Bragg Resonance

by Ripple Beds

Zoe Gumm (17017569)

Department of Mathematics

University of Reading

Supervisors: Dr N. Biggs and Dr P. Chamberlain

August 21, 2009

Abstract

The overall aim of this project is to gain an understanding of how asymptotic methods can be used in linear water-wave theory. Furthermore these methods will be used to develop a solution to a problem posed in Porter and Chamberlain (1997).

Porter and Chamberlain (1997) show the unsuccessful application of a regular series expansion to demonstrate the motion of plane harmonic waves. Here we have investigated how the method of multiple scales can be used to improve approximations made with a regular series expansions, exposing the phenomenon of Bragg resonance.

The examples considered in this paper are focused on first-order perturbation theory as this itself shows a significant improvement in results when comparing a regular series expansion to a multiple scale expansion.

Declaration

I confirm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

Zoe Gumm

Contents

Acknowledgments

I would like to acknowledge the financial assistance of the NERC and my Boohbah for the tireless hours it has spent with me in front of my computer. Also to Dr Biggs for designing my project and Dr Chamberlain for allowing me to work on his problem.

1 Introduction

Asymptotic expansions can provide us with relatively simple examples for studying certain systems in linear water-wave theory. In particular, given certain circumstances, different approaches within the region of asymptotics can be used where non-uniformities are apparent.

In this report, I have attempted to recreate and thus solve a problem given in Porter and Chamberlain (1997). Not only do I list the results and consequences, I have also tried to make connections to the original paper, Asfar and Nayfeh (1983), which gives the method to the problem posed. I assume a background knowledge in elementary asymptotics, as I should have explained anything more complicated in detail.

Firstly, in section two, I work through the problem posed in Porter and Chamberlain (1997). This will be done, omitting only details from the paper that are not relevant to the problem we are focusing on, in a purely mathematical context.

In section three, we rigorously go through the example given in Asfar and Nayfeh (1983) for electro-magnetics. We do this by first applying a regular series expansion to the governing equations set, and upon showing that this fails, we then show that by applying "multiple scales" that the non-uniformity will not affect the results.

In section four, we apply the method shown in Asfar and Nayfeh (1983) to the problem posed in Porter and Chamberlain (1997) and analyse the result found.

For a more advanced treatment of the material presented here, please refer to the works listed in the References section.

2 A Regular Asymptotic Expansion

2.1 General Problem

Porter and Chamberlain (1997) begin by considering a fairly general problem in linear water-wave theory. As a base for the problem, they consider a fluid that is incompressible, irrotational, inviscid and time harmonic. In order to set up this problem, we can suppose that x, y and z are cartesian coordinates with $z = 0$ being the position of the fluids free-surface. Our velocity potential $\phi(x, y, z)$ satisfies

$$\left.\begin{array}{ll} \nabla^2\phi = 0 & (-h < z < 0) \\ \phi_z - \nu\phi = 0 & (z = 0) \\ \phi_z + \nabla_h h \cdot \nabla_h \phi = 0 & (z = -h) \end{array}\right\},$$

where $\nabla_h = (\partial/\partial x, \partial/\partial y)$ and $\nu = \omega^2/g$, ω being the given angular frequency and g the acceleration due to gravity. The condition at $z = 0$, (the free-surface), actually satisfies a rectangular shape, therefore we know that the free surface we are modelling is flat with no perturbations. Also the bed condition at $z = -h$ tells us that there is no normal flow at the bottom.

If $h(x, y)$ is constant then

$$\begin{aligned} \nabla_h h &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) h, \\ &= 0, \end{aligned}$$

thus we can substitute it to get the boundary condition

$$\begin{aligned} \nabla_h h \cdot \nabla_h \phi &= 0 \cdot \nabla_h \phi, \\ &= 0. \end{aligned}$$

Therefore our governing system of equations becomes

$$\left.\begin{array}{ll} \nabla^2\phi = 0 & (-h < z < 0) \\ \phi_z - \nu\phi = 0 & (z = 0) \\ \phi_z = 0 & (z = -h) \end{array}\right\}. \tag{1}$$

The new bed condition also has the criteria that there would be no normal flow at $z = -h$.

2.2 Separation Solutions

Here we suppose that h is a constant and seek separation solutions. By using separation of variables, we can find a general solution. Begin by letting $\phi(x,y,z) = X(x,y)Z(z)$. For $-h < z < 0$, our substitution transforms our equation into

$$Z(z){\nabla_h}^2 X(x) + Z''(z)X(x) = 0,$$

and therefore

$$\frac{{\nabla_h}^2 X(x)}{X(x)} = -\frac{Z''(z)}{Z(z)} = \lambda,$$

where λ is the separation constant and prime denotes differentiation with respect to z.

If we first consider the case for the function $Z(z)$, we have

$$Z''(z) + \lambda Z(z) = 0. \tag{2}$$

There are now three separate cases to consider, according to the sign of λ.

When $\lambda = 0$, equation (2) becomes

$$Z''(z) = 0$$

and for this we obtain the general solution

$$Z(z) = Az + B,$$

where A and B are constants.

We can evaluate A and B using the boundary conditions. The boundary condition at $z = -h$ is

$$Z'(-h) = 0,$$

so we have

$$Z'(-h) = A.$$

Therefore A must be zero for the boundary condition to be satisfied at $z = -h$. Now we use the boundary condition at $z = 0$,

$$Z'(0) - \nu Z(0) = 0.$$

So we have

$$\begin{aligned} Z'(0) - \nu Z(0) &= A - \nu(A(0) + B), \\ &= -\nu B, \end{aligned}$$

therefore B must also equal zero for the condition at $z = 0$ to be satisfied. We have found $A = B = 0$. When putting these values into our general solution we will obtain

$$\begin{aligned} Z(z) &= (0)z + (0), \\ &= 0. \end{aligned}$$

We can therefore deduce that $\lambda \neq 0$ as it does not give us any nontrivial solutions.

Now we can consider when $\lambda > 0$. For this we will let $\lambda = k^2$, where $k \in \mathbb{R}$. Equation (2) becomes

$$Z''(z) + k^2 Z(z) = 0,$$

and the general solution for this equation is

$$Z(z) = A \cos kz + B \sin kz.$$

This time, we will manipulate the general solution to satisfy the boundary conditions. So if we use the addition formula, we can then express our general solution as

$$Z(z) = A \cos k(z + h),$$

which now also satisfies our boundary condition at $z = -h$. Next we use the boundary condition for $z = 0$ to get

$$Z'(0) - \nu Z(0) = 0.$$

So we have

$$Z'(0) - \nu Z(0) = -Ak \sin k(0 + h) - \nu A \cos k(0 + h),$$

which gives us

$$-Ak\sin kh - \nu A\cos kh = 0,$$

and therefore

$$\begin{aligned}\nu &= -\frac{k\sin kh}{\cos kh},\\ &= -k\tan kh.\end{aligned}$$

This equation describes a dispersion relation, as illustrated in the figure below.

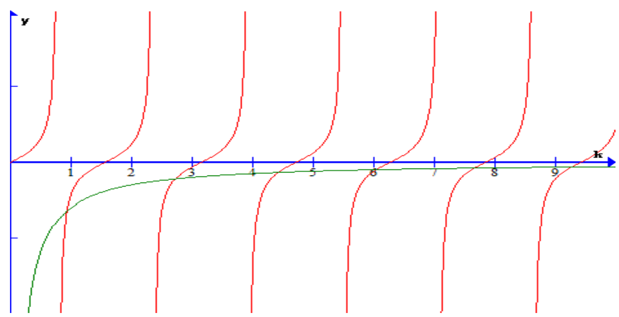


Figure 1: Graph of dispersion relation for $\lambda > 0$ where $\nu = 3$, $h = 2$.

For this graph, we have recognized that

$$-\frac{\nu}{k} = \tan kh,$$

therefore we have plotted

$$-\frac{\nu}{k} = y = \tan kh.$$

As we can see, this equation has an infinite number of roots, indicating an infinite number of waves, therefore we shall label k as k_n (k_n denoting the nth root). So our dispersion relation becomes

$$\nu = -k_n\tan k_n h. \tag{3}$$

We can also consider the horizontal component for when $\lambda > 0$. For this setup we have the general solution

$$X(x) = A_n e^{k_n x} + B_n e^{-k_n x}.$$

Note that because λ is positive and real in this case, then the terms are evanescent and can be neglected.

Now consider when $\lambda < 0$, for this condition we will let $\lambda = -k^2$, where $k \in \mathbb{R}$. Equation (2) becomes

$$Z''(z) - k^2 Z(z) = 0.$$

For this equation we have the general solution

$$Z(z) = A\cosh kz + B\sinh kz.$$

We can repeat the process of manipulating the solution to satisfy the boundary condition. Using the addition formula, we get

$$Z(z) = A\cosh k(z+h). \tag{4}$$

This satisfies the boundary condition at $z = -h$. Next we consider the boundary condition for $z = 0$,

$$Z'(0) - \nu Z(0) = 0.$$

So we have,

$$Z'(0) - \nu Z(0) = Ak\sinh k(0+h) - \nu A\cosh k(0+h),$$

which gives us

$$Ak\sinh kh - \nu A\cosh kh = 0,$$

and therefore

$$\nu = \frac{k\sinh kh}{\cosh kh},$$

which gives us

$$\nu = k\tanh kh. \tag{5}$$

This equation is illustrated below.

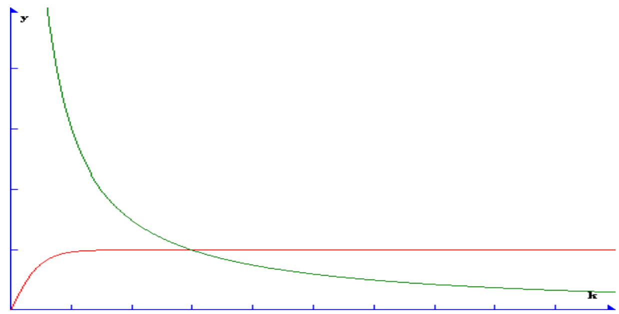


Figure 2: Graph of dispersion relation for $\lambda < 0$ where $\nu = 3$, $h = 2$.

As we can see from this graph, there is only one positive root. This tells us that there is one wave propagating for $k > 0$. Because this function is symmetric, we also have the same root for $-k$, $k < 0$. This tells us that there is the one wave, moving both to the left and to the right. To gain any information about how this wave is moving, we should consider the horizontal component of the wave. For the general solution concerning the $X(x)$ component we have

$$X(x) = Ae^{ikx} + Be^{-ikx}.$$

Because in this case $\lambda < 0$ and the terms are imaginary, we obtain two propagating modes. This accounts for the wave moving parallel to the x axis in both positive and negative directions.

