven in Section 2.5 and extends to $\alpha_0, \beta_0 \in \mathbb{C}$ via the same arguments as in Theorem 1.5.2.

Theorem 2.1.2. *Let $\alpha_0, \alpha_1 \in \mathbb{C}$, $\operatorname{Re} \alpha_0, \operatorname{Re} \alpha_1 > -1/2$, $\beta_0, \beta_1 \in \mathbb{C}$ be such that $\|\beta\| = 1$, where the seminorm is defined in (1.4.10). Let f be defined by (2.2.1), and consider the associated Toeplitz determinant denoted by $D_n(t)$. Denote by $\tilde{\beta}_0 = \beta_0 + n_0$, $\tilde{\beta}_1 = \beta_0 + n_1$ the only non-trivial F - H representation of the symbol f at $t = 0$. The following asymptotic expansion holds as $n \rightarrow \infty$ with*

the error term $o(1)$ uniform for $0 \leq t \leq t_0$, where t_0 is sufficiently small,

$$\begin{aligned} D_n(f) &= \mathcal{R}(f(z; \beta_0, \beta_1)) \tilde{\Omega}(2nt)(1 + o(1)) \\ &\quad + (z_1^{n_1})^n \mathcal{R}\left(f(z; \tilde{\beta}_0, \tilde{\beta}_1)\right) \tilde{\Omega}(2nt) \\ &\quad \times \frac{n^{-(2\beta_0+1)}}{\Gamma(1 + \alpha_0 + \beta_0)} \frac{K(2nt)}{e^{nt}} (1 - e^{-2t})^{-(2\beta_0+1)} \Sigma(t)(1 + o(1)), \end{aligned} \quad (2.1.3)$$

where, $\mathcal{R}\left(f(z; \hat{\beta}_0, \hat{\beta}_1)\right)$ corresponds to the RHS of (2.1.1) for a symbol f with β -parameters $\hat{\beta}_0, \hat{\beta}_1$, without the error term nor $\tilde{\Omega}(2nt)$, which is defined in (2.1.2). Further,

$$K(x) = e^{x/2} \int_x^\infty y^{\alpha_0 + \beta_0} e^{-y} dy, \quad (2.1.4)$$

has the following behaviour,

$$K(x) \sim \begin{cases} e^{-x/2} x^{\alpha_0 + \beta_0}, & \text{as } x \rightarrow \infty, \\ e^{x/2} \Gamma(\alpha_0 + \beta_0 + 1), & \text{as } x \rightarrow 0, \end{cases}$$

and

$$\begin{aligned} \Sigma(t) &= \left[\left(\frac{z_1 - e^t}{z_1 - e^{-t}} \right)^{\alpha_1 + \tilde{\beta}_1} \exp \left\{ 2 \sum_{k=1}^{\infty} V_k(\sinh(tk)) \right\} \left(\frac{2t}{1 - e^{-2t}} \right)^{\alpha_0 - \beta_0} \right. \\ &\quad \left. + \left(\frac{z_1 - e^t}{z_1 - e^{-t}} \right)^{\alpha_1 - \tilde{\beta}_1} \exp \left\{ -2 \sum_{k=1}^{\infty} V_{-k}(\sinh(tk)) \right\} \left(\frac{2t}{1 - e^{-2t}} \right)^{-(\alpha_0 + \beta_0)} \right]. \end{aligned} \quad (2.1.5)$$

2.2 The Symbol

We start with one F-H singularity positioned at the point $z_1 = e^{i\theta_1}$, $\theta_1 \in (0, 2\pi)$ and one that is just emerging, at $z_0 = e^{i\theta_0} = 1$, $\theta_0 = 0$. Locally, around the point $z = 1$ we have the situation which was described in [11]. We write the symbol in the following way,

$$\begin{aligned} f(z; t) &= e^{V(z)} z^{\beta_1} |z - z_1|^{2\alpha_1} g_{z_1, \beta_1}(z) z_1^{-\beta_1} \\ &\quad \times (z - e^t)^{\alpha_0 + \beta_0} (z - e^{-t})^{\alpha_0 - \beta_0} z^{-\alpha_0 + \beta_0} e^{-i\pi(\alpha_0 + \beta_0)}, \quad (z \in \mathbb{T}). \end{aligned} \quad (2.2.1)$$

We find the asymptotics for (2.2.1) when $t > 0$ is fixed using Theorem 1.4.2. We first notice that the analytic part of the symbol is no longer just $e^{V(z)}$ but the function (1.5.1) with $\alpha = \alpha_0$ and $\beta = \beta_0$.

We can write the symbol as,

$$f(z; t) = a(z; t) \times z^{\beta_1} |z - z_1|^{2\alpha_1} g_{z_1, \beta_1}(z) z_1^{-\beta_1}, \quad (z \in \mathbb{T}), \quad (2.2.2)$$

where $a(z; t)$ is an analytic function in the annulus containing the unit circle (in fact it is analytic in $\mathbb{C}/[0, e^{-t}] \cup [e^t, +\infty]$), see Section 1.5.1,

$$a(z; t) = e^{V(z)} (z - e^t)^{\alpha_0 + \beta_0} (z - e^{-t})^{\alpha_0 - \beta_0} z^{-\alpha_0 + \beta_0} e^{-i\pi(\alpha_0 + \beta_0)} \quad (z \in \mathbb{T}). \quad (2.2.3)$$

Recall the Fourier coefficients of $\log a(z; t)$ from (1.5.2). We can write this function in the following way,

$$\log a(z; t) = \sum_{k=-\infty}^{\infty} (\log a)_k z^k,$$

We compute the Wiener-Hopf factorisation of $a(z; t)$ which yields,

$$\log a(z; t) = \log a_+(z; t) + t(\alpha_0 + \beta_0) + V_0 + \log a_-(z; t),$$

$$\log a_+(z; t) = \sum_{k=1}^{\infty} (V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k}) z^k, \quad \log a_-(z; t) = \sum_{k=1}^{\infty} (V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k}) z^{-k}.$$

We can now use Theorem 1.4.2 to obtain the following expression for the asymptotics in the case $t > 0$ with t fixed,

$$\begin{aligned} D_n(f) &= \exp \{nV_0 + nt(\alpha_0 + \beta_0)\} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \right\} \\ &\times \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} (V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k}) z_1^k - (\alpha_1 + \beta_1) \sum_{k=1}^{\infty} (V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k}) z_1^{-k} \right\} \\ &\times n^{(\alpha_1^2 - \beta_1^2)} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1}(1 + o(1)), \end{aligned} \quad (2.2.4)$$

where $G_{\alpha_j + \beta_j, \alpha_j - \beta_j}$ is defined for a $j \in \mathbb{N}_0$ in (1.4.13).

At $t = 0$ the symbol is the symbol with 2 F-H singularities (1.4.1). The asymptotics in the case

of $\|\beta\| < 1$ are given by Theorem 1.4.2 straightforwardly,

$$\begin{aligned}
D_n(f) &= \exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ -(\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k - (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} \right\} \\
&\times \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^k - (\alpha_1 + \beta_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\times n^{\sum_{j=0}^1 (\alpha_j^2 - \beta_j^2)} |1 - z_1|^{2(\beta_0 \beta_1 - \alpha_0 \alpha_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{\alpha_0 \beta_1 - \alpha_1 \beta_0} \\
&\times \prod_{j=0}^1 G_{\alpha_j + \beta_j, \alpha_j - \beta_j} (1 + o(1)). \tag{2.2.5}
\end{aligned}$$

In the case when the seminorm $\|\beta\| = 1$, the symbol possesses F-H representations (see Section 1.4.2) and one uses Theorem 1.4.6. The symbol has only two β parameters, which implies it can only have two F-H representations. The trivial one corresponds to β_0 and β_1 . Without loss of generality let us assume $\operatorname{Re} \beta_0 < \operatorname{Re} \beta_1$, and so the one non-trivial representation will correspond to $\tilde{\beta}_0 = \beta_0 + 1$ and $\tilde{\beta}_1 = \beta_1 - 1$. The asymptotics are then given by (1.4.23),

$$\begin{aligned}
D_n(f) &= \left[\exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ -(\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k - (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} \right\} \right. \\
&\times \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^k - (\alpha_1 + \beta_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\times n^{\sum_{j=0}^1 (\alpha_j^2 - \beta_j^2)} |1 - z_1|^{2(\beta_0 \beta_1 - \alpha_0 \alpha_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{\alpha_0 \beta_1 - \alpha_1 \beta_0} \prod_{j=0}^1 G_{\alpha_j + \beta_j, \alpha_j - \beta_j} \\
&+ (z_1^{-1})^n \exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ -(\alpha_0 - \tilde{\beta}_0) \sum_{k=1}^{\infty} V_k - (\alpha_0 + \tilde{\beta}_0) \sum_{k=1}^{\infty} V_{-k} \right\} \\
&\times \exp \left\{ -(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} V_k z_1^k - (\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\left. \times n^{\sum_{j=0}^1 (\alpha_j^2 - \tilde{\beta}_j^2)} |1 - z_1|^{2(\tilde{\beta}_0 \tilde{\beta}_1 - \alpha_0 \alpha_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{\alpha_0 \tilde{\beta}_1 - \alpha_1 \tilde{\beta}_0} \times \prod_{j=0}^1 G_{\alpha_j + \tilde{\beta}_j, \alpha_j - \tilde{\beta}_j} \right] (1 + o(1)). \tag{2.2.6}
\end{aligned}$$

This case will be considered in detail in Section 2.6.

Notice that taking the limit as $t \rightarrow 0$ in (2.2.4) will not produce (2.2.5)—notice in particular the missing product of Barnes G -functions and the powers of n —and much less (2.2.6) for $\|\beta\| = 1$.

The question we want to answer is related to the work done in [11], [12], [14] and [16]. The aim

is to describe the transition asymptotics of the determinant of a Toeplitz matrix with symbol (2.2.1) as the parameter $t \rightarrow 0$. This finds various applications, one of them is described in Chapter 3. The problem will be tackled in a similar way to [16], we first set the problem for orthogonal polynomials (OPs) with the weight $f_t(z)$ given by (2.2.1) and (2.2.2). After that, we find differential identities concerning the Toeplitz determinant. These are given in terms of entries of the solution of the R-H problem for OPs with weight $f_t(z)$. We will proceed to evaluate the R-H problem in our case, substitute the asymptotics into the differential identity and integrating this expression will give us the answer.

2.3 Differential Identity

In this Section we will derive the differential identity that will link our Toeplitz determinant with symbol $f(z; t)$ (2.2.1) to the R-H problem for polynomials orthogonal with respect to the weight $f(z; t)$. While the identity is the same which appeared in [11], we would like to emphasise that we derive it via a different method, one which follows more naturally from our exposition in Chapter 1. In [11] the authors used integral and integrable Fredholm operators, here we present the same result using orthogonal polynomials (cf. Section 1.6). In what follows we write $f(z)$ for $f(z; t)$.

Lemma 2.3.1. *Let $t > 0$ and $n \in \mathbb{N}$. Suppose that the R-H problem for $Y(z; n, t)$ in Section 1.6.2 with $f(z)$ given by (2.2.1) is solvable. Then $D_n \neq 0$ and the following differential identity holds,*

$$\frac{\partial}{\partial t} \log D_n(t) = -(\alpha_0 + \beta_0) e^t \left(Y^{-1} \frac{dY}{dz} \right)_{22} (e^t) + (\alpha_0 - \beta_0) e^{-t} \left(Y^{-1} \frac{dY}{dz} \right)_{22} (e^{-t}), \quad (2.3.1)$$

where $\left(Y^{-1} \frac{dY}{dz} \right)_{22}(\xi)$ denotes the 22 entry of the matrix obtained by multiplying the two matrices $Y^{-1}(z)$ and $\frac{dY}{dz}(z)$ (each entry of Y is differentiated with respect to z) together, evaluated at $z = \xi$.

Proof. As has been derived in [16] and subsequently also used in [12], the following identity involving orthogonal polynomials and the determinant of the Toeplitz matrix holds for any parameter with respect to which the polynomials are differentiable,

$$\frac{\partial}{\partial t} \log D_n(f(z)) = 2n \frac{\frac{\partial \chi_n}{\partial t}}{\chi_n} + \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\partial}{\partial t} \left(\phi_n(z) \frac{d\hat{\phi}_n(z^{-1})}{dz} - \hat{\phi}_n(z^{-1}) \frac{d\phi_n(z)}{dz} \right) f(z) dz. \quad (2.3.2)$$

Using the Leibniz rule,

$$\frac{\partial}{\partial t} \int_{\mathbb{T}} F(z)f(z)dz = \int_{\mathbb{T}} \frac{\partial}{\partial t} (F(z)f(z)) dz, \quad (2.3.3)$$

as \mathbb{T} does not depend on t , and using the product rule one obtains that for an analytic function $F(z)$ and $\frac{\partial F(z)}{\partial t}$,

$$\int_{\mathbb{T}} \frac{\partial F(z)}{\partial t} f(z)dz = \frac{\partial}{\partial t} \int_{\mathbb{T}} F(z)f(z)dz - \int_{\mathbb{T}} F(z) \frac{\partial f(z)}{\partial t} dz. \quad (2.3.4)$$

Letting $F(z) = \phi_n(z) \frac{d\hat{\phi}_n(z^{-1})}{dz} - \hat{\phi}_n(z^{-1}) \frac{d\phi_n(z)}{dz}$ and noting that by linearity, and orthogonality conditions (1.6.1) (as $\frac{d\hat{\phi}_n(z^{-1})}{dz} = -n\chi_n z^{-(n+1)} + \dots$ and $\frac{d\phi_n(z)}{dz} = n\chi_n z^{n-1} + \dots$),

$$\frac{\partial}{\partial t} \left(\frac{1}{2\pi i} \int_{\mathbb{T}} \left(\phi_n(z) \frac{d\hat{\phi}_n(z^{-1})}{dz} - \hat{\phi}_n(z^{-1}) \frac{d\phi_n(z)}{dz} \right) z f(z) \frac{dz}{z} \right) = \frac{\partial}{\partial t} (-2n) = 0, \quad (2.3.5)$$

we now have,

$$\frac{\partial}{\partial t} \log D_n(f(z)) = 2n \frac{\frac{\partial \chi_n}{\partial t}}{\chi_n} - \frac{1}{2\pi i} \int_{\mathbb{T}} \left(\phi_n(z) \frac{d\hat{\phi}_n(z^{-1})}{dz} - \hat{\phi}_n(z^{-1}) \frac{d\phi_n(z)}{dz} \right) \frac{\partial f(z)}{\partial t} dz. \quad (2.3.6)$$

Computing the derivative gives,

$$\frac{\partial f(z)}{\partial t} = \left(-\frac{\alpha_0 + \beta_0}{z - e^t} e^t + \frac{\alpha_0 - \beta_0}{z - e^{-t}} e^{-t} \right) f(z),$$

thus we obtain (now using the notation $F(z)$ as mentioned before),

$$\frac{\partial}{\partial t} \log D_n(f(z)) = 2n \frac{\frac{\partial \chi_n}{\partial t}}{\chi_n} + (\alpha_0 + \beta_0) e^t \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{F(z)f(z)}{z - e^t} dz - (\alpha_0 - \beta_0) e^{-t} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{F(z)f(z)}{z - e^{-t}} dz.$$

We are left with evaluating the following integral,

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{F(z)f(z)}{z - \xi} dz = I_1 - I_2,$$

where $\xi = e^t$ or e^{-t} , and,

$$I_1 = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\phi_n(z) \frac{d\hat{\phi}_n(z^{-1})}{dz}}{z - \xi} f(z) dz \quad \text{and} \quad I_2 = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\hat{\phi}_n(z^{-1}) \frac{d\phi_n(z)}{dz}}{z - \xi} f(z) dz.$$

Note first that for any polynomial of degree $n \in \mathbb{Z}_+$, $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$,

$$\begin{aligned} \frac{p_n(z) - p_n(\xi)}{z - \xi} &= \frac{a_n(z^n - \xi^n) + a_{n-1}(z^{n-1} - \xi^{n-1}) + \dots + a_1(z - \xi)}{z - \xi} \\ &= \frac{a_n(z - \xi)(z^{n-1} + \xi^{n-1}) - z\xi^{n-1} + \xi z^{n-1} + \dots}{z - \xi} \\ &= a_n z^{n-1} + \mathcal{O}(z^{n-2}), \end{aligned} \tag{2.3.7}$$

which also holds for $n \in \mathbb{Z}_-$ by substituting $z \mapsto z^{-1}$ and $\xi \mapsto \xi^{-1}$.

Now, starting with I_1 , adding and subtracting $\frac{d\hat{\phi}_n(z^{-1})}{dz}|_{z=\xi}$ in the numerator and using orthogonality (1.6.1) gives,

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \int -\phi_n(z) \frac{\left(\frac{d\hat{\phi}_n(z^{-1})}{dz} - \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} \right)}{\frac{-(z-\xi)}{z\xi} \left(= \frac{1}{z} - \frac{1}{\xi} \right)} (z\xi)^{-1} f(z) dz + \frac{1}{2\pi i} \int \frac{\phi_n(z) \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi}}{z - \xi} f(z) dz \\ &= \frac{1}{2\pi i} \int -\phi_n(z) (-n\chi_n z^{-n} + \dots + \dots z^{-1}) (z\xi)^{-1} f(z) dz \\ &\quad + \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} \frac{1}{2\pi i} \int \frac{\phi_n(z)}{z - \xi} (z^n - \xi^n + \xi^n) \frac{f(z)}{z^n} dz \\ &= \frac{n\chi_n \xi^{-1}}{\chi_n} + \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} \frac{1}{2\pi i} \int \frac{\phi_n(z)}{z - \xi} (z^n - \xi^n) \frac{f(z)}{z^n} dz + \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} \frac{1}{2\pi i} \int \frac{\phi_n(z)}{z - \xi} \xi^n \frac{f(z)}{z^n} dz \\ &= n\xi^{-1} + \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} \frac{1}{2\pi i} \int \phi_n(z) (z^{n-1} + \dots) \frac{f(z)}{z^n} dz + \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} \frac{1}{2\pi i} \int \frac{\phi_n(z)}{z - \xi} \xi^n \frac{f(z)}{z^n} dz, \end{aligned}$$

and by comparing with the entries of the R-H problem (1.6.10),

$$I_1 = n\xi^{-1} + \xi^n \chi_n \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} Y_{12}(\xi).$$

We now look at I_2 , proceeding as before,

$$\begin{aligned} I_2 &= \frac{1}{2\pi i} \int \hat{\phi}_n(z^{-1}) \frac{\left(\frac{d\phi_n(z)}{dz} - \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \right)}{z - \xi} z f(z) \frac{dz}{z} + \frac{1}{2\pi i} \int \frac{\hat{\phi}_n(z^{-1}) \frac{d\phi_n(z)}{dz} \Big|_{z=\xi}}{z - \xi} f(z) dz \\ &= \frac{1}{2\pi i} \int \hat{\phi}_n(z^{-1}) (n\chi_n z^{n-1} + \dots) f(z) \frac{dz}{z} + \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \frac{1}{2\pi i} \int \frac{\hat{\phi}_n(z^{-1})}{z - \xi} f(z) dz. \end{aligned}$$

By orthogonality in the first term and by using the following recurrence relation:

$$\chi_n \hat{\phi}_n(z^{-1}) = \chi_{n-1} z^{-1} \hat{\phi}_{n-1}(z^{-1}) + \hat{\phi}_n(0) z^{-n} \phi_n(z)$$

in the second, we obtain

$$\begin{aligned} I_2 &= 0 + \frac{1}{2\pi i} \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \int \frac{\chi_{n-1} \hat{\phi}_{n-1}(z^{-1}) f(z)}{\chi_n} \frac{dz}{z-\xi} \frac{dz}{z} + \frac{1}{2\pi i} \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \int \frac{\hat{\phi}_n(0) \phi_n(z) f(z)}{\chi_n} \frac{dz}{z-\xi} \frac{dz}{z^n} \\ &= -\frac{dY_{11}(z)}{dz} \Big|_{z=\xi} Y_{22}(\xi) + \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \hat{\phi}_n(0) Y_{12}(\xi). \end{aligned}$$

We now combine the two results,

$$I_1 - I_2 = n\xi^{-1} + \xi^n \chi_n \frac{d\hat{\phi}_n(z^{-1})}{dz} \Big|_{z=\xi} Y_{12}(\xi) + \frac{dY_{11}(z)}{dz} \Big|_{z=\xi} Y_{22}(\xi) - \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \hat{\phi}_n(0) Y_{12}(\xi).$$

Using the same recurrence relation as above for $\hat{\phi}_n(z^{-1})$ and collecting the $Y_{12}(\xi)$ terms gives,

$$\begin{aligned} I_1 - I_2 &= n\xi^{-1} + \frac{dY_{11}(z)}{dz} \Big|_{z=\xi} Y_{22}(\xi) + \left\{ \xi^n \frac{d}{dz} \left(\chi_{n-1} z^{-1} \hat{\phi}_{n-1}(z^{-1}) + \hat{\phi}_n(0) z^{-n} \phi_n(z) \right) \Big|_{z=\xi} \right. \\ &\quad \left. - \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \hat{\phi}_n(0) \right\} Y_{12}(\xi) \\ &= n\xi^{-1} + \frac{dY_{11}(z)}{dz} \Big|_{z=\xi} Y_{22}(\xi) + \left\{ \xi^n \left(-\chi_{n-1} \xi^{-2} \hat{\phi}_{n-1}(\xi^{-1}) + \chi_{n-1} \xi^{-1} \frac{d\hat{\phi}_{n-1}(z^{-1})}{dz} \Big|_{z=\xi} \right. \right. \\ &\quad \left. \left. - n\hat{\phi}_n(0) \xi^{-(n+1)} \phi_n(\xi) + \hat{\phi}_n(0) \xi^{-n} \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \right) - \frac{d\phi_n(z)}{dz} \Big|_{z=\xi} \hat{\phi}_n(0) \right\} Y_{12}(\xi). \end{aligned}$$

After a cancellation and further manipulation we obtain,

$$\begin{aligned} I_1 - I_2 &= n\xi^{-1} + \frac{dY_{11}(z)}{dz} \Big|_{z=\xi} Y_{22}(\xi) + \left\{ \xi^{-1} \left(-\chi_{n-1} \xi^{n-1} \hat{\phi}_{n-1}(\xi^{-1}) \right) + \chi_{n-1} \xi^{n-1} \frac{d\hat{\phi}_{n-1}(z^{-1})}{dz} \Big|_{z=\xi} \right. \\ &\quad \left. - n\hat{\phi}_n(0) \xi^{-1} \phi_n(\xi) \right\} Y_{12}(\xi). \end{aligned}$$

Adding and subtracting $\chi_{n-1}(n-1)\xi^{n-2}\hat{\phi}_{n-1}(\xi^{-1})$,

$$\begin{aligned} I_1 - I_2 &= n\xi^{-1} + \frac{dY_{11}(z)}{dz} \Big|_{z=\xi} Y_{22}(\xi) + \left\{ \xi^{-1} \left(-\chi_{n-1} \xi^{n-1} \hat{\phi}_{n-1}(\xi^{-1}) - \chi_{n-1}(n-1)\xi^{n-1} \hat{\phi}_{n-1}(\xi^{-1}) \right) \right. \\ &\quad \left. + \chi_{n-1}(n-1)\xi^{n-2} \hat{\phi}_{n-1}(\xi^{-1}) + \chi_{n-1} \xi^{n-1} \frac{d\hat{\phi}_{n-1}(z^{-1})}{dz} \Big|_{z=\xi} - n\hat{\phi}_n(0) \xi^{-1} \phi_n(\xi) \right\} Y_{12}(\xi). \end{aligned}$$

Noting that $\frac{dY_{21}(z)}{dz} = -\chi_{n-1}(n-1)z^{n-2}\hat{\phi}_{n-1}(z^{-1}) - \chi_{n-1}(z^{n-1})\frac{d\hat{\phi}_{n-1}(z^{-1})}{dz}$ we rewrite using the R-H problem (1.6.10),

$$I_1 - I_2 = n\xi^{-1} + \frac{dY_{11}(z)}{dz}\Big|_{z=\xi} Y_{22}(\xi) + \left\{ n\xi^{-1} Y_{21}(\xi) - \frac{dY_{21}(z)}{dz}\Big|_{z=\xi} - n\hat{\phi}_n(0)\xi^{-1}\chi_n Y_{11}(\xi) \right\} Y_{12}(\xi).$$

Next, we look at the term $\frac{2n\frac{\partial\chi_n}{\partial t}}{\chi_n}$. Noting that

$$\frac{1}{2\pi i} \int \frac{\partial\phi_n(z)}{\partial t} \hat{\phi}_n(z^{-1}) f(z) \frac{dz}{z} = \frac{\partial\chi_n}{\chi_n} = \frac{1}{2\pi i} \int \frac{\partial\hat{\phi}_n(z^{-1})}{\partial t} \phi(z) f(z) \frac{dz}{z},$$

we have,

$$2\frac{\partial\chi_n}{\chi_n} = \frac{1}{2\pi i} \int \frac{\partial}{\partial t} \left(\phi_n(z) \hat{\phi}_n(z^{-1}) \right) f(z) \frac{dz}{z}.$$

Using (2.3.4) and that $\frac{\partial}{\partial t} \left[\frac{1}{2\pi i} \int_{\mathbb{T}} \hat{\phi}_n(z^{-1}) \phi_n(z) f(z) \frac{dz}{z} \right] = \frac{\partial}{\partial t} [1] = 0$,

$$\begin{aligned} 2\frac{\partial\chi_n}{\chi_n} &= -\frac{1}{2\pi i} \int \phi_n(z) \hat{\phi}_n(z^{-1}) \frac{\partial f(z)}{\partial t} \frac{dz}{z} \\ &= (\alpha_0 + \beta_0) e^t \frac{1}{2\pi i} \int \frac{\phi_n(z) \hat{\phi}_n(z^{-1}) f(z)}{z - e^t} \frac{dz}{z} - (\alpha_0 - \beta_0) e^{-t} \frac{1}{2\pi i} \int \frac{\phi_n(z) \hat{\phi}_n(z^{-1}) f(z)}{z - e^{-t}} \frac{dz}{z}. \end{aligned}$$

We now evaluate $\frac{1}{2\pi i} \int \frac{\phi_n(z) \hat{\phi}_n(z^{-1}) f(z)}{z - \xi} \frac{dz}{z}$, where $\xi = e^{\pm t}$. As before,

$$\frac{1}{2\pi i} \int \phi_n(z) \frac{\hat{\phi}_n(z^{-1})}{z - \xi} f(z) \frac{dz}{z} = \frac{1}{2\pi i} \int -\phi_n(z) \frac{\hat{\phi}_n(z^{-1})}{\frac{1}{z} - \frac{1}{\xi}} (z\xi)^{-1} f(z) \frac{dz}{z}. \quad (2.3.8)$$

Adding and subtracting $\hat{\phi}_n(\xi^{-1})$ in the numerator gives,

$$\text{RHS}(2.3.8) = \frac{1}{2\pi i} \int -\phi_n(z) \frac{\hat{\phi}_n(z^{-1}) - \hat{\phi}_n(\xi^{-1})}{\frac{1}{z} - \frac{1}{\xi}} (z\xi)^{-1} f(z) \frac{dz}{z} + \frac{1}{2\pi i} \int \phi_n(z) \frac{\hat{\phi}_n(\xi^{-1})}{z - \xi} f(z) \frac{dz}{z}.$$

Continuing with a similar argument as before using (2.3.7) in the first integral and manipulating z and adding and subtracting ξ^{n-1} in the second integral gives,

$$= \frac{1}{2\pi i} \int -\phi_n(z) \left(\chi_n z^{-(n-1)} + \dots \right) (z\xi)^{-1} f(z) \frac{dz}{z}$$

$$+ \hat{\phi}_n(\xi^{-1}) \frac{1}{2\pi i} \int \frac{\phi_n(z)}{z - \xi} (z^{n-1} - \xi^{n-1} + \xi^{n-1}) f(z) \frac{dz}{z^n},$$

by orthogonality,

$$\begin{aligned} &= -\xi^{-1} \chi_n \chi_n^{-1} + \xi^{n-1} \hat{\phi}_n(\xi^{-1}) \frac{1}{2\pi i} \int \frac{\phi_n(z)}{z - \xi} f(z) \frac{dz}{z^n} \\ &= -\xi^{-1} + \hat{\phi}_n(\xi^{-1}) \chi_n \xi^{n-1} Y_{21}(\xi), \quad \text{c.f. (1.6.10)}. \end{aligned} \quad (2.3.9)$$

Using the same recurrence relation for $\hat{\phi}_n(z^{-1})$ as before,

$$\begin{aligned} \text{RHS(2.3.9)} &= -\xi^{-1} + \xi^{n-1} Y_{21}(\xi) \chi_{n-1} \xi^{-1} \hat{\phi}_{n-1}(\xi^{-1}) + \xi^{n-1} Y_{21}(\xi) \hat{\phi}_n(0) \xi^{-n} \phi_n(\xi) \\ &= -\xi^{-1} - \xi^{-1} Y_{21}(\xi) Y_{12}(\xi) + \hat{\phi}_n(0) \chi_n \xi^{-1} Y_{11}(\xi) Y_{12}(\xi). \end{aligned}$$

Finally, we combine all of the results,

$$\begin{aligned} \frac{\partial}{\partial t} \log D_n(t) &= -(\alpha_0 + \beta_0) e^t \left(ne^{-t} + ne^{-t} Y_{21}(e^t) Y_{12}(e^t) - n \hat{\phi}_n(0) \chi_n e^{-t} Y_{11}(e^t) Y_{12}(e^t) \right) \\ &\quad + (\alpha_0 - \beta_0) e^{-t} \left(ne^t + ne^t Y_{21}(e^{-t}) Y_{12}(e^{-t}) - n \hat{\phi}_n(0) \chi_n e^t Y_{11}(e^{-t}) Y_{12}(e^{-t}) \right) \\ &\quad - (\alpha_0 + \beta_0) e^t \left(-ne^{-t} - \frac{dY_{11}(z)}{dz} \Big|_{z=e^t} Y_{22}(e^t) \right. \\ &\quad \left. + \left\{ -ne^{-t} Y_{21}(e^t) + \frac{dY_{21}(z)}{dz} \Big|_{z=e^t} + n \hat{\phi}_n(0) e^{-t} \chi_n Y_{11}(e^t) \right\} Y_{12}(e^t) \right) \\ &\quad + (\alpha_0 - \beta_0) e^{-t} \left(-ne^t - \frac{dY_{11}(z)}{dz} \Big|_{z=e^{-t}} Y_{22}(e^{-t}) \right. \\ &\quad \left. + \left\{ -ne^t Y_{21}(e^{-t}) + \frac{dY_{21}(z)}{dz} \Big|_{z=e^{-t}} + n \hat{\phi}_n(0) e^t \chi_n Y_{11}(e^{-t}) \right\} Y_{12}(e^{-t}) \right). \end{aligned}$$

After cancellations,

$$\begin{aligned} \frac{\partial}{\partial t} \log D_n(t) &= -(\alpha_0 + \beta_0) e^t \left(-\frac{dY_{11}(z)}{dz} \Big|_{z=e^t} Y_{22}(e^t) + \frac{dY_{21}(z)}{dz} \Big|_{z=e^t} Y_{12}(e^t) \right) \\ &\quad + (\alpha_0 - \beta_0) e^{-t} \left(-\frac{dY_{11}(z)}{dz} \Big|_{z=e^{-t}} Y_{22}(e^{-t}) + \frac{dY_{21}(z)}{dz} \Big|_{z=e^{-t}} Y_{12}(e^{-t}) \right). \end{aligned}$$

We thus obtain,

$$\frac{\partial}{\partial t} \log D_n(t) = -(\alpha_0 + \beta_0) e^t \left(-\frac{dY^{-1}}{dz} Y \right)_{22} (e^t) + (\alpha_0 - \beta_0) e^{-t} \left(-\frac{dY^{-1}}{dz} Y \right)_{22} (e^{-t}), \quad (2.3.10)$$

which is exactly (2.3.1), note that $0 = \frac{d}{dz}(Y^{-1}Y) = Y^{-1}\frac{dY}{dz} + \frac{dY^{-1}}{dz}Y$. \square

2.4 Asymptotic analysis of the Riemann-Hilbert problem

In this section we will solve the R-H problem which was posed in Section 1.6.2 with $f(z) = f(z; t)$ defined in (2.2.1). In order to evaluate the asymptotics for the solution to the R-H problem above, the problem needs to undergo a series of *reversible* transformations. We will be using the method of nonlinear steepest descent, which was introduced by Deift and Zhou in 1990's, see [20]. A very nice example of this method can be found in [13], where the author proves the SSLT (Theorem 1.3.4). Each transformation simplifies the problem further and brings us closer to the solution of the original R-H problem, i.e. the problem for $Y(z)$. The final goal of this is to arrive at a problem known as the small-norm problem. The jump matrix behaves asymptotically like an identity, and we can obtain a solution in terms of Neumann series. This culminating R-H problem will be called the R-H problem for $R(z)$. Then, after a series of reverse transformations, we will arrive at the solution we were ultimately looking for. We proceed with the analysis in a similar way to that in [11, 12, 14, 16].

2.4.1 Normalisation

The R-H problem for $Y(z)$ lacks the right behaviour at infinity. Defining a new function $T(z)$ in the following way normalises the problem at infinity,

$$T(z) = \begin{cases} Y(z)z^{-k\sigma_3}, & \text{as } |z| > 1, \\ Y(z), & \text{as } |z| < 1. \end{cases} \quad (2.4.1)$$

Recall one of the Pauli matrices,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.4.2)$$

it follows that $z^{-k\sigma_3} = \begin{pmatrix} z^{-k} & 0 \\ 0 & z^k \end{pmatrix}$.

We now have two equivalent R-H problems, if $Y(z)$ solves the R-H problem for orthogonal polynomials (1.6.10), then $T(z)$ solves the following R-H problem,

(T1) $T : \mathbb{C} \setminus \mathbb{T} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

$$(T2) \quad T_+(z) = T_-(z) \begin{pmatrix} z^k & f(z; t) \\ 0 & z^{-k} \end{pmatrix} \text{ for } z \in \mathbb{T}.$$

(T3) $T(z)$ has a the following asymptotic behaviour at ininity,

$$T(z) = I + \mathcal{O}(1/z) \quad \text{as } z \rightarrow \infty. \quad (2.4.3)$$

(T4) The asymptotic formulae close to point z_1 is the same as in the problem for $Y(z)$.

The two problems are equivalent in the sense that we can obtain a solution of one problem, using the solution to the other, and vice versa, via simple algebraic manipulation.

Notice that the diagonal entries of the jump matrix in point (T2)—let us denote it by $J_T(z)$ for future reference—oscillate rapidly on the unit circle for large k . The next transformation will turn this oscillatory behaviour on the unit circle into exponential decay on a new, deformed contour.