2.3 Localized Bed Perturbations

We can illustrate a continuous bed profile with the function

$$\left\{ \begin{array}{ll} h_0, & (x \leq 0, x \geq l) \\ h(x), & (0 < x < l) \end{array} \right\},$$

where l is the length of the region of the non-uniform bed. To consider a small perturbation, we write

$$h(x) = h_0 - \epsilon\zeta(x), \tag{6}$$

where $\zeta(x)$ is a given bounded function and $0 < \epsilon \ll 1$. The new bed profile specifies that we are only considering the setup in two dimensions, where we had been considering it in three.

Now we want to approximate the bed condition in (1) to $O(\epsilon)$. We will concentrate on the equation at $z = -h$, because it will be the most affected by the new perturbation. Note that because we are redefining h to be a function of x, we need to revert back to our previous boundary condition. To focus on the boundary condition at $z = -h(x)$, we have

$$\phi_z + \nabla_h h \cdot \nabla_h \phi = 0,$$

that is

$$\phi_z(x, -h(x)) + h_x \phi_x(x, -h(x)) = 0.$$

Now substitute in the bed perturbation, $h(x) = h_0 - \epsilon\zeta(x)$, to get

$$\phi_z(x, -h_0 + \epsilon\zeta) - \epsilon\zeta_x \phi_x(x, -h_0 + \epsilon\zeta) = 0. \tag{7}$$

In order to progress, we need to linearise the above expression. Since we have assumed that ϵ is small, we can use Taylor series to expand the terms, i.e.

$$\phi_z(x, -h_0 + \epsilon\zeta) = \phi_z(x, -h_0) + \epsilon\zeta\phi_{zz}(x, -h_0) + O(\epsilon^2)$$

and

$$\phi_x(x, -h_0 + \epsilon\zeta) = \phi_x(x, -h_0) + \epsilon\zeta\phi_{xz}(x, -h_0) + O(\epsilon^2).$$

We can substitute the expansions into equation (7) to obtain

$$\phi_z(x, -h_0) + \epsilon\zeta\phi_{zz}(x, -h_0) - \epsilon\zeta_x \left[\phi_x(x, -h_0) + \epsilon\zeta\phi_{xz}(x, -h_0)\right] = 0.$$

For ease of notation, write $\phi(x, -h_0) = \phi$, so we have

$$\phi_z + \epsilon\zeta\phi_{zz} - \epsilon\zeta_x\phi_x + O(\epsilon^2) = 0,$$

and so we can rearrange to obtain

$$\begin{aligned} \phi_z &= \epsilon(\zeta_x\phi_x - \zeta\phi_{zz}), \qquad (8) \\ &= \epsilon(\zeta_x\phi_x + \zeta\phi_{xx}), \end{aligned}$$

where we have used the fact that $\phi_{xx} = -\phi_{zz}$. In order to progress, we must consider a particular incident wave, a wave which will propagate and result in a transmitted and reflected wave. We can use the expansion $\phi = \phi_0 + \epsilon\phi_1 + O(\epsilon^2)$ to substitute into equation (1) to get a set of equations of $O(\epsilon)$. To help, we can define the incident wave as

$$\phi_0(x, z) = e^{ikx} Z_0(h_0, z), \tag{9}$$

where $Z_0(h_0, z)$ is the equivalent to the boundary condition found previously in the form $Z(h, z) = A \cosh k(z + h)$ in Porter and Chamberlain (1997). In $-h_0 < z < 0$, we have

$$\nabla^2 \phi = 0,$$

which will give us

$$\phi_{0xx} + \phi_{0zz} + \epsilon(\phi_{1xx} + \phi_{1zz}) + O(\epsilon^2) = 0.$$

The terms at order $O(\epsilon)$ are

$$\phi_{1xx} + \phi_{1zz} = 0. \tag{10}$$

For the boundary at $z = 0$, we have

$$\phi_z - \nu\phi = 0,$$

and after substituting in our asymptotic expansion, we obtain

$$\phi_{0z} + \epsilon\phi_{1z} - \nu\phi_0 - \nu\epsilon\phi_1 + O(\epsilon^2) = 0.$$

So the $O(\epsilon^1)$ terms are

$$\phi_{1z} - \nu\phi_1 = 0. \tag{11}$$

It is more complicated to determine the boundary condition satisfied by ϕ_1 at the bed, since it proves to be inhomogeneous and includes terms invoking ϕ_0. By substituting our regular expansion into the boundary condition

$$\phi_z + \nabla_h h \cdot \nabla_h \phi = 0,$$

we have

$$(\phi_0 + \epsilon\phi_1 + O(\epsilon^2))_z + \nabla_h h \cdot \nabla_h(\phi_0 + \epsilon\phi_1 + O(\epsilon^2)) = 0,$$

evaluated at $z = -h(x)$. We can be more explicit and say

$$\phi_{0z}(x, -h_0 + \epsilon\zeta) + \epsilon\phi_{1z}(x, -h_0 + \epsilon\zeta) - \epsilon\zeta_x\phi_{0x}(x, -h_0 + \epsilon\zeta) + O(\epsilon^2) = 0,$$

and from using Taylor series and taking the only terms necessary, this becomes

$$\phi_{0z}(x, -h_0) + \epsilon\zeta\phi_{0zz}(x, -h_0) + \epsilon\phi_{1z}(x, -h_0) - \epsilon\zeta_x\phi_{0x}(x, -h_0) + O(\epsilon^2) = 0.$$

The $O(\epsilon^1)$ terms are

$$\phi_{1z} = \zeta_x\phi_{0x} - \zeta\phi_{0zz} = \zeta_x\phi_{0x} + \zeta\phi_{0xx}. \tag{12}$$

Because equation (12) contains ϕ_0 we must use the incident wave, equation (9), to complete our expression. From equation (9) we know that

$$\phi_{0x} = ike^{ikx}Z_0(h_0, z),$$

and

$$\phi_{0xx} = -k^2e^{ikx}Z_0(h_0, z).$$

If we substitute the above expressions into equation (12) we obtain

$$\phi_{1z} = \zeta_x Z_0(h_0, z)ike^{ikx} - \zeta Z_0(h_0, z)k^2e^{ikx}.$$

From equation (4), we can work out that at $z = -h_0$, we have

$$Z_0(h_0, -h_0) = A_0 \cosh k(0) = A_0.$$

We can use this to obtain

$$\begin{aligned} \phi_{1z} &= \zeta_x\ A_0ike^{ikx} - \zeta A_0k^2e^{ikx}, \\ &= e^{ikx}A_0(\zeta_x ik - \zeta k^2), \\ &= ike^{ikx}A_0(\zeta_x + ik\zeta). \end{aligned} \tag{13}$$

Now we have a new set of governing equations at $O(\epsilon^1)$,

$$\left.\begin{array}{ll} \phi_{1xx} + \phi_{1zz} = 0 & (-h_0 < z < 0) \\ \phi_{1z} - \nu\phi_1 = 0 & (z = 0) \\ \phi_{1z} = ikA_0(h_0)(\zeta_x + ik\zeta)e^{ikx} & (z = -h_0) \end{array}\right\}. \tag{14}$$

To gain an explicit expression for ϕ_1 we must now go a step further and integrate the new boundary condition at $z = -h_0$. We do this using Green's function for $-h_0 < z < 0$ which satisfies

$$\left.\begin{array}{ll} \nabla^2 G = -\delta(x - x_0)\delta(z - z_0) & (-h_0 < z < 0, -\infty < x < \infty) \\ G_z - \nu G = 0 & (z = 0) \\ G_z = 0 & (z = -h_0) \end{array}\right\}, \tag{15}$$

as $x \to \infty$. The solution to Green's function, $G(x, z, x_0, z_0)$, would be

$$G(x, z, x_0, z_0) = \frac{i}{2k} Z_0(h_0, z) Z_0(h_0, z_0) e^{ik|x-x_0|} + \sum_{n=1}^{\infty} \frac{1}{2k_n} Z_n(h_0, z) Z_n(h_0, z_0) e^{-k_n|x-x_0|}.$$

If we consider this wave as it goes to $\pm\infty$, then for $n \geq 1$, this equation contributes very little and can therefore be neglected. So if we consider this change then we can express Green's function as

$$G(x, -h_0, x_0, z_0) \sim \frac{i}{2k} Z_0(h_0, -h_0) Z_0(h_0, z_0) e^{ik|x-x_0|}.$$

So the solution as $x \to \pm\infty$ becomes

$$G(x, -h_0, x_0, z_0) = \frac{i}{2k} A_0 Z_0(h_0, z_0) e^{ik|x-x_0|}. \tag{16}$$

Now from substituting Green's function into equation (13), we can obtain the integral

$$\begin{aligned} \phi_1(x_0, z_0) &= -ikc_0 \int_0^l G(x, -h_0, x_0, z_0)(\zeta'(x) + ik\zeta(x))e^{ikx} \, dx, \\ &\sim -ikc_0 \int_0^l \frac{i}{2k} A_0 Z_0(h_0, z_0) e^{ik|x-x_0|}(\zeta'(x) + ik\zeta(x))e^{ikx} \, dx, \\ &= \frac{1}{2} {c_0}^2 Z_0(h_0, z_0) \int_0^l (\zeta'(x) + ik\zeta(x)) e^{ikx+ik|x-x_0|} \, dx, \end{aligned} \tag{17}$$

as $x \to \pm\infty$.

2.4 Regular Series Expansion

An example of a small amplitude ripple bed were to be if we let

$$l = nl_0$$

and

$$\zeta(x) = \sin\left(\frac{2\pi x}{l_0}\right),$$

for $0 \leq x \leq l$ and where l_0 is the length of one ripple. Now we can consider what would happen for this example, by substituting the expressions for ζ and ζ_x into the integral and seeing what happens as $x_0 \to \pm\infty$.

As $x_0 \to \infty$ the integral in equation (17) would tend to zero. This is because the exponential term would simplify to e^{ikx_0}. This would then give us an integral with its only dependence on x in the part of $(\zeta'(x) + ik\zeta(x))$, which will equal zero.