2.4.2 Opening of the lenses

First, we note that we can factorise the jump matrix $J_T(z)$, in the following way

$$\begin{aligned} J_T(z) &= \begin{pmatrix} z^k & f(z; t) \\ 0 & z^{-k} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^{-k} f(z; t)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & f(z; t) \\ -f(z; t)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^k f(z; t)^{-1} & 1 \end{pmatrix} \\ &=: J_1(z) J_N(z) J_2(z), \text{ respectively.} \end{aligned} \quad (2.4.4)$$

The initial contour—the unit circle—is now deformed as shown in Figure 2.2. The matrix-valued functions resulting from the factorisation of the jump matrix $J_T(z)$ have some specific properties. Namely, $J_1(z)$ and $J_2(z)$ are invertible. They both have analytic continuations (which are also invertible) to the outside and inside of the unit disk, respectively—intersected with the annulus where the function $f(z, t)$ is analytic. We also note the desired decay of the off-diagonal terms in the

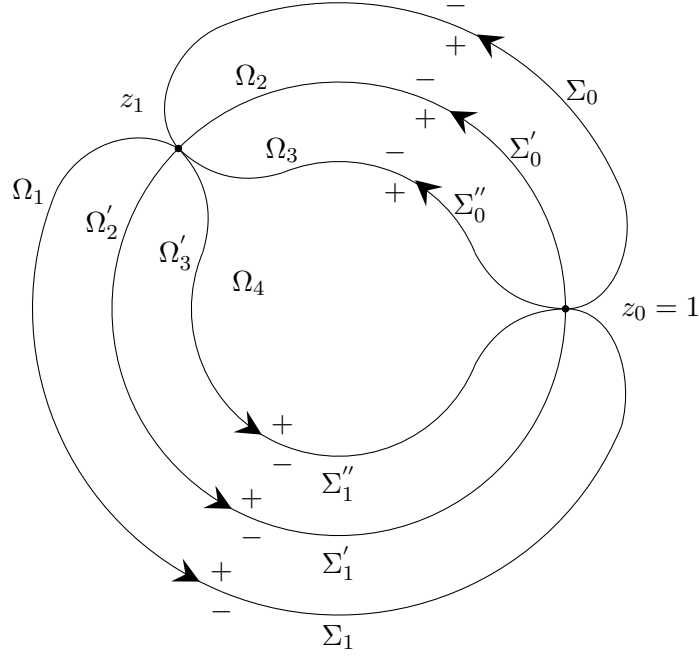


Figure 2.2: Deformed contour $\Sigma = \cup_{j=0}^1 (\Sigma_j \cup \Sigma'_j \cup \Sigma''_j)$, and regions Ω_j, Ω'_j for the Riemann-Hilbert problem for $S(z)$. Note well, $\mathbb{T} = \Sigma'_0 \cap \Sigma'_1$.

two matrices, in the respective regions of analytic continuation. Let us now define a new function,

$$S(z) = \begin{cases} T(z), & \text{for } z \in \Omega_1 \cup \Omega_4, \\ T(z)J_1(z), & \text{for } z \in \Omega_2 \cup \Omega'_2, \\ T(z)J_2^{-1}(z), & \text{for } z \in \Omega_3 \cup \Omega'_3. \end{cases} \quad (2.4.5)$$

Note well, that $\mathbb{T} = \Sigma'_0 \cup \Sigma'_1$. The function $S(z)$ solves the following R-H problem,

(S1) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$, where $\Sigma = \cup_{j=0}^1 (\Sigma_j \cup \Sigma'_j \cup \Sigma''_j)$.

(S2) The boundary values are related by the following jump conditions:

$$S_+(z) = S_-(z)J_1(z), \quad z \in \Sigma_0 \cup \Sigma_1.$$

$$S_+(z) = S_-(z)J_N(z), \quad z \in \Sigma'_0 \cup \Sigma'_1.$$

$$S_+(z) = S_-(z)J_2(z), \quad z \in \Sigma''_0 \cup \Sigma''_1.$$

(S3) $S(z) = T(z) \rightarrow I + \mathcal{O}(1/z)$ as $z \rightarrow \infty$.

(S4) As $z \rightarrow z_1$, $z \in \mathbb{C} \setminus \mathbb{T}$, outside the lense (i.e. in $\Omega_1 \cup \Omega_4$)

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|z - z_1|^{2\alpha_1}) + \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(|z - z_1|^{2\alpha_1}) + \mathcal{O}(1) \end{pmatrix} \quad \alpha_1 \neq 0, \quad (2.4.6)$$

and

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log |z - z_1|) \\ \mathcal{O}(1) & \mathcal{O}(\log |z - z_1|) \end{pmatrix} \quad \alpha_1 = 0, \beta_1 \neq 0. \quad (2.4.7)$$

The asymptotic behaviour in $\Omega_2 \cup \Omega_2'$ and $\Omega_3 \cup \Omega_3'$ is given by applying jump conditions to the expressions (2.4.6) and (2.4.7). We have the following,

$$S(z) = \begin{pmatrix} \mathcal{O}(1) + \mathcal{O}(|z - z_1|^{-2\alpha_1}) & \mathcal{O}(1) \\ \mathcal{O}(1) + \mathcal{O}(|z - z_1|^{-2\alpha_1}) & \mathcal{O}(1) \end{pmatrix} \quad \text{in } \Omega_2 \cup \Omega_2', \alpha_1 \neq 0, \quad (2.4.8)$$

$$S(z) = \begin{pmatrix} \mathcal{O}(1) + \mathcal{O}(|z - z_1|^{-2\alpha_1}) & \mathcal{O}(1) \\ \mathcal{O}(1) + \mathcal{O}(|z - z_1|^{-2\alpha_1}) & \mathcal{O}(1) \end{pmatrix} \quad \text{in } \Omega_2 \cup \Omega_2', \alpha_1 = 0, \beta_1 \neq 0, \quad (2.4.9)$$

and

$$S(z) = \begin{pmatrix} \mathcal{O}(1) + \mathcal{O}(\log |z - z_1|) & \mathcal{O}(\log |z - z_1|) \\ \mathcal{O}(1) + \mathcal{O}(\log |z - z_1|) & \mathcal{O}(\log |z - z_1|) \end{pmatrix} \quad \text{in } \Omega_3 \cup \Omega_3', \alpha_1 \neq 0, \quad (2.4.10)$$

$$S(z) = \begin{pmatrix} \mathcal{O}(1) + \mathcal{O}(\log |z - z_1|) & \mathcal{O}(\log |z - z_1|) \\ \mathcal{O}(1) + \mathcal{O}(\log |z - z_1|) & \mathcal{O}(\log |z - z_1|) \end{pmatrix} \quad \text{in } \Omega_3 \cup \Omega_3', \alpha_1 = 0, \beta_1 \neq 0. \quad (2.4.11)$$

The new R-H problem $(S(z), \Sigma)$ is called a deformation of the problem $(T(z), \mathbb{T})$ and can be compared to the deformation of a contour in the method of evaluating a classical steepest descent problem in complex analysis.

Let us encircle the points $z_0 = 1$ and z_1 by ε -small disks,

$$U_{z_j} = \{z : |z - z_j| < \varepsilon\}, \quad j = 0, 1. \quad (2.4.12)$$

We shall construct a global parametrix dealing with the jump condition over the unit circle and two local parametrices around the points of intersection, z_0 and z_1 .

2.4.3 Global Parametrix

Here, we consider a R-H problem for $N(z)$, ignoring $\cup_{j=0}^1 (\Sigma_j \cup \Sigma_j'')$ and the neighbourhoods $\cup_{j=0}^1 U_{z_j}$. The model problem is given by

$$(N1) \quad N : \mathbb{C} \setminus \mathbb{T} \rightarrow \mathbb{C}^{2 \times 2} \text{ is analytic,}$$

$$(N2) \quad N_+(z) = N_-(z)J_N(z), \text{ for } z \in \mathbb{T},$$

$$(N3) \quad N(z) = I + \mathcal{O}(1/z), \text{ as } z \rightarrow \infty.$$

Similarly to [11, 14], the problem is explicitly solvable and the solution is given using the Szegő function,

$$D(z) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log f_t(s)}{s-z} ds \right\}. \quad (2.4.13)$$

We have that,

$$N(z) = \begin{cases} D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } |z| < 1, \\ D(z)^{\sigma_3}, & \text{for } |z| > 1. \end{cases} \quad (2.4.14)$$

2.4.3.1 Computing the Szegő function

Lemma 2.4.1. *By evaluating the integral in (2.4.13) we can compute an explicit formula for the function $D(z)$,*

$$D(z) = \begin{cases} \left(\frac{z-z_1}{z_1 e^{i\pi}} \right)^{\alpha_1 + \beta_1} (z - e^t)^{\alpha_0 + \beta_0} e^{-i\pi(\alpha_0 + \beta_0)} \exp \left\{ \sum_{k=0}^{\infty} V_k z^k \right\}, & \text{for } |z| < 1, \\ \left(\frac{z-z_1}{z} \right)^{-\alpha_1 + \beta_1} (z - e^{-t})^{-\alpha_0 + \beta_0} z^{\alpha_0 - \beta_0} \exp \left\{ -\sum_{k=-\infty}^{-1} V_k z^k \right\}, & \text{for } |z| > 1. \end{cases} \quad (2.4.15)$$

Proof. First note that,

$$\begin{aligned} \log f(s; t) &= V(s) + \beta_1 \log s - \beta_1 \log z_1 + 2\alpha_1 \log |s - z_1| + \log g_{z_1, \beta_1}(s) \\ &+ (\alpha_0 + \beta_0) \log(s - e^t) + (\alpha_0 - \beta_0) \log(s - e^{-t}) + (-\alpha_0 + \beta_0) \log s - i\pi(\alpha_0 + \beta_0). \end{aligned} \quad (2.4.16)$$

Throughout, we use the following convergent series,

$$\frac{1}{s-z} = \begin{cases} \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \dots & \text{for } |z| < 1, \\ -\frac{1}{z} - \frac{s}{z^2} - \frac{s^2}{z^3} - \dots & \text{for } |z| > 1, \end{cases} \quad (2.4.17)$$

and

$$\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}, \text{ for } |z| < 1. \quad (2.4.18)$$

We begin with the first term in (2.4.16), the analytic function $V(z)$. Matching coefficients and using the residue theorem, we have for $|z| < 1$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{V(s)}{s-z} ds &= \frac{1}{2\pi i} \int_{\mathbb{T}} \left\{ \dots + V_{-1}s^{-1} + V_0 + V_1s + \dots \right\} \left\{ \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \dots \right\} ds \\ &= V_0 + V_1z + V_2z^2 + \dots = \sum_{k=0}^{\infty} V_k z^k \end{aligned}$$

and for $|z| > 1$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{V(s)}{s-z} ds &= \frac{1}{2\pi i} \int_{\mathbb{T}} \left\{ \dots + V_{-1}s^{-1} + V_0 + V_1s + \dots \right\} \left\{ -\frac{1}{z} - \frac{s}{z^2} - \frac{s^2}{z^3} - \dots \right\} ds \\ &= -V_{-1}z^{-1} - V_{-2}z^{-2} - V_{-3}z^{-3} - \dots = -\sum_{k=1}^{\infty} V_{-k} z^{-k}. \end{aligned}$$

In what follows, we use the substitution $s = e^{i\theta}$, integration by parts, and Residue theorem,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log s}{s-z} ds &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{\log(e^{i\theta}) i e^{i\theta}}{e^{i\theta} - z} d\theta \\ &= \frac{1}{2\pi i} \left[i\theta \log(e^{i\theta} - z) \right]_0^{2\pi} - \frac{1}{2\pi i} \int_0^{2\pi} i \log(e^{i\theta} - z) d\theta \\ &= \log(1-z) - \frac{1}{2\pi i} \int_{\mathbb{T}} \log(s-z) \frac{ds}{s}. \end{aligned}$$

For $|z| < 1$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log s}{s-z} ds &= \log(1-z) - \frac{1}{2\pi i} \int_{\mathbb{T}} \left\{ \log s - \frac{z}{s} - \frac{z^2}{2s^2} - \dots \right\} \left\{ \frac{1}{s} \right\} ds \\ &= \log(1-z) - \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log s}{s} ds - \frac{1}{2\pi i} \int_{\mathbb{T}} \left\{ -\frac{z}{s} - \frac{z^2}{2s^2} - \dots \right\} \left\{ \frac{1}{s} \right\} ds \end{aligned}$$

$$\begin{aligned}
&= \log(1-z) - \frac{1}{2\pi i} \int_0^{2\pi} i \log(e^{i\theta}) d\theta - 0 \\
&= \log(1-z) + \frac{1}{2\pi i} \int_0^{2\pi} \theta d\theta - 0 = \log(1-z) + \frac{1}{2\pi i} 2\pi^2 \\
&= \log(1-z) - i\pi,
\end{aligned} \tag{2.4.19}$$

and for $|z| > 1$,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log s}{s-z} ds &= i\pi + \log(z-1) - \frac{1}{2\pi i} \int_{\mathbb{T}} \left\{ i\pi + \log z - \frac{s}{z} - \frac{s^2}{2z^2} - \dots \right\} \left\{ \frac{1}{s} \right\} ds \\
&= i\pi + \log(z-1) - i\pi - \log z, \\
&= \log(z-1) - \log z.
\end{aligned} \tag{2.4.20}$$

Using the expansions from above and residue theorem, we have for $|z| < 1$,

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log z_1}{s-z} ds = \int_{\mathbb{T}} \log z_1 \left\{ \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \dots \right\} ds = \log z_1, \tag{2.4.21}$$

and for $|z| > 1$,

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log z_1}{s-z} ds = \int_{\mathbb{T}} \log z_1 \left\{ -\frac{1}{z} - \frac{s}{z^2} - \frac{s^2}{z^3} - \dots \right\} ds = 0. \tag{2.4.22}$$

In what follows, we use the function $h_{\alpha_1}(z) = |z - z_1|^{\alpha_1} = \frac{(z-z_1)^{\alpha_1}}{(zz_1 e^{i l_1})^{\alpha_1/2}}$, where $l_1 = 3\pi$ for $0 < \theta < \theta_1$ and $l_1 = \pi$ for $\theta_1 < \theta < 2\pi$ (recall that $z_1 = e^{i\theta_1}$), for more details and analyticity see (2.4.30). We write,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\alpha_1 \log |s - z_1|}{s-z} ds &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\alpha_1 \log(s - z_1) - \alpha_1 \log(s z_1 e^{i l_1})}{s-z} ds \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\alpha_1 \log(s - z_1)}{s-z} ds - \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\alpha_1 \log(s)}{s-z} ds - \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\alpha_1 \log(z_1)}{s-z} ds - \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\alpha_1 \log(e^{i l_1})}{s-z} ds.
\end{aligned}$$

We evaluate the first term in a similar way to (2.4.26) and (2.4.27) below to obtain,

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\alpha_1 \log(s - z_1)}{s-z} ds = \begin{cases} 2\alpha_1 \log(z - z_1) & \text{for } |z| < 1, \\ 0 & \text{for } |z| > 1. \end{cases} \tag{2.4.23}$$

The second and third term was obtained in (2.4.19), (2.4.20), (2.4.21) and (2.4.22). The last term we calculate using (2.4.25) as we note that,

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\alpha_1 \log(e^{il_j})}{s-z} ds = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\alpha_1 \log(e^{2\pi i})}{s-z} ds + \frac{\alpha_1}{\beta_1} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log g_{z_1, \beta_1}(s)}{s-z} ds,$$

which by (2.4.21), (2.4.22) and (2.4.25),

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\alpha_1 \log(e^{il_j})}{s-z} ds = \begin{cases} 2\pi i + \log(z - z_1) - \log(1 - z) & \text{for } |z| < 1, \\ \log(z - z_1) - \log(z - 1) & \text{for } |z| > 1. \end{cases}$$

And thus, for $|z| < 1$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\alpha_1 \log |s - z_1|}{s-z} ds &= 2\alpha_1 \log(z - z_1) - \alpha_1 \log(1 - z) + i\pi\alpha_1 - \alpha_1 \log z_1 \\ &\quad - 2i\pi\alpha_1 - \alpha_1 \log(z - z_1) + \alpha_1 \log(1 - z) \\ &= \alpha_1 \log(z - z_1) - \alpha_1 \log z_1 - i\pi\alpha_1, \end{aligned}$$

and for $|z| > 1$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\alpha_1 \log |s - z_1|}{s-z} ds &= 0 - \alpha_1 \log(z - 1) + \alpha_1 \log z - 0 - \alpha_1 \log(z - z_1) + \alpha_1 \log(z - 1) \\ &= \alpha_1 \log z - \alpha_1 \log(z - z_1) \end{aligned}$$

We now compute,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log g_{z_1, \beta_1}(s)}{s-z} ds &= \frac{1}{2\pi i} \int_0^{\theta_1} \frac{i\pi\beta_1}{s-z} ds - \frac{1}{2\pi i} \int_{\theta_1}^{2\pi} \frac{i\pi\beta_1}{s-z} ds \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{i\pi\beta_1}{s-z} ds - 2 \frac{1}{2\pi i} \int_{\theta_1}^{2\pi} \frac{i\pi\beta_1}{s-z} ds, \end{aligned} \tag{2.4.24}$$

by (2.4.21), (2.4.22),

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log g_{z_1, \beta_1}(s)}{s-z} ds = \begin{cases} i\pi\beta_1 - 2 \frac{1}{2\pi i} \int_{\theta_1}^{2\pi} \frac{i\pi\beta_1}{s-z} ds, & \text{for } |z| < 1, \\ -2 \frac{1}{2\pi i} \int_{\theta_1}^{2\pi} \frac{i\pi\beta_1}{s-z} ds, & \text{for } |z| > 1, \end{cases}$$

and

$$\begin{aligned}
2 \frac{1}{2\pi i} \int_{\theta_1}^{2\pi} \frac{i\pi\beta_1}{s-z} ds &= \beta_1 \int_{\theta_1}^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}-z} d\theta \\
&= \beta_1 \left[\log(e^{i\theta}-z) \right]_{\theta_1}^{2\pi} \\
&= \beta_1 (\log(1-z) - \log(z-z_1) - i\pi),
\end{aligned}$$

we have

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log g_{z_1, \beta_1}(s)}{s-z} ds = \begin{cases} \beta_1 \log(z-z_1) - \beta_1 \log(1-z), & \text{for } |z| < 1, \\ \beta_1 \log(z-z_1) - \beta_1 \log(z-1), & \text{for } |z| > 1. \end{cases} \quad (2.4.25)$$

Using the expansion from before, the residue theorem and using in addition the expansion of $\log(s-e^t) = \log(\exp(i\pi)) + \log(\exp(t)) - \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} s^k$,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(\alpha_0 + \beta_0) \log(s-e^t)}{s-z} ds &= \frac{(\alpha_0 + \beta_0)}{2\pi i} \int_{\mathbb{T}} \left\{ i\pi + t - e^{-t}s - \frac{e^{-2t}}{2}s^2 - \dots \right\} \left\{ \frac{1}{s} + \frac{z}{s^2} + \dots \right\} ds \\
&= (\alpha_0 + \beta_0) \left(i\pi + t - e^{-t}z - \frac{e^{-2t}}{2}z^2 - \dots \right) \\
&= (\alpha_0 + \beta_0) \left(i\pi + t - \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} s^k \right) = (\alpha_0 + \beta_0) \log(z - e^{-t}).
\end{aligned} \quad (2.4.26)$$

And for $|z| > 1$,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(\alpha_0 + \beta_0) \log(s-e^t)}{s-z} ds &= \frac{(\alpha_0 + \beta_0)}{2\pi i} \int_{\mathbb{T}} \left\{ i\pi + t - e^{-t}s - \frac{e^{-2t}}{2}s^2 - \dots \right\} \left\{ -\frac{1}{z} - \frac{s}{z^2} - \dots \right\} ds \\
&= 0.
\end{aligned} \quad (2.4.27)$$

Similarly to the last integral, but now using the expansion for $\log(z - e^{-t}) = \log(\exp(i\theta)) - \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} s^{-k}$, for $|z| < 1$,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(\alpha_0 - \beta_0) \log(s-e^{-t})}{s-z} ds &= \frac{(\alpha_0 - \beta_0)}{2\pi i} \int_{\mathbb{T}} \left\{ \log(s) - e^{-t}s^{-1} - \frac{e^{-2t}}{2}s^{-2} + \dots \right\} \frac{1}{s-z} ds \\
&= \frac{(\alpha_0 - \beta_0)}{2\pi i} \int_{\mathbb{T}} \frac{\log(s)}{s} ds + \frac{(\alpha_0 - \beta_0)}{2\pi i} \int_{\mathbb{T}} \left\{ -e^{-t}s^{-1} - \frac{e^{-2t}}{2}s^{-2} + \dots \right\} \left\{ \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \dots \right\} ds
\end{aligned}$$

$$= (\alpha_0 - \beta_0) \log(1 - z) - (\alpha_0 - \beta_0) i\pi + 0, \quad (\text{Using (2.4.19) and residue theorem, respectively}).$$

Now for $|z| > 1$, we proceed similarly, but using (2.4.20), we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(\alpha_0 - \beta_0) \log(s - e^{-t})}{s - z} ds &= \frac{(\alpha_0 - \beta_0)}{2\pi i} \int_{\mathbb{T}} \left\{ \log(s) - e^{-t} s^{-1} - \frac{e^{-2t}}{2} s^{-2} + \dots \right\} \frac{1}{s - z} ds \\ &= \frac{(\alpha_0 - \beta_0)}{2\pi i} \int_{\mathbb{T}} \frac{\log(s)}{s} ds + \frac{(\alpha_0 - \beta_0)}{2\pi i} \int_{\mathbb{T}} \left\{ -e^{-t} s^{-1} - \frac{e^{-2t}}{2} s^{-2} - \dots \right\} \left\{ -\frac{1}{z} - \frac{s}{z^2} - \dots \right\} ds \\ &= (\alpha_0 - \beta_0) \log(z - 1) - (\alpha_0 - \beta_0) \log z - (\alpha_0 - \beta_0) \left(-e^{-t} z^{-1} - \frac{e^{-2t}}{2} z^{-2} - \dots \right) \\ &= (\alpha_0 - \beta_0) \log(z - 1) - (\alpha_0 - \beta_0) \log(z - e^{-t}). \end{aligned}$$

We are left with the following terms, for which we use earlier results,

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(-\alpha_0 + \beta_0) \log s}{s - z} ds = \begin{cases} -(\alpha_0 - \beta_0) \log(1 - z) + (\alpha_0 - \beta_0) i\pi & \text{for } |z| < 1, \\ -(\alpha_0 - \beta_0) \log(z - 1) + (\alpha_0 - \beta_0) \log z & \text{for } |z| > 1, \end{cases}$$

using (2.4.19) and (2.4.20), and

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{-i\pi(\alpha_0 + \beta_0)}{s - z} ds = \begin{cases} -i\pi(\alpha_0 + \beta_0) & \text{for } |z| < 1, \\ 0 & \text{for } |z| > 1, \end{cases}$$

using (2.4.21) and (2.4.22), as both $\log z_1$ and $i\pi$ are just constants.

Finally, combining all the results above and exponentiating, we get the explicit formula for the Szegő function (2.4.15). \square

2.4.4 Local Parametrics

We go back to the R-H problem for $S(z)$ where we have opened the lens and therefore created a contour which possesses intersections. We will look at each intersection separately. On their own, the local parametrics at z_0 and z_1 are the same as in [14] and [11] respectively.

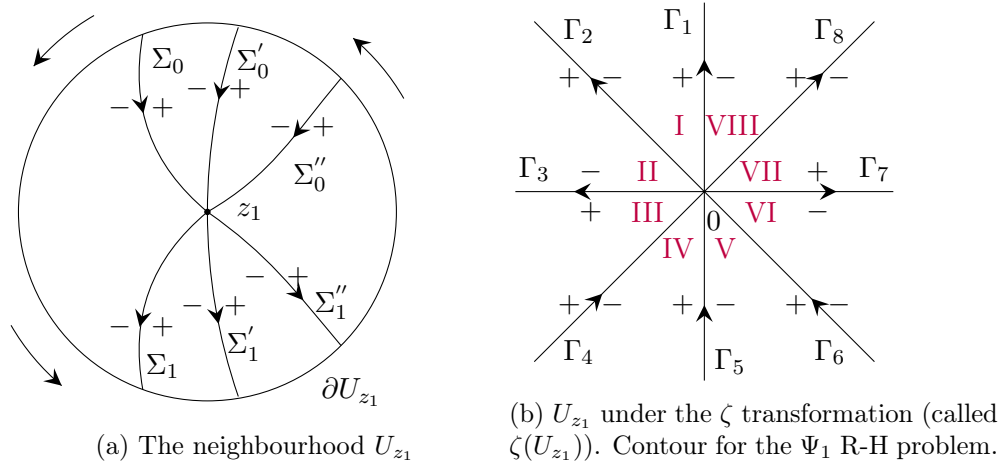


Figure 2.3

2.4.4.1 Parametrix at point z_1

In this section we are basing the analysis on the works of Deift, Its and Krasovsky in [14, 16]. First, we will construct the parametrix $P_{z_1}(z)$ in the neighbourhood U_{z_1} , see (2.4.12). We again look for a sectionally analytic matrix-valued function, this time in the neighbourhood U_{z_1} as opposed to the whole complex plane. This function will have the same jump conditions as $S(z)$, again only restricted to the intersection $\Sigma \cap U_{z_1}$. It will also have the same behaviour as $z \rightarrow z_1$ as the function $S(z)$, (2.4.6) - (2.4.11). However, instead of the condition of being normalised at infinity, the new function will satisfy the following matching condition,

$$P_{z_1}(z)N^{-1}(z) = I + o(1), \quad \text{as } n \rightarrow \infty. \quad (2.4.28)$$

Consider the following transformation (which is where the n appears),

$$\zeta = n \log \frac{z}{z_1}, \quad (2.4.29)$$

where $\log x > 0$ for $x > 0$, and the logarithm has a cut on the negative half of the real axis. This transformation maps the neighbourhood U_{z_1} , which can be seen in Figure 2.3a, into a neighbourhood of zero in the ζ -plane, Figure 2.3b. We choose the form of the $\Sigma \cap U_{z_1}$ to give straight lines under the transformation. The function $\zeta(z)$ is analytic and bijective, it takes an arc of the unit circle (recall that $\Sigma'_0 \cup \Sigma'_1 = \mathbb{T}$) to an interval of the imaginary axis (see Figure 2.3). The inside of the

unit circle corresponds to values of ζ in the sectors I, II, III, IV, whereas the outside corresponds to $\zeta \in V, VI, VII, VIII$. The pre-image of the rays Γ_3 and Γ_7 is added to the contour $\Sigma \cup U_{z_1}$ to deal with non-analyticity of $|z - z_1|^{\alpha_j}$, which we will now discuss. Let us write down the following function for $z \in \mathbb{T}$, as it was introduced in [14, (4.13)],

$$h_{\alpha_1}(z) = |z - z_1|^{\alpha_1} = (z - z_1)^{\alpha_1/2} (z^{-1} - z_1^{-1})^{\alpha_1/2} = \frac{(z - z_1)^{\alpha_1}}{(zz_1 e^{il_1})^{\alpha_1/2}}, \quad (2.4.30)$$

where $z = e^{i\theta}$, $z_1 = e^{i\theta_1}$, $0 \leq \theta < 2\pi$ and $\theta_1 \neq 0$. We fix the cut of $(z - z_1)^\alpha$ going along the ray $\theta = \theta_1$ from z_1 to infinity, and we fix the branch by the condition that on the line from z_1 to the right, perpendicular to the real axis, $\arg(z - z_1) = 2\pi$. For $z^{\alpha_1/2}$ in the denominator, $0 < \arg z < 2\pi$. We write,

$$(z - z_1)^{\alpha_1} = \exp\{\alpha_1 \log(z - z_1)\} = \exp\{\alpha_1 \log|z - z_1| + \alpha_1 i \arg(z - z_1)\}, \quad (2.4.31)$$

and

$$(zz_1 e^{il_1})^{-\alpha_1/2} = \exp\left\{-i \frac{\alpha_1}{2} (\theta + \theta_1 + l_1)\right\}. \quad (2.4.32)$$

Thus,

$$\frac{(z - z_1)^{\alpha_1}}{(zz_1 e^{il_1})^{\alpha_1/2}} = |z - z_1|^{\alpha_1} \exp\left\{i \alpha_1 \left(\arg(z - z_1) - \frac{\theta}{2} - \frac{\theta_1}{2} - \frac{l_1}{2}\right)\right\}, \quad (2.4.33)$$

and we need the power of the exponential above to be 0. The values of l_1 are,

$$l_1 = \begin{cases} 3\pi, & 0 < \theta < \theta_1, \\ \pi, & \theta_1 < \theta < 2\pi, \end{cases} \quad (2.4.34)$$

which can be seen by considering different triangles (see the the tables below which also consider different locations of z_1).

2nd Quadrant	1st Quadrant
At 0: $2\pi - \left(\frac{\pi-\theta_1}{2}\right) - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$	At 0: $2\pi - \left(\frac{\pi-\theta_1}{2}\right) - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$
At $(\pi - \theta_1)^-$: $2\pi - \left(\frac{\pi-\theta_1}{2}\right) - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$	At θ_1^- : $\frac{3\pi}{2} + \theta_1 - \frac{\theta_1}{2} - \frac{\theta_1}{2} \Rightarrow l_1 = 3\pi$
$(\pi - \theta_1)^+$: $0 - \left(\frac{\pi-\theta_1}{2}\right) - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$	At θ_1^+ : $\frac{\pi}{2} + \theta_1 - \frac{\theta_1}{2} - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = \pi$
At θ_1^- : $\theta_1 - \frac{\pi}{2} - \frac{\theta_1}{2} - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$	2π : $2\pi - \left(\frac{\pi-\theta_1}{2}\right) - \pi - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = \pi$
At θ_1^+ : $\theta_1 + \frac{\pi}{2} - \frac{\theta_1}{2} - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = \pi$	
At 2π : $2\pi - \left(\frac{\pi-\theta_1}{2}\right) - \pi - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = \pi$	
3rd Quadrant	4th Quadrant
At 0: $\frac{\theta_1-\pi}{2} - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$	At 0: $\left(\frac{\theta_1-\pi}{2}\right) - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$
At θ_1^- : $\theta_1 - \frac{\pi}{2} - \frac{\theta_1}{2} - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$	θ_1^- : $\theta_1 - \frac{\pi}{2} - \frac{\theta_1}{2} - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = 3\pi$
At θ_1^+ : $\theta_1 + \frac{\pi}{2} - \frac{\theta_1}{2} - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = \pi$	θ_1^+ : $\theta_1 - \frac{3\pi}{2} - \frac{\theta_1}{2} - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = \pi$
At $(3\pi - \theta_1)^-$: $2\pi - \left(\frac{3\pi-\theta_1}{2}\right) - \frac{\theta_1}{2} = \frac{l_1}{2} \Rightarrow l_1 = \pi$	At 2π : $\frac{\theta_1-\pi}{2} - \pi - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = \pi$
$(3\pi - \theta_1)^+$: $0 - \left(\frac{3\pi-\theta_1}{2}\right) - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = \pi$	
At 2π : $\frac{\theta_1-\pi}{2} - \pi - \frac{\theta_1}{2} + 2\pi = \frac{l_1}{2} \Rightarrow l_1 = \pi$	

Alternatively, we can write,

$$\frac{z - z_1}{(zz_1 e^{il_1})^{1/2}} = \exp\left\{i\left(\frac{\pi - l_1}{2} + \pi k\right)\right\} \mp |z - z_1| \begin{cases} - & \text{for } 0 \leq \theta < \theta_1 \\ + & \text{for } 0 \leq \theta_1 < \theta \end{cases} \quad (2.4.35)$$

we want this to equal to $|z - z_1|$, thus,

$$\begin{aligned} \exp\left\{i\left(\frac{\pi - l_1}{2} + \pi k\right)\right\} &= \begin{cases} -1 = e^{i\pi} & \text{for } \theta < \theta_1 \\ 1 = e^0 & \text{for } \theta_1 < \theta \end{cases} \\ \frac{\pi - l_1}{2} + \pi k &= \begin{cases} \pi \pmod{2\pi} & \text{for } \theta < \theta_1 \\ 0 \pmod{2\pi} & \text{for } \theta_1 < \theta \end{cases} \\ l_1 &= \begin{cases} (2k - 1)\pi \pmod{4\pi} & \text{for } \theta < \theta_1 \\ (2k + 1)\pi \pmod{4\pi} & \text{for } \theta_1 < \theta \end{cases} \end{aligned}$$

2nd Quadrant		1st Quadrant	
<p style="text-align: right;">$\frac{\pi}{2} < \theta_1 < \pi$</p>		<p style="text-align: right;">$0 < \theta_1 < \frac{\pi}{2}$</p>	
$0 < \theta < \pi - \theta_1$	$2\pi - (\frac{\pi - \theta_1}{2}) < \arg(z - z_1) < 2\pi$	$0 < \theta < \theta_1$	$2\pi - (\frac{\pi - \theta_1}{2}) < \arg(z - z_1) < \frac{3\pi}{2} + \theta_1$
$\pi - \theta_1 < \theta < \theta_1$	$0 < \arg(z - z_1) < \theta_1 - \frac{\pi}{2}$	$\theta_1 < \theta < 2\pi$	$\frac{\pi}{2} + \theta_1 < \arg(z - z_1) < 2\pi - (\frac{\pi - \theta_1}{2})$
$\theta_1 < \theta < 2\pi$	$\theta_1 + \frac{\pi}{2} < \arg(z - z_1) < 2\pi - (\frac{\pi - \theta_1}{2})$		
3rd Quadrant		4th Quadrant	
<p style="text-align: right;">$\pi < \theta_1 < \frac{3\pi}{2}$</p>		<p style="text-align: right;">$\frac{3\pi}{2} < \theta_1 < 2\pi$</p>	
$0 < \theta < \theta_1$	$\frac{\theta_1 - \pi}{2} < \arg(z - z_1) < \theta_1 - \frac{\pi}{2}$	$0 < \theta < \theta_1$	$\frac{\theta_1 - \pi}{2} < \arg(z - z_1) < \theta_1 - \frac{\pi}{2}$
$\theta_1 < \theta < 3\pi - \theta_1$	$\theta_1 + \frac{\pi}{2} < \arg(z - z_1) < 2\pi$	$\theta_1 < \theta < 2\pi$	$\theta_1 - \frac{3\pi}{2} < \arg(z - z_1) < \frac{\theta_1 - \pi}{2}$
$3\pi - \theta_1 < \theta < 2\pi$	$0 < \arg(z - z_1) < \frac{\theta_1 - \pi}{2}$		

We proceed in the same way and using the same notation as [14, Section 4.2], with $j = 1$. Let us define the following auxiliary function (cf. [14, (4.15)]),

$$F_1(z) = \exp \left\{ \frac{1}{2} \log a(z; t) \right\} \left(\frac{z}{z_1} \right)^{\frac{\beta_1}{2}} h_{\alpha_1}(z) \times \begin{cases} e^{-i\pi\alpha_1}, & \zeta \in \text{I, II, V, VI}, \\ e^{i\pi\alpha_1}, & \zeta \in \text{III, IV, VII, VIII}, \end{cases} \quad z \in U_{z_1}, \quad (2.4.36)$$

recall function $a(z; t)$ from (2.2.2) and the sectors in Figure 2.3b. It can be verified that $F_1(z)$ is analytic in the intersection of each quarter ζ -plane with $\zeta(U_{z_1})$ and has the following jumps,

$$F_{1,+}(z) = F_{1,-}(z)e^{-2\pi i\alpha_1} \quad \zeta \in \Gamma_1, \quad (2.4.37)$$

$$F_{1,+}(z) = F_{1,-}(z)e^{2\pi i\alpha_1} \quad \zeta \in \Gamma_5, \quad (2.4.38)$$

$$F_{1,+}(z) = F_{1,-}(z)e^{\pi i\alpha_1} \quad \zeta \in \Gamma_3 \cup \Gamma_7. \quad (2.4.39)$$

It is easy to see, after considering the analytic continuation ((2.4.30)) of $f_t(z)$ off the arcs between singularities, (2.2.2) and how we defined $F_1(z)$ above, that

$$F_1(z)^2 = f_t(z)e^{-2\pi i\alpha_j} g_{z_1, \beta_1}^{-1}(z), \quad \zeta \in \text{I, II, V, VI}, \quad (2.4.40)$$

$$F_1(z)^2 = f_t(z)e^{2\pi i\alpha_j} g_{z_1, \beta_1}^{-1}(z), \quad \zeta \in \text{III, IV, VII, VIII}. \quad (2.4.41)$$

We now look for $P_{z_1}(z)$ in the following form,

$$P_{z_1}(z) = E(z)P^{(1)}(z)F_1(z)^{-\sigma_3} z^{\pm n\sigma_3/2}, \quad (2.4.42)$$

where the plus sign is taken for $|z| < 1$ and minus for $|z| > 1$, which corresponds to $\zeta \in \text{I,II,III,IV}$, and $\zeta \in \text{V,VI,VII,VIII}$ respectively. The matrix $E(z)$ is analytic and invertible in the neighbourhood of U_{z_1} , thus does not affect the jump and analyticity conditions and is chosen in order for $P_{z_1}(z)$ to satisfy the matching condition on the boundary (2.4.28).

As $P_{z_1}(z)$ has the same jump conditions as $S(z)$, it can be verified straightforwardly that $P^{(1)}(z)$ satisfies jump conditions with constant jump matrices. As in [14], we set,

$$P^{(1)}(z) = \Psi_1(\zeta) \quad (2.4.43)$$

Then $\Psi_1(\zeta)$ satisfies a R-H problem on the contour $\Gamma := \cup_{k=1}^8 \Gamma_k$ given in Figure 2.3b,

(Ψ_1 1) $\Psi_1 : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(Ψ_1 2) $\Psi_1(z)$ satisfies the following jump conditions:

$$\Psi_{1,+}(\zeta) = \Psi_{1,-}(\zeta) \begin{pmatrix} 0 & e^{-i\pi\beta_1} \\ -e^{i\pi\beta_1} & 0 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_1, \quad (2.4.44)$$

$$\Psi_{1,+}(\zeta) = \Psi_{1,-}(\zeta) \begin{pmatrix} 0 & e^{i\pi\beta_1} \\ -e^{-i\pi\beta_1} & 0 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_5, \quad (2.4.45)$$

$$\Psi_{1,+}(\zeta) = \Psi_{1,-}(\zeta) e^{i\pi\alpha_1\sigma_3}, \quad \text{for } \zeta \in \Gamma_3 \cup \Gamma_7, \quad (2.4.46)$$

$$\Psi_{1,+}(\zeta) = \Psi_{1,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\pm i\pi(\beta_1 - 2\alpha_1)} & 1 \end{pmatrix}, \quad \text{with } \begin{cases} + \text{ in the exponent for } \zeta \in \Gamma_2, \\ - \text{ in the exponent for } \zeta \in \Gamma_4, \end{cases} \quad (2.4.47)$$

$$\Psi_{1,+}(\zeta) = \Psi_{1,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\pm i\pi(\beta_1 + 2\alpha_1)} & 1 \end{pmatrix}, \quad \text{with } \begin{cases} + \text{ in the exponent for } \zeta \in \Gamma_8, \\ - \text{ in the exponent for } \zeta \in \Gamma_6, \end{cases} \quad (2.4.48)$$

(Ψ_1 3) As $\zeta \rightarrow 0$, $\zeta \in \mathbb{C} \setminus \Gamma$ outside the lenses (i.e. sectors II, III, VI, VII), for $\alpha_1 \neq 0$,

$$\Psi_1(\zeta) = \begin{pmatrix} \mathcal{O}(\zeta^{\alpha_1}) & \mathcal{O}(\zeta^{\alpha_1}) + \mathcal{O}(\zeta^{-\alpha_1}) \\ \mathcal{O}(\zeta^{\alpha_1}) & \mathcal{O}(\zeta^{\alpha_1}) + \mathcal{O}(\zeta^{-\alpha_1}) \end{pmatrix}, \quad (2.4.49)$$

and if $\alpha_1 = 0, \beta_1 \neq 0$

$$\Psi_1(\zeta) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z|) \end{pmatrix}. \quad (2.4.50)$$

The behaviour of $\Psi_1(\zeta)$ for $\zeta \rightarrow 0$ in other sectors can be computed using the appropriate jump conditions.

This problem was solved explicitly in [14, Section 4.2] and this solutions using confluent hypergeometric functions (see appendix of [27]). Define the function $\psi(a, c; z)$ as a unique solution of the

confluent hypergeometric equation

$$zw'' + (c - a)w' - aw = 0, \quad (2.4.51)$$

satisfying the following asymptotic condition

$$\psi(a, c; z) \sim z^{-a} \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (1 + a - c)_n}{n! z^n}, \quad (2.4.52)$$

$$z \rightarrow \infty, \quad -\frac{3\pi}{2} < \arg z < \frac{3\pi}{2},$$

where

$$(a)_0 = 1, \quad (a)_n = a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \geq 1.$$

The authors of [14] have proved the following result.

Theorem 2.4.2. [14, Proposition 4.1] *Let $\alpha_1 \pm \beta_1 \neq -1, -2, \dots$. Then a solution to the above R-H problem (Ψ_11) - (Ψ_13) for $\Psi_1(\zeta)$, $0 < \arg \zeta < 2\pi$, is given by the following matrix-valued function in sector I:*

$$\begin{aligned} \Psi_1(\zeta) &= \Psi_1^{(I)}(\zeta) \\ &= \begin{pmatrix} \zeta^{\alpha_1} \psi(\alpha_1 + \beta_1, 1 + 2\alpha_1, \zeta) e^{i\pi(2\beta_1 + \alpha_1)} e^{-\zeta/2} \\ -\zeta^{-\alpha_1} \psi(1 - \alpha_1 + \beta_1, 1 - 2\alpha_1, \zeta) e^{i\pi(\beta_1 - 3\alpha_1)} e^{-\zeta/2} \frac{\Gamma(1 + \alpha_1 + \beta_1)}{\Gamma(\alpha_1 - \beta_1)} \\ -\zeta^{\alpha_1} \psi(1 + \alpha_1 - \beta_1, 1 + 2\alpha_1, e^{-i\pi}\zeta) e^{i\pi(\beta_1 + \alpha_1)} e^{\zeta/2} \frac{\Gamma(1 + \alpha_1 - \beta_1)}{\Gamma(\alpha_1 + \beta_1)} \\ \zeta^{-\alpha_1} \psi(-\alpha_1 - \beta_1, 1 - 2\alpha_1, e^{-i\pi}\zeta) e^{-i\pi\alpha_1} e^{\zeta/2} \end{pmatrix}, \end{aligned} \quad (2.4.53)$$

where $\psi(a, c; x)$ is the confluent hypergeometric function of the second kind defined above, and $\Gamma(x)$ is the Euler's Γ -function. The solution in the other sectors is given by successive application of the jump conditions from (Ψ_12) to (2.4.53).

The authors then match this solution with $N(z)$ (see (2.4.28)) on the boundary ∂U_{z_1} for large n . The limit $n \rightarrow \infty, z \in \partial U_{z_1}$, corresponds to $\zeta \rightarrow \infty$. Thus the authors find the asymptotic expansion of $\Psi_1(\zeta)$ using the classical result for the confluent hypergeometric function (which is (2.4.52) rewritten),

$$\psi(a, c; x) = x^{-a} [1 - a(1 + a - c)x^{-1} + \mathcal{O}(x^{-2})], \quad |x| \rightarrow \infty, \quad -3\pi/2 < \arg x < 3\pi/2. \quad (2.4.54)$$

These asymptotics can be taken both for $\psi(a, c; \zeta)$ and $\psi(a, c; e^{-i\pi}\zeta)$ for $\zeta \in \text{I}$ (recall the branches are fixed by the condition $0 < \arg \zeta < 2\pi$). They apply (2.4.54) to (2.4.53) to obtain the relevant asymptotics, which due to the triangular structure of the jump matrices remain the same in the sectors I and II,

$$\begin{aligned} \Psi_1^{(\text{I})}(\zeta) &= \Psi_1^{(\text{II})}(\zeta) \\ &= \left[I + \frac{1}{\zeta} \begin{pmatrix} \alpha_1^2 - \beta_1^2 & \frac{\Gamma(1+\alpha_1-\beta_1)}{\Gamma(\alpha_1+\beta_1)} e^{i\pi(\beta_1+4\alpha_1)} \\ -\frac{\Gamma(1+\alpha_1+\beta_1)}{\Gamma(\alpha_1-\beta_1)} e^{-i\pi(\beta_1+4\alpha_1)} & -(\alpha_1^2 - \beta_1^2) \end{pmatrix} + \mathcal{O}(\zeta^{-2}) \right] \\ &\quad \times \zeta^{-\beta_1\sigma_3} e^{-\zeta\sigma_3/2} \begin{pmatrix} e^{i\pi(2\beta_1+\alpha_1)} & 0 \\ 0 & e^{-i\pi(\beta_1+2\alpha_1)} \end{pmatrix}, \end{aligned} \tag{2.4.55}$$

$$\zeta \rightarrow \infty, \quad \zeta \in \text{I, II}, \quad \alpha_1 \pm \beta_1 \neq -1, -2, \dots$$

The asymptotics in the remaining sectors can be found using the relevant jump matrices. Substituting these asymptotics into the condition on $E(z)$ (see (2.4.28)),

$$P_{z_1}(z)N^{-1}(z) = E(z)\Psi_1(\zeta)F_1(z)^{-\sigma_3} z^{\pm n\sigma_3/2} N^{-1}(z) = I + o(1), \tag{2.4.56}$$

(with $+$ for $|z| < 1$, and $-$ for $|z| > 1$), gives

$$E(z) = N(z)\zeta^{\beta_1\sigma_3} F_1^{\sigma_3}(z) z_1^{-n\sigma_3/2} \begin{pmatrix} e^{-i\pi(2\beta_1+\alpha_1)} & 0 \\ 0 & e^{i\pi(\beta_1+2\alpha_1)} \end{pmatrix}, \quad \text{for } \zeta \in \text{I, II}. \tag{2.4.57}$$

The matrix $E(z)$ for the remaining sectors can be computed using the relevant asymptotics which are obtained using (2.4.55), see [14, Equations (4.42)-(4.50)] for those details.

Now we follow the authors to obtain the expansions in $u = z - z_1$, as $u \rightarrow 0$, which are unique to our problem. From (2.4.29), (2.4.15), (2.4.36), (2.4.30), recall also (2.2.2) and the Wiener-Hopf factorisation of $V(z) = \exp \left\{ \sum_{k=0}^{\infty} V_k z^k \right\} e^{V_0} \exp \left\{ \sum_{k=0}^{\infty} V_{-k} z^{-k} \right\}$,

$$F_1(z) = \exp \left\{ \log a(z_1; t)/2 \right\} e^{-3i\pi\alpha_1/2} z_1^{-\alpha_1} u^{\alpha_1} (1 + \mathcal{O}(u)), \quad \zeta \in \text{I}, \tag{2.4.58}$$

$$D(z) = u^{\alpha_1+\beta_1} z_1^{-(\alpha_1+\beta_1)} e^{-i\pi(\alpha_1+\beta_1)} (z_1 - e^t)^{(\alpha_0+\beta_0)} e^{-i\pi(\alpha_0+\beta_0)}$$

$$\times \exp \left\{ \sum_{k=0}^{\infty} V_k z_1^k \right\} (1 + \mathcal{O}(u)), \quad (2.4.59)$$

$$\zeta(z) = n \log \frac{z}{z_1} = n \frac{u}{z_1} + \mathcal{O}(u^2) = n \frac{u}{z_1} (1 + \mathcal{O}(u)). \quad (2.4.60)$$

Putting them together we obtain the following,

$$\begin{aligned} \left(\frac{D(z)}{\zeta^{\beta_1} F_1(z)} \right)^2 &= e^{V_0} \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k z_1^k \right\}}{\exp \left\{ \sum_{k=1}^{\infty} V_{-k} z_k^{-k} \right\}} (z_1 - e^t)^{(\alpha_0 + \beta_0)} (z_1 - e^{-t})^{-(\alpha_0 - \beta_0)} z_1^{\alpha_0 - \beta_0} \\ &\quad \times e^{-i\pi(\alpha_0 + \beta_0)} e^{i\pi(\alpha_1 - 2\beta_1)} n^{-2\beta_1} (1 + \mathcal{O}(u)), \quad \zeta \in \mathbf{I}. \\ &= e^{V_0} \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k z_1^k \right\}}{\exp \left\{ \sum_{k=1}^{\infty} V_{-k} z_k^{-k} \right\}} (1 - e^{-t} z_1)^{(\alpha_0 + \beta_0)} (1 - e^{-t} z_1^{-1})^{-(\alpha_0 - \beta_0)} \\ &\quad \times e^{t(\alpha_0 + \beta_0)} e^{i\pi(\alpha_1 - 2\beta_1)} n^{-2\beta_1} (1 + \mathcal{O}(u)), \quad \zeta \in \mathbf{I}. \end{aligned} \quad (2.4.61)$$

From (2.4.57) it can be seen that $\det E(z) = e^{i\pi(\alpha_1 - \beta_1)}$ (this holds for all sections I-VIII of the ζ -plane in fact). We also note that $\det \Psi_1(\zeta) = e^{-i\pi(\alpha_1 - \beta_1)}$. This can be seen from Liouville's theorem, the function $\det \Psi_1(\zeta)$ has no jumps (verifying directly from the jump conditions in $(\Psi_1 2)$ above), the singularity at 0 is removable as $\operatorname{Re} \alpha_1 > -1/2$ (by looking at the determinant of (2.4.49)) and the value of the function follows from computing the determinant of (2.4.55). Those two facts, (2.4.42) and that $\det A^{\sigma_3} = 1$ for any matrix A , give $\det P_{z_1}(z) = 1$, making $P_{z_1}(z)$ unique (indeed, by the usual argument, if $\det P_{z_1}(z) = 1$ its inverse exists and it follows that $P_{z_1}(z) \tilde{P}_{z_1}(z) \equiv I$ by Liouville's). We also note that $S(z)P_{z_1}(z)^{-1}$ is analytic in the neighbourhood of U_{z_1} , we will need this for the final R-H problem in Section 2.4.5. By looking at (2.4.49), (2.4.50) and (2.4.6)-(2.4.11), it can be seen that the singularity at z_1 is at most $\mathcal{O}(|z - z_1|^{2\alpha_1})$ or $\mathcal{O}(\log |z - z_1|)$, but by the construction of $P_{z_1}(z)$ (which as you recall, had the same jumps as $S(z)$), the matrix-valued function $S(z)P_{z_1}(z)^{-1}$ has no jumps in the neighbourhood of z_1 , thus the singularity is removable.

Note that the error term in (2.4.56), $o(1) = n^{-\operatorname{Re} \beta_1 \sigma_3} \mathcal{O}(n^{-1}) n^{\operatorname{Re} \beta_1 \sigma_3}$, which is $o(1)$ if $-1/2 < \operatorname{Re} \beta_1 < 1/2$. We now compute the first correction term $\Delta_1(z)$ in the asymptotic series in inverse powers of n of (2.4.56),

$$P_{z_1}(z) N^{-1}(z) = I + \Delta_1(z) + n^{-\operatorname{Re} \beta_1 \sigma_3} \mathcal{O}(n^{-2}) n^{\operatorname{Re} \beta_1 \sigma_3} \quad (2.4.62)$$

A full series can be found by considering further terms in (2.4.55). We start by denoting by E_{ij} and $\Psi_{1,ij}$, $i, j = 1, 2$, the matrix elements of $E(z)$ and the asymptotic expansion of $\Psi_1(\zeta)$, (2.4.55). In what follows, we only compute using the values valid in sector I (equivalently, where appropriate $|z| < 1$); the expression for the asymptotic series extends to the whole boundary ∂U_{z_1} by analytic continuation by a consideration of other sectors. Multiplying out the matrices in (2.4.57) we get $E_{11} = E_{22} = 0$, and

$$E_{12} = D(z)\zeta^{-\beta_1}F_1^{-1}z_1^{n/2}e^{i\pi(\beta_1+2\alpha_1)}, \quad (2.4.63)$$

$$E_{21} = -E_{12}^{-1}e^{i\pi(\alpha_1-\beta_1)}. \quad (2.4.64)$$

Further, multiplying the matrices in (2.4.56),

$$P_{z_1}(z)N^{-1}(z) = \begin{pmatrix} E_{12}\Psi_{1,22}F_1z^{-n/2}D(z)^{-1} & -E_{12}\Psi_{1,21}F_1^{-1}z^{n/2}D(z) \\ E_{21}\Psi_{1,12}F_1z^{-n/2}D(z)^{-1} & -E_{21}\Psi_{1,11}F_1^{-1}z^{n/2}D(z) \end{pmatrix}, \quad (2.4.65)$$

and substituting in (2.4.63),(2.4.64) and using (2.4.55) for $\Psi_{1,ij}$ gives,

$$\Delta_1(z) = \frac{1}{\zeta} \begin{pmatrix} -(\alpha_1^2 - \beta_1^2) & z_1^n \frac{\Gamma(1+\alpha_1+\beta_1)}{\Gamma(\alpha_1-\beta_1)} \left(\frac{D(z)}{\zeta^{\beta_1}F_1} \right)^2 e^{i\pi(2\beta_1-\alpha_1)} \\ -z_1^n \frac{\Gamma(1+\alpha_1-\beta_1)}{\Gamma(\alpha_1+\beta_1)} \left(\frac{D(z)}{\zeta^{\beta_1}F_1} \right)^{-2} e^{-i\pi(2\beta_1-\alpha_1)} & (\alpha_1^2 - \beta_1^2) \end{pmatrix}. \quad (2.4.66)$$

Later on we will need the 12 element of the correction term and so we write it down in detail here,

$$\begin{aligned} (\Delta_1(z))_{12} &= \frac{1}{\zeta} z_1^n e^{V_0} (1 - z_1 e^{-t})^{(\alpha_0+\beta_0)} (1 - e^{-t} z_1^{-1})^{-(\alpha_0-\beta_0)} \exp \left\{ \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} z_k^{-k} \right\} \\ &\quad \times e^{t(\alpha_0-\beta_0)} n^{-2\beta_1} \frac{\Gamma(1+\alpha_1+\beta_1)}{\Gamma(\alpha_1-\beta_1)} (1 + \mathcal{O}(u)), \end{aligned} \quad (2.4.67)$$

and we also note,

$$\frac{1}{\zeta} = \frac{z_1}{n(z-z_1)} + \frac{1}{2n} + \mathcal{O}(z-z_1), \quad z \rightarrow z_1. \quad (2.4.68)$$

2.4.4.2 Parametrix at point $z_0 = 1$

Before we proceed with the local parametrix near the point z_0 , we first look at the R-H problem for $\Psi(z)$ —the problem for Painlevé V—which will play a crucial role in finding the solution to the problem for the local parametrix. This problem was presented and solved in [11, Section 1.3], we provide a short summary of it here for the benefit of the reader. We will also use some of the

details from [11, Section 4.1,4.2] to compute some additional details that are relevant to solving our problem.

The parametrix P_{z_0} will be constructed for $0 < t < t_0$, for t_0 sufficiently small. Similarly to the parametrix at z_1 , it will satisfy the same jump conditions as the R-H problem for $S(z)$ in the neighbourhood U_{z_0} of z_0 , which has a sufficiently small, fixed radius. And we will have a matching condition with the R-H problem for $N(z)$ on the boundary of U_{z_0} (referred to as ∂U_{z_0} , which will be determined by an analytic (in U_{z_0}) matrix function $E(z)$).

Riemann-Hilbert Problem for Painlevé V

Here, we will be looking at the solution of the following second order ODE,

$$\left(x \frac{d^2 \sigma}{dx^2}\right)^2 = \left(\sigma - x \frac{d\sigma}{dx} + 2 \left(\frac{d\sigma}{dx}\right)^2 + 2\alpha_0 \frac{d\sigma}{dx}\right)^2 - 4 \left(\frac{d\sigma}{dx}\right)^2 \left(\frac{d\sigma}{dx} + \alpha_0 + \beta_0\right) \left(\frac{d\sigma}{dx} + \alpha_0 - \beta_0\right), \quad (2.4.69)$$

which is the σ -form of the Painlevé V equation,

$$u_{xx} = \left(\frac{1}{2u} + \frac{1}{u-1}\right) u_x^2 - \frac{1}{x} u_x + \frac{(u-1)^2}{x^2} \left(Au + \frac{b}{u}\right) + \frac{Cu}{x} + D \frac{u(u+1)}{u-1}, \quad (2.4.70)$$

which was produced by Jimbo, Miwa, Okamoto (see [28, 29]). With regards to the R-H problem we are considering, we have the following parameters for the equation above,

$$A = \frac{1}{2}(\alpha_0 - \beta_0)^2, \quad B = -\frac{1}{2}(\alpha_0 + \beta_0)^2, \quad C = 1 + 2\beta_0, \quad D = -\frac{1}{2}. \quad (2.4.71)$$

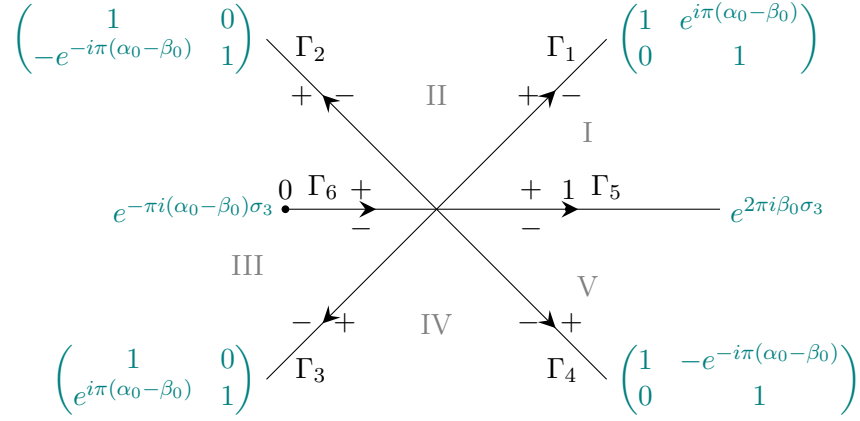
The solution of (2.4.69), the function $\sigma(x)$, can be constructed explicitly in terms of the R-H problem below. We consider $\Gamma = \cup_{j=1}^6 \Gamma_j \subset \mathbb{C}$ as the contour for this problem (see Figure 2.4), where

$$\begin{aligned} \Gamma_1 &= \frac{1}{2} + e^{i\pi/4}\mathbb{R}^+, & \Gamma_2 &= \frac{1}{2} + e^{i3\pi/4}\mathbb{R}^+, & \Gamma_3 &= \frac{1}{2} + e^{i5\pi/4}\mathbb{R}^+, \\ \Gamma_4 &= \frac{1}{2} + e^{i7\pi/4}\mathbb{R}^+, & \Gamma_5 &= (1, +\infty), & \Gamma_6 &= (0, 1). \end{aligned} \quad (2.4.72)$$

The contours $\Gamma_1, \dots, \Gamma_5$ are oriented towards infinity and Γ_6 is oriented to the right. As before, we assume that $\operatorname{Re} \alpha_0 > -1/2$. Consider the following R-H problem.

Riemann-Hilbert problem for Ψ_0

$$(\Psi_01) \quad \Psi_0 : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{2 \times 2} \text{ is analytic.}$$

Figure 2.4: Jump contour and jump matrices for the Painleve V R-H problem for Ψ_0

(Ψ_0 2) Ψ_0 has continuous boundary values on $\Gamma \setminus \{0, 1/2, 1\}$, and they are related by the following jump conditions,

$$\Psi_{0,+}(\zeta) = \Psi_{0,-}(\zeta) \begin{pmatrix} 1 & e^{\pi i(\alpha_0-\beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_1, \quad (2.4.73)$$

$$\Psi_{0,+}(\zeta) = \Psi_{0,-}(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{-\pi i(\alpha_0-\beta_0)} & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_2, \quad (2.4.74)$$

$$\Psi_{0,+}(\zeta) = \Psi_{0,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\pi i(\alpha_0-\beta_0)} & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_3, \quad (2.4.75)$$

$$\Psi_{0,+}(\zeta) = \Psi_{0,-}(\zeta) \begin{pmatrix} 1 & -e^{-\pi i(\alpha_0-\beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_4, \quad (2.4.76)$$

$$\Psi_{0,+}(\zeta) = \Psi_{0,-}(\zeta) e^{2\pi i \beta_0 \sigma_3}, \quad \text{for } \zeta \in \Gamma_5, \quad (2.4.77)$$

$$\Psi_{0,+}(\zeta) = \Psi_{0,-}(\zeta) e^{-\pi i(\alpha_0-\beta_0) \sigma_3}, \quad \text{for } \zeta \in \Gamma_6, \quad (2.4.78)$$

(Ψ_0 3) Ψ_0 has the following asymptotic behaviour as $\zeta \rightarrow \infty$, for some matrices $C_1 = C_1(x, \alpha_0, \beta_0)$, $C_2 = C_2(x, \alpha_0, \beta_0)$,

$$\Psi_0(\zeta) = \left(I + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \mathcal{O}(\zeta^{-3}) \right) \zeta^{-\beta_0 \sigma_3} e^{-(x/2)\zeta \sigma_3}. \quad (2.4.79)$$

(Ψ_0 4) Ψ_0 has the following asymptotic behaviour close to the following points,

$$(\Psi_0 4.1) \quad \Psi_0(\zeta) = \mathcal{O} \begin{pmatrix} |\zeta|^{(\alpha_0 - \beta_0)/2} & |\zeta|^{-(\alpha_0 - \beta_0)/2} \\ |\zeta|^{(\alpha_0 - \beta_0)/2} & |\zeta|^{-(\alpha_0 - \beta_0)/2} \end{pmatrix}, \quad \text{as } \zeta \rightarrow 0, \quad (2.4.80)$$

$$(\Psi_0 4.2) \quad \Psi_0(\zeta) = \mathcal{O} \begin{pmatrix} |\zeta - 1|^{-(\alpha_0 + \beta_0)/2} & |\zeta - 1|^{(\alpha_0 + \beta_0)/2} \\ |\zeta - 1|^{-(\alpha_0 + \beta_0)/2} & |\zeta - 1|^{(\alpha_0 + \beta_0)/2} \end{pmatrix}, \quad \text{as } \zeta \rightarrow 1, \quad (2.4.81)$$

and Ψ_0 is bounded near $\zeta = 1/2$.

Let L_j denote the jump matrices in the jump conditions above, for $j = 1, \dots, 6$, corresponding to each Γ_j . In all cases we have that $\det L_j = 1$, and so,

$$\begin{aligned} (\det \Psi_0)_+(\zeta) &= (\det \Psi_0)_-(\zeta) \det L_j, \quad \text{for each } j = 1, \dots, 6 \\ &= (\det \Psi_0)_-(\zeta). \end{aligned}$$

Thus, $\det \Psi_0(\zeta)$ is analytic in \mathbb{C} . And, $\det \Psi_0(\zeta) = 1$ since $\det \Psi_0(\zeta) = 1 + \mathcal{O}(\zeta^{-1})$ as $\zeta \rightarrow \infty$, thus using (2.4.79), we have that $\text{tr} C_1 = 0$. In light of this, let us denote the elements of the matrix C_1 by,

$$C_1(x) = \begin{pmatrix} q(x) & r(x) \\ t(x) & -q(x) \end{pmatrix}. \quad (2.4.82)$$

Now, define the following functions, v and u in terms of the matrix elements of C_1 ,

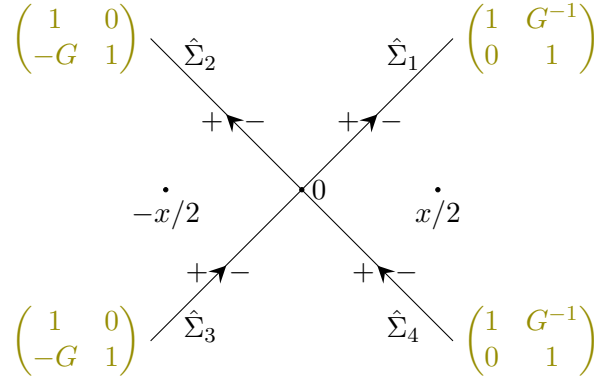
$$v(x) = \frac{\alpha_0 + \beta_0}{2} - q(x) - xr(x)t(x) \quad (2.4.83)$$

$$u(x) = 1 + \frac{xt}{(2\beta_0 + 1 - x)t(x) + xt'(x)}. \quad (2.4.84)$$

It was shown in [11, Section 4.3] that the solution to the Painlevé V equation above can be written as,

$$\sigma(x) = \int_x^{+\infty} v(\xi) d\xi. \quad (2.4.85)$$

This function plays a crucial role in describing transition asymptotics, as we will find out later on.

Figure 2.5: Jump contour and jump matrices for the Φ -R-H problem**The Parametrix at z_0**

Let us assume that $\Psi_0(\zeta)$ solves the R-H problem for Ψ_0 above and define

$$\Phi(\lambda; x) = e^{\frac{\pi}{4}\sigma_3} x^{-\beta_0\sigma_3} \Psi_0\left(\frac{\lambda}{x} + \frac{1}{2}; x\right) G(\lambda; x)^{\frac{1}{2}\sigma_3} e^{\pm\frac{\pi i}{2}(\alpha_0 - \beta_0)\sigma_3}, \quad \text{for } \pm \operatorname{Im} \lambda > 0, \quad (2.4.86)$$

respectively. The function $G(\lambda; x)$ is defined to be,

$$G(\lambda; x) = \left(\lambda + \frac{x}{2}\right)^{-(\alpha_0 - \beta_0)} \left(\lambda - \frac{x}{2}\right)^{\alpha_0 + \beta_0} e^{\lambda e^{-\pi i(\alpha_0 - \beta_0)}}, \quad x > 0, \quad (2.4.87)$$

and is analytic in $\mathbb{C} \setminus ((-\infty, -x/2] \cup [x/2, +\infty))$. We choose $-\pi < \arg(\lambda + x/2) < \pi$ and $0 < \arg(\lambda - x/2) < 2\pi$. The matrix function $\Phi = \Phi(\lambda; x)$ defined above, solves the R-H problem for $x > 0$ below.

Riemann-Hilbert problem for Φ

($\Phi 1$) $\Phi : \mathbb{C} \setminus \cup_{j=1}^4 e^{\pi i(2j-1)/4} \mathbb{R}^+ \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, with the rays $e^{\pi i(2j-1)/4} \mathbb{R}^+$ oriented as shown in Figure 2.5.

($\Phi 2$) Φ has continuous boundary values on $\cup_{j=1}^4 e^{\pi i(2j-1)/4} \mathbb{R}^+ \setminus \{0\}$, and they are related by the following jump conditions,

$$\Phi_+(\lambda) = \Phi_-(\lambda) \begin{pmatrix} 1 & G(\lambda; x)^{-1} \\ 0 & 1 \end{pmatrix}, \quad \text{as } \lambda \in e^{\pi i/4} \mathbb{R}^+ \cup e^{7\pi i/4} \mathbb{R}^+, \quad (2.4.88)$$

$$\Phi_+(\lambda) = \Phi_-(\lambda) \begin{pmatrix} 1 & 0 \\ -G(\lambda; x) & 1 \end{pmatrix}, \quad \text{as } \lambda \in e^{3\pi i/4}\mathbb{R}^+ \cup e^{5\pi i/4}\mathbb{R}^+ \quad (2.4.89)$$

($\Phi 3$) Φ has the following behaviour as $\lambda \rightarrow \infty$,

$$\Phi(\lambda) = I + \mathcal{O}(\lambda^{-1}). \quad (2.4.90)$$

($\Phi 4$) Φ is bounded near 0.

The following result was proven by Claeys, Its and Krasovsky and can be found in [11, Proposition 3.1]. It reads as follows.

Proposition 2.4.3.

- If $\operatorname{Re} \alpha_0 > -1/2$, the R-H problem for Φ is uniquely solvable for all but possibly a finite number of positive x -values $\{x_1, \dots, x_k\}$, where $x_j = x_j(\alpha_0, \beta_0)$ and $k = k(\alpha, \beta)$
- If $\alpha_0 > -1/2$ ($\operatorname{Im} \alpha_0 = 0$) and $\operatorname{Re} \beta_0 = 0$, the R-H problem for Φ is uniquely solvable for all $x > 0$.
- If $\operatorname{Re} \alpha_0 > -1/2$, the asymptotic condition (2.4.90) for Φ is valid uniformly for $x \in (0, +\infty)$ provided that x remains bounded away from the set $\{x_1, \dots, x_k\}$.

We now transform the jump matrices for Φ into the jump matrices for S near $z_0 = 1$. First of all, notice that the off-diagonal entries of the jump matrices for Φ have branch points at $\pm x/2$, and the ones for S have the branch points at $e^{\pm t}$. We thus define the following conformal mapping in the neighbourhood of z_0 ,

$$\lambda(z) = \frac{x}{2t} \log(z), \quad z \in U_{z_0}, \quad (2.4.91)$$

which maps e^{-t} to $-x/2$, e^t to $x/2$ and 1 to 0. Again, we take the branch of the logarithm such that $\log z > 0$ for $z > 1$, and the branch cut is along the negative real axis. We also need that $e^\lambda(z) = z^n$ and therefore,

$$x = 2nt. \quad (2.4.92)$$

We choose the contours Σ_1 and Σ_2 near 1, such that λ maps $\Sigma_1 \cup \Sigma_2$ onto the jump contour $\cup_{j=1}^4 e^{\pi i(2j-1)/4} \mathbb{R}^+$ for Φ . We look for the parametrix P_{z_0} of the following form,

$$P_{z_0}(z) = E(z)\Phi(\lambda(z); 2nt)W(z), \quad (2.4.93)$$

where $E(z)$ is an analytic function in U_{z_0} and $W(z)$ is given by

$$W(z) = \begin{cases} -G(\lambda(z))^{-\frac{1}{2}\sigma_3} z^{\frac{n}{2}\sigma_3} f(z)^{-\frac{1}{2}\sigma_3} \sigma_3, & \text{for } |z| < 1, \\ G(\lambda(z))^{-\frac{1}{2}\sigma_3} z^{\frac{n}{2}\sigma_3} f(z)^{\frac{1}{2}\sigma_3} \sigma_1, & \text{for } |z| > 1. \end{cases} \quad (2.4.94)$$

The function $W(z)$ is analytic in $U_{z_0} \setminus \mathbb{T}$, as $\lambda(e^t) = x/2$ and $\lambda(e^{-t}) = -x/2$, and the branch points of G cancel out with the branch points of f .