We should also consider the case when $x_0 \to -\infty$. We would then have to evaluate the integral in more depth. We begin with

$$I_n = e^{-ikx_0} \int_0^{nl_0} (\zeta_x + ik\zeta)\, e^{2ikx} dx.$$

After substituting the ζ terms into I_n, we get

$$I_n = e^{-ikx_0} \int_0^{nl_0} \left[\frac{2\pi}{l_0} \cos\left(\frac{2\pi x}{l_0}\right) + ik \sin\left(\frac{2\pi x}{l_0}\right)\right] e^{2ikx} dx.$$

Then from integration by parts we can obtain

$$\begin{aligned} I_n &= e^{-ikx_0} \left\{ \frac{2\pi}{l_0} \left[\frac{e^{2ikx} {l_0}^2}{-4k^2 l_0^2 + 4\pi^2} \left(2ik \cos\left(\frac{2\pi x}{l_0}\right) + \frac{2\pi}{l_0} \sin\left(\frac{2\pi x}{l_0}\right)\right)\right]_0^{nl_0} \right. \\ &+ \left. ik \left[\frac{e^{2ikx} {l_0}^2}{-4k^2 l_0^2 + 4\pi^2} \left(2ik \sin\left(\frac{2\pi x}{l_0}\right) - \frac{2\pi}{l_0} \cos\left(\frac{2\pi x}{l_0}\right)\right)\right]_0^{nl_0} \right\}, \end{aligned}$$

which gives us

$$I_n = \frac{e^{-ikx_0}}{-4k^2 {l_0}^2 + 4\pi^2} \left(\frac{2\pi}{l_0}\left[e^{2iknl_0} {l_0}^2 2ik - {l_0}^2 2ik\right] - ik\left[e^{2iknl_0} {l_0}^2 \frac{2\pi}{l_0} - {l_0}^2 \frac{2\pi}{l_0}\right]\right).$$

We should next collect common terms of e^{2iknl_0} to get

$$I_n = \frac{e^{-ikx_0}}{-4k^2 {l_0}^2 + 4\pi^2} \frac{2\pi}{l_0} \left[e^{2iknl_0}\left({l_0}^2 2ik - ik{l_0}^2\right) - \left({l_0}^2 2ik - {l_0}^2 ik\right)\right].$$

This can be simplified to become

$$I_n = \frac{e^{-ikx_0}}{-4k^2 {l_0}^2 + 4\pi^2} \pi l_0 ik \left[e^{2iknl_0} - 1\right]. \tag{18}$$

Note that after some heavy manipulation $e^{2iknl_0} - 1$ can be written as

$$e^{2iknl_0} - 1 = 2\left(i\cos(knl_0) - \sin(knl_0)\right)\sin(knl_0). \tag{19}$$

Expression (19) can then be substituted into equation (18) to give us

$$
\begin{aligned}
I_n &= \frac{e^{-ikx_0}}{-4k^2{l_0}^2+4\pi^2}\pi l_0 ik\left[2\left(i\cos(knl_0)-\sin(knl_0)\right)\sin(knl_0)\right],\\
&= \frac{e^{-ikx_0}}{-4k^2{l_0}^2+4\pi^2}\pi l_0 ik\left[-2\sin(knl_0)e^{iknl_0}\right].
\end{aligned}
$$

Now if we substitute out integral back into the equation (17), we obtain

$$
\phi_1(x,z)\sim\frac{1}{2}{c_0}^2(h_0)Z_0(h_0,z)e^{-ik(x-nl_0)}\frac{\pi kl_0\sin(nkl_0)}{(kl_0)^2-\pi^2}\qquad(x\to-\infty). \tag{20}
$$

Now we can go ahead and consider what problems may lie within this equation, this will tell us whether our regular series expansion has been a good approximation. The weakness in this equation clearly lies in the denominator, so if we were to let $kl_0=\pi$, then equation (20) becomes

$$
\phi_1(x,z)=\frac{1}{2}{c_0}^2(h_0)Z_0(h_0,z)e^{-ik(x-nl_0)}\frac{\pi kl_0\sin(n\pi)}{(\pi)^2-\pi^2}\qquad(x\to-\infty),
$$

which gives us

$$
\phi_1(x,z)=\frac{1}{2}{c_0}^2(h_0)Z_0(h_0,z)e^{-ik(x-nl_0)}\left(\frac{0}{0}\right).
$$

To evaluate this further, we should use L'Hopitals rule, giving

$$
\begin{aligned}
\lim_{kl_0\to\pi}\frac{\sin(nkl_0)}{(kl_0)^2-\pi^2} &= \lim_{kl_0\to\pi}\frac{n\cos(nkl_0)}{2kl_0},\\
&= \frac{n\cos(n\pi)}{2\pi},
\end{aligned}
$$

which can be expressed as

$$
\lim_{kl_0\to\pi}\frac{\sin(nkl_0)}{(kl_0)^2-\pi^2}=\frac{(-1)^n n}{2\pi}.
$$

So we have ϕ_1 in the form of

$$
\phi_1=\frac{1}{2}{c_0}^2(h_0)Z_0(h_0,z)e^{-ik(x-nl_0)}\frac{(-1)^n n}{2\pi}. \tag{21}
$$

Here we should remind ourselves of our regular series expansion, which was

$$
\phi=\phi_0+\epsilon\phi_1+O(\epsilon^2).
$$

So if we consider as the number of ripples, $n\to\infty$, the ratio of the second term (ϕ_1) to the first term (ϕ_0) is no longer small for when $n=O(\epsilon^{-1})$. Therefore the expansion is *only* valid for $n \ll \epsilon^{-1}$. Now we know that the regular series expansion is not uniform and therefore there is a known non-uniformity for $k \sim \pi/l_0$. This result exhibits Bragg resonance. Because a regular series expansion has not worked, we must now try a different method to approximate the governing equation and boundary conditions.

3 Two-Dimensional Electromagnetic Waveguides

3.1 Regular Series Expansion

To find an improved approximation to the problem posed in Porter and Chamberlain (1997), we shall consult the paper by Asfar and Nayfeh (1983) which deals with several types of cases of wave propagation within ducts with periodic wall perturbations. The example we are interested in is an electromagnetic case with multiple transverse magnetic propagating waves, which can be applied equally well to a water-wave case. Note that for Porter and Chamberlain (1997) we have only the one wave which travels in opposite directions to consider, whilst in Asfar and Nayfeh (1983) there will be n-many. The setup for the problem in this paper is to have two parallel wall distortion functions: one on the right, $x = 1 + \delta \sin(k_w z + \tau)$ and one on the left, $x = \delta \sin(k_w z)$ where τ is the phase shift and δ is the dimensionless amplitude of the wall undulations, assumed to be much smaller than unity.

The governing equation for the flow in this problem is the Helmholtz equation in the form of

$$\nabla^2 \psi + k^2 \psi = 0, \tag{22}$$

where $\nabla = \partial/\partial x + \partial/\partial z$, k is the free-space dimensionless wavenumber and ψ is the wave function. The boundary conditions at the two perturbed walls are

$$\left(\frac{\partial^2}{\partial z^2} + k^2 \right) \psi + \delta k_w \cos(k_w z + \tau) \frac{\partial^2 \psi}{\partial x \partial z} = 0 \tag{23}$$

for $x = 1 + \delta \sin(k_w z + \tau)$, and

$$\left(\frac{\partial^2}{\partial z^2} + k^2 \right) \psi + \delta k_w \cos(k_w z) \frac{\partial^2 \psi}{\partial x \partial z} = 0 \tag{24}$$

for $x = \delta \sin(k_w z)$.

Now, as we did in Porter and Chamberlain (1997), we use an asymptotic expansion of the form

$$\psi(x, z) = \psi_0(x, z) + \delta \psi_1(x, z) + ...$$

where δ has taken the place of ϵ in section 2. We use this expansion in order to find out what happens at $O(\delta^1)$ to the propagating waves.

As we had shown in Porter and Chamberlain (1997), an asymptotic expansion may give invalid results. Asfar and Nayfeh (1983) recreates this and shows how to solve the problem.

First we substitute our asymptotic expansion into equation (22), to get

$$\nabla^2(\psi_0 + \delta\psi_1 + ...) + k^2(\psi_0 + \delta\psi_1 + ...) = 0.$$

Therefore at leading order we obtain

$$\nabla^2\psi_0 + k^2\psi_0 = 0, \tag{25}$$

and at $O(\delta^1)$ we obtain

$$\nabla^2\psi_1 + k^2\psi_1 = 0. \tag{26}$$

Now we should do the same thing to the left and right plates. To do this we must linearise the boundaries at $x = 0$ and $x = 1$ by developing ψ and its derivatives using Taylor series. For the right plate, at $x = 1+\delta\sin(k_w z+\tau)$, we substitute the asymptotic expansion into equation (24), to obtain

$$\begin{aligned} F(1+\delta\sin(k_w z+\tau), z) &\equiv \left(\frac{\partial^2}{\partial z^2} + k^2\right)(\psi_0 + \delta\psi_1 + ...) \\ &+ \delta k_w \cos(k_w z + \tau)\frac{\partial^2}{\partial x \partial z}(\psi_0 + \delta\psi_1 + ...) = 0. \end{aligned}$$

To linearise this equation, we use

$$F(1+\delta\sin(k_w z+\tau), z) = F(1, z) + \delta\sin(k_w z+\tau)F_x(1, z) + O(\delta^2).$$

Because there is no dependence on x, $F(1, z)$ is equivalent to $F(1+\delta\sin(k_w z+\tau), z)$. Therefore we can work out $F_x(1, z)$ in a similar way.

$$\begin{aligned} F_x(1, z) &= F_x(1+\delta\sin(k_w z+\tau), z) \\ &= \left(\frac{\partial^2}{\partial z^2} + k^2\right)\frac{\partial}{\partial x}(\psi_0 + \delta\psi_1 + ...) + \delta k_w \cos(k_w z + \tau)\frac{\partial^3}{\partial x^2 \partial z}(\psi_0 + \delta\psi_1 + ...). \end{aligned}$$

If we substitute this into our expression for $F(1+\delta\sin(k_w z+\tau), z)$, we get

$$
\begin{aligned}
F(1+\delta\sin(k_w z+\tau), z) &\equiv \left(\frac{\partial^2}{\partial z^2}+k^2\right)(\psi_0+\delta\psi_1+...) \\
+ \quad & \delta k_w\cos(k_w z+\tau)\frac{\partial^2}{\partial x\partial z}(\psi_0+\delta\psi_1+...) \\
+ \quad & \delta\sin(k_w z+\tau)\left[\left(\frac{\partial^2}{\partial z^2}+k^2\right)\frac{\partial}{\partial x}(\psi_0+\delta\psi_1+...)\right. \\
+ \quad & \left.\delta k_w\cos(k_w z+\tau)\frac{\partial^3}{\partial x^2\partial z}(\psi_0+\delta\psi_1+...)\right] = 0.
\end{aligned}
$$

We can equate coefficients of leading order to obtain the equation

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_0 = 0. \tag{27}$$

To $O(\delta^1)$ we obtain

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_1 = -\sin(k_w z+\tau)\left(\frac{\partial^2}{\partial z^2}+k^2\right)\frac{\partial\psi_0}{\partial x} - k_w\cos(k_w z+\tau)\frac{\partial^2\psi_0}{\partial x\partial z}. \tag{28}$$

The left plate can be solved using the same method. We again equate the coefficients of leading order to obtain the equation

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_0 = 0, \tag{29}$$

and of $O(\delta^1)$ we have

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_1 = -\sin(k_w z)\left(\frac{\partial^2}{\partial z^2}+k^2\right)\frac{\partial\psi_0}{\partial x} - k_w\cos(k_w z)\frac{\partial^2\psi_0}{\partial x\partial z}, \tag{30}$$

for $x=0$. A general solution can be found for ψ_0 from considering equations (25), (27) and (29). We begin by considering equation (25), then by writing $\psi_0(x,z) = X(x)Z(z)$, we can express equation (25) as

$$X''(x)Z(z) + Z''(z)X(x) + k^2X(x)Z(z) = 0,$$

to get

$$X(x)\left[Z''(z)+k^2Z(z)\right] = -X''(x)Z(z),$$

and therefore

$$\frac{Z''(z)+k^2Z(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \lambda,$$

where λ is the separation constant. For $X(x)$ and $Z(z)$ we have

$$X''(x) + \lambda X(x) = 0 \tag{31}$$

and

$$Z''(z) + (k^2 - \lambda)Z(z) = 0. \tag{32}$$

There are now three separate cases to consider,

1. $\lambda = 0$
2. $\lambda < 0$
3. $\lambda > 0$.