If $E(z)$ is analytic in U_{z_0} then $P_{z_0}(z)$ satisfies the same jump conditions as the matrix $S(z)$ with the jump conditions given in the R-H problem for $S(z)$.

The R-H problem for Φ is not solvable for a finite set of values $\{x_1, \dots, x_k\}$ and thus we need the condition that $x = 2nt$ does not belong to this set of problematic values of x , for more detail see Theorem 1.5.2.

We now look to fix the matrix $E(z)$ in a way that makes the parametrix $P_{z_0}(z)$ as defined in (2.4.93) agree with the R-H problem for $N(z)$ on the boundary of U_{z_0} . Let us then consider the behaviour of $P_{z_0}(z)$ on ∂U_{z_0} , starting with (2.4.91) we have that,

$$\begin{aligned} |\lambda(z)| &= |n \log z| \\ &> n |\log z|, \quad |\log z| < c \text{ for } z \in \partial U_{z_0}, \text{ and } c > 0 \\ &> cn. \end{aligned}$$

Thus using Proposition 2.4.3, as $n \rightarrow \infty$ and if $2nt$ stays bounded away from $\{x_1, \dots, x_k\}$, we can use the asymptotic behaviour of Φ in (2.4.90) to give,

$$P_{z_0}(z) = E(z)(I + \mathcal{O}(n^{-1}))W(z), \quad \text{as } z \rightarrow \infty, \quad (2.4.95)$$

uniformly for $0 < t < t_0$ and $z \in \partial U_{z_0}$. For a t_0 sufficiently small, it is safe to assume that the points $e^{\pm t}$ lie inside U_{z_0} , it is also safe to assume that these points lie at a distance which is bounded from

below from ∂U_{z_0} . From (2.4.94) and (2.4.87) we obtain the following,

$$W(z) = n^{-\beta_0} \begin{cases} \begin{pmatrix} \mathcal{O}(1) & 0 \\ 0 & \mathcal{O}(1) \end{pmatrix}, & |z| < 1, \\ \begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & 0 \end{pmatrix}, & |z| > 1, \end{cases} \quad (2.4.96)$$

as $n \rightarrow \infty$ uniformly for $0 < t < t_0$ and uniformly for $z \in \partial U_{z_0} \setminus \mathbb{T}$. We set,

$$E(z) = N(z)W(z)^{-1}, \quad (2.4.97)$$

it can be verified using the jumps for $N(z)$ and $W(z)$ across \mathbb{T} , that $E(z)$ is analytic in the whole neighbourhood \bar{U}_{z_0} of 1. Using (2.4.14) and (2.4.96), we have that,

$$E(z) = \begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & 0 \end{pmatrix} n^{\beta_0 \sigma_3} \quad (2.4.98)$$

as $n \rightarrow \infty$ uniformly for $0 < t < t_0$ and $z \in \partial U_{z_0}$. Using this and (2.4.95), we have the following asymptotics for the matching condition for $z \in \partial U_{z_0}$ as $n \rightarrow \infty$,

$$P_{z_0}(z)N(z)^{-1} = E(z)(I + \mathcal{O}(n^{-1}))E(z)^{-1} = I + n^{-\beta_0 \sigma_3} \mathcal{O}(n^{-1})n^{\beta_0 \sigma_3}, \quad (2.4.99)$$

uniformly for $0 < t < t_0$ and if $2nt$ is away from the set $\{x_1, \dots, x_k\}$.

We also note here that $S(z)P_{z_0}^{-1}(z)$ is analytic in the neighbourhood of U_{z_0} , which we need for the final R-H problem in Section 2.4.5. As $P_{z_0}(z)$ has the same jumps as $S(z)$ inside U_{z_0} it follows that any singularity at $z = 1$ is removable.

Similarly to the parametrix at z_1 , we will now compute the first correction term $\Delta_1(z)$ in the asymptotic series in inverse powers of n ,

$$P_{z_0}(z)N^{-1}(z) = I + \Delta_1(z) + n^{-\operatorname{Re} \beta_0 \sigma_3} \mathcal{O}(n^{-2})n^{\operatorname{Re} \beta_0 \sigma_3}. \quad (2.4.100)$$

Using (2.4.93), (2.4.97) and (2.4.14),

$$P_{z_0}(z)N^{-1}(z) = \begin{cases} D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W(z)^{-1} \Phi(\lambda(z); x) W(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D(z)^{-\sigma_3}, & \text{for } |z| < 1, \\ D(z)^{\sigma_3} W(z)^{-1} \Phi(\lambda(z); x) W(z) D(z)^{-\sigma_3}, & \text{for } |z| > 1, \end{cases} \quad (2.4.101)$$

Recall (2.4.94) and denote the matrix elements of $\Phi(\lambda(z); x) =: \Phi_{ij}$ for $i, j = 1, 2$ and $G(\lambda(z), x) =: G$ to save space,

$$P_{z_0}(z)N^{-1}(z) = \begin{cases} \begin{pmatrix} \Phi_{22} & D(z)^2 G^{-1} z^n f(z)^{-1} \Phi_{21} \\ D(z)^{-2} G z^{-n} f(z) \Phi_{12} & \Phi_{11} \end{pmatrix} & \text{for } |z| < 1 \\ \begin{pmatrix} \Phi_{22} & D(z)^2 G^{-1} z^n f(z) \Phi_{21} \\ D(z)^{-2} G z^{-n} f(z)^{-1} \Phi_{12} & \Phi_{11} \end{pmatrix} & \text{for } |z| > 1 \end{cases} \quad (2.4.102)$$

We now look at the behaviour of these as $z \rightarrow e^{\mp t}$. We denote by $u = z - e^{\mp t}$, choose the appropriate region $|z| \lesssim 1$ and take $u \rightarrow 0$. We obtain the following, recall the Szegő function (2.4.13) and (2.4.15),

$$\begin{aligned} D(z) &= \begin{cases} \left(\frac{e^{-t} - z_1}{z_1 e^{i\pi}} \right)^{\alpha_1 + \beta_1} (e^{-t} - e^t)^{\alpha_0 + \beta_0} e^{-i\pi(\alpha_0 + \beta_0)} \exp \left\{ \sum_{k=0}^{\infty} V_k e^{-tk} \right\} (1 + \mathcal{O}(u)), & \begin{matrix} z \rightarrow e^{-t} \\ |z| < 1, \end{matrix} \\ \left(\frac{e^t - z_1}{e^t} \right)^{-(\alpha_1 - \beta_1)} (e^t - e^{-t})^{-(\alpha_0 - \beta_0)} e^{t(\alpha_0 - \beta_0)} \exp \left\{ - \sum_{k=-\infty}^{-1} V_k e^{tk} \right\} (1 + \mathcal{O}(u)), & \begin{matrix} z \rightarrow e^t \\ |z| > 1, \end{matrix} \end{cases} \\ &= \begin{cases} (1 - z_1^{-1} e^{-t})^{\alpha_1 + \beta_1} (1 - e^{-2t})^{\alpha_0 + \beta_0} e^{t(\alpha_0 + \beta_0)} \exp \left\{ \sum_{k=0}^{\infty} V_k e^{-tk} \right\} (1 + \mathcal{O}(u)), & \begin{matrix} z \rightarrow e^{-t} \\ |z| < 1, \end{matrix} \\ (1 - z_1 e^{-t})^{-(\alpha_1 - \beta_1)} (1 - e^{-2t})^{-(\alpha_0 - \beta_0)} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\} (1 + \mathcal{O}(u)), & \begin{matrix} z \rightarrow e^t \\ |z| > 1. \end{matrix} \end{cases} \end{aligned} \quad (2.4.103)$$

By (2.4.87),

$$\begin{aligned} G(\lambda(z); x = 2nt) &= n^{2\beta_0} \left(\log \frac{z}{e^{-t}} \right)^{-(\alpha_0 - \beta_0)} \left(\log \frac{z}{e^t} \right)^{\alpha_0 + \beta_0} z^n e^{-i\pi(\alpha_0 - \beta_0)}, \\ &= \begin{cases} n^{2\beta_0} u^{-(\alpha_0 - \beta_0)} (2t)^{\alpha_0 + \beta_0} e^{-tn} e^{-t(\alpha_0 - \beta_0)} e^{-2\pi i \beta_0} (1 + \mathcal{O}(u)), & \begin{matrix} u \rightarrow 0 \\ z \rightarrow e^{-t}, \end{matrix} \\ n^{2\beta_0} u^{\alpha_0 + \beta_0} (2t)^{-(\alpha_0 - \beta_0)} e^{tn} e^{-t(\alpha_0 + \beta_0)} e^{-i\pi(\alpha_0 - \beta_0)} (1 + \mathcal{O}(u)), & \begin{matrix} u \rightarrow 0 \\ z \rightarrow e^t. \end{matrix} \end{cases} \end{aligned} \quad (2.4.104)$$

Now considering the symbol (2.2.1) and (2.4.30),

$$f(z; t) = \begin{cases} e^{V(e^{-t})}(1 - z_1^{-1}e^{-t})^{2\alpha_1}(1 - e^{-2t})^{\alpha_0+\beta_0}u^{\alpha_0-\beta_0}e^{t(\alpha_1-\beta_1)} \\ \quad \times e^{2t\alpha_0}e^{i\pi(\alpha_1-\beta_1)}z_1^{\alpha_1-\beta_1}(1 + \mathcal{O}(u)), & \begin{matrix} u \rightarrow 0 \\ z \rightarrow e^{-t}, \end{matrix} \\ e^{V(e^t)}(1 - z_1e^{-t})^{2\alpha_1}(1 - e^{-2t})^{\alpha_0+\beta_0}u^{\alpha_0+\beta_0}e^{t(\alpha_1+\beta_1)} \\ \quad \times e^{-i\pi(\alpha_1+\beta_1)}e^{-i\pi(\alpha_1+\beta_1)}z_1^{-(\alpha_1+\beta_1)}(1 + \mathcal{O}(u)), & \begin{matrix} u \rightarrow 0 \\ z \rightarrow e^t. \end{matrix} \end{cases} \quad (2.4.105)$$

For what follows, we only need the 12 matrix element of $\Delta_1(z)$, which we give in detail. The other elements can be easily computed if needed, using the information above. Combining the above with (2.4.93), as $z \rightarrow e^{-t}$,

$$\begin{aligned} (\Delta_1(z))_{12} &= (1 - z_1^{-1}e^{-t})^{2\beta_1}(1 - e^{-2t})^{\alpha_0+\beta_0} \exp \left\{ \sum_{k=0}^{\infty} V_k e^{-tk} \right\} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{tk} \right\} \\ &\quad \times e^{t(\alpha_0+\beta_0)}e^{-t(\alpha_1-\beta_1)}e^{-2\pi i\beta_0}e^{-i\pi(\alpha_1-\beta_1)}z_1^{-(\alpha_1-\beta_1)}(2t)^{-(\alpha_0+\beta_0)}n^{-2\beta_0} \\ &\quad \times \Phi_{21}(\lambda(z \rightarrow e^{-t}); x = 2nt) (1 + \mathcal{O}(u)), \end{aligned} \quad (2.4.106)$$

and as $z \rightarrow e^t$,

$$\begin{aligned} (\Delta_1(z))_{12} &= (1 - z_1e^{-t})^{2\beta_1}(1 - e^{-2t})^{-(\alpha_0-\beta_0)} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\} \exp \left\{ \sum_{k=0}^{\infty} V_k e^{tk} \right\} \\ &\quad \times e^{t(\alpha_0+\beta_0)}e^{t(\alpha_1+\beta_1)}e^{-2\pi i\beta_0}e^{-i\pi(\alpha_1+\beta_1)}z_1^{-(\alpha_1+\beta_1)}(2t)^{\alpha_0-\beta_0}n^{-2\beta_0} \\ &\quad \times \Phi_{21}(\lambda(z \rightarrow e^t); x = 2nt) (1 + \mathcal{O}(u)). \end{aligned} \quad (2.4.107)$$

All that is left to do now is to find the appropriate asymptotics for $\Phi(\lambda, x)$. We will find two asymptotic expansions below, both as $\lambda' \rightarrow \infty$,

$$\Phi(\lambda') = I + \frac{C_1}{\lambda'} + \mathcal{O}((\lambda')^{-2}), \quad (2.4.108)$$

one will hold uniformly for $0 < x < \delta$ and the other will hold uniformly for $x > C$, where $\delta, C > 0$ and $\lambda'(z) = \lambda(z) \pm \frac{x}{2}$. After that, judging by the similarities in the two asymptotic expansions, we will attempt to give a C_1 for the expansion above that will asymptotically match both cases.

Asymptotics for Φ as $x \rightarrow \infty$

For this case we do not require any additional details. The authors have found an expansion in [11, Section 4.1], uniform for $x > C$, $C > 0$, which has sufficient detail for our use. Specifically they have found an expansion in ζ for a function,

$$\tilde{\Phi}(\zeta) = \Phi(x\zeta = \lambda; x). \quad (2.4.109)$$

They obtained the first coefficient \tilde{C}_1 ,

$$\tilde{\Phi}(\zeta) = I + \frac{\tilde{C}_1}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \text{as } \zeta \rightarrow \infty, \quad (2.4.110)$$

which is given by,

$$\tilde{C}_1 = \begin{pmatrix} \frac{x^{-2+2\alpha}e^{-x}}{\Gamma(\alpha_0-\beta_0)\Gamma(\alpha_0+\beta_0)} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) & -x^{-1+\alpha_0-\beta_0}e^{-x/2} \frac{e^{-2\pi i\beta_0}}{\Gamma(\alpha_0+\beta_0)} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) \\ x^{-1+\alpha_0+\beta_0}e^{-x/2} \frac{e^{2\pi i\beta_0}}{\Gamma(\alpha_0-\beta_0)} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) & \frac{-x^{-2+2\alpha}e^{-x}}{\Gamma(\alpha_0-\beta_0)\Gamma(\alpha_0+\beta_0)} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) \end{pmatrix}. \quad (2.4.111)$$

Via the transformation $\zeta \mapsto \frac{\lambda}{x}$ we obtain the following for the 21 element of the expansion for Φ ,

$$\Phi_{21}(\lambda; x) = x^{\alpha_0+\beta_0}e^{-x/2} \frac{e^{2\pi i\beta_0}}{\Gamma(\alpha_0-\beta_0)} \lambda^{-1} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) + \mathcal{O}\left(\frac{x^2}{\lambda^2}\right), \quad \text{as } \lambda \rightarrow \infty. \quad (2.4.112)$$

uniformly for $x > C > 0$. Translating λ by $\pm \frac{x}{2}$ makes no difference, thus we also have,

$$\Phi_{21}(\lambda'; x) = x^{\alpha_0+\beta_0}e^{-x/2} \frac{e^{2\pi i\beta_0}}{\Gamma(\alpha_0-\beta_0)} (\lambda')^{-1} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) + \mathcal{O}\left(\left(\frac{x}{\lambda'}\right)^2\right), \quad \text{as } \lambda' \rightarrow \infty. \quad (2.4.113)$$

uniformly for $x > C > 0$.

Asymptotics for Φ as $x \rightarrow 0$

For this case we will use the analysis that was performed in [11, Sections 4.2] to find asymptotics of Ψ_0 and Φ as $x \rightarrow 0$. We go back to the R-H problem for Ψ_0 , (Ψ_01) - (Ψ_04) —equations (2.4.73)-(2.4.81)—which has an intimate connection to the function $\Phi(\lambda(z); x)$ via (2.4.86).

We denote by $\Psi_{0,I}, \dots, \Psi_{0,V}$ the analytic continuation of Ψ_0 from the relevant sectors I, \dots, V in

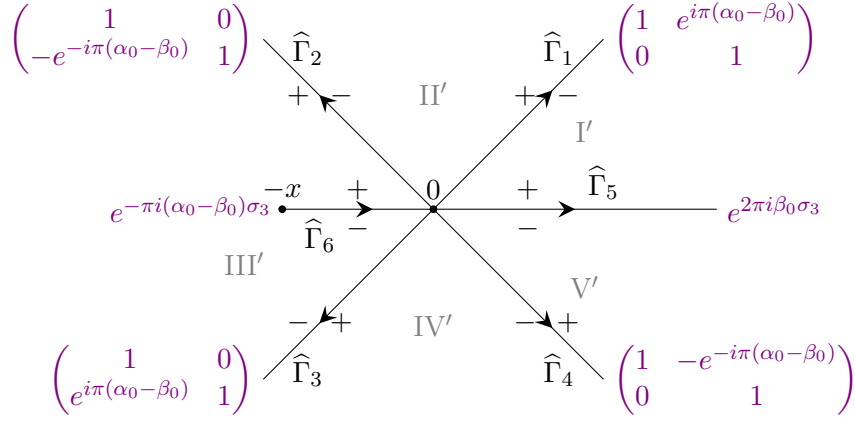
Figure 2.6: Jump contour and jump matrices for the $\widehat{\Psi}$ R-H problem

Figure 2.4 to $\mathbb{C} \setminus [0, +\infty)$ and consider the following function,

$$\widehat{\Psi}(\widehat{\lambda}; x) := e^{(x/2)\sigma_3} x^{-\beta_0\sigma_3} \times \begin{cases} \Psi_{0,\text{I}}(\frac{\lambda}{x} + 1; x) & \text{for } \widehat{\lambda} \text{ in region I}', \\ \Psi_{0,\text{II}}(\frac{\lambda}{x} + 1; x) & \text{for } \widehat{\lambda} \text{ in region II}', \\ \Psi_{0,\text{III}}(\frac{\lambda}{x} + 1; x) & \text{for } \widehat{\lambda} \text{ in region III}', \\ \Psi_{0,\text{IV}}(\frac{\lambda}{x} + 1; x) & \text{for } \widehat{\lambda} \text{ in region IV}', \\ \Psi_{0,\text{V}}(\frac{\lambda}{x} + 1; x) & \text{for } \widehat{\lambda} \text{ in region V}'. \end{cases} \quad (2.4.114)$$

We denote $\widehat{\lambda}(z) := \lambda(z) - \frac{x}{2}$. The contour is translated by a half in the ζ -plane and then by realising that $\zeta(z) = \frac{\lambda(z)}{x}$ (Note well that the $\zeta(z)$ here is not the same as in the R-H problem for Ψ_1), it is transformed into a contour in the $\widehat{\lambda}$ -plane (see Figure 2.6 and compare with Figure 2.4). The main thing to note here is that the intersection of the contour lines is now at $\widehat{\lambda} = 0$ as opposed to $\zeta = \frac{1}{2}$, i.e. $\lambda = \frac{x}{2}$.

The following R-H problem for $\widehat{\Psi}$ follows straightforwardly from the Ψ_0 problem (Ψ_{01}) - (Ψ_{04}) .

Riemann-Hilbert Problem for $\widehat{\Psi}$

($\widehat{\Psi}1$) $\widehat{\Psi} : \mathbb{C} \setminus \widehat{\Gamma} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic. Where $\widehat{\Gamma} := \cup_{j=1}^6 \widehat{\Gamma}_j$ and

$$\begin{aligned} \widehat{\Gamma}_1 &= e^{i\pi/4}\mathbb{R}^+, & \widehat{\Gamma}_2 &= e^{3i\pi/4}\mathbb{R}^+, & \widehat{\Gamma}_3 &= e^{5i\pi/4}\mathbb{R}^+, \\ \widehat{\Gamma}_4 &= e^{7i\pi/4}\mathbb{R}^+, & \widehat{\Gamma}_5 &= \mathbb{R}^+, & \widehat{\Gamma}_6 &= (-x, 0). \end{aligned}$$

($\widehat{\Psi}2$) $\widehat{\Psi}$ has continuous boundary values on $\widehat{\Gamma} \setminus \{-x, 0\}$, and they are related by the following jump conditions,

$$\widehat{\Psi}_+(\widehat{\lambda}) = \widehat{\Psi}_-(\widehat{\lambda}) \begin{pmatrix} 1 & e^{\pi i(\alpha_0 - \beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_1, \quad (2.4.115)$$

$$\widehat{\Psi}_+(\widehat{\lambda}) = \widehat{\Psi}_-(\widehat{\lambda}) \begin{pmatrix} 1 & 0 \\ -e^{-\pi i(\alpha_0 - \beta_0)} & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_2, \quad (2.4.116)$$

$$\widehat{\Psi}_+(\widehat{\lambda}) = \widehat{\Psi}_-(\widehat{\lambda}) \begin{pmatrix} 1 & 0 \\ e^{\pi i(\alpha_0 - \beta_0)} & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_3, \quad (2.4.117)$$

$$\widehat{\Psi}_+(\widehat{\lambda}) = \widehat{\Psi}_-(\widehat{\lambda}) \begin{pmatrix} 1 & -e^{-\pi i(\alpha_0 - \beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_4, \quad (2.4.118)$$

$$\widehat{\Psi}_+(\widehat{\lambda}) = \widehat{\Psi}_-(\widehat{\lambda}) e^{2\pi i\beta_0\sigma_3}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_5, \quad (2.4.119)$$

$$\widehat{\Psi}_+(\widehat{\lambda}) = \widehat{\Psi}_-(\widehat{\lambda}) e^{-\pi i(\alpha_0 - \beta_0)\sigma_3}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_6, \quad (2.4.120)$$

($\widehat{\Psi}3$) $\widehat{\Psi}_0$ has the following asymptotic behaviour as $\widehat{\lambda} \rightarrow \infty$,

$$\widehat{\Psi}(\widehat{\lambda}) = \left(I + \mathcal{O}(\widehat{\lambda}^{-1}) \right) \widehat{\lambda}^{-\beta\sigma_3} e^{-(1/2)\widehat{\lambda}\sigma_3}. \quad (2.4.121)$$

($\widehat{\Psi}4$) $\widehat{\Psi}$ has the following asymptotic behaviour close to the following points,

$$(\widehat{\Psi}4.1) \quad \widehat{\Psi}(\widehat{\lambda}) = \mathcal{O} \begin{pmatrix} |\widehat{\lambda} + x|^{(\alpha_0 - \beta_0)/2} & |\widehat{\lambda} + x|^{-(\alpha_0 - \beta_0)/2} \\ |\widehat{\lambda} + x|^{(\alpha_0 - \beta_0)/2} & |\widehat{\lambda} + x|^{-(\alpha_0 - \beta_0)/2} \end{pmatrix}, \quad \text{as } \widehat{\lambda} \rightarrow -x, \quad (2.4.122)$$

$$(\widehat{\Psi}4.2) \quad \widehat{\Psi}(\widehat{\lambda}) = \mathcal{O} \begin{pmatrix} |\widehat{\lambda}|^{-(\alpha_0 + \beta_0)/2} & |\widehat{\lambda}|^{(\alpha_0 + \beta_0)/2} \\ |\widehat{\lambda}|^{-(\alpha_0 + \beta_0)/2} & |\widehat{\lambda}|^{(\alpha_0 + \beta_0)/2} \end{pmatrix}, \quad \text{as } \widehat{\lambda} \rightarrow 0, \widehat{\lambda} \in I', V', \quad (2.4.123)$$

and for $\widehat{\lambda}$ in other sectors we apply the appropriate jump conditions to (2.4.123).

In [11] the authors then solve the problem for $\widehat{\Psi}$ using the same steepest descent techniques as are being used to solve the main problem (notice the 'Inception' style situation). For small values of x , they construct a global and a local parametrix, match them on the boundary of a small neighbourhood of $\widehat{\lambda} = 0$, denoted by U_ε (this neighbourhood contains also $[-x, 0]$), and show the

final R-H problem is a small norm R-H problem, thus solvable. We will summarise the key points here, only providing the details that we will need for our own computation.

Riemann-Hilbert Problem for M (Global Parametrix for the $\widehat{\Psi}$ R-H problem)

(M1) $M : \mathbb{C} \setminus (\cup_{j=1}^5 \widehat{\Gamma}_j) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(M2) M has continuous boundary values on $(\cup_{j=1}^5 \widehat{\Gamma}_j) \setminus \{0\}$ related by the following jump conditions,

$$M_+(\widehat{\lambda}) = M_-(\widehat{\lambda}) \begin{pmatrix} 1 & e^{\pi i(\alpha_0 - \beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_1, \quad (2.4.124)$$

$$M_+(\widehat{\lambda}) = M_-(\widehat{\lambda}) \begin{pmatrix} 1 & 0 \\ -e^{-\pi i(\alpha_0 - \beta_0)} & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_2, \quad (2.4.125)$$

$$M_+(\widehat{\lambda}) = M_-(\widehat{\lambda}) \begin{pmatrix} 1 & 0 \\ e^{\pi i(\alpha_0 - \beta_0)} & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_3, \quad (2.4.126)$$

$$M_+(\widehat{\lambda}) = M_-(\widehat{\lambda}) \begin{pmatrix} 1 & -e^{-\pi i(\alpha_0 - \beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_4, \quad (2.4.127)$$

$$M_+(\widehat{\lambda}) = M_-(\widehat{\lambda}) e^{2\pi i \beta_0 \sigma_3}, \quad \text{for } \widehat{\lambda} \in \widehat{\Gamma}_5, \quad (2.4.128)$$

(M3) M has the following behaviour at infinity,

$$M(\widehat{\lambda}) = \left(I + \mathcal{O}(\widehat{\lambda}^{-1}) \right) \widehat{\lambda}^{-\beta_0 \sigma_3} e^{-(1/2)\widehat{\lambda} \sigma_3}, \quad \text{as } \widehat{\lambda} \rightarrow \infty. \quad (2.4.129)$$

The problem is explicitly solvable in terms of the confluent hypergeometric function (2.4.51). We give the solution as it was presented in [11, Section 4.2.1], but only for sector I' (compare with the analysis performed in Section 2.4.4.1), details of other sectors can be found in [11], or by applying the appropriate jumps. We define the following matrix-valued function,

$$\begin{aligned} H(\widehat{\lambda}) &:= \begin{pmatrix} e^{-i\pi(2\beta_0 + \alpha_0)} & 0 \\ 0 & e^{i\pi(\beta_0 + 2\alpha_0)} \end{pmatrix} e^{-(i\pi/2)\alpha_0 \sigma_3} \\ &\times \begin{pmatrix} \widehat{\lambda}^{\alpha_0} \psi(\alpha_0 + \beta_0, 1 + 2\alpha_0, \widehat{\lambda}) e^{i\pi(2\beta_0 + \alpha_0)} \\ \widehat{\lambda}^{-\alpha_0} \psi(1 - \alpha_0 + \beta_0, 1 - 2\alpha_0, \widehat{\lambda}) e^{i\pi(\beta_0 - 3\alpha_0)} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} & \left. \begin{aligned} & \widehat{\lambda}^{\alpha_0} \psi(1 + \alpha_0 - \beta_0, 1 + 2\alpha_0, e^{-i\pi\widehat{\lambda}}) e^{i\pi(\beta_0 + \alpha_0) \frac{\Gamma(1 + \alpha_0 - \beta_0)}{\Gamma(\alpha_0 + \beta_0)}} \\ & \widehat{\lambda}^{-\alpha_0} \psi(-\alpha_0 - \beta_0, 1 - 2\alpha_0, e^{-i\pi\widehat{\lambda}}) e^{i\pi\alpha_0} \end{aligned} \right) \\ & \times e^{(i\pi\alpha_0/2)\sigma_3} e^{-\widehat{\lambda}\sigma_3/2}, \quad \alpha_0 \pm \beta_0 \neq -1, -2, \dots, \end{aligned} \quad (2.4.130)$$

where $\psi(a, b, x)$ is the confluent hypergeometric function (2.4.51), and $\Gamma(x)$ is the Euler's Γ -function. The solution to the R-H problem for M in the sector I' is then given by,

$$M(\widehat{\lambda}) = M_1(\widehat{\lambda}) := H(\widehat{\lambda}) \begin{pmatrix} 1 & -e^{i\pi(\alpha_0 - \beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } 0 < \arg \widehat{\lambda} < \frac{\pi}{4}, \quad (2.4.131)$$

and we denote the matrix elements of $M(\widehat{\lambda})$ by M_{ij} , $i, j = 1, 2$ for future use.

The authors then proceed to the construction of a local parametrix at $\widehat{\lambda} = 0$, details of which we do not need here. In particular they prove that for $\widehat{\lambda} \in \partial U_\varepsilon$,

$$P(\widehat{\lambda}; x) M(\widehat{\lambda})^{-1} = \begin{cases} I + \mathcal{O}(x) + \mathcal{O}(x^{1+2\alpha_0}), & \text{as } x \rightarrow 0, \text{ if } 2\alpha_0 \notin \mathbb{Z}, \\ I + \mathcal{O}(x \log x), & \text{as } x \rightarrow 0, \text{ if } 2\alpha_0 \in \mathbb{Z}. \end{cases} \quad (2.4.132)$$

$$= I + o(1) \quad \text{as } x \rightarrow 0, \quad (2.4.133)$$

which makes the following final R-H problem small-norm for sufficiently small x .

Riemann-Hilbert problem for R

(R1) $R : \mathbb{C} \setminus \partial U_\varepsilon \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(R2) R has the following jump condition,

$$R_+(\widehat{\lambda}) = R_-(\widehat{\lambda}) J(\widehat{\lambda}), \quad \text{for } \widehat{\lambda} \in \partial U_\varepsilon, \quad (2.4.134)$$

where $J(\widehat{\lambda}) = P(\widehat{\lambda}) M(\widehat{\lambda})^{-1}$.

(R3) $R(\widehat{\lambda}) = I + \mathcal{O}(\widehat{\lambda}^{-1})$ as $\widehat{\lambda} \rightarrow \infty$.

The solvability of this problem implies also through the invertible transformations $\Psi_0 \rightarrow \widehat{\Psi} \rightarrow R$ and $\Psi_0 \rightarrow \Phi$ that the R-H problems for Φ and Ψ_0 are solvable for $0 < x < \delta$. Further, from the fact that $J(\widehat{\lambda}) = I + o(1)$ and (R2), it follows that $R(\widehat{\lambda}) = I + o(1)$ uniformly for $\widehat{\lambda} \in \mathbb{C} \setminus U_\varepsilon$ as $x \rightarrow 0$.

This holds in particular at infinity, thus we have that $R(\widehat{\lambda}) = I + \mathcal{O}(\widehat{\lambda}^{-1})$ as $\widehat{\lambda} \rightarrow \infty$ uniformly for small x .

The following function solves the R-H problem for $R(z)$ above,

$$R(\widehat{\lambda}) = \begin{cases} \widehat{\Psi}(\widehat{\lambda})M(\widehat{\lambda})^{-1} & \text{for } \widehat{\lambda} \in \mathbb{C} \setminus U_\varepsilon, \\ \widehat{\Psi}(\widehat{\lambda})P(\widehat{\lambda})^{-1} & \text{for } \widehat{\lambda} \in U_\varepsilon. \end{cases} \quad (2.4.135)$$

We will now use this to find asymptotics of $\widehat{\Psi}(\widehat{\lambda})$ as $\widehat{\lambda} \rightarrow \infty$. We rewrite (2.4.86) and (2.4.87) using $\lambda = \widehat{\lambda} + \frac{x}{2}$ (see the discussion below (2.4.114)). We obtain,

$$\Phi\left(\lambda = \widehat{\lambda} + \frac{x}{2}; x\right) = e^{x/4\sigma_3} x^{-\beta_0\sigma_3} \Psi\left(\frac{\widehat{\lambda}}{x} + 1; x\right) G\left(\widehat{\lambda} + \frac{x}{2}, x\right)^{\frac{1}{2}\sigma_3} e^{\frac{i\pi}{2}(\alpha_0 - \beta_0)\sigma_3}, \quad (2.4.136)$$

and

$$G\left(\widehat{\lambda} + \frac{x}{2}; x\right) = \left(\widehat{\lambda} + x\right)^{-(\alpha_0 - \beta_0)} \widehat{\lambda}^{\alpha_0 + \beta_0} e^{\widehat{\lambda}x/2} e^{-i\pi(\alpha_0 - \beta_0)}. \quad (2.4.137)$$

Now using (2.4.114) and (2.4.135),

$$\begin{aligned} \Phi(\widehat{\lambda}; x) &= e^{-x/4\sigma_3} \widehat{\Psi}(\widehat{\lambda}; x) G\left(\widehat{\lambda} + \frac{x}{2}; x\right)^{\frac{1}{2}\sigma_3} e^{\frac{i\pi}{2}(\alpha_0 - \beta_0)\sigma_3} \\ &= e^{-x/4\sigma_3} R(\widehat{\lambda})M(\widehat{\lambda})G\left(\widehat{\lambda} + \frac{x}{2}; x\right)^{\frac{1}{2}\sigma_3} e^{\frac{i\pi}{2}(\alpha_0 - \beta_0)\sigma_3} \quad \text{for } \widehat{\lambda} \in \mathbb{C} \setminus U_\varepsilon. \end{aligned}$$

Using the asymptotic behaviour of $R(\widehat{\lambda}) = I + \mathcal{O}(\widehat{\lambda}^{-1})$ as $\widehat{\lambda} \rightarrow \infty$ and using $G := G(\widehat{\lambda} + \frac{x}{2})$ to save space, we obtain,

$$\begin{aligned} \Phi(\widehat{\lambda}; x) &= \begin{pmatrix} e^{-x/4} \left(M_{11}(1 + \mathcal{O}(\widehat{\lambda}^{-1})) + M_{21}\mathcal{O}(\widehat{\lambda}^{-1}) \right) & e^{-x/4} \left(M_{12}(1 + \mathcal{O}(\widehat{\lambda}^{-1})) + M_{22}\mathcal{O}(\widehat{\lambda}^{-1}) \right) \\ e^{x/4} \left(M_{11}(1 + \mathcal{O}(\widehat{\lambda}^{-1})) + M_{21}\mathcal{O}(\widehat{\lambda}^{-1}) \right) & e^{x/4} \left(M_{12}(1 + \mathcal{O}(\widehat{\lambda}^{-1})) + M_{22}\mathcal{O}(\widehat{\lambda}^{-1}) \right) \end{pmatrix} \\ &\quad \times \begin{pmatrix} G^{\frac{1}{2}} e^{\frac{i\pi}{2}(\alpha_0 - \beta_0)} & 0 \\ 0 & G^{-\frac{1}{2}} e^{-\frac{i\pi}{2}(\alpha_0 - \beta_0)} \end{pmatrix} \\ &= \begin{pmatrix} e^{-x/4} G^{\frac{1}{2}} e^{\frac{i\pi}{2}(\alpha_0 - \beta_0)} M_{11} \left(\left(1 + \frac{M_{21}}{M_{11}} \mathcal{O}(\widehat{\lambda}^{-1})\right) \right) & e^{-x/4} G^{-\frac{1}{2}} e^{-\frac{i\pi}{2}(\alpha_0 - \beta_0)} M_{12} \left(\left(1 + \frac{M_{22}}{M_{12}} \mathcal{O}(\widehat{\lambda}^{-1})\right) \right) \\ e^{x/4} G^{\frac{1}{2}} e^{\frac{i\pi}{2}(\alpha_0 - \beta_0)} M_{21} \left(\left(1 + \frac{M_{11}}{M_{21}} \mathcal{O}(\widehat{\lambda}^{-1})\right) \right) & e^{x/4} G^{-\frac{1}{2}} e^{-\frac{i\pi}{2}(\alpha_0 - \beta_0)} M_{22} \left(\left(1 + \frac{M_{12}}{M_{22}} \mathcal{O}(\widehat{\lambda}^{-1})\right) \right) \end{pmatrix}. \end{aligned} \quad (2.4.138)$$

We can find the asymptotics of each M_{ij} as $\widehat{\lambda} \rightarrow \infty$, but we only concentrate on finding the details of M_{21} (finding details of every M_{ij} shows that the ratios above are $\mathcal{O}(1)$). From (2.4.131) and (2.4.130),

$$M_{21} = e^{i\pi(\beta_0+3\alpha_0)} e^{-\widehat{\lambda}/2} \widehat{\lambda}^{-\alpha_0} \psi \left(1 - \alpha_0 + \beta_0, 1 - 2\alpha_0, \widehat{\lambda} \right) e^{i\pi(\beta_0-3\alpha_0)} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)}. \quad (2.4.139)$$

Using the asymptotics for the confluent hypergeometric function ψ from (2.4.54) we obtain,

$$M_{21} = e^{2i\pi\beta_0} e^{-\widehat{\lambda}/2} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} \widehat{\lambda}^{-\beta_0} \widehat{\lambda}^{-1} \left(1 + \mathcal{O}(\widehat{\lambda}^{-1}) \right). \quad (2.4.140)$$

We thus obtain the following asymptotics for the 21 matrix element of the function $\Phi(\lambda; x)$ after combining (2.4.138), (2.4.140) and (2.4.137),

$$\begin{aligned} \Phi_{21}(\widehat{\lambda}) &= e^{x/4} e^{2i\pi\beta_0} (\widehat{\lambda} + x)^{\left(\frac{\alpha_0-\beta_0}{2}\right)} \widehat{\lambda}^{\left(\frac{\alpha_0+\beta_0}{2}\right)} e^{\widehat{\lambda}/2} e^{x/4} e^{-\widehat{\lambda}/2} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} \widehat{\lambda}^{-\beta_0} \widehat{\lambda}^{-1} \left(1 + \mathcal{O}(\widehat{\lambda}^{-1}) \right) \\ &= e^{x/2} e^{2i\pi\beta_0} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} \left(\widehat{\lambda}^{-\left(\frac{\alpha_0-\beta_0}{2}\right)} \left(1 + \frac{x}{\widehat{\lambda}} \right)^{-\left(\frac{\alpha_0-\beta_0}{2}\right)} \widehat{\lambda}^{\frac{\alpha_0-\beta_0}{2}} \right) \widehat{\lambda}^{-1} \left(1 + \mathcal{O}(\widehat{\lambda}^{-1}) \right) \\ &= e^{x/2} e^{2i\pi\beta_0} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} \widehat{\lambda}^{-1} \left(1 + \mathcal{O}(\widehat{\lambda}^{-1}) \right), \quad \text{as } \widehat{\lambda} \rightarrow \infty, \end{aligned} \quad (2.4.141)$$

uniformly for $0 < x < \delta$, $\delta > 0$. A simple translation $\widehat{\lambda} \mapsto \tilde{\lambda} - x$ verifies also that as $\tilde{\lambda} \rightarrow \infty$ we also have,

$$\begin{aligned} \Phi_{21}(\tilde{\lambda}) &= e^{x/4} G^{1/2} \left(\tilde{\lambda} - x; x \right) e^{i\pi/2(\alpha_0-\beta_0)} M_{21}(\tilde{\lambda} - x) \left(1 + \mathcal{O}(\tilde{\lambda}^{-1}) \right) \\ &= e^{x/4} \left(\tilde{\lambda}^{-\left(\frac{\alpha_0-\beta_0}{2}\right)} (\tilde{\lambda} - x)^{\left(\frac{\alpha_0+\beta_0}{2}\right)} e^{\tilde{\lambda}/2} e^{-x/4} e^{-i\pi/2(\alpha_0-\beta_0)} \right) e^{i\pi/2(\alpha_0-\beta_0)} e^{2i\pi\beta_0} e^{-\tilde{\lambda}/2} \\ &\quad \times \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} (\tilde{\lambda} - x)^{-1-\beta_0} \left(1 + \mathcal{O}(\tilde{\lambda}^{-1}) \right) \\ &= e^{x/2} e^{2i\pi\beta_0} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} \left(\tilde{\lambda}^{-\left(\frac{\alpha_0-\beta_0}{2}\right)} \left(\tilde{\lambda} \left(1 - \frac{x}{\tilde{\lambda}} \right) \right)^{\left(\frac{\alpha_0-\beta_0}{2}\right)} \tilde{\lambda}^{-1} \left(1 - \frac{x}{\tilde{\lambda}} \right)^{-1} \right) \left(1 + \mathcal{O}(\tilde{\lambda}^{-1}) \right) \\ &= e^{x/2} e^{2i\pi\beta_0} \frac{\Gamma(1 + \alpha_0 + \beta_0)}{\Gamma(\alpha_0 - \beta_0)} \tilde{\lambda}^{-1} \left(1 + \mathcal{O}(\tilde{\lambda}^{-1}) \right), \quad \text{as } \tilde{\lambda} \rightarrow \infty, \end{aligned} \quad (2.4.142)$$

uniformly for $0 < x < \delta$, $\delta > 0$.

Expression for a fixed x

Let us introduce a new function,

$$K(x) = e^{x/2} \int_x^\infty y^{\alpha_0 + \beta_0} e^{-y} dy. \quad (2.4.143)$$

It has the following behaviour,

$$K(x) \sim \begin{cases} e^{-x/2} x^{\alpha_0 + \beta_0}, & \text{as } x \rightarrow \infty, \\ e^{x/2} \Gamma(\alpha_0 + \beta_0 + 1), & \text{as } x \rightarrow 0. \end{cases} \quad (2.4.144)$$

Thus we could use $K(x)$ to write down an expression for $\Phi_{21}(\lambda'; x)$ with fixed x , that is asymptotically still valid for x small or large. From (2.4.113), (2.4.141), (2.4.142) we obtain,

$$\Phi_{21}(\lambda'; x) = \frac{e^{2\pi i \beta_0}}{\Gamma(\alpha_0 - \beta_0)} K(x) (\lambda')^{-1} + \mathcal{O}\left((\lambda')^{-2}\right) \quad \text{as } \lambda' \rightarrow \infty. \quad (2.4.145)$$

We also note that if $x = 2nt$, $\widehat{\lambda} = n \log \frac{z}{e^t}$ and $\widetilde{\lambda} = n \log \frac{z}{e^{-t}}$ and for t fixed,

$$\frac{1}{\widehat{\lambda}} = \frac{e^t}{n(z - e^t)} + \frac{1}{2n} + \mathcal{O}(z - e^t), \quad z \rightarrow e^t, \quad (2.4.146)$$

and similarly,

$$\frac{1}{\widetilde{\lambda}} = \frac{e^{-t}}{n(z - e^{-t})} + \frac{1}{2n} + \mathcal{O}(z - e^{-t}), \quad z \rightarrow e^{-t}. \quad (2.4.147)$$

Combining the results from (2.4.106), (2.4.107) and (2.4.141), (2.4.142), we obtain the final expression for $(\Delta_1(z))_{12}$ in the expansion of $P_{z_0}(z)N^{-1}(z)$ in inverse powers of n , for $z \rightarrow e^{-t}$,

$$\begin{aligned} (\Delta_1(z))_{12} &= \widehat{\lambda}^{-1} (1 - z_1^{-1} e^{-t})^{2\beta_1} (1 - e^{-2t})^{\alpha_0 + \beta_0} \exp \left\{ \sum_{k=0}^{\infty} V_k e^{-tk} \right\} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{tk} \right\} \\ &\times e^{t(\alpha_0 + \beta_0)} e^{-t(\alpha_1 - \beta_1)} e^{-i\pi(\alpha_1 - \beta_1)} z_1^{-(\alpha_1 - \beta_1)} (2t)^{-(\alpha_0 + \beta_0)} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0 - \beta_0)} K(2nt), \end{aligned} \quad (2.4.148)$$

and for $z \rightarrow e^t$,

$$(\Delta_1(z))_{12} = \widetilde{\lambda}^{-1} (1 - z_1 e^{-t})^{2\beta_1} (1 - e^{-2t})^{-(\alpha_0 - \beta_0)} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\} \exp \left\{ \sum_{k=0}^{\infty} V_k e^{tk} \right\}$$

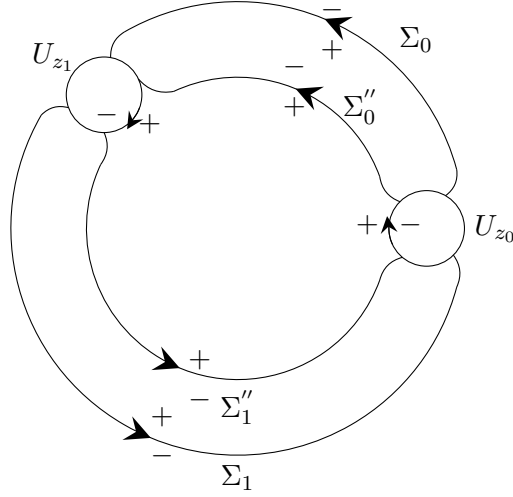


Figure 2.7: Contour $\Gamma = \cup_{j=0}^1 \partial U_{z_j} \cup_{j=0}^1 \Sigma_j \cup_{j=0}^1 \Sigma'_j$ for the R and \tilde{R} R-H problem.

$$\times e^{t(\alpha_0+\beta_0)} e^{t(\alpha_1+\beta_1)} e^{-i\pi(\alpha_1+\beta_1)} z_1^{-(\alpha_1+\beta_1)} (2t)^{\alpha_0-\beta_0} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0-\beta_0)} K(2nt). \quad (2.4.149)$$

2.4.5 Final Riemann-Hilbert Problem

We assume $\alpha_j \pm \beta_j \neq -1, -2, \dots \forall j = 0, 1$. Let

$$R(z) = \begin{cases} S(z)N^{-1}(z), & z \in \mathbb{C} \setminus \{U_{z_1} \cup U_{z_0} \cup \Gamma\}, \\ S(z)P_{z_1}^{-1}(z), & z \in U_{z_1}, \\ S(z)P_{z_0}^{-1}(z), & z \in U_{z_0}. \end{cases} \quad (2.4.150)$$

As was mentioned in Sections 2.4.4.1 and 2.4.4.2, $S(z)P_{z_0}^{-1}(z)$ and $S(z)P_{z_1}^{-1}(z)$ are analytic in the neighbourhoods U_{z_0} and U_{z_1} respectively. The function $R(z)$ satisfies the following Riemann-Hilbert problem, for which the contour Γ is defined in Figure 2.7,

(R1) $R(z) : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(R2) $R(z)$ satisfies the following jump conditions,

$$R_+(z) = R_-(z)J_R(z) \quad \text{for } z \in \Gamma. \quad (2.4.151)$$

The jump matrices are defined in the following way,

$$J_R(z) = N(z)J_k(z)N(z)^{-1}, \quad \text{with} \quad \begin{cases} k = 1 \text{ for } z \in \Sigma := \Sigma_0 \cup \Sigma_1, \\ k = 2 \text{ for } z \in \Sigma'' := \Sigma''_0 \cup \Sigma''_1, \end{cases} \quad (2.4.152)$$

and

$$J_R(z) = P_{z_1}(z)N^{-1}(z), \quad \text{for } z \in \partial U_{z_1}, \quad (2.4.153)$$

$$J_R(z) = P_{z_0}(z)N^{-1}(z), \quad \text{for } z \in \partial U_{z_0}. \quad (2.4.154)$$

Recall from (2.4.4) that $J_1(z) = \begin{pmatrix} 1 & 0 \\ z^{-n}f(z;t)^{-1} & 1 \end{pmatrix}$, $J_2(z) = \begin{pmatrix} 1 & 0 \\ z^n f(z;t)^{-1} & 1 \end{pmatrix}$.

$$(R3) \quad R(z) = I + \mathcal{O}(1/z) \text{ as } z \rightarrow \infty.$$

To find the solution of the R-H problem for $R(z)$, we look at the asymptotic behaviour of the jump matrices as $n \rightarrow \infty$. The jump matrices have $I + o(1)$ behaviour at infinity, the problem is a so-called small norm problem and solution is given in terms of a Neumann series (see [17, Theorem 7.8]).

The jump matrices $J_R(z)$ on Σ and Σ'' can be estimated uniformly as $I + \mathcal{O}(e^{-\varepsilon n})$, for a positive constant ε using the J_k above (2.4.14) and (2.4.15) for $0 < t < t_0$ and $x = 2nt$ bounded away from the set $\{x_1, \dots, x_k\}$. Thus these jump matrices already behave the way we require.

On the contour ∂U_{z_1} , the jump matrix admits a uniform expansion in the inverse powers of n , conjugated by $n^{\beta_1 \sigma_3} z_1^{-n \sigma_3 / 2}$,

$$J_R(z) = I + \Delta_1(z) + \Delta_2(z) + \dots + \Delta_k(z) + \Delta_{k+1}^{(r)}(z) \quad \text{for } z \in \partial U_{z_1}. \quad (2.4.155)$$

Each term $\Delta_k(z) = z_1^{\sigma_3 n / 2} n^{-\sigma_3 \beta_1} \mathcal{O}(1/n) n^{\sigma_3 \beta_1} z_1^{-\sigma_3 n / 2} = \mathcal{O}(n^{2 \max_1 |\operatorname{Re} \beta_1| - k})$. To obtain the solution, we require that $J_R = I + o(1)$, i.e. $n^{2 \max_1 |\operatorname{Re} \beta_1| - 1} = o(1)$, which means that $\operatorname{Re} \beta_1 \in (-1/2, 1/2)$.

We have found the asymptotics for $P_{z_0}(z)N^{-1}(z)$ in (2.4.99),

$$\begin{aligned} P_{z_0}(z)N(z)^{-1} &= I + n^{-\beta_0 \sigma_3} \mathcal{O}(n^{-1}) n^{\beta_0 \sigma_3} \\ &= I + \Delta_1(z) + \dots + \Delta_k(z) + \Delta_{k+1}^{(r)}(z) \quad \text{for } z \in \partial U_{z_0}. \end{aligned}$$

Again, for this to be of order $I + o(1)$, we need $-\frac{1}{2} < \operatorname{Re} \beta_0 < \frac{1}{2}$.

However, we need not stipulate that $\operatorname{Re} \beta_j \in (-\frac{1}{2}, \frac{1}{2})$. It is possible to find a solution for $\operatorname{Re} \beta_j \in (q - \frac{1}{2}, q + \frac{1}{2})$, for some $q \in \mathbb{R}$; i.e. the more general condition when $|||\beta||| < 1$. To accommodate these cases, we consider the following transformation of the R-H problem for $R(z)$ which was used in [14, Equation (4.63)],

$$\tilde{R}(z) = n^{\omega\sigma_3} R(z) n^{-\omega\sigma_3} \quad (2.4.156)$$

where

$$\omega = \frac{1}{2} \left(\min_{j=0,1} \operatorname{Re} \beta_j + \max_{j=0,1} \operatorname{Re} \beta_j \right). \quad (2.4.157)$$

This transformation moves all β_j into the strip $(-\frac{1}{2}, \frac{1}{2})$ making the above asymptotics of the jump matrices of the order $I + o(1)$. Note well, that the β_j are moved only in the conjugation by n^{β_j} , and not in the actual Fisher-Hartwig symbol $f(z; t)$.

Remark 2.4.4. As mentioned before, the solution obtained here only works for the case when the strength of the jump singularities is contained in the strip of the width strictly less than 1, i.e. $|||\beta||| < 1$. Later on, we consider the case when $|||\beta||| = 1$, which is of interest in some applications, one of which is outlined in Chapter 3. Note that, considering only these two cases does not render the case $|||\beta||| > 1$ unsolvable. On the contrary, this case is already covered if you recall the Tracy-Basor conjecture from Section 1.4.2. It tells us that through Fisher-Hartwig representation, we can reduce this to only these two mutually exclusive cases.

Let us look at how the transformation affects the jump conditions of the problem for $R(z)$;

$$\begin{aligned} \tilde{R}_+(z) &= n^{\omega\sigma_3} R_+(z) n^{-\omega\sigma_3} = n^{\omega\sigma_3} R_-(z) J_R(z) n^{-\omega\sigma_3} \\ &= n^{\omega\sigma_3} R_-(z) n^{-\omega\sigma_3} n^{\omega\sigma_3} J_R(z) n^{-\omega\sigma_3} \\ &= \tilde{R}_-(z) n^{\omega\sigma_3} J_R(z) n^{-\omega\sigma_3}. \end{aligned}$$

The asymptotic behaviour of the jump matrices on Σ and Σ'' remains unchanged by the transformation, $\tilde{J}_R(z) = I + \mathcal{O}(e^{-\varepsilon n})$, but with a different ε .

The jump matrices on ∂U_{z_0} and ∂U_{z_1} are now of the form,

$$I + n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} + n^{\omega\sigma_3} \Delta_2(z) n^{-\omega\sigma_3} + \dots + n^{\omega\sigma_3} \Delta_k(z) n^{-\omega\sigma_3} + n^{\omega\sigma_3} \Delta_{k+1}^{(r)}(z) n^{-\omega\sigma_3},$$

and the order of each of the terms is $\mathcal{O}(n^{2\max_j |\beta_j - \omega| - k})$, which behaves as $I + o(1)$ as we have that $-\frac{1}{2} < \beta_0 - \omega < \frac{1}{2}$.

This means that we can find the solution to the problem for $\tilde{R}(z)$ for β -parameters in the range $\operatorname{Re} \beta_j \in (q - 1/2, q + 1/2)$ for any $q \in \mathbb{R}$.

Now that we have all the jump matrices of the right order, we can use [17, Theorem 7.8] to get the Neumann series solution of the problem $\tilde{R}(z)$. We have that,

$$\tilde{R}(z) = I + \sum_{p=1}^k \tilde{R}_p(z) + \tilde{R}_{k+1}^{(r)}(z). \quad (2.4.158)$$

Each $\tilde{R}_p(z)$, is computed recursively via separate, additive R-H problems. The conditions are that each $\tilde{R}_p(z)$ is analytic outside $\partial U = \partial U_{z_0} \cup \partial U_{z_1}$, $\tilde{R}_p(z) \rightarrow 0$ as $z \rightarrow \infty$ for all p and satisfies the following jump condition,

$$\tilde{R}_{p,+}(z) = \tilde{R}_{p,-}(z) + \sum_{i=1}^p \tilde{R}_{p-i,-}(z) n^{\omega\sigma_3} \Delta_i(z) n^{-\omega\sigma_3}, \quad (2.4.159)$$

where we set $\tilde{R}_0(z) = I$. The first R-H problem for $\tilde{R}_1(z)$ satisfies the following conditions:

(\tilde{R}_1 1) $\tilde{R}_1(z) : \mathbb{C} \setminus \partial U \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(\tilde{R}_1 2) $\tilde{R}_1(z)$ satisfies the following jump condition,

$$\tilde{R}_{1,+}(z) = \tilde{R}_{1,-}(z) + n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} \quad \text{for } z \in \partial U.$$

(\tilde{R}_1 3) $\tilde{R}_1(z) \rightarrow 0$, as $z \rightarrow \infty$.

First, we recall the transformation (2.4.156), and denote,

$$R_p(z) = n^{-\omega\sigma_3} \tilde{R}_p(z) n^{\omega\sigma_3}, \quad R_p^{(r)}(z) = n^{-\omega\sigma_3} \tilde{R}_p^{(r)}(z) n^{\omega\sigma_3}. \quad (2.4.160)$$

We find the solution to this additive R-H problem using the Plemelj formulae (see Theorem 1.2.3) and the Residue Theorem,

$$R_1(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\Delta_1(x) dx}{x - z} \quad (2.4.161)$$

$$= \begin{cases} \frac{A_1}{z-z_1} + \frac{A_{e^t}}{z-e^t} + \frac{A_{e^{-t}}}{z-e^{-t}}, & z \in \mathbb{C} \setminus \cup_{j=0}^1 U_{z_j}, \\ \frac{A_1}{z-z_1} + \frac{A_{e^t}}{z-e^t} + \frac{A_{e^{-t}}}{z-e^{-t}} - \Delta_1(z), & z \in U_{z_j}, j = 0, 1, \end{cases} \quad (2.4.162)$$

where the contours in the integral are oriented in the negative direction (as seen in Figure 2.7) and $A_1, A_{e^{\pm t}}$ are the coefficients in the Laurent expansion of $\Delta_1(z)$,

$$\Delta_1(z) = \frac{A_k}{z-z_1} + B_1 + \mathcal{O}(z-z_1), \quad z \rightarrow z_1, \quad (2.4.163)$$

and

$$\Delta_1(z) = \frac{A_{e^{\pm t}}}{z-e^{\pm t}} + B_1 + \mathcal{O}(z-e^{\pm t}), \quad z \rightarrow e^{\pm t}, \quad (2.4.164)$$

The coefficients $A_1, A_{e^{\pm t}}$ are given below and $B_1, B_{e^{\pm t}}$ can also be computed explicitly if needed. In Sections 2.4.4.1 and 2.4.4.2 we computed the 12 entries of $\Delta_1(z)$ of each parametrix at the points z_1, e^{-t} and e^t , they are given by (2.4.67), (2.4.148) and (2.4.149). Using those together with (2.4.68), (2.4.147) and (2.4.146) respectively, we obtain the 12 elements of the matrices $A_{z_1}, A_{e^{-t}}, A_{e^t}$,

$$\begin{aligned} A_{z_1} &= \frac{z_1}{n} z_1^n e^{V_0} (1-z_1 e^t)^{(\alpha_0+\beta_0)} (1-e^{-t} z_1^{-1})^{-(\alpha_0-\beta_0)} \exp \left\{ \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\ &\quad \times e^{t(\alpha_0+\beta_0)} n^{-2\beta_1} \frac{\Gamma(1+\alpha_1+\beta_1)}{\Gamma(\alpha_1-\beta_1)} (1+\mathcal{O}(u)), \end{aligned} \quad (2.4.165)$$

$$\begin{aligned} A_{e^{-t}} &= \frac{e^{-t}}{n} (1-z_1^{-1} e^{-t})^{2\beta_1} (1-e^{-2t})^{\alpha_0+\beta_0} e^{V_0} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{-tk} \right\} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{tk} \right\} \\ &\quad \times e^{t(\alpha_0+\beta_0)} e^{-t(\alpha_1-\beta_1)} e^{-i\pi(\alpha_1-\beta_1)} z_1^{-(\alpha_1-\beta_1)} (2t)^{-(\alpha_0+\beta_0)} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0-\beta_0)} K(2nt) (1+\mathcal{O}(n^{-1})), \end{aligned} \quad (2.4.166)$$

$$\begin{aligned} A_{e^t} &= \frac{e^t}{n} (1-z_1 e^{-t})^{2\beta_1} (1-e^{-2t})^{-(\alpha_0-\beta_0)} e^{V_0} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{tk} \right\} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\} \\ &\quad \times e^{t(\alpha_0+\beta_0)} e^{t(\alpha_1+\beta_1)} e^{-i\pi(\alpha_1+\beta_1)} z_1^{-(\alpha_1+\beta_1)} (2t)^{\alpha_0-\beta_0} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0-\beta_0)} K(2nt) (1+\mathcal{O}(n^{-1})). \end{aligned} \quad (2.4.167)$$

Next we look at the R-H problem for $\tilde{R}_2(z)$:

$$(\tilde{R}_21) \quad \tilde{R}_2(z) : \mathbb{C} \setminus \partial U \rightarrow \mathbb{C}^{2 \times 2} \text{ is analytic.}$$

$$(\tilde{R}_22) \quad \tilde{R}_2(z) \text{ satisfies the following jump condition,}$$

$$\tilde{R}_{2,+}(z) = \tilde{R}_{2,-}(z) + \tilde{R}_{1,-}(z)n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_2(z)n^{-\omega\sigma_3} \quad \text{for } z \in \partial U.$$

$$(\tilde{R}_23) \quad \tilde{R}_2(z) \rightarrow 0, \quad \text{as } z \rightarrow \infty .$$

We again use the Plemelj formulae (Theorem 1.2.3), the solutions is given by evaluating the following integral,

$$\tilde{R}_2(z) = \frac{1}{2\pi i} \int_{\partial U} \left(\tilde{R}_{1,-}(z)n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_2(z)n^{-\omega\sigma_3} \right) \frac{dx}{x-z}. \quad (2.4.168)$$

By noting that each $\Delta_k(z) = \mathcal{O}(n^{2\max_j |\operatorname{Re} \beta_j| - k})$, just as in the paper by [14], and by the fact that we are using the same transformation also, we can conclude that our error term is in fact the same [14, Equations (4.66),(4.74)], we have that $\tilde{R}_{k+1}^{(r)}(z) = \mathcal{O}(|\tilde{R}_{k+1}(z)|) + \mathcal{O}(|\tilde{R}_{k+2}(z)|)$ and,

$$R_3^{(r)}(z) = \begin{pmatrix} \mathcal{O}(\delta/n) + \mathcal{O}(\delta^2) & \mathcal{O}\left(\delta \max_k \frac{n^{-2\operatorname{Re} \beta_k}}{n}\right) \\ \mathcal{O}\left(\delta \max_k \frac{n^{2\operatorname{Re} \beta_k}}{n}\right) & \mathcal{O}(\delta/n) + \mathcal{O}(\delta^2) \end{pmatrix}, \quad (2.4.169)$$

where

$$\delta = \max_{j,k} n^{2\operatorname{Re}(\beta_j - \beta_k - 1)}. \quad (2.4.170)$$

2.5 Asymptotics for the determinant if $\|\beta\| < 1$

To obtain the asymptotics of the Toeplitz determinant if $\|\beta\| < 1$, we now need to go through reverse transformations. We are only interested in the solution near $z = e^t$ and $z = e^{-t}$ as this is what the differential identity (2.3.1) calls for. Thus, in this section we only restrict ourselves to only those regions. In the Section 2.4.5 we have solved the final R-H problem, providing details for the first terms in the series $R(z) = I + \sum_{p=1}^k R_p(z) + R_{k+1}^{(r)}(z)$. These details will be utilised in finding asymptotics for the determinant if the seminorm $\|\beta\| = 1$. Here however, we only require that $\tilde{R}(z) = I + \mathcal{O}(n^{-1})$ as $n \rightarrow \infty$ uniformly for $z \in \mathbb{C} \setminus \Gamma$, which follows from the solution obtained in

Section 2.4.5. Following the transformations we obtain,

$$\begin{aligned}
Y(z) &= \begin{cases} T(z)z^{n\sigma_3}, & z = e^t, \\ T(z), & z = e^{-t}, \end{cases} \\
&= \begin{cases} S(z)z^{n\sigma_3}, & z = e^t, \\ S(z), & z = e^{-t}, \end{cases} \\
&= \begin{cases} R(z)P_{z_0}(z)z^{n\sigma_3}, & z = e^t, \\ R(z)P_{z_0}(z), & z = e^{-t}, \end{cases} \\
&= \begin{cases} n^{-\omega\sigma_3}\tilde{R}(z)n^{\omega\sigma_3}P_{z_0}(z)z^{n\sigma_3}, & z = e^t, \\ n^{-\omega\sigma_3}\tilde{R}(z)n^{\omega\sigma_3}P_{z_0}(z), & z = e^{-t}, \end{cases} \\
&= \begin{cases} n^{-\omega\sigma_3}(I + \mathcal{O}(n^{-1}))n^{\omega\sigma_3}P_{z_0}(z)z^{n\sigma_3}, & z = e^t, \\ n^{-\omega\sigma_3}(I + \mathcal{O}(n^{-1}))n^{\omega\sigma_3}P_{z_0}(z), & z = e^{-t}, \end{cases} \tag{2.5.1}
\end{aligned}$$

$$= \begin{cases} n^{-\omega\sigma_3}(I + \mathcal{O}(n^{-1}))n^{\omega\sigma_3}D(z)^{\sigma_3}W(z)^{-1}\Phi(z)W(z)z^{n\sigma_3}, & z = e^t, \\ n^{-\omega\sigma_3}(I + \mathcal{O}(n^{-1}))n^{\omega\sigma_3}D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W(z)^{-1}\Phi(z)W(z), & z = e^{-t}. \end{cases} \tag{2.5.2}$$

It follows from the R-H problem for $\tilde{R}(z)$ that the asymptotics for $Y(z)$ as $n \rightarrow \infty$ are uniform for $0 < t < t_0$ for a sufficiently small t_0 , as long as $2nt$ remains bounded away from the set of numbers $\{x_0, x_1, \dots, x_k\}$.

It now remains to substitute the asymptotics into the differential identity. We need to evaluate $Y^{-1}Y'_z$ at the points $z = e^t$ and $z = e^{-t}$. Differentiating the expressions in (2.5.1) we obtain,

$$Y^{-1}Y'_z = \begin{cases} \frac{n\sigma_3}{z} + z^{-n\sigma_3}P^{-1}P'_z z^{n\sigma_3} + \\ + z^{-n\sigma_3}P^{-1}(z)n^{-\omega\sigma_3}(I + \mathcal{O}(n^{-1}))^{-1}\mathcal{O}(n^{-1})'_z n^{\omega\sigma_3}P(z)z^{n\sigma_3}, & \text{near } e^t, \\ P^{-1}P'_z + P^{-1}(z)n^{-\omega\sigma_3}(I + \mathcal{O}(n^{-1}))^{-1}n^{\omega\sigma_3}P(z), & \text{near } e^{-t}. \end{cases} \tag{2.5.3}$$

Let us define $A(z)$ as follows,

$$A(z) = \begin{cases} G(\lambda(z))^{-1/2} z^{n/2} f(z)^{1/2}, & \text{for } |z| > 1, \\ -G(\lambda(z))^{-1/2} z^{n/2} f(z)^{-1/2}, & \text{for } |z| < 1, \end{cases} \quad (2.5.4)$$

and as such, we can write,

$$W(z) = \begin{cases} A(z)^{\sigma_3} \sigma_1, & \text{for } |z| > 1, \\ A(z)^{\sigma_3} \sigma_3, & \text{for } |z| < 1. \end{cases} \quad (2.5.5)$$

We now find, using (2.5.2),

$$P^{-1}P'_z = \begin{cases} -\sigma_3 \frac{A'_z}{A} + W^{-1}\Phi^{-1}\Phi'_z W - W^{-1}\Phi^{-1}\sigma_3\Phi W \left(\frac{A'_z}{A} + \frac{D'_z}{D} \right), & \text{near } e^t, \\ \sigma_3 \frac{A'_z}{A} + W^{-1}\Phi^{-1}\Phi'_z W - W^{-1}\Phi^{-1}\sigma_3\Phi W \left(\frac{A'_z}{A} + \frac{D'_z}{D} \right), & \text{near } e^{-t}. \end{cases} \quad (2.5.6)$$

$$\begin{aligned} \left(G^{-\frac{1}{2}}(\lambda(z), 2nt) \right)'_z &= [(\alpha_0 - \beta_0)(n \log(z) + nt)^{-1} - (\alpha_0 + \beta_0)(n \log(z) - nt)^{-1} - 1] \\ &\quad \times \frac{1}{2} \left(\frac{n}{z} \right) G(\lambda(z))^{-\frac{1}{2}} \end{aligned} \quad (2.5.7)$$

$$A'_z(z) = \left[\frac{1}{2}(\alpha_0 - \beta_0)(n \log(z) + nt)^{-1} - \frac{1}{2}(\alpha_0 + \beta_0)(n \log(z) - nt)^{-1} \right] \left(\frac{n}{z} \right) A(z) \quad (2.5.8)$$

$$+ \frac{1}{2} f'_z(z) f^{-1}(z) A(z) \quad (2.5.9)$$

$$f'_z(z) = V'(z)f(z) - (\alpha_1 - \beta_1)z^{-1}f(z) + 2\alpha_1(z - z_1)^{-1}f(z) \quad (2.5.10)$$

$$+ (\alpha_0 + \beta_0)(z - e^t)^{-1}f(z) + (\alpha_0 - \beta_0)(z - e^{-t})^{-1}f(z) + (-\alpha_0 + \beta_0)z^{-1}f(z) \quad (2.5.11)$$

We compute the derivative of $|z - z_1|^{2\alpha}$ near $e^{\pm t}$ using the function $h_{\alpha_j}(z)$ from (2.4.30). We have that $(h_{\alpha_j}^2)'_z = 2(h_{\alpha_j})'_z h_{\alpha_j}$, where

$$\begin{aligned} (h_{\alpha_j})'_z(z) &= \frac{(zz_j e^{il_j})^{\alpha_j/2} \alpha_j (z - z_j)^{\alpha_j - 1} - (z - z_j)^{\alpha_j} \left(\frac{\alpha_j}{2} \right) (z_j e^{il_j}) (zz_j e^{il_j})^{\alpha_j/2 - 1}}{(zz_j e^{il_j})^{\alpha_j}} \\ &= \alpha_j (z - z_j)^{-1} h_{\alpha_j}(z) - \frac{\alpha_j}{2} z^{-1} h_{\alpha_j}(z), \end{aligned} \quad (2.5.12)$$

and the derivative of $g_{\beta_1, z_1}(z)$ is 0 as it's a piecewise constant function. We proceed, from (2.5.8) and (2.5.12),

$$\frac{A'_z(z)}{A} = \begin{cases} \frac{\alpha_0 + \beta_0}{4} e^{-t} + \frac{\alpha_0 - \beta_0}{4} e^{-t} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \frac{1}{2} V'_z(e^t) - \frac{\alpha_1 - \beta_1}{2} e^{-t} + 2\alpha_1 (e^t - z_1)^{-1} & \text{near } e^t, \\ \frac{\alpha_0 + \beta_0}{4} e^t \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \frac{\alpha_0 - \beta_0}{4} e^t - \frac{1}{2} V'_z(e^{-t}) + \frac{\alpha_1 - \beta_1}{2} e^t - 2\alpha_1 (e^{-t} - z_1)^{-1} & \text{near } e^{-t}. \end{cases} \quad (2.5.13)$$