When $\lambda = 0$, equation (31) becomes

$$X''(x) = 0$$

and for this we obtain the general solution

$$X(x) = Ax + B,$$

where A and B are constants. Also for equation (32), we obtain

$$Z''(z) + k^2 Z(z) = 0$$

which has a general solution of the form

$$Z(z) = Ce^{ikz} + De^{-ikz},$$

where C and D are also constants. This then tells us that ψ is of the form

$$\psi_0 = X(x)Z(z) = (Ax + B)(Ce^{ikz} + De^{-ikz}).$$

We can evaluate A, B, C and D using the boundary conditions. The boundary condition at $x = 0$ is

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_0 = 0,$$

so we have

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi(0, z) &= -(A(0) + B)k^2(Ce^{ikz} + De^{-ikz}) \\ &+ k^2(A(0) + B)(Ce^{ikz} + De^{-ikz}).\end{aligned}$$

As you can see this does equal zero and therefore satisfies the boundary condition at $x = 0$. This is also the case for the boundary condition at $x = 1$ as it is the same condition. By considering different values of x, we can then conclude that the boundary condition is satisfied for all values of x.

If we consider when $\lambda < 0$, we can let $\lambda = -m^2$, where $m \in \mathbb{R}$, to gain the general equation

$$X''(x) - m^2 X(x) = 0,$$

and for this we obtain the general solution

$$X(x) = A\cosh mx + B\sinh mx.$$

We can also gain the equation

$$Z''(z) + (k^2 + m^2)Z(z) = 0$$

which has the general solution

$$Z(z) = C\cosh(\sqrt{k^2+m^2}z) + D\sinh(\sqrt{k^2+m^2}z).$$

This then gives us ψ_0 of the form

$$\psi_0 = X(x)Z(z) = (A\cosh mx + B\sinh mx)(C\cosh(\sqrt{k^2+m^2}z) + D\sinh(\sqrt{k^2+m^2}z)).$$

To find the four arbitrary constants A, B, C and D we should consider the boundary conditions at the right and left plates. The boundary condition at $x = 0$ is

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_0 = 0,$$

so we have

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_0(0,z) = (A\cosh m(0) + B\sinh m(0))(k^2+m^2)(C\cosh(\sqrt{k^2+m^2}z)\\
+\quad & D\sinh(\sqrt{k^2+m^2}z)) + k^2(A\cosh m(0) + B\sinh m(0))(C\cosh(\sqrt{k^2+m^2}z)\\
+\quad & D\sinh(\sqrt{k^2+m^2}z)).
\end{aligned}$$

This can be simplified to

$$(2k^2 + m^2)\psi_0(0,z) = 0.$$

Both k and m cannot equal zero, therefore $\psi_0 = 0$, so we have

$$A(C\cosh(\sqrt{k^2+m^2}z) + D\sinh(\sqrt{k^2+m^2}z)) = 0.$$

For the expression above to be satisfied, $A = 0$, then the boundary condition at $x = 0$ will be satisfied. Now if we consider when $x = 1$, we have

$$B\sinh m(1)(C\cosh(\sqrt{k^2+m^2}z) + D\sinh(\sqrt{k^2+m^2}z)) = 0$$

Since we have defined $m \neq 0$, we must therefore make $B = 0$ so the boundary condition remains valid,thus not giving us a general solution. So we must investigate further, considering the case when $\lambda > 0$. When this is the case we can make $\lambda = m^2$, equation (31) becomes

$$X''(x) + m^2X(x) = 0,$$

which gives us a general solution of the form

$$X(x) = A\sin mx + B\cos mx.$$

From equation (32) we also have

$$Z''(z) + (k^2 - m^2)Z(z) = 0,$$

which has a general solution of the form

$$Z(z) = Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z},$$

therefore we know that ψ_0 is of the form

$$\psi_0 = X(x)Z(z) = (A\sin mx + B\cos mx)(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}).$$

To find more information about the constants A, B, C and D we must again investigate further using the boundary conditions. The boundary condition at $x = 0$ is

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_0(0,z) = 0,$$

so we get

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_0(0,z) &= -(k^2-m^2)(A\sin(m(0)) + B\cos(m(0)))(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}) \\ &+ k^2(A\sin m(0) + B\cos m(0))(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}), \\ &= m^2\psi_0(0,z).\end{aligned}$$

This tells us that $\psi_0(0,z) = 0$, $\forall x \in \mathbb{R}$ since $m \neq 0$. This then gives us

$$\begin{aligned}\psi_0(0,z) &= (A\sin m(0) + B\cos m(0))(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}), \\ &= B(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}).\end{aligned}$$

Therefore $B = 0$ for the boundary condition to remain valid at $x = 0$. Now we consider when $x = 1$. We begin with

$$\psi_0(1,z) = 0,$$

so we get

$$\psi_0(1,z) = A\sin m(1)(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}).$$

For this boundary condition to be satisfied, $m = n\pi$, for $n \in \mathbb{Z}$. For this we obtain a general solution of the form

$$\psi_0 = \sum_{n=0}^{\infty} \sin(n\pi x)\left\{C_n e^{i\sqrt{k^2-(n\pi)^2}z} + D_n e^{-i\sqrt{k^2-(n\pi)^2}z}\right\}. \tag{33}$$

Asfar and Nayfeh (1983) does not have a general solution that accounts for the waves that are moving to the left, instead the general solution is

$$\psi_0(x,z) = \sum_{n=-\infty}^{\infty} A_n \sin(n\pi x) e^{ik_n z}. \tag{34}$$

We can only assume that k_n is defined in the following way,

$$k_n = \begin{cases} \sqrt{k^2-(n\pi)^2}, & n > 0 \text{ and } k^2 > (n\pi)^2 \\ i\sqrt{(n\pi)^2-k^2}, & n > 0 \text{ and } k^2 < (n\pi)^2 \\ -\sqrt{k^2-(n\pi)^2}, & n < 0 \text{ and } k^2 > (n\pi)^2 \\ -i\sqrt{(n\pi)^2-k^2}, & n < 0 \text{ and } k^2 < (n\pi)^2. \end{cases} \tag{35}$$

Because the exponential term has only imaginary powers, both forms of k_n containing 'i' will result in a real power, and therefore the mode will decay and not propagate.

Now we have our expression for ψ_0, we can generate new boundary conditions for ψ_1. We begin by focusing on the right plate. We generate the new boundary condition by substituting equation (34) into equation (28). We have

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 = -\sin(k_w z + \tau)\left[\sum_{n=-\infty}^{\infty}(-{k_n}^2 n\pi A_n \cos(n\pi x)e^{ik_n z}) \right.$$
$$\left. + k^2 \sum_{n=-\infty}^{\infty} n\pi A_n \cos(n\pi x)e^{ik_n z}\right] - k_w\cos(k_w z + \tau)\sum_{n=-\infty}^{\infty} ik_n n\pi A_n \cos(n\pi x)e^{ik_n z}.$$

The linearised position of the right plate is $x = 1$, so using the result $\cos(n\pi) = (-1)^n$, we have

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 = \sum_{n=-\infty}^{\infty}\Big[\sin(k_w z + \tau){k_n}^2 n\pi A_n(-1)^n e^{ik_n z}$$
$$-\quad \sin(k_w z + \tau)k^2 n\pi A_n(-1)^n e^{ik_n z} - k_w\cos(k_w z + \tau)ik_n n\pi A_n(-1)^n e^{ik_n z}\Big],$$

and by substituting identities for sine and cosine, we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 &= \sum_{n=-\infty}^{\infty}\frac{i}{2}\Big[(e^{i(k_w z+\tau)} - e^{-i(k_w z+\tau)})k^2 n\pi A_n(-1)^n e^{ik_n z}\\ &- (e^{i(k_w z+\tau)} - e^{-i(k_w z+\tau)}){k_n}^2 n\pi A_n(-1)^n e^{ik_n z}\\ &- (e^{i(k_w z+\tau)} + e^{-i(k_w z+\tau)})ik_w k_n n\pi A_n(-1)^n e^{ik_n z}\Big].\end{aligned}$$

By collecting common terms, we can obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 &= \frac{i}{2}\sum_{n=-\infty}^{\infty}(-1)^n n\pi A_n[e^{i(k_n+k_w)z+i\tau}(k^2 - {k_n}^2 - k_w k_n)\\ &- e^{i(k_n-k_w)z-i\tau}(k^2 - {k_n}^2 + k_w k_n)].\end{aligned}$$

Here we can use ${k_n}^2 = k^2 - (n\pi)^2$, to rearrange the above equation to get

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 &= \frac{i}{2}\sum_{n=-\infty}^{\infty}(-1)^n n\pi A_n\left[e^{i(k_n+k_w)z+i\tau}((n\pi)^2 - k_w k_n)\right. \qquad (36)\\ &- \left.e^{i(k_n-k_w)z-i\tau}((n\pi)^2 + k_n k_w)\right],\end{aligned}$$

for the right plate at $x = 1$.

The boundary condition for the left plate, at $x = 0$, uses the same method and is of the simpler form

$$\begin{aligned}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 &= \frac{i}{2}\sum_{n=-\infty}^{\infty}n\pi A_n\left[e^{i(k_n+k_w)z}((n\pi)^2 - k_w k_n)\right. \qquad (37)\\ &- \left.e^{i(k_n-k_w)z}((n\pi)^2 + k_n k_w)\right].\end{aligned}$$

The two boundary conditions suggest that ψ_1 has a particular solution of the form

$$\begin{aligned}\psi_1(x,z) = \frac{i}{2}\sum_{n=-\infty}^{\infty}n\pi A_n&\Big\{((n\pi)^2 - k_n k_w)\Phi_1(x)e^{i(k_n+k_w)z}\\ &-((n\pi)^2 + k_n k_w)\Phi_2(x)e^{i(k_n-k_w)z}\Big\}. \qquad (38)\end{aligned}$$

By substituting equation (38) into equation (26) and our recently derived boundary conditions, equations (36) and (37), we yield a governing equation and boundary conditions for both $\Phi_1(x)$ and $\Phi_2(x)$. Equation (26) will become

$$\begin{aligned}
& \frac{i}{2}\sum_{n=-\infty}^{\infty} n\pi A_n \Big\{((n\pi)^2 - k_n k_w)\frac{\partial^2}{\partial x^2}\Phi_1(x)e^{i(k_n+k_w)z} - ((n\pi)^2 + k_n k_w)\frac{\partial^2}{\partial x^2}\Phi_2(x)e^{i(k_n-k_w)z} \\
- \quad & (k_n + k_w)^2((n\pi)^2 - k_n k_w)\Phi_1(x)e^{i(k_n+k_w)z} + (k_n - k_w)^2((n\pi)^2 + k_n k_w)\Phi_2(x)e^{i(k_n-k_w)z} \\
+ \quad & k^2\Big[((n\pi)^2 - k_n k_w)\Phi_1(x)e^{i(k_n+k_w)z} - ((n\pi)^2 + k_n k_w)\Phi_2(x)e^{i(k_n-k_w)z}\Big]\Big\} = 0.
\end{aligned}$$

Now we collect coefficients of the exponential terms to obtain

$$((n\pi)^2 - k_n k_w)\left[\frac{\partial^2}{\partial x^2}\Phi_1(x) + \Phi_1(x)(k^2 - (k_n + k_w)^2)\right] = 0$$

and

$$-((n\pi)^2 + k_n k_w)\left[\frac{\partial^2}{\partial x^2}\Phi_2(x) + \Phi_2(x)(k^2 - (k_n - k_w)^2)\right] = 0,$$

which is true for all $n \in \mathbb{R}$. This tells us that

$$\frac{\partial^2}{\partial x^2}\Phi_j(x) + {\alpha_j}^2\Phi_j(x) = 0, \tag{39}$$

where $j \in \{1, 2\}$ and ${\alpha_j}^2 = k^2 - (k_n \pm k_w)^2$.

We can find Φ_1 and Φ_2 for the right plate, at $x = 1$, by first using the same method as we had for equation (26) to (28). We have

$$\begin{aligned}
& \frac{i}{2}\sum_{n=-\infty}^{\infty} n\pi A_n \Big[(k_n - k_w)^2((n\pi)^2 + k_n k_w)\Phi_2(1)e^{i(k_n-k_w)z} \\
- \quad & (k_n + k_w)^2((n\pi)^2 - k_n k_w)\Phi_1(1)e^{i(k_n+k_w)z} + k^2\{((n\pi)^2 - k_n k_w)\Phi_1(1)e^{i(k_n+k_w)z} \\
- \quad & ((n\pi)^2 + k_n k_w)\Phi_2(1)e^{i(k_n-k_w)z}\}\Big] - \frac{i}{2}\sum_{n=-\infty}^{\infty} n\pi A_n(-1)^n\{((n\pi)^2 + k_n k_w)e^{i(k_n+k_w)z+i\tau} \\
- \quad & ((n\pi)^2 + k_n k_w)e^{i(k_n-k_w)z-i\tau}\} = 0,
\end{aligned}$$

and by combining common exponential terms, we obtain

$$\begin{aligned}
& \Phi_1(1)(k^2 - (k_n + k_w)^2)((n\pi)^2 - k_n k_w)e^{i(k_n+k_w)z} \\
+ \quad & \Phi_2(1)((k_n - k_w)^2 - k^2)((n\pi)^2 + k_n k_w)e^{i(k_n-k_w)z} \\
= \quad & (-1)^n\{((n\pi)^2 - k_n k_w)e^{i(k_n+k_w)z}e^{i\tau} - ((n\pi)^2 + k_n k_w)e^{i(k_n-k_w)z}e^{-i\tau}\}.
\end{aligned}$$

By allowing $j \in \{1,2\}$ we can write this equation more succinctly as

$$\Phi_j(1)(k^2 - (k_n \pm k_w)^2) = (-1)^n e^{\pm i\tau},$$

and by again letting ${\alpha_j}^2 = k^2 - (k_n \pm k_w)^2$ we get

$$\Phi_j(1){\alpha_j}^2 = (-1)^n e^{\pm i\tau},$$

where the plus (minus) sign corresponds to Φ_1 and α_1, (Φ_2 and α_2). By dividing through by ${\alpha_j}^2$ we obtain

$$\Phi_j(1) = (-1)^n {\alpha_j}^{-2} e^{\pm i\tau}. \tag{40}$$

The boundary condition at the left plate can be found using the same method. We have

$$\begin{aligned} &(k_n - k_w)^2((n\pi)^2 + k_n k_w)\Phi_2(0)e^{i(k_n-k_w)z} - (k_n + k_w)^2((n\pi)^2 - k_n k_w)\Phi_1(0)e^{i(k_n+k_w)z} \\ + \quad &k^2[((n\pi)^2 - k_n k_w)\Phi_1(0)e^{i(k_n+k_w)z} - ((n\pi)^2 + k_n k_w)\Phi_2(0)e^{i(k_n-k_w)z}] = 0, \end{aligned}$$

which becomes

$$\begin{aligned} &\Phi_2\left[(k_n - k_w)^2((n\pi)^2 + k_n k_w) + k^2((n\pi)^2 + k_n k_w)\right] \\ = &\Phi_1\left[(k_n + k_w)^2((n\pi)^2 - k_n k_w) - k^2((n\pi)^2 - k_n k_w)\right]. \end{aligned}$$

So we obtain the equation

$$\Phi_j(0)((n\pi)^2 - k_n k_w)e^{i(k_n+k_w)z}(k^2 - (k_n + k_w)^2) = ((n\pi)^2 + k_n k_w)e^{i(k_n-k_w)z},$$

which simplifies to

$$\Phi_j(0) = {\alpha_j}^{-2}. \tag{41}$$

If we combine equations (39), (40) and (41), we can work out a solution for Φ_j. We do this by first considering the governing equation (39). The general solution for equation (39) would be

$$\Phi_j(x) = A \sin \alpha_j x + B \cos \alpha_j x. \tag{42}$$

Now we can use the boundary conditions to find the constants A and B. We begin by considering the left plate boundary at $x = 0$,

$$\Phi_j(0) = {\alpha_j}^{-2},$$

so we get

$$\begin{aligned}\Phi_j(0) &= A\sin\alpha_j(0) + B\cos\alpha_j(0),\\ &= B.\end{aligned}$$

This implies that $B = {\alpha_j}^{-2}$ for the boundary condition at $x = 0$ to be satisfied. Now we should continue by considering the boundary condition at $x = 1$,

$$\Phi_j(1) = (-1)^n {\alpha_j}^{-2} e^{\pm i\tau},$$

so we get

$$\begin{aligned}\Phi_j(1) &= A\sin\alpha_j(1) + B\cos\alpha_j(1),\\ &= A\sin\alpha_j + {\alpha_j}^{-2}\cos\alpha_j.\end{aligned}$$

We can obtain A by rearrangement, to obtain

$$\begin{aligned}A\sin\alpha_j &= (-1)^n {\alpha_j}^{-2} e^{\pm i\tau} - {\alpha_j}^{-2}\cos\alpha_j,\\ &= \left[(-1)^n e^{\pm i\tau} - \cos\alpha_j\right] {\alpha_j}^{-2},\end{aligned}$$

and by dividing both sides of the equation by $\sin\alpha_j$ we obtain

$$A = \frac{(-1)^n e^{\pm i\tau} - \cos\alpha_j}{{\alpha_j}^2 \sin\alpha_j}.$$

Now we can substitute the two constants A and B into the general solution, equation (42), to obtain

$$\Phi_j(x) = \left(\frac{(-1)^n e^{\pm i\tau} - \cos\alpha_j}{{\alpha_j}^2 \sin\alpha_j}\right)\sin\alpha_j x + {\alpha_j}^{-2}\cos\alpha_j x,$$

which can be expressed as

$$\Phi_j(x) = \frac{[(-1)^n e^{-i\tau\cos(j\pi)} - \cos\alpha_j]\sin\alpha_j x + \sin\alpha_j\cos\alpha_j x}{{\alpha_j}^2\sin\alpha_j}. \tag{43}$$

Now we want to show that equation (43) is not valid for certain parameter combinations, as we had for Porter and Chamberlain (1997). We do this by considering what would happen when we let $\alpha_j \to m\pi$, where $m \in \mathbb{R}$. We would obtain

$$\lim_{\alpha_j\to m\pi}\Phi_j(x) = \frac{[(-1)^n e^{-i\tau\cos(j\pi)} - \cos(m\pi)]\sin(m\pi x) + \sin(m\pi)\cos(m\pi x)}{(m\pi)^2\sin(m\pi)}.$$

Since $\sin(m\pi) = 0$, this would mean that $\Phi_j \to \infty$ as $\alpha_j \to m\pi$, therefore making the expression invalid. Unfortunately the regular series expansion is giving us a singularity as $\alpha_j \to m\pi$. Therefore we have demonstrated that expression (43) breaks down for $\alpha_j = m\pi$.

As we had defined ψ_1 as a combination of Φ_1 and Φ_2 in equation (38), this would then imply that $\psi_1 \to \infty$ as $\alpha_j \to m\pi$. This would mean again that the term of $O(\delta^1)$ in the asymptotic expansion is no longer small, and therefore the amplitude of the wave is now unbounded. This result demonstrates that a wave has been transmitted through a body of fluid within our boundaries but has not been reflected back whatsoever. This is the criteria for Bragg's condition to be satisfied, whilst in the neighborhood of

$$k_m = k_n \pm k_w.$$

3.2 Multiple Scale Expansion

We can now conclude that our regular asymptotic expansion is not sufficient to obtain a uniform representation of what happens to our governing equations at further order. To rectify this, we seek an asymptotic expansion of the form

$$\psi(x, z) = \psi_0(x, z_0, z_1) + \delta\psi_1(x, z_0, z_1) + ...$$

where $z_0 = z$ is in effect, a length scale which represents the distance of a wavelength and $z_1 = \delta z$ would be the corresponding long length scale defining the order of amplitude, so that when $z = O(1/\delta)$, $z_1 = O(1)$. The method of using long and short/fast and slow variables in order to gain a more accurate approximation to the waves propagation is called *multiple scales.*

We now begin by repeating the process we used for the regular asymptotic expansion. We should begin by calculating an $O(\delta^0)$ approximation to the governing equation

$$\nabla^2\psi + k^2\psi = 0.$$

After substituting in the asymptotic expansion, we can collect the terms of $O(\delta^0)$ to get

$$\frac{\partial^2 \psi_0}{\partial x^2} + \frac{\partial^2 \psi_0}{\partial {z_0}^2} + k^2 \psi_0 = 0. \tag{44}$$

Using the above result, we also readily deduce the boundary conditions

$$\left(\frac{\partial^2}{\partial {z_0}^2} + k^2 \right) \psi_0 = 0, \tag{45}$$

for both $x = 0$ and $x = 1$. Now, as before, we want to look for the equations of $O(\delta^1)$. when considering the governing equation these terms can be picked out, and we obtain

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial {z_0}^2} + k^2 \psi_1 = -2 \frac{\partial^2 \psi_0}{\partial z_0 \partial z_1}. \tag{46}$$

We can find the right plate boundary condition of $O(\delta^1)$, via the same method as equation (45). We can rearrange to get

$$\left(\frac{\partial^2}{\partial {z_0}^2} + k^2 \right) \psi_1 = -\sin(k_w z_0 + \tau) \left(\frac{\partial^2}{\partial {z_0}^2} + k^2 \right) \frac{\partial \psi_0}{\partial x} - k_w \cos(k_w z_0 + \tau) \frac{\partial^2 \psi_0}{\partial x \partial z_0} - 2 \frac{\partial^2 \psi_0}{\partial z_1 \partial z_0},$$

for $x = 1$. We can also work out the left plate in a very similar way, to obtain

$$\left(\frac{\partial^2}{\partial {z_0}^2} + k^2 \right) \psi_1 = -\sin(k_w z_0) \left(\frac{\partial^2}{\partial {z_0}^2} + k^2 \right) \frac{\partial \psi_0}{\partial x} - k_w \cos(k_w z_0) \frac{\partial^2 \psi_0}{\partial x \partial z_0} - 2 \frac{\partial^2 \psi_0}{\partial z_1 \partial z_0},$$

for $x = 0$.