Differentiating (2.4.15) gives,

$$\frac{D'_z(z)}{D} = \begin{cases} (-\alpha_1 + \beta_1) \left(\frac{z_1 e^{-t}}{e^t - z_1} \right) - (\alpha_0 - \beta_0) \frac{e^{-2t}}{\sinh t} - \sum_{k=-\infty}^{-1} k V_k e^{t(k-1)} & \text{for } z = e^t, \\ (\alpha_1 + \beta_1) \frac{1}{e^{-t} - z_1} - (\alpha_0 + \beta_0) \frac{1}{2 \sinh t} + \sum_{k=0}^{\infty} k V_k e^{-t(k-1)} & \text{for } z = e^{-t}. \end{cases} \quad (2.5.14)$$

From (2.5.6) we obtain the 22 entry on the matrix,

$$\left(P^{-1} P'_z \right)_{22}(e^t) = \frac{A'_z(e^t)}{A} + \left(\Phi^{-1} \Phi'_z \right)_{11}(e^t) - \left(\Phi^{-1} \sigma_3 \Phi'_z \right)_{11}(e^t) \left[\frac{A'_z(e^t)}{A} + \frac{D'_z(e^t)}{D} \right],$$

and

$$\left(P^{-1} P'_z \right)_{22}(e^{-t}) = -\frac{A'_z(e^{-t})}{A} + \left(\Phi^{-1} \Phi'_z \right)_{22}(e^{-t}) - \left(\Phi^{-1} \sigma_3 \Phi'_z \right)_{22}(e^{-t}) \left[\frac{A'_z(e^{-t})}{A} + \frac{D'_z(e^{-t})}{D} \right].$$

By substituting the results from above in (2.5.6) we obtain,

$$\begin{aligned} e^t \left(P^{-1} P'_z \right)_{22}(e^t) &= \frac{\alpha_0 + \beta_0}{4} + \frac{\alpha_0 - \beta_0}{4} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \frac{1}{2} e^t V'_z(e^t) - \frac{\alpha_1 - \beta_1}{2} + \alpha_1 \frac{e^t}{(e^t - z_1)} \\ &+ e^t \left(\Phi^{-1} \Phi'_z \right)_{11}(e^t) - \left(\Phi^{-1} \sigma_3 \Phi'_z \right)_{11}(e^t) \times \left[\frac{\alpha_0 + \beta_0}{4} + \frac{\alpha_0 - \beta_0}{4} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) \right. \\ &\left. - (\alpha_1 - \beta_1) \left(\frac{e^t + z_1}{2(e^t - z_1)} \right) + \alpha_1 \frac{e^t}{e^t - z_1} + \frac{1}{2} e^t V'_z(e^t) - \sum_{-\infty}^{-1} k V_k e^{tk} \right], \end{aligned}$$

and,

$$\begin{aligned} e^{-t} \left(P^{-1} P'_z \right)_{22}(e^{-t}) &= -\frac{\alpha_0 + \beta_0}{4} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) - \frac{\alpha_0 - \beta_0}{4} + \frac{1}{2} e^{-t} V'_z(e^{-t}) - \frac{\alpha_1 - \beta_1}{2} \\ &+ \alpha_1 \frac{e^{-t}}{(e^{-t} - z_1)} + e^{-t} \left(\Phi^{-1} \Phi'_z \right)_{22}(e^{-t}) - \left(\Phi^{-1} \sigma_3 \Phi'_z \right)_{22}(e^{-t}) \times \left[\frac{\alpha_0 - \beta_0}{4} \right. \end{aligned}$$

$$\left. + \frac{\alpha_0 + \beta_0}{4} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) - \left(\frac{\alpha_1 - \beta_1}{2} \right) + \beta_1 \frac{e^{-t}}{e^{-t} - z_1} - \frac{1}{2} e^{-t} V'_z(e^{-t}) + \sum_{k=0}^{\infty} k V_k e^{-tk} \right].$$

Now, from (2.5.3),

$$\begin{aligned} e^t \left(Y^{-1} Y'_z \right)_{22} (e^t) &= -n + e^t (P^{-1} P'_z)_{22} (e^t) + (\hat{\Phi}^{-1}(t) \mathcal{O}(1/n) \hat{\Phi}(t))_{22} \\ &= -n + \frac{\alpha_0 + \beta_0}{4} + \frac{\alpha_0 - \beta_0}{4} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \frac{1}{2} e^t V'_z(e^t) - \frac{\alpha_1 - \beta_1}{2} + \alpha_1 \frac{e^t}{e^t - z_1} + e^t \left(\Phi^{-1} \Phi'_z \right)_{11} (e^t) \\ &\quad - \left\{ \frac{\alpha_0 + \beta_0}{4} + \frac{\alpha_0 - \beta_0}{4} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) - (\alpha_1 - \beta_1) \left(\frac{e^t + z_1}{2(e^t - z_1)} \right) + \alpha_1 \frac{e^t}{e^t - z_1} + \frac{1}{2} e^t V'_z(e^t) \right. \\ &\quad \left. - \sum_{k=-\infty}^{-1} k V_k e^{tk} \right\} \left(\Phi^{-1} \sigma_3 \Phi \right)_{11} (e^t) + (\hat{\Phi}^{-1}(t) \mathcal{O}(1/n) \hat{\Phi}(t))_{22} \end{aligned}$$

and

$$\begin{aligned} e^{-t} \left(Y^{-1} Y'_z \right)_{22} (e^{-t}) &= e^{-t} (P^{-1} P'_z)_{22} (e^{-t}) + (\hat{\Phi}^{-1}(t) \mathcal{O}(1/n) \hat{\Phi}(t))_{22} \\ &= -\frac{\alpha_0 - \beta_0}{4} - \left(\frac{\alpha_0 + \beta_0}{4} \right) \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \frac{1}{2} e^{-t} V'_z(e^{-t}) - \frac{\alpha_1 - \beta_1}{2} + \alpha_1 \frac{e^{-t}}{e^{-t} - z_1} \\ &\quad + e^{-t} \left(\Phi^{-1} \Phi'_z \right)_{22} (e^{-t}) - \left\{ \frac{\alpha_0 - \beta_0}{4} + \frac{\alpha_0 + \beta_0}{4} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) - \frac{\alpha_1 - \beta_1}{2} + \beta_1 \frac{e^{-t}}{e^{-t} - z_1} \right. \\ &\quad \left. - \frac{1}{2} e^{-t} V'_z(e^{-t}) + \sum_{k=0}^{\infty} k V_k e^{-tk} \right\} \left(\Phi^{-1} \sigma_3 \Phi \right) (e^{-t}) + (\hat{\Phi}^{-1}(t) \mathcal{O}(1/n) \hat{\Phi}(t))_{22}. \end{aligned}$$

Now recall the differential identity (2.3.1),

$$\begin{aligned} \frac{d}{dt} \log D_n(t) &= -(\alpha_0 + \beta_0) e^t \left(Y^{-1} Y'_z \right)_{22} (e^t) + (\alpha_0 - \beta_0) e^{-t} \left(Y^{-1} Y'_z \right)_{22} (e^{-t}) \\ &= (\alpha_0 + \beta_0) n - \frac{\alpha_0^2 + \beta_0^2}{2} - \frac{\alpha_0^2 - \beta_0^2}{2} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \beta_0 (\alpha_1 - \beta_1) \\ &\quad + \alpha_1 \left[-(\alpha_0 + \beta_0) \frac{e^t}{e^t - z_1} + (\alpha_0 - \beta_0) \frac{e^{-t}}{e^{-t} - z_1} \right] - \frac{\alpha_0 + \beta_0}{2} e^t V'_z(e^t) \\ &\quad + \frac{\alpha_0 - \beta_0}{2} e^{-t} V'_z(e^{-t}) + 2nw(x) + (\Phi^{-1} \sigma_3 \Phi'_z)_{11} (e^t) \times \left(\frac{\alpha_0 + \beta_0}{2} \right) \\ &\quad \times \left(\frac{\alpha_0 + \beta_0}{2} + \frac{\alpha_0 - \beta_0}{2} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) + (\alpha_1 - \beta_1) + \frac{\beta_1 e^t}{e^t - z_1} + \sum_{k=0}^{\infty} k V_k e^{tk} - \sum_{k=-\infty}^{-1} k V_k e^{tk} \right) \\ &\quad - (\Phi^{-1} \sigma_3 \Phi'_z)_{22} (e^{-t}) \times \left(\frac{\alpha_0 - \beta_0}{2} \right) \left(-\frac{\alpha_0 - \beta_0}{2} - \frac{\alpha_0 + \beta_0}{2} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) \right) \end{aligned}$$

$$-(\alpha_1 - \beta_1) + \frac{\beta_1 e^{-t}}{e^{-t} - z_1} + \sum_{k=0}^{\infty} k V_k e^{-tk} - \sum_{k=-\infty}^{-1} k V_k e^{-tk} \Big) + \hat{\Phi}^{-1} \mathcal{O}(1/n) \hat{\Phi}(t)$$

where we denoted $w(x)$ by

$$w(x) = \frac{\alpha_0 + \beta_0}{2} \left(\Phi^{-1} \Phi'_{\lambda} \right)_{11} (x/2) + \frac{\alpha_0 - \beta_0}{2} \left(\Phi^{-1} \Phi'_{\lambda} \right) (-x/2). \quad (2.5.15)$$

We now refer to the results that were obtained in [11] concerning the Painlevé V function $\sigma(x)$.

Proposition 2.5.1. [11, Proposition 4.4] *Set*

$$a(\zeta; x) = \left(\Psi(\zeta; x) \sigma_3 \Psi^{-1}(\zeta; x) \right)_{11} = - \left(\Psi(\zeta; x) \sigma_3 \Psi^{-1}(\zeta; x) \right)_{22} \quad (2.5.16)$$

Then we have the following identities,

$$\frac{\alpha_0 - \beta_0}{2} a(0; x) = A_{0,11} = -v(x) + \frac{\alpha_0 - \beta_0}{2}, \quad (2.5.17)$$

$$\frac{\alpha_0 + \beta_0}{2} a(1; x) = -A_{1,11} = -v(x) + \frac{\alpha_0 + \beta_0}{2}. \quad (2.5.18)$$

It is also worth nothing that $[\Psi(\zeta; x) \sigma_3 \Psi^{-1}(\zeta; x)]_{diag} = [\Phi(\zeta; x) \sigma_3 \Phi^{-1}(\zeta; x)]_{diag}$.

Proposition 2.5.2. [11, Proposition 4.5] *If we have $w(x)$ given by (2.5.15) then,*

$$v(x) = -(xw(x))' \quad (2.5.19)$$

$$\sigma(x) = xw(x) \quad (2.5.20)$$

$$\sigma(x) = \int_x^{+\infty} v(\xi) d\xi \quad (2.5.21)$$

Now, using the fact that $(\Phi(\zeta; x) \sigma_3 \Phi^{-1})_{11}(e^t) = a(1; x)$ and $(\Phi(\zeta; x) \sigma_3 \Phi^{-1})_{22}(e^t) = -a(0; x)$ we obtain the following,

$$\begin{aligned} \frac{d}{dt} \log D_n(t) &= (\alpha_0 + \beta_0)n - \frac{\alpha_0^2 + \beta_0^2}{2} - \frac{\alpha_0^2 - \beta_0^2}{2} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \beta_0(\alpha_1 - \beta_1) \\ &+ \alpha_1 \left(-(\alpha_0 + \beta_0) \frac{e^t}{e^t - z_1} + (\alpha_0 - \beta_0) \frac{e^{-t}}{e^{-t} - z_1} \right) - \frac{\alpha_0 + \beta_0}{2} e^t V'_z(e^t) + \frac{\alpha_0 - \beta_0}{2} e^{-t} V'_z(e^t) \\ &+ \frac{1}{t} \sigma(x) + \left\{ \frac{\alpha_0 + \beta_0}{2} + \frac{\alpha_0 - \beta_0}{2} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) + (\alpha_1 - \beta_1) + 2\beta_1 \frac{e^t}{e^t - z_1} \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} k \left(V_k e^{tk} + V_{-k} e^{-tk} \right) \left\{ -v(x) + \frac{\alpha_0 + \beta_0}{2} \right\} + \left\{ \frac{\alpha_0 - \beta_0}{2} + \frac{\alpha_0 + \beta_0}{2} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) \right. \\
& + (\alpha_1 - \beta_1) + 2\beta_1 \frac{e^{-t}}{e^{-t} - z_1} + \sum_{k=1}^{\infty} k \left(V_k e^{-tk} + V_{-k} e^{tk} \right) \left. \right\} \left(-v(x) + \frac{\alpha_0 - \beta_0}{2} \right) \\
& = (\alpha_0 + \beta_0)n - \frac{\alpha_0^2 + \beta_0^2}{2} - \frac{\alpha_0^2 - \beta_0^2}{2} \left(\frac{1}{t} + \frac{e^{-t}}{\sinh t} \right) + \beta_0(\alpha_1 - \beta_1) \\
& + \alpha_1 \left(-(\alpha_0 + \beta_0) \frac{e^t}{e^t - z_1} + (\alpha_0 - \beta_0) \frac{e^{-t}}{e^{-t} - z_1} \right) - \frac{\alpha_0 + \beta_0}{2} \sum_{k=-\infty}^{\infty} k V_k e^{tk} + \frac{\alpha_0 - \beta_0}{2} \sum_{k=-\infty}^{\infty} k V_k e^{-tk} \\
& + \frac{1}{t} \sigma(x) - v(x) \left\{ \frac{\alpha_0 + \beta_0}{2} + \frac{\alpha_0 - \beta_0}{2} + \left(\frac{\alpha_0 - \beta_0}{2} + \frac{\alpha_0 + \beta_0}{2} \right) \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) \right. \\
& + (\alpha_1 - \beta_1) - (\alpha_1 - \beta_1) + 2\beta_1 \left(\frac{e^t}{e^t - z_1} + \frac{e^{-t}}{e^{-t} - z_1} \right) + \sum_{k=1}^{\infty} k \left(V_k (e^{tk} + e^{-tk}) + V_{-k} (e^{tk} + e^{-tk}) \right) \left. \right\} \\
& + \left\{ \frac{\alpha_0^2 + \beta_0^2}{4} + \frac{2\alpha_0\beta_0}{4} + \frac{\alpha_0^2 - \beta_0^2}{4} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) \right. \\
& + \frac{(\alpha_1 - \beta_1)(\alpha_0 + \beta_0)}{2} + \beta_1(\alpha_0 + \beta_0) \frac{e^t}{e^t - z_1} + \frac{\alpha_0 + \beta_0}{2} \sum_{k=1}^{\infty} k \left(V_k e^{tk} + V_{-k} e^{-tk} \right) \left. \right\} \\
& + \left\{ \frac{\alpha_0^2 + \beta_0^2}{4} - \frac{2\alpha_0\beta_0}{4} + \frac{\alpha_0^2 - \beta_0^2}{4} \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) \right. \\
& + \frac{(\alpha_1 - \beta_1)(\alpha_0 - \beta_0)}{2} + \beta_1(\alpha_0 - \beta_0) \frac{e^{-t}}{e^{-t} - z_1} + \frac{\alpha_0 - \beta_0}{2} \sum_{k=1}^{\infty} k \left(V_k e^{-tk} + V_{-k} e^{tk} \right) \left. \right\} \\
& = (\alpha_0 + \beta_0)n - (\alpha_0^2 - \beta_0^2) \frac{e^{-t}}{\sinh t} + (\alpha_0 + \beta_0)(\alpha_1 - \beta_1) - (\alpha_1 - \beta_1)(\alpha_0 + \beta_0) \frac{e^t}{e^t - z_1} \\
& + (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \frac{e^{-t}}{e^{-t} - z_1} + (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} k V_{-k} e^{-tk} + (\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k e^{-tk} + \frac{1}{t} \sigma(x) \\
& - v(x) \left\{ \alpha_0 + \alpha_0 \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) + 2\beta_1 \left(\frac{e^t}{e^t - z_1} + \frac{e^{-t}}{e^{-t} - z_1} \right) + 2 \sum_{k=1}^{\infty} k \cosh(kt) (V_k + V_{-k}) \right\} \\
& + \hat{\Phi}^{-1} \mathcal{O}(1/n) \hat{\Phi}(t).
\end{aligned}$$

We proceed to integrating the expression above. Let us mention beforehand the following integral, which we will use and the form in which we will use them,

$$\int_{\varepsilon}^t \frac{e^{\tau}}{e^{\tau} - z_1} d\tau = \log(1 - z_1 e^{-t}) - \log(1 - z_1 e^{-\varepsilon}) + t - \varepsilon, \quad (2.5.22)$$

$$\int_{\varepsilon}^t \frac{e^{-\tau}}{e^{-\tau} - z_1} d\tau = -\log\left(\frac{z_1}{e^{t\varepsilon}}\right) - \log(1 - z_1^{-1} e^{-t}) + \log(e^{-\varepsilon} - z_1). \quad (2.5.23)$$

We integrate from $\varepsilon > 0$ to some t , where $0 < t < t_0$, and $0 < \varepsilon < t$. Noting that $\int_\varepsilon^t \frac{d}{dt} \log D_n(\tau) d\tau = \log D_n(t) - \log D_n(\varepsilon)$, we obtain,

$$\begin{aligned}
\log D_n(t) &= \log D_n(\varepsilon) + (\alpha_0 + \beta_0)n(t - \varepsilon) - (\alpha_0^2 - \beta_0^2) (\log(1 - e^{-2t}) - \log(1 - e^{-2\varepsilon})) \\
&+ (\alpha_0 + \beta_0)(\alpha_1 - \beta_1)(t - \varepsilon) - (\alpha_1 - \beta_1)(\alpha_0 + \beta_0) (\log(1 - z_1 e^{-t}) - \log(1 - z_1 e^{-\varepsilon}) + t - \varepsilon) \\
&+ (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \left(-\log\left(\frac{z_1}{e^{i\pi}}\right) - \log(1 - z_1^{-1} e^{-t}) + \log(e^{-\varepsilon} - z_1) \right) \\
&+ (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} k \left(\frac{V_{-k} e^{-tk}}{-k} \right) + (\alpha_0 - \beta_0) \sum_{k=1}^{\infty} k \left(\frac{V_k e^{-tk}}{-k} \right) + (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} e^{-\varepsilon k} \\
&+ (\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k e^{-\varepsilon k} + \int_{2n\varepsilon}^{2nt} \frac{\sigma(x)}{x} dx + R_n(t) + \mathcal{O}(1/n) \\
&= \log D_n(\varepsilon) + (\alpha_0 + \beta_0)n(t - \varepsilon) + \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \\
&- \sum_{k=1}^{\infty} k V_k V_{-k} + (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} e^{-\varepsilon k} + (\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k e^{-\varepsilon k} \\
&+ (\alpha_0 + \beta_0 - \alpha_0 - \beta_0)(\alpha_1 - \beta_1)(t - \varepsilon) - (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \log\left(\frac{z_1}{e^{i\pi}}\right) \\
&+ (\alpha_1 - \beta_1)(\alpha_0 + \beta_0) \left(\sum_{k=1}^{\infty} \frac{z_1^k e^{-tk}}{k} \right) + (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \left(\sum_{k=1}^{\infty} \frac{z_1^{-k} e^{-tk}}{k} \right) \\
&+ (\alpha_1 - \beta_1)(\alpha_0 + \beta_0) \log(1 - z_1 e^{-\varepsilon}) + (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \log(e^{-\varepsilon} - z_1) \\
&+ \left[\int_{2n\varepsilon}^{2nt} \frac{\sigma(x) - (\alpha_0^2 - \beta_0^2)}{x} dx + (\alpha_0^2 - \beta_0^2) \log(2nt) + (\alpha_0^2 - \beta_0^2) \log\left(\frac{n(1 - e^{-2\varepsilon})}{2n\varepsilon}\right) \right] \\
&- (\alpha_0^2 - \beta_0^2) \log n + R_n(t) + \mathcal{O}(1/n),
\end{aligned}$$

where

$$\begin{aligned}
R_n(t) &= - \int_\varepsilon^t v(2nt) \left\{ \alpha_0 + \alpha_0 \left(\frac{1}{t} - \frac{e^{-t}}{\sinh t} \right) + 2\beta_1 \left(\frac{e^t}{e^t - z_1} + \frac{e^{-t}}{e^{-t} - z_1} \right) \right. \\
&\quad \left. + 2 \sum_{k=1}^{\infty} k \cosh(kt) (V_k + V_{-k}) \right\}, \tag{2.5.24}
\end{aligned}$$

and just as in [11, (5.3)] we have,

$$|R_n(t)| < C \int_0^t |v(2nu)| du = \mathcal{O}(1/n), \quad \text{as } n \rightarrow \infty, \quad 0 < t < t_0, \tag{2.5.25}$$

and we used the following,

$$-(\alpha_0^2 - \beta_0^2) \log(1 - e^{-2t}) = (\alpha_0^2 + \beta_0^2) \sum_{k=1}^{\infty} \frac{e^{-2tk}}{k} = \sum_{k=1}^{\infty} k \left[(\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \left[(\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right]. \quad (2.5.26)$$

We now take the limit as $\varepsilon \rightarrow 0$. We note that $\lim_{\varepsilon \rightarrow 0} D_n(\varepsilon)$ is the expression for the determinant with two Fisher-Hartwig singularities. It follows that, using the L'Hopital's rule for the limit of $\log\left(\frac{n(1-e^{-2\varepsilon})}{2n\varepsilon}\right)$,

$$\begin{aligned} \log D_n(t) &= nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} - (\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k - (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} - (\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^{-k} \\ &\quad + (\alpha_0^2 - \beta_0^2) \log n + (\alpha_1^2 - \beta_1^2) \log n + 2(\beta_0\beta_1 - \alpha_0\alpha_1) \log(1 - z_1) \\ &\quad + (\alpha_0\alpha_1 - \beta_0\beta_1)(\log z_1 + \log e^{i\pi}) + (\alpha_0\beta_1 - \alpha_1\beta_0) \log\left(\frac{z_1}{e^{i\pi}}\right) + \log G_{\alpha_0+\beta_0, \alpha_0-\beta_0} \\ &\quad + \log G_{\alpha_1+\beta_1, \alpha_1-\beta_1} + (\alpha_0 + \beta_0)nt + \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \\ &\quad - \sum_{k=1}^{\infty} kV_kV_{-k} + (\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} + (\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k - (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \log\left(\frac{z_1}{e^{i\pi}}\right) \\ &\quad + (\alpha_1 - \beta_1)(\alpha_0 + \beta_0) \left(\sum_{k=1}^{\infty} \frac{z_1^k e^{-tk}}{k} \right) + (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \left(\sum_{k=1}^{\infty} \frac{z_1^{-k} e^{-tk}}{k} \right) \\ &\quad + (\alpha_1 - \beta_1)(\alpha_0 + \beta_0) \log(1 - z_1) + (\alpha_1 + \beta_1)(\alpha_0 - \beta_0) \log(1 - z_1) \\ &\quad + \left[\int_0^{2nt} \frac{\sigma(x) - (\alpha_0^2 - \beta_0^2)}{x} dx + (\alpha_0^2 - \beta_0^2) \log(2nt) \right] - (\alpha_0^2 - \beta_0^2) \log n + \mathcal{O}(1/n). \end{aligned}$$

After cancellations, and recalling that we take the branch of $\log z$ to be the negative real line, we are left with the following expression,

$$\begin{aligned} \log D_n(t) &= nV_0 + nt(\alpha_0 + \beta_0) + \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \\ &\quad - (\alpha_1 - \beta_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right) z_1^k \right] - (\alpha_1 + \beta_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \\ &\quad + (\alpha_1^2 - \beta_1^2) \log n + \log G_{\alpha_0+\beta_0, \alpha_0-\beta_0} + \log G_{\alpha_1+\beta_1, \alpha_1-\beta_1} \\ &\quad + \left[\int_0^{2nt} \frac{\sigma(x) - (\alpha_0^2 - \beta_0^2)}{x} dx + (\alpha_0^2 - \beta_0^2) \log(2nt) \right] + \mathcal{O}(1/n). \quad (2.5.27) \end{aligned}$$

Thus we have obtained an expression for the asymptotics of the Toeplitz determinant from Sec-

tion 2.2, uniform for $0 < t < t_0$, for t_0 sufficiently small.

2.6 Asymptotics for the determinant if $\|\beta\| = 1$

In this section we consider the asymptotics of the Toeplitz determinant with the symbol given by (2.2.1) when the seminorm (see (1.4.10)) is equal to 1. This means that that we can write $\operatorname{Re} \beta_0 = q - 1/2$ and $\operatorname{Re} \beta_1 = q + 1/2$, for some $q \in \mathbb{R}$. We also assumed here without any loss in generality that $\operatorname{Re} \beta_0 < \operatorname{Re} \beta_1$. In this section we will make use of a lemma and the approach that was presented in [14] to prove the Tracy-Basor conjecture (Theorem 1.4.6). We only give the particular case of this lemma here. However, note well that the symbol (2.2.1) has only one F-H singularity for $t > 0$, but two β parameters. If we translate β 's we will not get a F-H representation as it was presented in Definition 1.4.4. The symbol we obtain by shifting β 's will vary by more than just a multiplicative constant,

$$\begin{aligned}
f(z; t) &= e^{V(z)} z^{\beta_1} |z - z_1|^{2\alpha_1} g_{z_1, \beta_1}(z) z_1^{-\beta_1} \\
&\quad \times (z - e^t)^{\alpha_0 + \beta_0} (z - e^{-t})^{\alpha_0 - \beta_0} z^{-\alpha_0 + \beta_0} e^{-i\pi(\alpha_0 + \beta_0)}, \quad (z \in \mathbb{T}). \\
&= e^{V(z)} z^{(\beta_1 - 1) + 1} |z - z_1|^{2\alpha_1} g_{z_1, (\beta_1 - 1) + 1}(z) z_1^{-(\beta_1 - 1) - 1} \\
&\quad \times (z - e^t)^{\alpha_0 + (\beta_0 + 1) - 1} (z - e^{-t})^{\alpha_0 - (\beta_0 + 1) + 1} z^{-\alpha_0 + (\beta_0 + 1) - 1} e^{-i\pi(\alpha_0 + (\beta_0 + 1) - 1)} \\
&= \frac{(z - e^{-t})}{(z - e^t)} z_1^{-1} \tilde{f}(z; t)
\end{aligned} \tag{2.6.1}$$

Where $\tilde{f}(z; t)$ is (2.2.1) with β_0 and β_1 replaced by $\tilde{\beta}_0 = \beta_0 + 1$ and $\tilde{\beta}_1 = \beta_1 - 1$ respectively. Notice that shifting β_1 , which is associated with the F-H singularity still produced the multiplicative constant we know. Following the idea of the proof that was considered in [14, Section 6], we will define a new symbol $\hat{f}(z; t)$, which is given by (2.2.1) but whose β -parameters are replaced by $\hat{\beta}_j$'s where:

- $\hat{\beta}_0 = \beta_0$,
- $\hat{\beta}_1 = \beta_1 - 1$.

For this new symbol $\hat{f}(z; t)$, we have $\|\beta\| < 1$ and we have computed the asymptotics of the corresponding Toeplitz matrix, which are given by (2.5.27)—with the β_j parameters replaced by $\hat{\beta}_j$. We will simply try to relate the two symbols, the original symbol $f(z; t)$ with $\|\beta\| = 1$ and $\hat{f}(z; t)$,

and make use of the asymptotics we already know for $\hat{f}(z; t)$ in order to compute the asymptotics for $f(z; t)$. We can obtain the original symbol by shifting $\hat{\beta}_1$ in $\hat{f}(z; t)$ back by +1, this is what we would call trivial a F-H 'representation'. Alternatively, we can shift β_0 by +1 to obtain the 'representation' corresponding to $\tilde{\beta}_j$'s from (2.6.1) above. We thus have,

$$f(z; t) = (-1)z_1^{-1}z\hat{f}(z; t), \quad (2.6.2)$$

and

$$\tilde{f}(z; t) = (-1)\frac{z(z - e^t)}{(z - e^{-t})}\hat{f}(z; t). \quad (2.6.3)$$

It is sufficient for us to consider only one of the above relations, we pick (2.6.2) and make use of the following lemma.

Lemma 2.6.1. [14, Lemma 2.4] *Let the Toeplitz determinants $D_n(f)$ with symbol $f(z)$ be non-zero for all $n \geq N_0$ with a fixed $N_0 \geq 0$. If $\phi_k(0) \neq 0$, $k = N_0, N_0 + 1, \dots, n - 1$, we have*

$$D_n(zf(z)) = (-1)^n \frac{\phi_n(0)}{\chi_n} D_n(f(z)), \quad n \geq N_0, \quad (2.6.4)$$

where χ_n is the leading coefficient of the polynomial $\phi(z)$, see (1.6.1).

The proof uses Christoffel's formula [43, Theorem 2.5] to represent new orthogonal polynomials, say $q_n(z)$, orthogonal with respect to some weight $\rho(z)f(z)$ (where $\rho(z)$ is a polynomial), in terms of polynomials $\phi_n(z)$, which are orthogonal with respect to the weight $f(z)$. Using orthogonality conditions (1.6.1) and relating the leading coefficients χ_n , via (1.6.6) one can link the Toeplitz determinants with the weights that vary by a polynomial $\rho(z)$.

Thus, using (2.6.3) and (2.6.4), we can express the Toeplitz determinant with $\|\beta\| = 1$ using the uniform asymptotics we computed in Section 2.5 and asymptotics of the polynomials orthogonal with respect to $\hat{f}(z; t)$,

$$\begin{aligned} D_n(f) &= D_n((-1)z_1^{-1}z\hat{f}) \\ &= (-1)^n z_1^{-n} D_n(z\hat{f}) \\ &= (-1)^{2n} z_1^{-n} \frac{\phi_n(0)}{\chi_n} D_n(\hat{f}) \\ &= z_1^{-n} \frac{\phi_n(0)}{\chi_n} D_n(\hat{f}) \end{aligned} \quad (2.6.5)$$

Remark 2.6.2. Note that choosing (2.6.3) instead of (2.6.2) should produce the same end result. The reason it was not chosen is partly because of convenience of having Lemma 2.6.1 already, and partly because I could not find an expression for polynomials with respect to the weight $\rho(z)f(z)$, where $\rho(z) = (z - a)(z - b)^{-1}$. In fact, even $(z - b)^{-1}f(z)$ posed an obstacle which I could not overcome. I feel fairly confident that it is plausible however to find an expression relating these orthogonal polynomials—that is, if it does not already exist somewhere within the Orthogonal Polynomial community.

2.6.1 Asymptotics for the Orthogonal Polynomials

Lemma 2.6.3. *Let $t > 0$ and $n \in \mathbb{N}$. Suppose that the R-H problem for $Y(z; n, t)$ in Section 1.6.2 is solvable with $\hat{f}(z)$ given by (2.2.1) by replacing β -parameters by $\hat{\beta}$, $\|\hat{\beta}\| < 1$ and $\alpha_j \pm \hat{\beta}_j \neq -1, -2, \dots$, $j = 0, 1$. Let $\phi(z)$ and $\hat{\phi}(z)$ be the OPs associated to the weight $\hat{f}(z)$ (see Section 1.6). Then as $n \rightarrow \infty$,*

$$\begin{aligned} \frac{\phi_n(0)}{\chi_n} = & \left[z_1^n (1 - z_1 e^{-t})^{(\alpha_0 + \hat{\beta}_0)} (1 - e^{-t} z_1^{-1})^{-(\alpha_0 - \hat{\beta}_0)} \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k z_1^k \right\}}{\exp \left\{ \sum_{k=1}^{\infty} V_{-k} z_k^{-k} \right\}} \right. \\ & \times n^{-2\hat{\beta}_1 - 1} \frac{\Gamma(1 + \alpha_1 + \hat{\beta}_1)}{\Gamma(\alpha_1 - \hat{\beta}_1)} \\ & + (1 - z_1 e^{-t})^{2\hat{\beta}_1} (1 - e^{-2t})^{-(\alpha_0 - \hat{\beta}_0)} \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k e^{tk} \right\}}{\exp \left\{ \sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\}} e^{t(\alpha_1 + \hat{\beta}_1)} \\ & \times e^{-i\pi(\alpha_1 + \hat{\beta}_1)} z_1^{-(\alpha_1 + \hat{\beta}_1)} (2t)^{\alpha_0 - \hat{\beta}_0} \frac{n^{-2\hat{\beta}_0 - 1}}{\Gamma(\alpha_0 - \hat{\beta}_0)} K(2nt) \\ & + (1 - z_1^{-1} e^{-t})^{2\hat{\beta}_1} (1 - e^{-2t})^{\alpha_0 + \hat{\beta}_0} \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k e^{-tk} \right\}}{\exp \left\{ \sum_{k=1}^{\infty} V_{-k} e^{tk} \right\}} e^{-t(\alpha_1 - \hat{\beta}_1)} \\ & \left. \times e^{-i\pi(\alpha_1 - \hat{\beta}_1)} z_1^{-(\alpha_1 - \hat{\beta}_1)} (2t)^{-(\alpha_0 + \hat{\beta}_0)} \frac{n^{-2\hat{\beta}_0 - 1}}{\Gamma(\alpha_0 - \hat{\beta}_0)} K(2nt) \right] (1 + o(1)). \end{aligned} \quad (2.6.6)$$

Proof. We find the asymptotics for the orthogonal polynomials to be used in (2.6.5). We recall the matrix-valued function which is the solution of the R-H problem for the polynomials orthogonal on the unit circle with respect to the weight $\hat{f}(z)$ (1.6.10) and note that,

$$Y_{11}(0) = \frac{\phi_n(0)}{\chi_n}, \quad (2.6.7)$$

where $Y_{11}(0)$ is the 11 element of the solution matrix evaluated at 0.