As we did before, we are looking for a general solution for ψ_0, but this time we have two wave numbers, k_m and k_n. This is because the two waves that we are concentrating on are dominant in comparison to the other waves. We have shown that the interaction between the k_m and k_n wave numbers is of importance, regardless of whether we include the other wave numbers, so they can be neglected. Instead of having the one term for our general solution, we now have a two-term general solution, in the form of

$$\psi_0 = A_n(z_1) \sin(n\pi x) e^{i k_n z_0} + A_m(z_1) \sin(m\pi x) e^{i k_m z_0}, \tag{47}$$

thus incorporating both wave numbers. A_m and A_n are the amplitudes to be found at the next order of approximation.

Now we have a new expression for ψ_0 we can substitute this into our equations of $O(\delta^1)$. First for our governing equation (46), we see that

$$\frac{\partial^2\psi_0}{\partial z_0 \partial z_1} = A_n'(z_1)ik_n \sin(n\pi x)e^{ik_n z_0} + A_m'(z_1)ik_m \sin(m\pi x)e^{ik_m z_0},$$

giving

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial {z_0}^2} + k^2\psi_1 = -2i\sum_{j=m,n} A_j'(z_1)k_j \sin(j\pi x)e^{ik_j z_0}. \tag{48}$$

We shall now do the same thing for the boundary condition for the right plate at $x = 1$. Firstly

$$\begin{aligned}
\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 &= -\sin(k_w z_0 + \tau)\left[-{k_n}^2 A_n(z_1)n\pi(-1)^n e^{ik_n z_0}\right.\\
-\quad & \left.{k_m}^2 A_m(z_1)m\pi(-1)^m e^{ik_m z_0} + k^2(A_n(z_1)n\pi(-1)^n e^{ik_n z_0} + A_m(z_1)m\pi(-1)^m e^{ik_m z_0})\right]\\
-\quad & k_w \cos(k_w z_0 + \tau)\left[ik_n A_n(z_1)n\pi(-1)^n e^{ik_n z_0} + ik_m A_m(z_1)m\pi(-1)^m e^{ik_m z_0}\right],
\end{aligned}$$

and by using identities for sine and cosine, we can obtain

$$\begin{aligned}
\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 &= -\left(\frac{e^{i(k_w z_0+\tau)} - e^{-i(k_w z_0+\tau)}}{2i}\right)\left[-{k_n}^2 A_n(z_1)n\pi(-1)^n e^{ik_n z_0}\right.\\
-\quad & \left.{k_m}^2 A_m(z_1)m\pi(-1)^m e^{ik_m z_0} + k^2(A_n(z_1)n\pi(-1)^n e^{ik_n z_0} + A_m(z_1)m\pi(-1)^m e^{ik_m z_0})\right]\\
-\quad & \left(\frac{e^{i(k_w z_0+\tau)} + e^{-i(k_w z_0+\tau)}}{2}\right)k_w\left[ik_n A_n(z_1)n\pi(-1)^n e^{ik_n z_0} + ik_m A_m(z_1)m\pi(-1)^m e^{ik_m z_0}\right].
\end{aligned}$$

By collecting like terms, we can proceed to get

$$\begin{aligned}
\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 &= \frac{i\pi}{2}\left\{(e^{i(k_w z_0+\tau)} - e^{-i(k_w z_0+\tau)})\left[-{k_n}^2 A_n(z_1)n(-1)^n e^{ik_n z_0}\right.\right.\\
-\quad & \left.{k_m}^2 A_m(z_1)m(-1)^m e^{ik_m z_0} + k^2(A_n(z_1)n(-1)^n e^{ik_n z_0} + A_m(z_1)m(-1)^m e^{ik_m z_0})\right]\\
-\quad & \left.k_w(e^{i(k_w z_0+\tau)} + e^{-i(k_w z_0+\tau)})\left[k_n A_n(z_1)n(-1)^n e^{ik_n z_0} + k_m A_m(z_1)m(-1)^m e^{ik_m z_0}\right]\right\}.
\end{aligned}$$

Again, by collecting the coefficients of the exponential terms and summing over j, we can obtain

$$\begin{aligned}
\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 &= \frac{i\pi}{2}\left\{\sum_{j=m,n} e^{i(k_j+k_w)z_0+i\tau}\left[-{k_j}^2 A_j(z_1)j(-1)^j + k^2 A_j(z_1)j(-1)^j\right.\right.\\
-\quad & \left.k_w k_j A_j(z_1 j(-1)^j\right] - \sum_{j=m,n} e^{i(k_j-k_w)z_0-i\tau}\left[-{k_j}^2 A_j(z_1)j(-1)^j + k^2 A_j(z_1)j(-1)^j\right.\\
+\quad & \left.\left.k_w k_j A_j(z_1 j(-1)^j\right]\right\},
\end{aligned}$$

which becomes

$$\left(\frac{\partial^2}{\partial {z_0}^2}+k^2\right)\psi_1 = \frac{i\pi}{2}\left\{\sum_{j=m,n} e^{i(k_j+k_w)z_0+i\tau} j(-1)^j A_j(-{k_j}^2+k^2-k_w k_j)\right.$$

$$\left. - \sum_{j=m,n} e^{i(k_j-k_w)z_0-i\tau} j(-1)^j A_j(-{k_j}^2+k^2+k_w k_j)\right\}.$$

We already know that ${k_n}^2 = k^2-(n\pi)^2$ and ${k_m}^2 = k^2-(m\pi)^2$. We can then assume that

$${k_j}^2 = k^2-(j\pi)^2 \tag{49}$$

is also true. If we substitute equation (49) into our calculations, we then obtain for $x=1$, that

$$\left(\frac{\partial^2}{\partial {z_0}^2}+k^2\right)\psi_1 = \frac{i\pi}{2}\left\{\sum_{j=m,n} (-1)^j j((j\pi)^2-k_j k_w)A_j e^{i(k_j+k_w)z_0+i\tau}\right. \tag{50}$$

$$\left. - \sum_{j=m,n} (-1)^j j((j\pi)^2+k_j k_w)A_j e^{i(k_j-k_w)z_0-i\tau}\right\}.$$

The appropriate boundary conditions for the left plate can be shown in the same way to obtain

$$\left(\frac{\partial^2}{\partial {z_0}^2}+k^2\right)\psi_1 = \frac{i\pi}{2}\left\{\sum_{j=m,n} j((j\pi)^2-k_j k_w)A_j e^{i(k_j+k_w)z_0}\right. \tag{51}$$

$$\left. - \sum_{j=m,n} j((j\pi)^2+k_j k_w)A_j e^{i(k_j-k_w)z_0}\right\}.$$

From considering equations (48), (50) and (51), we can seek a particular solution for ψ_1 in the form

$$\psi_1 = i\sum_{j=m,n} \Phi_j(x,z_1)e^{ik_j z_0}. \tag{52}$$

We can now form a relationship between the wave number and the wall undulation. To do this we must consider information we found in section (3.1). We now know that there is a uniformity at

$$\alpha_j = m\pi,$$

where

$${\alpha_j}^2 = k^2-(k_n-(-1)^j k_w)^2.$$

This implies that

$$(m\pi)^2 = k^2 - (k_n - (-1)^j k_w)^2$$

and from rearrangement we can obtain

$$(k_n - (-1)^j k_w)^2 = {k_m}^2,$$

where

$${k_m}^2 = k^2 - (m\pi)^2.$$

Therefore we know that

$$\begin{aligned} k_m &= k_n - (-1)^j k_w, \\ &= k_n \pm k_w. \end{aligned}$$

Unfortunately this equation, as shown above, has a non-uniformity where $k_m = k_n \pm k_w$, therefore we need to add a detuning parameter, $\sigma = O(1)$, so we have

$$k_m = k_n \pm k_w + \delta\sigma.$$

This means we can consider the case close to the perfectly tuned case which corresponds to the Bragg condition.

As in Asfar and Nayfeh (1983), we focus on the case $k_m = k_n - k_w + \delta\sigma$, (the other follows in a similar way). From this relationship we can deduce that

$$e^{i(k_n - k_w)z_0} = e^{i(k_m - \delta\sigma)z_0} = e^{ik_m z_0 - i\sigma z_1}, \tag{53}$$

and

$$e^{i(k_m + k_w)z_0} = e^{i(k_n + \delta\sigma)z_0} = e^{ik_n z_0 + i\sigma z_1}. \tag{54}$$

Now, as before, we need to find the governing equation and boundary conditions (at $x = 0$ and $x = 1$) for $\Phi_j(x, z_1)$. We begin by substituting equation (52) and its derivatives into (46) which yields

$$\begin{aligned} & i\frac{\partial^2}{\partial x^2}\Phi_m(x,z_1)e^{ik_m z_0} + i\frac{\partial^2}{\partial x^2}\Phi_n(x,z_1)e^{ik_n z_0} - ik_m\Phi_m(x,z_1)e^{ik_m z_0} \\ - & ik_n\Phi_n(x,z_1)e^{ik_n z_0} + k^2[i\Phi_m(x,z_1)e^{ik_m z_0} i\Phi_n(x,z_1)e^{ik_n z_0}] \\ = & -2i[k_m A_m'(z_1)\sin(m\pi x)e^{ik_m z_0} + k_n A_n'(z_1)\sin(n\pi x)e^{ik_n z_0}]. \end{aligned}$$

We now equate the coefficients of $e^{ik_n z_0}$ to obtain

$$\frac{\partial^2 \Phi_n}{\partial x^2} + (n\pi)^2 \Phi_n = -2k_n A_n'(z_1)\sin(n\pi x), \tag{55}$$

where we have used the relationship ${k_n}^2 = k^2 - (n\pi)^2$. Similarly

$$\frac{\partial^2 \Phi_m}{\partial x^2} + (m\pi)^2 \Phi_m = -2k_m A_m'(z_1)\sin(m\pi x). \tag{56}$$