Now, going through the transformations $R(z) \mapsto S(z) \mapsto T(z) \mapsto Y(z)$, and using the Neumann series solution to $R(z)$, we obtain,

$$Y(z) = \left[I + R_1(z) + R_2(z) + R_3^{(r)}(z) \right] D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.6.8)$$

which in turn leads to,

$$\frac{\phi_n(0)}{\chi_n} = Y_{11}(0) = -D(0)^{-1} \left[R_{1,12}(0) + R_{2,12}(0) + \mathcal{O} \left(\delta \max_k \frac{n^{-2\operatorname{Re} \beta_k}}{n} \right) \right], \quad (2.6.9)$$

where we used (2.4.169) and δ is given by (2.4.170). We have by (2.4.13) or (2.4.15),

$$D(0) = e^{V_0} e^{t(\alpha_0 + \hat{\beta}_0)}. \quad (2.6.10)$$

Now by (2.4.162),

$$R_1(0) = -\frac{A_1}{z_1} - \frac{A_e t}{e^t} - \frac{A_{e^{-t}}}{e^{-t}}, \quad (2.6.11)$$

and by (2.4.163), (2.4.164) and (2.4.165), (2.4.166), (2.4.167),

$$\begin{aligned} R_{1,12}(0) = & e^{V_0} e^{t(\alpha_0 + \hat{\beta}_0)} \left[-\frac{1}{n} z_1^n (1 - z_1 e^{-t})^{(\alpha_0 + \hat{\beta}_0)} (1 - e^{-t} z_1^{-1})^{-(\alpha_0 - \hat{\beta}_0)} \right. \\ & \times \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k z_1^k \right\}}{\exp \left\{ -\sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\}} n^{-2\hat{\beta}_1} \frac{\Gamma(1 + \alpha_1 + \hat{\beta}_1)}{\Gamma(\alpha_1 - \hat{\beta}_1)} (1 + o(1)) \\ & - \frac{1}{n} (1 - z_1 e^{-t})^{2\hat{\beta}_1} (1 - e^{-2t})^{-(\alpha_0 - \hat{\beta}_0)} \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k e^{tk} \right\}}{\exp \left\{ -\sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\}} e^{t(\alpha_1 + \hat{\beta}_1)} \\ & \times e^{-i\pi(\alpha_1 + \hat{\beta}_1)} z_1^{-(\alpha_1 + \hat{\beta}_1)} (2t)^{\alpha_0 - \hat{\beta}_0} \frac{n^{-2\hat{\beta}_0}}{\Gamma(\alpha_0 - \hat{\beta}_0)} K(2nt) (1 + o(1)) \\ & - \frac{1}{n} (1 - z_1^{-1} e^{-t})^{2\hat{\beta}_1} (1 - e^{-2t})^{\alpha_0 + \hat{\beta}_0} \frac{\exp \left\{ \sum_{k=1}^{\infty} V_k e^{-tk} \right\}}{\exp \left\{ -\sum_{k=1}^{\infty} V_{-k} e^{tk} \right\}} e^{-t(\alpha_1 - \hat{\beta}_1)} \\ & \left. \times e^{-i\pi(\alpha_1 - \hat{\beta}_1)} z_1^{-(\alpha_1 - \hat{\beta}_1)} (2t)^{-(\alpha_0 + \hat{\beta}_0)} \frac{n^{-2\hat{\beta}_0}}{\Gamma(\alpha_0 - \hat{\beta}_0)} K(2nt) (1 + o(1)) \right]. \quad (2.6.12) \end{aligned}$$

We thus obtain the asymptotics in (2.6.6). \square

2.6.2 Asymptotics for the determinant

We now use the relation between two Toeplitz determinants we established in (2.6.5). We again use (2.4.18) to express the terms of the form $\exp \{ \log(1-z)^{\alpha \pm \beta} \} = \exp \left\{ -(\alpha \pm \beta) \sum_{k=1}^{\infty} \frac{z^k}{k} \right\}$ for $|z| < 1$ —which is the case for all such terms in (2.6.6). Furthermore, we note that by the properties of the Barnes G -function (see (1.4.8) and paragraph below, $G(z+1) = \Gamma(z)G(z)$) and by how we defined the shorthand (1.4.13) we have that,

$$\begin{aligned} G_{\alpha_j + \beta_j + 1, \alpha_j - \beta_j - 1} &= \frac{G(1 + \alpha_j + \beta_j + 1)G(1 + \alpha_j - \beta_j - 1)}{G(1 + 2\alpha_j)} \\ &= \frac{\Gamma(1 + \alpha_j + \beta_j)}{\Gamma(\alpha_j - \beta_j)} G_{\alpha_j + \beta_j, \alpha_j - \beta_j}. \end{aligned} \quad (2.6.13)$$

We also denote by,

$$\tilde{\Omega}(2nt) = \exp \{ \Omega(2nt) \} = \exp \left\{ \int_0^{2nt} \frac{\sigma(x) - (\alpha_0^2 - \beta_0^2)}{x} dx + (\alpha_0^2 - \beta_0^2) \log(2nt) \right\}, \quad (2.6.14)$$

in (2.5.27) ($\Omega(2nt)$ is also the same as in (1.5.6)) and it should be noted that it sees no shift in $\beta_0 = \hat{\beta}_0$.

As many terms come together in this computation, to make this somewhat easier to see, refer to the Table 2.1 below, where matching terms are colour-coded. We obtain from (2.6.5) and (2.6.6),

$$\begin{aligned} D_n(f) &= \exp \left\{ nV_0 + nt(\alpha_0 + \hat{\beta}_0) \right\} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \right\} \\ &\times \exp \left\{ -(\alpha_1 - \hat{\beta}_1 - 1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} \\ &\times \exp \left\{ -(\alpha_1 + \hat{\beta}_1 + 1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} n^{(\alpha_1^2 - \hat{\beta}_1^2 - 2\hat{\beta}_1 - 1)} \\ &\times G_{\alpha_0 + \hat{\beta}_0, \alpha_0 - \hat{\beta}_0} G_{\alpha_1 + \hat{\beta}_1 + 1, \alpha_1 - \hat{\beta}_1 - 1} \tilde{\Omega}(2nt) (1 + o(1)) \\ &+ \exp \left\{ nV_0 + nt(\alpha_0 + \hat{\beta}_0) \right\} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \hat{\beta}_0 + 1) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \right\} \\ &\times \exp \left\{ -(\alpha_1 - \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \hat{\beta}_0 + 1) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} (1 - e^t z_1^{-1})^{\alpha_1 + \hat{\beta}_1} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{tk} \right\} \\ &\times \exp \left\{ -(\alpha_1 + \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} n^{(\alpha_1^2 - \hat{\beta}_1^2)} n^{-2\beta_0 - 1} \end{aligned}$$

$$\begin{aligned}
& \times z_1^{-n} (2t)^{\alpha_0 - \beta_0} \frac{G_{\alpha_0 + \hat{\beta}_0 + 1, \alpha_0 - \hat{\beta}_0 - 1}}{\Gamma(1 + \alpha_0 + \hat{\beta}_0)} G_{\alpha_1 + \hat{\beta}_1, \alpha_1 - \hat{\beta}_1} \tilde{\Omega}(2nt) K(2nt) (1 + o(1)) \\
& + \exp \left\{ nV_0 + nt(\alpha_0 + \hat{\beta}_0) \right\} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \hat{\beta}_0 - 1) \frac{e^{-tk}}{k} \right] \right\} \\
& \times \exp \left\{ -(\alpha_1 - \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} (1 - e^t z_1)^{-(\alpha_1 - \hat{\beta}_1)} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{tk} \right\} \\
& \times \exp \left\{ -(\alpha_1 + \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \hat{\beta}_0 - 1) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} n^{(\alpha_1^2 - \hat{\beta}_1^2)} n^{-2\beta_0 - 1} \\
& \times z_1^{-n} (2t)^{-(\alpha_0 + \beta_0)} \frac{G_{\alpha_0 + \hat{\beta}_0 + 1, \alpha_0 - \hat{\beta}_0 - 1}}{\Gamma(1 + \alpha_0 + \hat{\beta}_0)} G_{\alpha_1 + \hat{\beta}_1, \alpha_1 - \hat{\beta}_1} \tilde{\Omega}(2nt) K(2nt) (1 + o(1)). \tag{2.6.15}
\end{aligned}$$

Recalling that $\hat{\beta}_0 = \beta_0$ and $\hat{\beta}_1 = \beta_1 - 1$, and using $\tilde{\beta}_0 = \beta_0 + 1$, $\tilde{\beta}_1 = \beta_1 - 1$ from (2.6.1)—a ‘would-be non-trivial F-H representation’ gives the answer,

$$\begin{aligned}
D_n(f) &= \exp \{ nV_0 + nt(\alpha_0 + \beta_0) \} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \right\} \\
& \times \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} \\
& \times \exp \left\{ -(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} n^{(\alpha_1^2 - \beta_1^2)} \\
& \times G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} \tilde{\Omega}(2nt) (1 + o(1)) \\
& + \exp \{ nV_0 + nt(\alpha_0 + \beta_0) \} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \tilde{\beta}_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \right\} \\
& \times \exp \left\{ -(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \tilde{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} (1 - e^t z_1^{-1})^{\alpha_1 + \tilde{\beta}_1} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{tk} \right\} \\
& \times \exp \left\{ -(\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} n^{(\alpha_1^2 - \tilde{\beta}_1^2)} n^{-2\beta_0 - 1} \\
& \times z_1^{-n} (2t)^{\alpha_0 - \beta_0} \frac{G_{\alpha_0 + \tilde{\beta}_0, \alpha_0 - \tilde{\beta}_0}}{\Gamma(1 + \alpha_0 + \beta_0)} G_{\alpha_1 + \tilde{\beta}_1, \alpha_1 - \tilde{\beta}_1} \tilde{\Omega}(2nt) K(2nt) (1 + o(1)) \\
& + \exp \{ nV_0 + nt(\alpha_0 + \beta_0) \} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \tilde{\beta}_0) \frac{e^{-tk}}{k} \right] \right\} \\
& \times \exp \left\{ -(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} (1 - e^t z_1)^{-(\alpha_1 - \tilde{\beta}_1)} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{tk} \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \exp \left\{ -(\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \tilde{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} n^{(\alpha_1^2 - \tilde{\beta}_1^2)} n^{-2\beta_0 - 1} \\
& \times z_1^{-n} (2t)^{-(\alpha_0 + \beta_0)} \frac{G_{\alpha_0 + \tilde{\beta}_0, \alpha_0 - \tilde{\beta}_0}}{\Gamma(1 + \alpha_0 + \beta_0)} G_{\alpha_1 + \tilde{\beta}_1, \alpha_1 - \tilde{\beta}_1} \tilde{\Omega}(2nt) K(2nt) (1 + o(1)). \tag{2.6.16}
\end{aligned}$$

By manipulating the above, we arrive at a more compact expression resulting in Theorem 2.1.3.

Table 2.1: Colour-coded similar terms for the product $\frac{\phi_n(0)}{X^n} D_n(f)$

$\frac{\phi_n(0)}{X^n} = \Theta_1 + \Theta_2 + \Theta_3$	$D_n(f)$
$\Theta_1 = z_1^n \exp \left\{ -(\alpha_0 + \hat{\beta}_0) \sum_{k=1}^{\infty} \frac{z_1^k e^{-kt}}{k} \right\}$ $\exp \left\{ (\alpha_0 - \hat{\beta}_0) \sum_{k=1}^{\infty} \frac{e^{-kt} z_1^{-k}}{k} \right\} \exp \left\{ \sum_{k=1}^{\infty} V_k z_1^k \right\}$ $\exp \left\{ -\sum_{k=1}^{\infty} V_{-k} z_k^{-k} \right\} n^{-2\hat{\beta}_1 - 1} \frac{\Gamma(1 + \alpha_1 + \hat{\beta}_1)}{\Gamma(\alpha_1 - \hat{\beta}_1)} (1 + o(1))$	$\exp \left\{ nV_0 + nt(\alpha_0 + \hat{\beta}_0) \right\}$ $\exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \right\}$ $\exp \left\{ -(\alpha_1 - \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\}$ $\exp \left\{ -(\alpha_1 + \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\}$ $n^{(\alpha_1^2 - \hat{\beta}_1^2)} G_{\alpha_0 + \hat{\beta}_0, \alpha_0 - \hat{\beta}_0} G_{\alpha_1 + \hat{\beta}_1, \alpha_1 - \hat{\beta}_1} \tilde{\Omega}(2nt) (1 + o(1))$
$\Theta_2 = \exp \left\{ -2\hat{\beta}_1 \sum_{k=1}^{\infty} \frac{z_1^k e^{-tk}}{k} \right\}$ $\exp \left\{ (\alpha_0 - \hat{\beta}_0) \sum_{k=1}^{\infty} \frac{e^{-2tk}}{k} \right\} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{tk} \right\}$ $\exp \left\{ -\sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\} e^{t(\alpha_1 + \hat{\beta}_1)} e^{-i\pi(\alpha_1 + \hat{\beta}_1)} z_1^{-(\alpha_1 + \hat{\beta}_1)}$ $(2t)^{\alpha_0 - \hat{\beta}_0} n^{-2\hat{\beta}_0 - 1} \frac{1}{\Gamma(\alpha_0 - \hat{\beta}_0)} K(2nt) (1 + o(1))$	$\exp \left\{ nV_0 + nt(\alpha_0 + \hat{\beta}_0) \right\}$ $\exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \right\}$ $\exp \left\{ -(\alpha_1 - \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\}$ $\exp \left\{ -(\alpha_1 + \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\}$ $n^{(\alpha_1^2 - \hat{\beta}_1^2)} G_{\alpha_0 + \hat{\beta}_0, \alpha_0 - \hat{\beta}_0} G_{\alpha_1 + \hat{\beta}_1, \alpha_1 - \hat{\beta}_1} \tilde{\Omega}(2nt) (1 + o(1))$
$\Theta_3 = \exp \left\{ -2\hat{\beta}_1 \sum_{k=1}^{\infty} \frac{z_1^{-k} e^{-tk}}{k} \right\}$ $\exp \left\{ -(\alpha_0 + \hat{\beta}_0) \sum_{k=1}^{\infty} \frac{e^{-2tk}}{k} \right\} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{-tk} \right\}$ $\exp \left\{ -\sum_{k=1}^{\infty} V_{-k} e^{tk} \right\} e^{-t(\alpha_1 - \hat{\beta}_1)} e^{-i\pi(\alpha_1 - \hat{\beta}_1)} z_1^{-(\alpha_1 - \hat{\beta}_1)}$ $(2t)^{-(\alpha_0 + \hat{\beta}_0)} n^{-2\hat{\beta}_0 - 1} \frac{1}{\Gamma(\alpha_0 - \hat{\beta}_0)} K(2nt) (1 + o(1))$	$\exp \left\{ nV_0 + nt(\alpha_0 + \hat{\beta}_0) \right\}$ $\exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right] \right\}$ $\exp \left\{ -(\alpha_1 - \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\}$ $\exp \left\{ -(\alpha_1 + \hat{\beta}_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \hat{\beta}_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\}$ $n^{(\alpha_1^2 - \hat{\beta}_1^2)} G_{\alpha_0 + \hat{\beta}_0, \alpha_0 - \hat{\beta}_0} G_{\alpha_1 + \hat{\beta}_1, \alpha_1 - \hat{\beta}_1} \tilde{\Omega}(2nt) (1 + o(1))$

2.7 Verifying transitions

In this section we will verify that we can recover the asymptotics (2.2.4), (2.2.5) and (2.2.6) from the uniform asymptotics we obtained in (2.5.27) for the seminorm $\|\beta\| < 1$ and (2.6.16) for $\|\beta\| = 1$.

It is important to notice and draw a clear distinction between the two cases. In the case when $\|\beta\| < 1$ we expect to recover asymptotics for a Toeplitz determinant with one singularity with β parameters β_0 and β_1 when we keep t fixed. For the uniform asymptotics when $\|\beta\| = 1$, by keeping t fixed, we should be able to obtain the same asymptotics for a Toeplitz determinant with one singularity with β parameters β_0 and β_1 , but at the same time we might also be able to quantify the difference made by the function that arises from shifting β_0 and β_1 in (2.6.1). Perhaps we could see something like this,

$$\begin{aligned} D_n(f(z;t)) &= z_1^{-n} D_n\left(\frac{(z - e^{-t})}{(z - e^t)} \tilde{f}(z;t)\right), \\ &= z_1^{-n} E(t, \tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\beta}_0, \tilde{\beta}_1) D_n(\tilde{f}(z;t)), \end{aligned}$$

where $E(t, \tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\beta}_0, \tilde{\beta}_1)$ would tell us more about what is happening when we change the symbol of the Toeplitz matrix by multiplication by $\frac{(z - e^{-t})}{(z - e^t)}$. It most definitely is not equal to $D_n\left(\frac{(z - e^{-t})}{(z - e^t)}\right)$, as it is known that in general $T_n(fg) \neq T_n(f)T_n(g)$ where f, g are two symbols for a Toeplitz matrix T_n .

As for the recovery of asymptotics as $t \rightarrow 0$ there is only one simple answer in both cases. If $\|\beta\| < 1$ then we expect to recover (2.2.5) and for $\|\beta\| = 1$ we expect the Tracy-Basor generalised F-H asymptotics (2.2.6).

2.7.1 Case $\|\beta\| < 1$

2.7.1.1 Asymptotics keeping t fixed, $n \rightarrow \infty$

We look at (2.5.27) keeping t fixed. We know from (1.5.13) and (2.6.14) that,

$$\tilde{\Omega}(+\infty) = G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0}^{-1}, \quad (2.7.1)$$

using notation defined in (1.4.13). We thus obtain from (2.5.27),

$$\begin{aligned}
D_n(t) &= \exp \{nV_0 + nt(\alpha_0 + \beta_0)\} \exp \left\{ \sum_{k=1}^{\infty} k \left[V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right] \right\} \\
&\quad \times \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} \\
&\quad \times \exp \left\{ -(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} \\
&\quad \times n^{(\alpha_1^2 - \beta_1^2)} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} (1 + o(1)),
\end{aligned}$$

which is exactly (2.2.4).

2.7.1.2 Asymptotics as $t \rightarrow 0$, n fixed

We first rewrite (2.5.27) expanding the factors inside the sums,

$$\begin{aligned}
D_n(t) &= \exp \{nV_0 + nt(\alpha_0 + \beta_0)\} \exp \left\{ \sum_{k=1}^{\infty} k V_k V_{-k} \right\} \exp \left\{ -(\alpha_0 + \beta_0) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} V_{-k} \right\} \\
&\quad \exp \left\{ -(\alpha_0 - \beta_0) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} V_k \right\} \exp \left\{ (\alpha_0^2 - \beta_0^2) \sum_{k=1}^{\infty} \frac{e^{-2tk}}{k} \right\} \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \\
&\quad \exp \left\{ (\alpha_0 + \beta_0)(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} z_1^k \right\} \exp \left\{ -(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\quad \exp \left\{ (\alpha_0 - \beta_0)(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} z_1^{-k} \right\} n^{(\alpha_1^2 - \beta_1^2)} G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} \tilde{\Omega}(2nt) (1 + o(1)).
\end{aligned}$$

Now, again using (2.4.18) to convert the following,

$$\exp \left\{ (\alpha_0^2 - \beta_0^2) \sum_{k=1}^{\infty} \frac{e^{-2tk}}{k} \right\} = \exp \left\{ -(\alpha_0^2 - \beta_0^2) \log(1 - e^{-2t}) \right\}, \quad (2.7.2)$$

$$\exp \left\{ (\alpha_0 + \beta_0)(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} z_1^k \right\} = \exp \left\{ -(\alpha_0 + \beta_0)(\alpha_1 - \beta_1) \log(1 - e^{-t} z_1) \right\}, \quad (2.7.3)$$

$$\exp \left\{ (\alpha_0 - \beta_0)(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} z_1^{-k} \right\} = \exp \left\{ -(\alpha_0 - \beta_0)(\alpha_1 + \beta_1) \log(1 - e^{-t} z_1^{-1}) \right\}, \quad (2.7.4)$$

we obtain,

$$\begin{aligned}
D_n(t) &= \exp \{nV_0 + nt(\alpha_0 + \beta_0)\} \exp \left\{ \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ -(\alpha_0 + \beta_0) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} V_{-k} \right\} \\
&\exp \left\{ -(\alpha_0 - \beta_0) \sum_{k=1}^{\infty} \frac{e^{-tk}}{k} V_k \right\} \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&(1 - e^{-2t})^{-(\alpha_0^2 - \beta_0^2)} (1 - e^{-t} z_1)^{-(\alpha_0 + \beta_0)(\alpha_1 - \beta_1)} (1 - e^{-t} z_1^{-1})^{-(\alpha_0 - \beta_0)(\alpha_1 + \beta_1)} \\
&n^{(\alpha_1^2 - \beta_1^2)} G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} \tilde{\Omega}(2nt) (1 + o(1)).
\end{aligned}$$

We obtained in Section 1.5.3 that if n is fixed and $t \rightarrow 0$, $\tilde{\Omega}(2nt) = \exp \{(\alpha_0^2 - \beta_0^2) \log(2nt) + o(1)\}$.

Using this and the fact that $\left(\frac{2nt}{1 - e^{-2t}}\right) \rightarrow n$ as $t \rightarrow 0$, we have,

$$\begin{aligned}
D_n(t \rightarrow 0) &= \exp \{nV_0\} \exp \left\{ \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ -(\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} \right\} \exp \left\{ -(\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k \right\} \\
&\exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&n^{(\alpha_0^2 - \beta_0^2) + (\alpha_1^2 - \beta_1^2)} (1 - z_1)^{-(\alpha_0 + \beta_0)(\alpha_1 - \beta_1)} (1 - z_1^{-1})^{-(\alpha_0 - \beta_0)(\alpha_1 + \beta_1)} \\
&G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} (1 + o(1)).
\end{aligned}$$

Rewriting

$$(1 - z_1^{-1}) = e^{i\pi(2k-1)} z_1^{-1} (1 - z_1), \quad \text{for } k \in \mathbb{Z}, \quad (2.7.5)$$

recalling the function $h_\alpha(z)$ from (2.4.30) and using the right k to suit our branch cuts, we obtain,

$$(1 - z_1)^{-(\alpha_0 + \beta_0)(\alpha_1 - \beta_1)} \left(e^{i\pi(2k-1)} z_1^{-1} (1 - z_1) \right)^{-(\alpha_0 - \beta_0)(\alpha_1 + \beta_1)} = \quad (2.7.6)$$

$$= (1 - z_1)^{-2(\alpha_0 \alpha_1 - 2\beta_0 \beta_1)} \left(e^{i\pi(2k-1)} z_1^{-1} \right)^{-(\alpha_0 \alpha_1 - \beta_0 \beta_1)} \left(e^{i\pi(2k-1)} z_1^{-1} \right)^{-(\alpha_0 \beta_1 - \beta_0 \alpha_1)} \quad (2.7.7)$$

$$= \left(\frac{1 - z_1}{(e^{-i\pi(2k-1)} z_1)^{1/2}} \right)^{-2(\alpha_0 \alpha_1 - \beta_0 \beta_1)} \left(\frac{z_1}{e^{i\pi(2k-1)}} \right)^{(\alpha_0 \beta_1 - \beta_0 \alpha_1)} \quad (2.7.8)$$

$$= |1 - z_1|^{-2(\alpha_0 \alpha_1 - \beta_0 \beta_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{(\alpha_0 \beta_1 - \beta_0 \alpha_1)}. \quad (2.7.9)$$

Finally,

$$\begin{aligned}
D_n(0) &= \exp \{nV_0\} \exp \left\{ \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ -(\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} \right\} \exp \left\{ -(\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k \right\} \\
&\quad \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\quad n^{(\alpha_0^2 - \beta_0^2) + (\alpha_1^2 - \beta_1^2)} |1 - z_1|^{-2(\alpha_0 \alpha_1 - \beta_0 \beta_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{(\alpha_0 \beta_1 - \beta_0 \alpha_1)} \\
&\quad G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} (1 + o(1)),
\end{aligned}$$

which is exactly (2.2.5).

2.7.2 Case $\|\beta\| = 1$

2.7.2.1 Asymptotics keeping t fixed, $n \rightarrow \infty$

Here we fix $t > 0$ and send $n \rightarrow \infty$, as we take $x = 2nt$, we thus need to recall the asymptotics for $K(x)$ as $x \rightarrow \infty$, see (2.4.144),

$$K(2nt) \sim e^{-nt} (2nt)^{\alpha_0 + \beta_0} \quad \text{as } 2nt \rightarrow \infty. \quad (2.7.10)$$

What we do next is in many ways similar to undoing the factorisations from (2.6.15) and the Table 2.1 should again be helpful in following this computation. Cancelling the product of Barnes G -functions $G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0}$ in the first term of the sum using (2.7.1) and then taking the factor of $D_n(f(z; t, \alpha_0, \alpha_1, \beta_0, \beta_1))$ (2.2.4) out in (2.6.16) gives,

$$\begin{aligned}
D_n(f) &= D_n(f(z; t, \alpha_0, \alpha_1, \beta_0, \beta_1)) \left[1 + z_1^{-n} n^{-2\beta_0 - 1} n^{2\beta_1 - 1} e^{-nt} (2nt)^{\alpha_0 + \beta_0} \right. \\
&\quad \times \exp \left\{ - \sum_{k=1}^{\infty} \left[\left(V_k - (\alpha_0 + \beta_0) \frac{e^{-tk}}{k} \right) z_1^k \right] \right\} \exp \left\{ \sum_{k=1}^{\infty} \left[\left(V_{-k} - (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} \right) z_1^{-k} \right] \right\} \\
&\quad \times \frac{\Gamma(1 + \alpha_1 - \beta_1)}{\Gamma(\alpha_0 - \beta_0) \Gamma(\alpha_1 + \beta_1)} \left((1 - z_1^{-1} e^{-t})^{2\hat{\beta}_1} (1 - e^{-2t})^{\alpha_0 + \beta_0} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{-tk} \right\} \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{tk} \right\} \right. \\
&\quad \times e^{-t(\alpha_1 - \hat{\beta}_1)} e^{-i\pi(\alpha_1 - \hat{\beta}_1)} z_1^{-(\alpha_1 - \hat{\beta}_1)} (2t)^{-(\alpha_0 + \beta_0)} + (1 - z_1 e^{-t})^{2\hat{\beta}_1} (1 - e^{-2t})^{-(\alpha_0 - \beta_0)} \\
&\quad \left. \times \exp \left\{ - \sum_{k=1}^{\infty} V_{-k} e^{-tk} \right\} \exp \left\{ \sum_{k=1}^{\infty} V_k e^{tk} \right\} e^{t(\alpha_1 + \hat{\beta}_1)} e^{-i\pi(\alpha_1 + \hat{\beta}_1)} z_1^{-(\alpha_1 + \hat{\beta}_1)} (2t)^{\alpha_0 - \beta_0} \right) \left. \right].
\end{aligned}$$

We take note only of the terms containing n . It is clear that $z_1^{-n} = \mathcal{O}(1)$ as z_1 is a point on the unit circle. We could also note that $n^{-2\operatorname{Re}\beta_0-1+2\operatorname{Re}\beta_1-1} = n^{-2q+2q}$ for some $q \in \mathbb{R}$ for $\operatorname{Re}\beta_0 = q - 1/2$, $\operatorname{Re}\beta_1 = q + 1/2$, so $n^{-2\beta_0+2\operatorname{Re}\beta_1-2} = \mathcal{O}(1)$. For the rest we note that $e^{-nt}(2nt)^{\alpha_0+\beta_0} \rightarrow 0$ as $n \rightarrow \infty$ and t fixed. All this and the fact that the rest of the expression behaves like a constant for a fixed t gives us exactly what we expect,

$$D_n(f) = D_n(f(z; t, \alpha_0, \alpha_1, \beta_0, \beta_1))(1 + o(1)).$$

2.7.2.2 Asymptotics as $t \rightarrow 0$, n fixed

To aid the presentation of this case we will denote the terms in the sum of (2.6.16) in the following way,

$$D_n(f) = \Xi_{z_1} + \Xi_t + \Xi_{-t}. \quad (2.7.11)$$

Throughout this section we keep n fixed and take the limit as $t \rightarrow 0$. The case of Ξ_{z_1} is identical to what we considered in Section 2.7.1.2 for $\|\beta\| < 1$. We obtain $\Xi_{z_1} = (2.2.5)$.

Next, we recall that $\tilde{\Omega}(2nt) = \exp\{(\alpha_0^2 - \beta_0^2)\log(2nt) + o(1)\}$ as $t \rightarrow 0$ and n fixed. We also recall the asymptotics for $K(x)$ as $x \rightarrow 0$ are given by (2.4.144),

$$K(2nt) \sim e^{nt}\Gamma(1 + \alpha_0 + \beta_0), \quad \text{as } 2nt \rightarrow 0. \quad (2.7.12)$$

Expanding out the sums in Ξ_t , we send t to 0,

$$\begin{aligned} \Xi_{t \rightarrow 0} &= \exp\{nV_0\} \exp\left\{\sum_{k=1}^{\infty} kV_k V_{-k}\right\} \exp\left\{\sum_{k=1}^{\infty} -(\alpha_0 + \tilde{\beta}_0)V_{-k} - \sum_{k=1}^{\infty} (\alpha_0 - \beta_0 - 1)V_k\right\} \\ &\quad \times \exp\left\{(\alpha_0 + \tilde{\beta}_0)(\alpha_0 - \beta_0) \lim_{t \rightarrow 0} \sum_{k=1}^{\infty} \frac{e^{-2tk}}{k}\right\} \exp\left\{-(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} V_k z_1^k\right\} \\ &\quad \times \exp\left\{(\alpha_1 - \tilde{\beta}_1) \lim_{t \rightarrow 0} \sum_{k=1}^{\infty} (\alpha_0 + \tilde{\beta}_0) \frac{e^{-tk}}{k} z_1^k\right\} \lim_{t \rightarrow 0} (1 - e^t z_1^{-1})^{\alpha_1 + \tilde{\beta}_1} \\ &\quad \times \exp\left\{-(\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k}\right\} \exp\left\{(\alpha_1 + \tilde{\beta}_1) \lim_{t \rightarrow 0} \sum_{k=1}^{\infty} (\alpha_0 - \beta_0) \frac{e^{-tk}}{k} z_1^{-k}\right\} n^{(\alpha_1^2 - \tilde{\beta}_1^2)} \\ &\quad \times n^{-2\beta_0-1} z_1^{-n} G_{\alpha_0 + \tilde{\beta}_0, \alpha_0 - \tilde{\beta}_0} G_{\alpha_1 + \tilde{\beta}_1, \alpha_1 - \tilde{\beta}_1} \lim_{t \rightarrow 0} \left[(2t)^{\alpha_0 - \beta_0} \exp\{(\alpha_0^2 - \beta_0^2)\log(2nt)\} \right] (1 + o(1)) \end{aligned}$$

We once again use (2.4.18) (see also (2.7.2),(2.7.3),(2.7.4)), noting that $\tilde{\beta}_0 = \beta_0 + 1$ in the first and second lines above,

$$\begin{aligned}
\Xi_{t \rightarrow 0} &= z_1^{-n} \exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ \sum_{k=1}^{\infty} -(\alpha_0 + \tilde{\beta}_0)V_{-k} - \sum_{k=1}^{\infty} (\alpha_0 - \tilde{\beta}_0)V_k \right\} \\
&\times \exp \left\{ -(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\times \lim_{t \rightarrow 0} \exp \left\{ (\alpha_0^2 - \beta_0^2) \log \left(\frac{2nt}{1 - e^{-2t}} + \right) + (\alpha_0 - \beta_0) \log \left(\frac{2t}{1 - e^{-2t}} \right) \right\} \\
&\times \lim_{t \rightarrow 0} \left[(1 - e^{-t} z_1)^{-(\alpha_1 - \tilde{\beta}_1)(\alpha_0 + \tilde{\beta}_0)} (1 - e^t z_1^{-1})^{\alpha_1 + \tilde{\beta}_1} (1 - e^{-t} z_1^{-1})^{-(\alpha_1 + \tilde{\beta}_1)(\alpha_0 - \beta_0)} \right] \\
&\times n^{(\alpha_1^2 - \tilde{\beta}_1^2)} n^{-2\beta_0 - 1} G_{\alpha_0 + \tilde{\beta}_0, \alpha_0 - \tilde{\beta}_0} G_{\alpha_1 + \tilde{\beta}_1, \alpha_1 - \tilde{\beta}_1} (1 + o(1)) \\
&= z_1^{-n} \exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ \sum_{k=1}^{\infty} -(\alpha_0 + \tilde{\beta}_0)V_{-k} - \sum_{k=1}^{\infty} (\alpha_0 - \tilde{\beta}_0)V_k \right\} \\
&\times \exp \left\{ -(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\times \exp \left\{ (\alpha_0^2 - \beta_0^2) \log n + (\alpha_0 - \beta_0) \log 1 \right\} \\
&\times \left[(1 - z_1)^{-(\alpha_1 - \tilde{\beta}_1)(\alpha_0 + \tilde{\beta}_0)} (1 - z_1^{-1})^{-(\alpha_1 + \tilde{\beta}_1)(\alpha_0 - \beta_0 - 1)} \right] \tag{2.7.13} \\
&\times n^{(\alpha_1^2 - \tilde{\beta}_1^2)} n^{-2\beta_0 - 1} G_{\alpha_0 + \tilde{\beta}_0, \alpha_0 - \tilde{\beta}_0} G_{\alpha_1 + \tilde{\beta}_1, \alpha_1 - \tilde{\beta}_1} (1 + o(1))
\end{aligned}$$

In line (2.7.13) we proceed in the exact same way as in (2.7.5)-(2.7.9) (note well different β -parameters), we factorise the power of n relating to $\alpha_0, \beta_0, \tilde{\beta}_0$, we thus obtain,

$$\begin{aligned}
\Xi_{t=0} &= z_1^{-n} \exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ \sum_{k=1}^{\infty} -(\alpha_0 + \tilde{\beta}_0)V_{-k} - \sum_{k=1}^{\infty} (\alpha_0 - \tilde{\beta}_0)V_k \right\} \\
&\times \exp \left\{ -(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\times |1 - z_1|^{2(\tilde{\beta}_1 \tilde{\beta}_0 - \alpha_0 \alpha_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{\alpha_0 \tilde{\beta}_1 - \tilde{\beta}_0 \alpha_1} \\
&\times n^{(\alpha_1^2 - \tilde{\beta}_1^2) + (\alpha_0^2 - \tilde{\beta}_0^2)} G_{\alpha_0 + \tilde{\beta}_0, \alpha_0 - \tilde{\beta}_0} G_{\alpha_1 + \tilde{\beta}_1, \alpha_1 - \tilde{\beta}_1} (1 + o(1)),
\end{aligned}$$

which is exactly the RHS of (2.2.5) with β_0, β_1 replaced by $\tilde{\beta}_0, \tilde{\beta}_1$ respectively, multiplied by z_1^{-n} . Taking the limit as $t \rightarrow 0$ in Ξ_{-t} follows the exact same route as the limit of Ξ_t and produces the same result. We arrive at $\Xi_{t=0} = \Xi_{-t=0}$.

Putting the terms in the sum back together,

$$\begin{aligned}
D_n(f) &= \exp \{nV_0\} \exp \left\{ \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ -(\alpha_0 + \beta_0) \sum_{k=1}^{\infty} V_{-k} \right\} \exp \left\{ -(\alpha_0 - \beta_0) \sum_{k=1}^{\infty} V_k \right\} \\
&\quad \times \exp \left\{ -(\alpha_1 - \beta_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \beta_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\quad \times n^{(\alpha_0^2 - \beta_0^2) + (\alpha_1^2 - \beta_1^2)} |1 - z_1|^{-2(\alpha_0 \alpha_1 - \beta_0 \beta_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{(\alpha_0 \beta_1 - \beta_0 \alpha_1)} \\
&\quad \times G_{\alpha_0 + \beta_0, \alpha_0 - \beta_0} G_{\alpha_1 + \beta_1, \alpha_1 - \beta_1} (1 + o(1)) \\
&+ 2z_1^{-n} \exp \left\{ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right\} \exp \left\{ \sum_{k=1}^{\infty} -(\alpha_0 + \tilde{\beta}_0) V_{-k} - \sum_{k=1}^{\infty} (\alpha_0 - \tilde{\beta}_0) V_k \right\} \\
&\quad \times \exp \left\{ -(\alpha_1 - \tilde{\beta}_1) \sum_{k=1}^{\infty} V_k z_1^k \right\} \exp \left\{ -(\alpha_1 + \tilde{\beta}_1) \sum_{k=1}^{\infty} V_{-k} z_1^{-k} \right\} \\
&\quad \times n^{(\alpha_1^2 - \tilde{\beta}_1^2) + (\alpha_0^2 - \tilde{\beta}_0^2)} |1 - z_1|^{2(\tilde{\beta}_1 \tilde{\beta}_0 - \alpha_0 \alpha_1)} \left(\frac{z_1}{e^{i\pi}} \right)^{\alpha_0 \tilde{\beta}_1 - \tilde{\beta}_0 \alpha_1} \\
&\quad \times G_{\alpha_0 + \tilde{\beta}_0, \alpha_0 - \tilde{\beta}_0} G_{\alpha_1 + \tilde{\beta}_1, \alpha_1 - \tilde{\beta}_1} (1 + o(1)),
\end{aligned}$$

which is (2.2.6) up to a constant in the second term.

Chapter 3

Applications

3.1 Statistical Mechanics

Statistical mechanics is a branch of physics which concerns itself with studying the average properties of a mechanical system of a large number of particles, often looking at these properties as the number of the particles in the system grows to infinity. Of special interest in the study of statistical mechanics is the existence of phase transitions and the behaviour of particles close to the critical point—when the transition occurs. A simple example of a phase transition related to temperature in the system is boiling water, turning it into steam at $100^{\circ}C$, or freezing it at $0^{\circ}C$. The most famous problem in the study of statistical mechanics is the Ising model, which was mentioned in Sections 1.3 and 1.4. This is because the Ising model is extremely varied. In one dimension, the Ising model does not undergo any phase transitions at all. The case of the Ising model in two dimensions has been studied extensively and it not only has a ferromagnetic phase transition, but also exhibits many physical properties which can be computed explicitly. In three dimensions the Ising model is extremely complicated and no exact computation exists. See [5] and [37] for thorough introductions to the Ising model and statistical mechanics.

In what follows, we will be looking at a particular case of what is known as a Heisenberg spin chain. This is a one-dimensional model of magnetism or model of spin- $\frac{1}{2}$ particles that have a spin-spin interaction. The one-dimensional Ising model is also a particular example of this. The spin chain consists of a number of sites, N , on some lattice (in one dimension the lattice is typically just \mathbb{Z}) and we consider a spin- $\frac{1}{2}$ particle on each site. This is described using spin operators $s_i^{x,y,z}$, two

dimensional unit vectors which prescribe an angle to each site of the lattice. For example in the Ising model we consider spins that only point up or down, $\sigma_i = \pm 1$ and in the problem below we take the spin operators to be the Pauli matrices which are the following,

$$\sigma_1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.1.1)$$

$$\sigma_2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (3.1.2)$$

$$\sigma_3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.1.3)$$

In a lot of problems we are only interested in the case when only the nearest neighbouring spins interact with each other which is written down in the following way,

$$J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z, \quad (3.1.4)$$

where J_α , $\alpha = x, y, z$, are called the interaction constants and describe the interactions between the two spins at sites i and $i + 1$. The whole system is governed by the Hamiltonian, H , which is the collection of all interactions in the system,

$$H = \sum_{i=1}^N [J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z] - h \sum_{i=1}^N \sigma_i^z, \quad (3.1.5)$$

the last term for a $h \neq 0$ adds the effects of a non-zero external magnetic field acting on the system. This is a Hamiltonian for a so-called XYZ spin chain model. If $J_x = J_y$, it would be called the XXZ spin chain model and in the case below $J_z = 0$ making it a XY spin chain model. As we want to consider this for a large or growing system, i.e. as $N \rightarrow \infty$, we need to set boundary conditions. Here the matrices are periodic $\sigma_i^\alpha = \sigma_{i+N}^\alpha$. The next natural step is to compute a correlation function which models physical phenomena such as phase transitions. There is no unique correlation function for each mechanical system, these functions are computed to highlight the particular average properties of the system that one is interested in. Linking back to the example of boiling or freezing water, one would be interested in the density of the water molecules for instance. In the Ising model

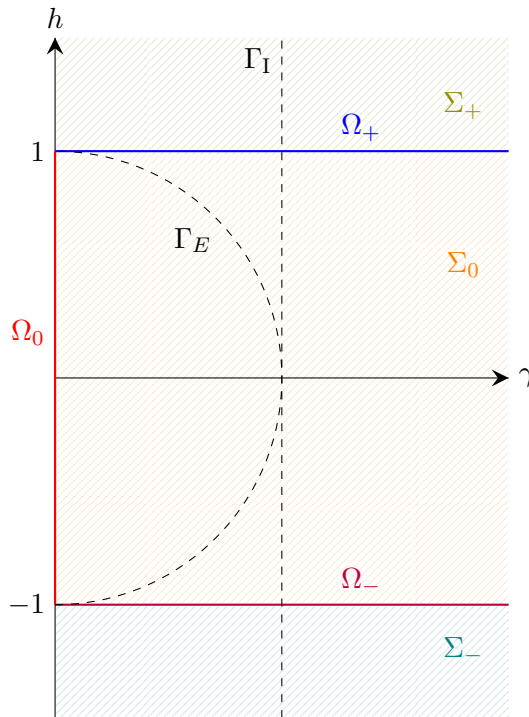


Figure 3.1: Phase diagram of the XY model for $\gamma \geq 0$. There are three critical lines: Ω_+ : $h = 1$, Ω_0 : $\gamma = 0$, $|h| < 1$, Ω_- : $h = -1$ and three non-critical domains: Σ_+ : $h < -1$, Σ_0 : $-1 < h < 1$, Σ_- : $h > 1$. The line Γ_I when $\gamma = 1$ represent the Ising model in transverse magnetic field. On the line Γ_E when $\gamma^2 + h^2 = 1$ the ground state of the theory is a product of single spin states.

we are interested in magnetisation and in the problem below, a correlator called the Emptiness Formation Probability.

3.2 XY Spin Chains

The XY model was first introduced by Matsubara and Matsuda in 1956 [36], as a model of a quantum lattice gas. The critical behaviour of this model has since been investigated in detail between 1968 and 1974 by Betts and his collaborators, who have also emphasised the relevance of this model to the study of insulating ferromagnets. The XY model has gathered a lot of attention within groups studying quantum entanglement with works such as Vidal et al [44], Jin and Korepin [30] or Keating and Mezzadri [31].

In [25] Franchini and Abanov look at how the change between regions over critical lines in the phase diagram (see Figure 3.1) influences the asymptotics of a special correlator called the Emptiness Formation Probability (EFP) for the 1-dimensional, anisotropic XY spin-1/2 chain in a transverse

magnetic field h . The Hamiltonian for this model is given by,

$$H = \sum_{i=1}^N \left[\left(\frac{1+\gamma}{2} \right) \sigma_i^x \sigma_{i+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_i^y \sigma_{i+1}^y \right] - h \sum_{i=1}^N \sigma_i^z, \quad (3.2.1)$$

where σ_i^α , with $\alpha = x, y, z$ are the Pauli matrices (see Equations (3.1.1)-(3.1.3)), which describe the spin operators on the i -th lattice site of the spin chain and the boundary conditions are chosen to be periodic: $\sigma_i^\alpha = \sigma_{i+N}^\alpha$ with ($N \gg 1$). The EFP is given by,

$$P(n) \equiv \frac{1}{Z} \text{Tr} \left\{ e^{H/T} \prod_{j=1}^n \frac{1 - \sigma_j^z}{2} \right\}, \quad (3.2.2)$$

where the partition function is given by $Z \equiv \text{Tr} \{ e^{H/T} \}$ and T is the temperature. The majority of the paper [25] deals with the case when $T = 0$. In that case,

$$P(n) \equiv \langle 0 | \prod_{i=1}^n \frac{1 - \sigma_i^z}{2} | 0 \rangle,$$

and $P(n)$ is then the probability that n consecutive spins are all aligned downward in the ground state $|0\rangle$.

After reformulating (3.2.1) using ‘spinless fermions’ and other transformation techniques frequently used by the statistical mechanics community, the authors arrive at fermionic correlators in the thermodynamic limit, these are given by

$$F_{jk} \equiv i \langle \psi_j \psi_k \rangle = -i \langle \psi_j^\dagger \psi_k^\dagger \rangle = \int_0^{2\pi} \frac{1}{2} \sin \vartheta_q e^{iq(j-k)} \frac{dq}{2\pi}, \quad (3.2.3)$$

$$G_{jk} \equiv \langle \psi_j \psi_k^\dagger \rangle = \int_0^{2\pi} \frac{1 + \cos \vartheta_q}{2} e^{iq(j-k)} \frac{dq}{2\pi}, \quad (3.2.4)$$

where ψ_i are the spinless fermions (c.f. Jordan-Wigner transformation) and

$$e^{i\vartheta_q} = \frac{1}{\varepsilon_q} (\cos q - h + i\gamma \sin q), \quad (3.2.5)$$

$$\varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}. \quad (3.2.6)$$

The EFP is given as

$$P(n) = \text{Pf}(\mathbf{M}),$$

where

$$\text{Pf}(M) \equiv \sum_P (-1)^P M_{p_1 p_2} M_{p_3 p_4} \cdots M_{p_{2n-1} p_{2n}},$$

is called the Pfaffian. Here the sum is taken over all possible permutations $P = \{p_1, p_2, \dots, p_{2n}\}$ of the set $\{1, 2, \dots, n\}$ and $(-1)^P$ denotes the parity of the permutation. The matrix \mathbf{M} is a $2n \times 2n$ skew-symmetric matrix of correlation functions given by

$$\mathbf{M} = \begin{pmatrix} -i\mathbf{F} & \mathbf{G} \\ -\mathbf{G} & i\mathbf{F} \end{pmatrix},$$

where \mathbf{F} and \mathbf{G} are $n \times n$ matrices with entries given by F_{jk} and G_{jk} in (3.2.3) and (3.2.4). One of the properties of the Pfaffian¹ gives that

$$P(n) = \text{Pf}(\mathbf{M}) = \sqrt{\det(\mathbf{M})}.$$

After performing a unitary transformation one arrives at

$$\mathbf{M}' = \mathbf{U}\mathbf{M}\mathbf{U}^\dagger = \begin{pmatrix} \mathbf{0} & \mathbf{S}_n \\ -\mathbf{S}_n^\dagger & \mathbf{0} \end{pmatrix}, \quad \mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix},$$

where \mathbf{I} is a unit $n \times n$ matrix and $\mathbf{S}_n = \mathbf{G} + i\mathbf{F}$, $\mathbf{S}_n^\dagger = \mathbf{G} - i\mathbf{F}$. The ‘dagger’ \dagger denotes the adjoint of that matrix in the Hilbert space. The unitary transformation does not change the determinant and so we get that

$$\det(\mathbf{M}) = \det(\mathbf{M}') = \det(\mathbf{S}_n) \det(\mathbf{S}_n^\dagger) = |\det(\mathbf{S}_n)|^2.$$

Thus the EFP is given by calculating,

$$P(n) = |\det(\mathbf{S}_n)|,$$

¹Let A be a skew symmetric matrix, i.e. when $-A = A^T$. The determinant of A can always be written as the square of a polynomial in the matrix entries. Moreover, this polynomial has integer coefficients which only depend upon the size of the matrix. The value of this polynomial, when applied to the coefficients of A , is called the Pfaffian of A . Thus, for a skew-symmetric matrix A , $\text{Pf}(A)^2 = \det A$.

where the matrix \mathbf{S}_n is an $n \times n$ Toeplitz matrix with the following symbol,

$$\sigma(q) = \frac{1}{2} + \frac{\cos q - h + i\gamma \sin q}{2\sqrt{(\cos q - h)^2 + \gamma^2 \sin^2 q}}, \quad (3.2.7)$$

where $q \in (0, 2\pi]$ and h is the external magnetic field, the values of which affect the analyticity of the above function. This can be seen by critical lines in the phase diagram (Figure 3.1). Let us assume that $\gamma \neq 0^2$. To find the symbol of the Toeplitz matrix \mathbf{S}_n recall that by definition the entries are given by

$$(\mathbf{S}_n)_{jk} = \int_0^{2\pi i} \sigma(q) e^{iq(j-k)} \frac{dq}{2\pi}, \quad (3.2.8)$$

for some $\sigma(q)$. The function $\sigma(q)$ can then be found by recalling that $\mathbf{S}_n = \mathbf{G} + i\mathbf{F}$ and by (3.2.3) and (3.2.4),

$$\sigma(q) = \frac{1}{2} \left(1 + e^{i\vartheta_q} \right), \quad e^{i\vartheta_q} = \cos \vartheta_q + i \sin \vartheta_q. \quad (3.2.9)$$

Further, by recalling (3.2.5) and (3.2.6) we obtain (3.2.7).

Region Σ_- corresponding to $h < -1$

In this region, for $\gamma \neq 0$ and $h < -1$, the symbol (3.2.7) is analytic for all q and we can use SSLT (Theorem 1.3.4) to get the asymptotics. From now on let us use $q \equiv \theta$ to simplify comparison with the work of this thesis and let us emphasise the dependence on h or t by writing $\sigma(\theta, h)$ or $\sigma(\theta, t)$ respectively. In this region we can think of $h = -e^t$ with $t > 0$, and $t \rightarrow 0$ corresponds to the limit as h approaches -1 from $h < -1$. By taking the limit $t \rightarrow 0$ we arrive at the case of region Σ_0 below.

We can factorise another analytic function (for $t > 0$) out of $\sigma(\theta, t)$,

$$\mathcal{T}_{-1}(\theta, t) = \left(e^{i\theta} + e^t \right) e^{-3\pi i/2}, \quad \theta \in (0, 2\pi), \quad (3.2.10)$$

to write simply,

$$\sigma(\theta, t) = \mathcal{T}_{-1}(\theta, t) e^{V(\theta, t)}, \quad (3.2.11)$$

where $e^{V(\theta, t)}$ is an analytic function left after the factorisation. Function $e^{V(\theta, t)}$ depends on t , but

²For $\gamma = 0$, the problem is isotropic and instead of exponential, we see a Gaussian behaviour using Widom's Theorem with a power law prefactor, cf [39].

the limit $t \rightarrow 0$ does not affect its analyticity. We have that the function $\mathcal{T}_{-1}(\theta, t) \rightarrow \mathcal{F}_{-1}(\theta)$ as $t \rightarrow 0$ where

$$\mathcal{F}_{-1}(\theta) = |e^{i\theta} + 1| e^{i\theta/2} g_{-1,1/2}(e^{i\theta}) e^{-i\pi/2}, \quad (3.2.12)$$

and function $g_{z_j, \beta_j}(z)$ is the one defined in (1.4.2). Compare $\mathcal{F}_{-1}(\theta, t)$ with (1.4.1) and $z = e^{i\theta}$, this is the so-called pure F-H singularity at $z = -1$ or $\theta = \pi$ with $\alpha_1 = 1/2$ and $\beta_1 = 1/2$. Indeed we see that

$$\begin{aligned} \mathcal{T}_{-1}(\theta, 0) &= (e^{i\theta} + 1) e^{-3\pi i/2} \\ &= (e^{i\theta} + 1)^{2(1/2)} e^{-i\theta/2} e^{-\pi i} g_{-1,-1/2}(e^{i\theta}) g_{-1,1/2}(e^{i\theta}) e^{-\pi i/2} e^{i\theta/2} \\ &= \frac{(e^{i\theta} + 1)^{2(1/2)}}{(e^{-i\theta} e^{i1})^{1/2}} g_{-1,1/2}(e^{i\theta}) e^{-\pi i/2} e^{i\theta/2} \\ &= \mathcal{F}_{-1}(\theta) \end{aligned}$$

where we used (2.4.30) and (2.4.34) with $\theta_1 = \pi$ as

$$e^{i1(-1/2)} = e^{-\pi i} g_{-1,-1/2}(e^{i\theta}) = \begin{cases} e^{-i\pi} e^{i\pi(-1/2)} = e^{3\pi i(-1/2)}, & 0 \leq \theta < \pi, \\ e^{-i\pi} e^{-i\pi(-1/2)} = e^{\pi i(-1/2)}, & \pi \leq \theta < 2\pi. \end{cases}$$

We thus cross the critical line $h = -1$ and arrive at the next non-critical region Σ_0 .

Region Σ_0 corresponding to $-1 < h < 1$

For $-1 < h < 1$ the symbol has one F-H singularity at $\theta = \pi$ with strength $\beta = \frac{1}{2}, \alpha = \frac{1}{2}$. This is evident by using earlier notation (3.2.12). We can factorise the singularity out of $\sigma(\theta, h)$. The function $\mathcal{F}_{-1}^{-1}(\theta)\sigma(\theta, h)$ is analytic for $-1 < h < 1$ as shown on the plots in Figure 3.3.

As $h \rightarrow 1$ we see an emergence of a second singularity at $z = 1$ or $\theta = 0$. In this region we can think of $h = e^{-t}$ with $t > 0$ (note that t here is different to the one in the previous region), and $t \rightarrow 0$ corresponds to the limit as h approaches 1 from $h < 1$. By taking the limit $t \rightarrow 0$ we arrive at the region Σ_+ . Similarly as before, we can define a new function

$$\mathcal{T}_1(\theta, t) = (e^{i\theta} - e^{-t}) e^{-i\theta}, \quad \theta \in (0, 2\pi). \quad (3.2.13)$$

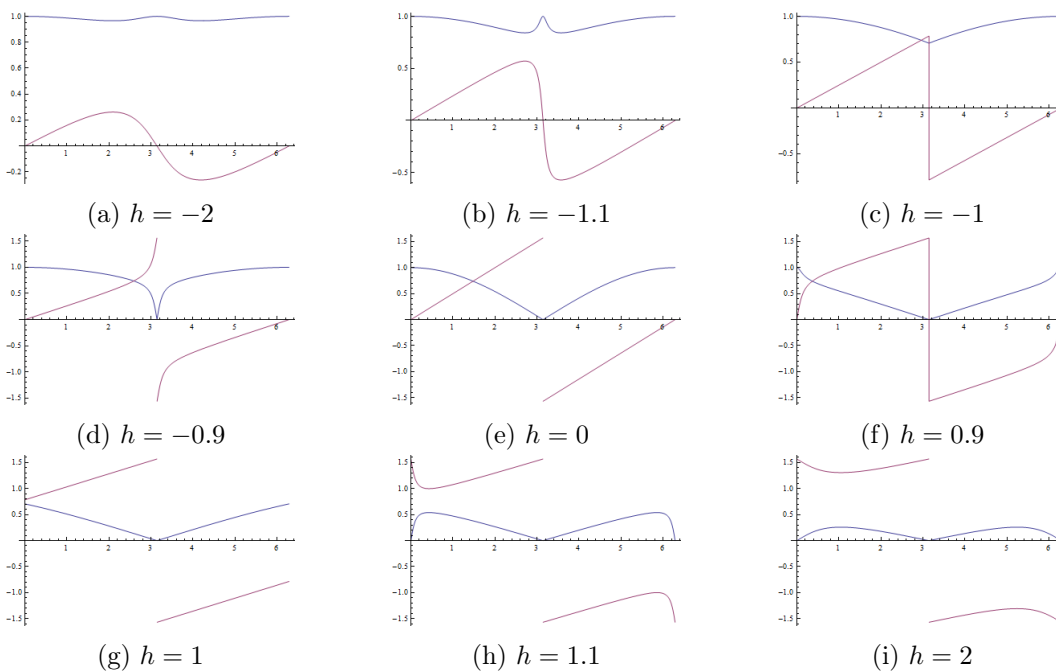


Figure 3.2: Plots of absolute value (blue) and argument (red) of the function $\sigma(\theta, h)$ for different values of $-2 \leq h \leq 2$ with $\gamma = 1$; showing jumps in argument and vanishing in the absolute value forming at $\theta = \pi$ as $h \geq -1$ and at $\theta = 0$ as $h \geq 1$.

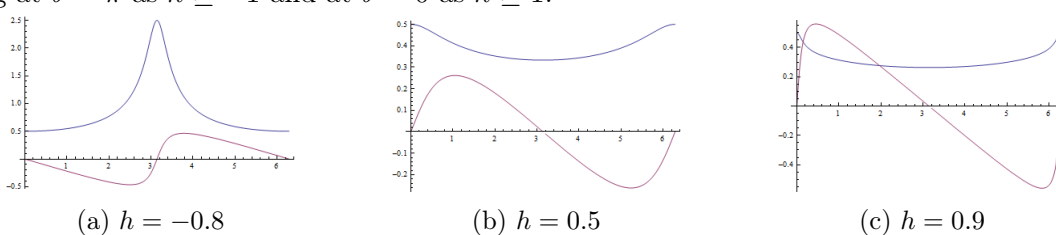


Figure 3.3: Plots of absolute value (blue) and argument (red) of the function $\mathcal{F}_{-1}^{-1}(\theta)\sigma(\theta, h)$ with $\gamma = 1$; showing no jumps in argument and boundedness and non-vanishing in absolute value for $-1 < h < 1$.

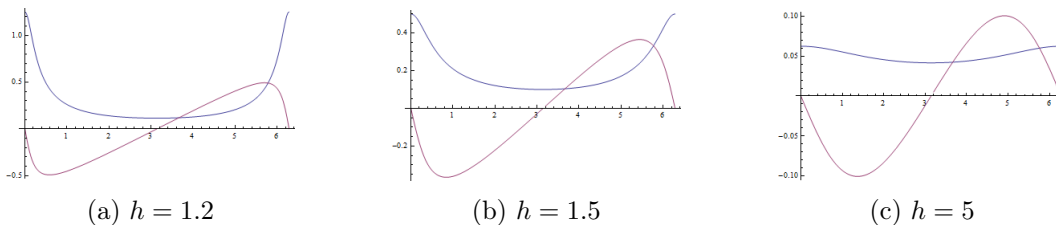


Figure 3.4: Plots of absolute value (blue) and argument (red) of the function $\mathcal{F}_{-1}^{-1}(\theta)\mathcal{F}_1^{-1}(\theta)\sigma(\theta, h)$ with $\gamma = 1$; showing no jumps in argument and boundedness and non-vanishing in absolute value for $h > 1$.

Compare this function to (2.2.3) with $\alpha_0 = 1/2$ and $\beta_0 = -1/2$. We can write

$$\sigma(\theta, t) = \mathcal{T}_1(\theta, t)\mathcal{F}_{-1}(\theta)e^{\tilde{V}(\theta, t)}, \quad (3.2.14)$$

where $e^{\tilde{V}(\theta, t)}$ is an analytic function left after the factorisation. Notice that this is exactly (2.2.2) with $z_1 = -1$, $\alpha_0 = 1/2$, $\beta_0 = -1/2$, $\alpha_1 = 1/2$, $\beta_1 = 1/2$. Again, $e^{\tilde{V}(\theta, t)}$ depends on t , but the limit $t \rightarrow 0$ does not affect its analyticity. We have that the function $\mathcal{T}_1(\theta, t) \rightarrow \mathcal{F}_1(\theta)$ as $t \rightarrow 0$ where

$$\mathcal{F}_1(\theta) = |e^{i\theta} - 1|e^{-i\theta/2}e^{i\pi/2}. \quad (3.2.15)$$

Compare $\mathcal{F}_1(\theta, t)$ with (1.4.1) and $z = e^{i\theta}$, this is now a pure singularity at $z = 1$ or $\theta = 0$ with $\alpha_0 = 1/2$ and $\beta_0 = -1/2$. Indeed we see that

$$\begin{aligned} \mathcal{T}_1(\theta, 0) &= (e^{i\theta} - 1) e^{-i\theta} \\ &= \frac{(e^{i\theta} + 1)^{2(1/2)}}{(e^{i\theta} e^{i\pi})^{1/2}} e^{-i\theta/2} e^{i\pi/2} \\ &= \mathcal{F}_1(\theta) \end{aligned}$$

where we have used (2.4.30) again. We thus cross the critical line $h = 1$ and arrive at the next non-critical region Σ_+ .

Region Σ_+ corresponding to $h > 1$

For $h > 1$, the symbol $\sigma(\theta, h)$ has two F-H singularities at $\theta = 0$ and $\theta = \pi$. Using our notation from Chapters 1 and 2, denote $\theta_0 = 0$ and $\theta_1 = \pi$. The corresponding strengths are $\beta_0 = -\frac{1}{2}$, $\alpha_0 = \frac{1}{2}$, $\beta_1 = \frac{1}{2}$, $\alpha_1 = \frac{1}{2}$. Using earlier notation (3.2.12), (3.2.15), we can write the symbol as

$$\sigma(\theta, t) = \mathcal{F}_{-1}(\theta, t)\mathcal{F}_1(\theta, t)e^{\hat{V}(\theta, h)}. \quad (3.2.16)$$

To see $e^{\hat{V}(\theta, h)}$ is analytic for $h > 1$, we look at the plot of $\mathcal{F}_{-1}^{-1}(\theta)\mathcal{F}_1^{-1}(\theta)\sigma(\theta, h)$ in Figure 3.4 and notice the function has no jumps in argument and its absolute value does not vanish or blow up.

However, this symbol has another representation, corresponding to the seminorm of β -parameters, recall from Sections 1.4.2 and (1.4.10) that when $\|\beta\| = \max_{i,k} |\operatorname{Re} \beta_i - \operatorname{Re} \beta_k| = 1$ we can write the

symbol using different β -parameters and the resulting function will only vary from the original by a multiplicative constant. By denoting

$$\mathcal{F}_{-1}^{\hat{\beta}}(\theta) = |e^{i\theta} + 1|e^{-i\theta/2}g_{-1,-1/2}(e^{i\theta})e^{i\pi/2}, \quad (3.2.17)$$

and

$$\mathcal{F}_1^{\hat{\beta}}(\theta) = |e^{i\theta} - 1|e^{i\theta/2}e^{-i\pi/2}, \quad (3.2.18)$$

we have that

$$\sigma(\theta, t) = e^{-i\pi} \mathcal{F}_{-1}^{\hat{\beta}}(\theta, t) \mathcal{F}_1^{\hat{\beta}}(\theta, t) e^{\widehat{V}(\theta, h)}.$$

By the Tracy-Basor conjecture (or the generalised F-H asymptotics), the asymptotics of the Toeplitz determinant are given using contributions from both representations, see Theorem 1.4.6.

It is natural to ask about the transition between each region. In fact, it is of great interest in problems of statistical mechanics to see what happens close to the critical points. In [11] (see also Section 1.5) Claeys, Its and Krasovsky obtain a uniform expression for the asymptotics of the Toeplitz determinant with a symbol that can be written in the form (3.2.11) after rotating the problem in [11] by π . This transition can thus be used to describe the change between the regions Σ_- and Σ_0 . This thesis describes the transition between Σ_0 and Σ_+ , i.e. the emergence of an additional singularity with the resulting F-H symbol satisfying $\|\beta\| = \max_{i,k} |\operatorname{Re} \beta_i - \operatorname{Re} \beta_k| = 1$. We can write the symbol (3.2.14) as (2.2.1) with $z_1 = -1$, $\alpha_0 = 1/2$, $\beta_0 = -1/2$, $\alpha_1 = 1/2$, $\beta_1 = 1/2$ and use Theorem 2.1.2 to give the uniform asymptotics in this case.

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Notation

Greek Alphabet in the following order:

$\alpha, \beta, \gamma, \Gamma, \delta, \Delta, \varepsilon, \zeta, \eta, \theta, \vartheta, \Theta, \kappa, \lambda, \mu, \nu, \xi, \Xi, \pi, \Pi, \rho, \sigma, \Sigma, \tau, \Upsilon, \phi, \varphi, \Phi, \chi, \psi, \Psi, \omega, \Omega$

α_j	α -singularity, Equation (1.4.3), page 21
β_j	β -singularity, Equation (1.4.3), page 21
$\ \ \beta\ \ $	β seminorm, Equation (1.4.10), page 24
Γ	Euler's Γ -function
δ	Equation (2.4.170), page 96
$\Delta_1(z)$	Term in the expansions of P_{z_1} and P_{z_0} , Equation (2.4.100), page 80
ε_q	Equation 3.2.6, page 124
ζ	Equation (2.4.29), page 64
ϑ_q	Equation 3.2.5, page 124
$\lambda(z)$	Equation (2.4.91), page 78
Ξ_j	Equation (2.7.11), page 117
$\sigma_1, \sigma_2, \sigma_3$	Pauli matrices, Equation(2.4.2), page 54, Equations (3.1.1)-(3.1.3), page 122
$\sigma(x)$	Solution to a Painlevé V equation. Equation (1.5.9), page 33
$\sigma(q)$	Equation (3.2.7), page 126
Σ_{\mp}, Σ_0	Non-critical regions of the phase diagram for the XY spin-1/2 chain, Equation (3.1), page 123
$\Sigma(t)$	Equation (2.1.5), page 45
$\phi_n, \hat{\phi}_n$	Orthogonal polynomials, Equation (1.6.1), page 36
$\varphi_{z_j, \beta_j}(z)$	Equation (1.4.7), page 22
Φ^{\pm}	Equation (1.2.1), page 17
$\Phi(\lambda; x)$	Equation (2.4.86), page 77
$\tilde{\Phi}(\zeta)$	Equation (2.4.109), page 83
χ_n	Leading coefficient of ϕ_n and $\hat{\phi}_n$ (1.6.6), page 37
$\psi(a, c; z)$	Confluent hypergeometric function, Equation (2.4.51), page 70

$\Psi_0(\zeta)$	Riemann-Hilbert problem for $\Psi_0(\zeta)$, page 75
$\Psi_1(\zeta)$	Solution to the Riemann-Hilbert problem for $\Psi_1(\zeta)$, Equation (2.4.53), page 70
$\widehat{\Psi}(\widehat{\lambda})$	Equation (2.4.114), page 84
ω	Equation (2.4.156), page 93
$\Omega(2nt)$	Equation (1.5.6), page 32
$\tilde{\Omega}(2nt)$	Equation (2.6.14), page 109

Roman Alphabet

$a(z; t)$	Symbol in [11], Equation (1.5.1), page 29
$A(z)$	Equation (2.5.4), page 98
b_{\pm}	Wiener-Hopf factorisation of the function b , Equation (1.4.11), page 24
D_n	Toeplitz Determinant, Equation (1.1.4), page 15
$D(z)$	Szegő function, Equation (2.4.13), page 58
$f(z; t)$	Toeplitz matrix symbol, Equation (2.2.1), page 45
F_{jk}	Equation (3.2.3), page 124
$F_1(z)$	Auxiliary function, Equation (2.4.36), page 68
$\mathcal{F}_{-1}(\theta)$	Equation (3.2.12), page 127
$\mathcal{F}_1(\theta)$	Equation (3.2.15), page 129
$\mathcal{F}_{-1}^{\hat{\beta}}(\theta)$	Equation (3.2.17), page 130
$\mathcal{F}_1^{\hat{\beta}}(\theta)$	Equation (3.2.18), page 130
g_k	Fourier coefficients of a function $g(z) \in L^1(\mathbb{T})$, Equation (1.1.1), page 13
$g_{z_j, \beta_j}(z)$	Equation (1.4.2), page 21
$G_{\alpha_j + \beta_j, \alpha_j - \beta_j}$	Product of Barnes G -function, Equation (1.4.13), page 24
G_{jk}	Equation (3.2.4), page 124
$G(\lambda; x)$	Equation (2.4.87), page 77
$G(z)$	Barnes G -function, Equation (1.4.8), page 23
h_{α_j}	Equation (2.4.30) page 65
H	Equation (3.1.5), page 122, Equation (3.2.1), page 124
$H^p(\mathbb{D})$	Hardy space on the open unit disk, Definition 1.1.2, page 13

$H^p(\mathbb{T})$	Hardy space on the circle, Definition 1.1.1, page 13
$H(\widehat{\lambda})$	Solution to the Riemann-Hilbert problem for $H(\widehat{\lambda})$, Equation (2.4.130), page 87
$J_1(z), J_2(z), J_N(z), J_T(z)$	Jump matrices, Equation(2.4.4), page 55
$K(x)$	Equation (2.4.143), page 90
l_1	Equation (2.4.34), page 65
$M(\widehat{\lambda})$	Solution to the Riemann-Hilbert problem for $M(\widehat{\lambda})$, Equation (2.4.131), page 87
$N(z)$	Solution to the Riemann-Hilbert problem for $N(z)$, Equation (2.4.14), page 58
O_β	Orbit of β 's, Equation (1.4.20), page 27
\mathcal{O}, o	Big and Small 'o' notation, Definition 1.3.1, page 18
P	Orthogonal Projection, Definition 1.1.3: Orthogonal Projection, page 14
$P(n)$	Equation (3.2.2), page 124
$P_{z_0}(z)$	Parametrix at z_0 , Equation (2.4.93), page 79
$P_{z_1}(z)$	Parametrix at z_1 , Equation (2.4.42), page 68
Q	Complementary Projection, Equation (1.2.2), page 17
$R(z)$	Solution to the Riemann-Hilbert problem for $R(z)$, Equation (2.4.150), page 91
$R_1(z)$	Solution to the Riemann-Hilbert problem for $R_1(z)$, Equation (2.4.162), page 95
$R_n(t)$	Equation (2.5.24), page 103
$R(\widehat{\lambda})$	Solution to the Riemann-Hilbert problem for $R(\widehat{\lambda})$, Equation (2.4.135), page 88
$\tilde{R}(z)$	Solution to the Riemann-Hilbert problem for $\tilde{R}(z)$, Equation (2.4.156), page 93
$\tilde{R}_2(z)$	Solution to the Riemann-Hilbert problem for $\tilde{R}_2(z)$, Equation (2.4.168), page 96
$\tilde{R}_3(z)$	Solution to the Riemann-Hilbert problem for $\tilde{R}_3(z)$, Equation (2.4.169), page 96

$\mathcal{R}(f(z; n_0, \dots, n_m))$	RHS of (1.4.12) without error term, Equation (1.4.23), page 28, 24
S	Cauchy singular integral operator (SIO), Definition 1.2.2, page 16
\mathbf{S}_n	Toeplitz matrix with symbol(3.2.9), page 126
$S(z)$	Solution to the Riemann-Hilbert problem for $S(z)$, Equation (2.4.5), page 56
T_f	Toeplitz Operator, Definition 1.1.4, page 14
$T(f)$	Toeplitz Matrix, Definition 1.1.5, page 14
$T(z)$	Solution to the Riemann-Hilbert problem for $T(z)$, Equation (2.4.1), page 54
$T_n(f)$	Truncated Toeplitz Matrix, Definition 1.1.6, page 15
$\mathcal{T}_{-1}(\theta, t)$	Equation (3.2.10), page 126
$\mathcal{T}_1(\theta, t)$	Equation (3.2.13), page 127
$u(x)$	Equation 2.4.84, page 76
U_{z_j}	Small neighbourhood around z_j , Equation (2.4.12), page 57
$v(x)$	Equation (2.4.83), page 76
$w(x)$	Equation (2.5.15), page 101
$W(z)$	Equation (2.4.94), page 79
$Y(z)$	Solution to the Riemann-Hilbert problem for $Y(z)$, Equation (1.6.10), page 39

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