Now we find the right plate boundary conditions for Φ_m and Φ_n. To do this we are going to substitute equations (52), (53) and (54) into equation (51). After differentiating ψ_1 twice with respect to z_0, we get

$$\begin{aligned}\frac{\partial^2 \psi_1}{\partial {z_0}^2} &= -i\sum_{j=m,n} \Phi_j(1,z_1){k_j}^2 e^{ik_j z_0},\\ &= -i\Phi_m(x,z_1){k_m}^2 e^{ik_m z_0} - i\Phi_n(1,z_1){k_n}^2 e^{ik_n z_0},\end{aligned}$$

so equation (51) becomes

$$\begin{aligned}&- i\Phi_m(1,z_1){k_m}^2 e^{ik_m z_0} - i\Phi_n(1,z_1){k_n}^2 e^{ik_n z_0} + k^2(i\Phi_m(1,z_1)e^{ik_m z_0} + i\Phi_n(1,z_1)e^{ik_n z_0})\\ &= \frac{i\pi}{2}[(-1)^m m((m\pi)^2 - k_m k_w)A_m e^{i(k_m+k_w)z_0+i\tau} + (-1)^n n((n\pi)^2 - k_n k_w)A_n e^{i(k_n+k_w)z_0+i\tau}\\ &- (-1)^m m((m\pi)^2 + k_m k_w)A_m e^{i(k_m-k_w)z_0-i\tau} - (-1)^n n((n\pi)^2 + k_n k_w)A_n e^{i(k_n-k_w)z_0-i\tau}],\end{aligned}$$

and here we can use the relationships (53) and (54), to obtain

$$\begin{aligned}&- i\Phi_m(1,z_1){k_m}^2 e^{ik_m z_0} - i\Phi_n(1,z_1){k_n}^2 e^{ik_n z_0} + k^2(i\Phi_m(1,z_1)e^{ik_m z_0} + i\Phi_n(1,z_1)e^{ik_n z_0})\\ &= \frac{i\pi}{2}[(-1)^m m((m\pi)^2 - k_m k_w)A_m e^{ik_n z_0+i\sigma z_1+i\tau} + (-1)^n n((n\pi)^2 - k_n k_w)A_n e^{i(k_n+k_w)z_0+i\tau}\\ &- (-1)^m m((m\pi)^2 + k_m k_w)A_m e^{i(k_m-k_w)z_0-i\tau} - (-1)^n n((n\pi)^2 + k_n k_w)A_n e^{ik_m z_0-i\sigma z_1-i\tau}.\end{aligned}$$

Now we can equate coefficients of $e^{ik_n z_0}$ to yield

$$\Phi_n(1,z_1) = \frac{m}{2(n^2\pi)}(-1)^m((m\pi)^2 - k_m k_w)A_m e^{i(\sigma z_1+\tau)}, \tag{57}$$

where again we have used the relationship ${k_n}^2 = k^2 - (n\pi)^2$. The equation for the coefficients of $e^{ik_m z_0}$ can be found in a similar way to be

$$\Phi_m(1,z_1) = -\frac{n}{2(m^2\pi)}(-1)^n((n\pi)^2 + k_n k_w)A_n e^{-i(\sigma z_1+\tau)}. \tag{58}$$

We can find the boundary condition at $x = 0$, via the same method and we obtain the equations

$$\Phi_n(0,z_1) = \frac{1}{2}\frac{m}{n^2\pi}A_m((m\pi)^2 - k_m k_w)e^{i\sigma z_1}, \tag{59}$$

and

$$\Phi_m(0, z_1) = -\frac{1}{2}\frac{n}{m^2\pi}A_n((n\pi)^2 - k_n k_w)e^{i\sigma z_1}. \tag{60}$$

Now we want to look at equation (55), this is to couple the two boundary conditions for $x = 0$ and $x = 1$.

$$\frac{\partial^2 \Phi_n}{\partial x^2} + (n\pi)^2\Phi_n = -2k_n A_n'(z_1)\sin(n\pi x).$$

Multiplying this equation by $\sin(n\pi x)$ and integrating between $x = 0$ and 1 yields the relationship

$$\Phi_n(0, z_1) - \Phi_n(1, z_1)(-1)^n = -\frac{A_n'(z_1)k_n}{(n\pi)}. \tag{61}$$

Now we have expressions for $\Phi_n(0, z_1)$ and $\Phi_n(1, z_1)$, we can now move forward in finding equations for the amplitudes $A_n(z_1)$ and $A_m(z_1)$. We have, from equations (57), (59) and (61), that

$$\begin{aligned}-\left(\frac{k_n}{n\pi}\right)A_n'(z_1) &= \frac{1}{2}\left(\frac{m}{n^2\pi}\right)A_m((m\pi)^2 - k_m k_w)e^{i(\sigma z_1+\tau)} \\ &-(-1)^n\left\{\frac{1}{2}\left(\frac{m}{n^2\pi}\right)(-1)^m A_m((m\pi)^2 - k_m k_w)e^{i(\sigma z_1+\tau)}\right\},\end{aligned}$$

from which

$$A_n'(z_1) = \frac{1}{2}\left(\frac{m}{k_n n}\right)A_m(k_m k_w - (m\pi)^2)\left[1 - (-1)^{m+n}e^{i\tau}\right]e^{i\sigma z_1}. \tag{62}$$

We can also find $A_m'(z_1)$ using the same method. We obtain

$$A_m'(z_1) = \frac{1}{2}\left(\frac{n}{k_m m}\right)A_n(k_n k_w - (n\pi)^2)\left[1 - (-1)^{m+n}e^{-i\tau}\right]e^{-i\sigma z}. \tag{63}$$

We now want the solutions for equations (62) and (63). To do this we define

$$A_m = a_m\ e^{s z_1}, \tag{64}$$

and

$$A_n = a_n\ e^{(s+i\sigma)z_1}, \tag{65}$$

where a_m, a_n and s are constants. We now want to substitute equations (64) and (65) into equations (62) and (63).

Note that

$$A_n'(z_1) = \frac{dA_n}{dz_1} = a_n(s + i\sigma)e^{(s+i\sigma)z_1},$$

and

$$A'_m(z_1) = \frac{dA_m}{dz_1} = a_m s e^{sz_1}.$$

So equation (62) and (63) become

$$a_n(s+i\sigma)e^{(s+i\sigma)z_1} = \frac{1}{2}\left(\frac{m}{k_n n}\right) a_m e^{sz_1}(k_m k_w - (m\pi)^2)\left[1-(-1)^{m+n}e^{i\tau}\right]e^{i\sigma z}, \quad (66)$$

and also

$$a_m e^{sz_1} = \frac{1}{2}\left(\frac{n}{k_m m}\right) a_n \ e^{(s+i\sigma)z_1}(k_n k_w - (n\pi)^2)\left[1-(-1)^{m+n}e^{-i\tau}\right]e^{-i\sigma z}. \quad (67)$$

If we rearrange equation (66) to get

$$a_n = \frac{1}{2}\left(\frac{m}{nk_n}\right)\frac{(k_m k_w - (m\pi)^2)a_m e^{sz_1}}{(s+i\sigma)e^{(s+i\sigma)z_1}}\left[1-(-1)^{m+n}e^{i\tau}\right]e^{i\sigma z_1},$$

then we can substitute this into equation (67) to obtain

$$\begin{aligned} a_m s e^{sz_1} &= \frac{1}{2}\left(\frac{n}{mk_m}\right)(k_n k_w + (n\pi)^2)\left[\frac{1}{2}\left(\frac{m}{nk_n}\right)\frac{(k_m k_w - (m\pi)^2)}{(s+i\sigma)e^{(s+i\sigma)z_1}}\right. \\ &\times \left.[1-(-1)^{m+n}e^{i\tau}]a_m e^{sz_1}e^{i\sigma z_1}\right]e^{(s+i\sigma)z_1}[1-(-1)^{m+n}e^{-i\tau}]e^{-i\sigma z_1}. \end{aligned}$$

This can be simplified by cancelling constants a_m and a_n to become

$$\begin{aligned} s(s+i\sigma) &= \frac{1}{2}(k_n k_w + (n\pi)^2)(k_m k_w + (m\pi)^2)\frac{1}{2}[1-(-1)^{m+n}(e^{i\tau}+e^{-i\tau})], \\ &= \frac{1}{2k_m k_n}(k_n k_w + (n\pi)^2)(k_m k_w + (m\pi)^2)[1-(-1)^{m+n}\cos\tau]. \end{aligned}$$

So we have

$$s(s+i\sigma) = \Omega, \quad (68)$$

where

$$\Omega = \frac{1}{2k_m k_n}(k_n k_w + (n\pi)^2)(k_m k_w + (m\pi)^2)[1-(-1)^{m+n}\cos\tau]. \quad (69)$$

So we have gained a quadratic in s,

$$s^2 + si\sigma - \Omega = 0,$$

which we can solve for, giving us

$$s = \frac{i}{2}[-\sigma \pm (\sqrt{\sigma^2 - 4\Omega})]. \quad (70)$$

We can begin analysing this equation by first considering which way the modes are moving. Firstly consider when $4\Omega > \sigma^2$, this tells us that the modes are moving in opposite directions. By using equation (70) we can see that s is complex and of the form $(a + bi)$, where a, $b \in \mathbb{R}$.

By allowing s to be complex, we can determine that equations (64) and (65) are telling us that the modes are quickly decaying. As the modes are propagating along the two wave guides, they are reducing in strength. This is called a 'stop band'.

Now we can consider when the modes are moving in the same direction, this occurs when $4\Omega < \sigma^2$ and s is completely imaginary with no real parts. Because of this, equations (64) and (65) are bounded. Therefore we know that as the waves propagate they cannot grow in size beyond their bounded limit, i.e. not shoot off to infinity. Also from considering the exponentials in equations (64) and (65), if s is imaginary then the exponential terms will not decay. Therefore we know that the modes are propagating in the same direction without loss of strength, meaning that energy in the waves and between the waves is conserved. This is called a passband interaction.

If $4\Omega = \sigma^2$, then this implies that the movement of the modes is changing from one example to the other. This is called a 'transition frequency'.

4 Multiple Scales Applied to the Water-Wave Problem

Now we can apply the method in Asfar and Nayfeh (1983) to the problem posed in Porter and Chamberlain (1997). Since we have shown that, in Porter and Chamberlain (1997), an asymptotic expansion of the form

$$\phi(x,z) = \phi_0(x,z) + \epsilon\phi_1(x,z) + ...$$

is not valid for certain parameter combinations, we now want to see what happens for an asymptotic expansion of the form

$$\phi(x_0,x_1,z) = \phi_0(x_0,x_1,z) + \epsilon\phi_1(x_0,x_1,z) + ...$$

as in the electromagnetic case. Note that the fast variable is $x_0 = x$ and the corresponding slow variable is $x_1 = \epsilon x$, ϵ being the equivalent to δ in Asfar and Nayfeh (1983). We begin by applying the new asymptotic expansion to the governing equation and boundary conditions

$$\left.\begin{array}{ll} \nabla^2\phi = 0 & (-h_0 < z < 0) \\ \phi_z - \nu\phi = 0 & (z = 0) \\ \phi_z - \epsilon\zeta_x\phi_x = 0 & (z = -h_0) \end{array}\right\},$$

where our bounded function ζ is $\zeta = \sin(2\pi x/l_0)$, so $\zeta_x = (2\pi/l_0)\cos(2\pi x/l_0)$.

The governing equation and boundary conditions at leading order are

$$\left.\begin{array}{ll} \phi_{0x_0x_0} + \phi_{0zz} = 0 & (-h_0 < z < 0) \\ \phi_{0z} - \nu\phi_0 = 0 & (z = 0) \\ \phi_{0z} = 0 & (z = -h_0) \end{array}\right\}. \tag{71}$$

If we refer back to section (2.1), we can see that we can write ϕ_0 as

$$\phi_0 = A_R(x_1)\cosh k(z+h_0)e^{ikx_0} + A_L(x_1)\cosh k(z+h_0)e^{-ikx_0}, \tag{72}$$

where now the amplitudes A_R and A_L are functions of the slow variable. In order to progress, we now want the $O(\epsilon^1)$ terms. The equations of $O(\epsilon^1)$ are

$$\left.\begin{array}{ll}\phi_{1x_0x_0}+\phi_{1zz}=-2\phi_{0x_0x_1} & (-h_0<z<0)\\ \phi_{1z}-\nu\phi_1=0 & (z=0)\\ \phi_{1z}=\frac{2\pi}{l_0}\cos\left(\frac{2\pi x}{l_0}\right)\phi_{0x_0}-\sin\left(\frac{2\pi x}{l_0}\right)\phi_{0zz} & (z=-h_0)\end{array}\right\}. \tag{73}$$

Now we can focus on obtaining a solution for ϕ_1. We do this by substituting equation (72) into (73). First we differentiate ϕ_0 with respect to both x_0 and x_1 to yield

$$\frac{\partial^2\phi_0}{\partial x_0\partial x_1}=ikA_R'(x_1)\cosh k(z+h_0)e^{ikx_0}-ikA_L'(x_1)\cosh k(z+h_0)e^{-ikx_0}$$

which gives

$$\frac{\partial^2\phi_1}{\partial {x_0}^2}+\frac{\partial^2\phi_1}{\partial z^2}=-2\left[ikA_R'(x_1)\cosh k(z+h_0)e^{ikx_0}-ikA_L'(x_1)\cosh k(z+h_0)e^{-ikx_0}\right], \tag{74}$$

for $(-h_0<z<0)$. The condition at the free surface will not change at this point, since it has no dependence on ϕ_0. So focusing on (73) for $z=-h_0$, we do the same as we did for $(-h_0<z<0)$. From (72), we have

$$\frac{\partial\phi_0}{\partial x_0}=ikA_R(x_1)\cosh k(z+h_0)e^{ikx_0}-ikA_L(x_1)\cosh k(z+h_0)e^{-ikx_0},$$

and

$$\frac{\partial^2\phi_0}{\partial z^2}=k^2A_R(x_1)\cosh k(z+h_0)e^{ikx_0}+k^2A_L(x_1)\cosh k(z+h_0)e^{-ikx_0},$$

so the bed condition becomes

$$\begin{aligned}\frac{\partial\phi_1}{\partial z}=\frac{2\pi}{l_0}\cos\left(\frac{2\pi x_0}{l_0}\right)&\left[A_R(x_1)ik\cosh k(z+h_0)e^{ikx_0}-A_L(x_1)ik\cosh(z+h_0)e^{-ikx_0}\right]\\ -\sin\left(\frac{2\pi x_0}{l_0}\right)&\left[k^2A_R(x_1)\cosh k(z+h_0)e^{ikx_0}+k^2A_L(x_1)\cosh k(z+h_0)e^{-ikx_0}\right],\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{\partial\phi_1}{\partial z}=\frac{2\pi}{l_0}\cos\left(\frac{2\pi x_0}{l_0}\right)&\left[A_R(x_1)ike^{ikx_0}-A_L(x_1)ike^{-ikx_0}\right]\\ -\sin\left(\frac{2\pi x_0}{l_0}\right)&\left[k^2A_R(x_1)e^{ikx_0}+k^2A_L(x_1)e^{-ikx_0}\right],\end{aligned}$$

since $z=-h_0$. To be consistent with Asfar and Nayfeh (1983) we write $2\pi/l_0=k_w$, the wall undulation parameter. Using the identities

$$\cos(k_wx_0)=\frac{e^{ik_wx_0}+e^{-ik_wx_0}}{2},$$

and

$$\sin(k_w x_0) = \frac{e^{ik_w x_0} - e^{-ik_w x_0}}{2i},$$

we obtain

$$\frac{\partial \phi_1}{\partial z} = \frac{i}{2}\Big\{e^{i(k_w+k)x_0}\left[kk_w A_R(x_1) + k^2 A_R(x_1)\right] - e^{i(k_w-k)x_0}\left[k_w k A_L(x_1) - k^2 A_L(x_1)\right]$$
$$+ \quad e^{i(k-k_w)x_0}\left[k_w k A_R(x_1) - k^2 A_R(x_1)\right] - e^{-i(k_w+k)x_0}\left[k_w k A_L(x_1) + k^2 A_L(x_1)\right]\Big\}.$$

We are now looking for a particular solution for ϕ_1 of the form

$$\phi_1 = i\Phi_+(x_1, z)e^{ikx_0} + i\Phi_-(x_1, z)e^{-ikx_0}. \tag{75}$$

This equation includes the exponential terms to demonstrate the one wave propagating both to the left and to the right, that we had shown to be the case in section (2.2).

We can now form a relationship between the wave number and the wall undulation. We have shown how to produce this relationship in section (3.2) noting that k represents the wave propagating to the right as k_n did and similarly $-k$ represents the wave k_m propagating to the left. We have

$$k = -k \pm k_w,$$

but this equation, as shown before, has a non-uniformity where $2k = \pm k_w$, therefore we need to add a detuning parameter, $\sigma = O(1)$, so we have

$$k = -k \pm k_w + \epsilon\sigma.$$

Note that we will be taking the 'minus case' as in Asfar and Nayfeh (1983), to give us the relationship

$$k = -k - k_w + \epsilon\sigma. \tag{76}$$

Now from equation (76) we gain the two equalities

$$k + k_w = -k + \epsilon\sigma,$$

and

$$-(k + k_w) = k - \epsilon\sigma.$$

We can then express these in an exponential form, giving us the two equations

$$e^{i(k+k_w)x_0} = e^{i(-k+\epsilon\sigma)x_0} = e^{-ikx_0+i\sigma x_1}, \tag{77}$$

and

$$e^{-i(k+k_w)x_0} = e^{i(k-\epsilon\sigma)x_0} = e^{ikx_0-i\sigma x_1}. \tag{78}$$

To progress we need to substitute our form of the solution, equation (75), into equation (74), which will give us

$$\begin{aligned}
-ik^2\Phi_+(x_1,z)e^{ikx_0} - ik^2\Phi_-(x_1,z)e^{-ikx_0} + i\frac{\partial^2}{\partial z^2}\Phi_+(x_1,z)e^{ikx_0} + i\frac{\partial^2}{\partial z^2}\Phi_-(x_1,z) \\
= -2\left[ikA_R'(x_1)\cosh k(z+h_0)e^{ikx_0} - ikA_L'(x_1)\cosh k(z+h_0)e^{-ikx_0}\right].
\end{aligned}$$

To analyse the two different waves we firstly consider the coefficients of e^{ikx_0} by equating them, we obtain

$$\frac{\partial^2}{\partial z^2}\Phi_+(x_1,z) - k^2\Phi_+(x_1,z) = -2kA_R'(x_1)\cosh k(z+h_0), \tag{79}$$

and the coefficients of e^{-ikx_0} are

$$\frac{\partial^2}{\partial z^2}\Phi_-(x_1,z) - k^2\Phi_-(x_1,z) = 2kA_L'(x_1)\cosh k(z+h_0). \tag{80}$$

We can also use this method, for the boundary condition at $z = 0$. From substituting equation (75) into the boundary condition at $z = 0$, we have

$$\frac{\partial\phi_1}{\partial z} = ie^{ikx_0}\frac{\partial}{\partial z}\Phi_+(x_1,0) + ie^{-ikx_0}\frac{\partial}{\partial z}\Phi_-(x_1,0)$$

and hence the free-surface boundary condition becomes

$$ie^{ikx_0}\frac{\partial}{\partial z}\Phi_+(x_1,0) + ie^{-ikx_0}\frac{\partial}{\partial z}\Phi_-(x_1,0) - \nu(i\Phi_+(x_1,0)e^{ikx_0} + i\Phi_-(x_1,0)e^{-ikx_0}) = 0.$$

As before we equate the coefficients of the exponential terms, yielding

$$\frac{\partial}{\partial z}\Phi_+(x_1,0) - \nu\Phi_+(x_1,0) = 0, \tag{81}$$

and

$$\frac{\partial}{\partial z}\Phi_-(x_1,0) - \nu\Phi_-(x_1,0) = 0. \tag{82}$$

Finally we can use the same method for the boundary condition at $z = -h_0$. For this calculation, we must plug equation (75) into the boundary condition at $z = -h_0$

and also use both equation (77) and (78), then, (omitting the details), we equate the coefficients of e^{ikx_0} to get

$$\frac{\partial}{\partial z}\Phi_+(x_1,-h_0) = -\frac{1}{2}e^{-i\sigma x_1}A_L(x_1)[k_w k + k^2], \tag{83}$$

and for e^{-ikx_0} we obtain

$$\frac{\partial}{\partial z}\Phi_-(x_1,-h_0) = \frac{1}{2}e^{i\sigma x_1}A_R(x_1)[k_w k + k^2]. \tag{84}$$

For a short time, we shall concentrate solely on equation (79). This is so we can combine the equations for Φ_+ and Φ_-. To repeat the calculations in Asfar and Nayfeh (1983), we multiply both sides of equation (79) by $\cosh k(z+h_0)$ and integrate them from $z=-h_0$ to $z=0$, eventually yielding

$$\cosh kh_0\frac{\partial}{\partial z}\Phi_+(x_1,0) - \frac{\partial}{\partial z}\Phi_+(x_1,-h) - k\sinh kh_0\Phi_+(x_1,0) = -\frac{A'_R(x_1)\sinh 2kh_0}{2} \\ -A'_R(x_1)kh_0.$$

In order to understand what the integral is telling us we can use equation (83). Firstly consider that equation (81) says that

$$\frac{\partial}{\partial z}\Phi_+(x_1,0) = \nu\Phi_+(x_1,0),$$

therefore we know that

$$\cosh kh_0\frac{\partial}{\partial z}\Phi_+(x_1,0) = \nu\cosh kh_0\Phi_+(x_1,0).$$

Note that Porter and Chamberlain (1997) states that $\nu = k\tanh kh_0$, therefore we can conclude that

$$\begin{aligned}\cosh kh_0\frac{\partial}{\partial z}\Phi_+(x_1,0) &= k\tanh kh_0\cosh kh_0\Phi_+(x_1,0),\\ &= k\frac{\sinh kh_0}{\cosh kh_0}\cosh kh_0\Phi_+(x_1,0),\\ &= k\sinh kh_0\Phi_+(x_1,0).\end{aligned}$$

So the entire integral becomes

$$k\sinh kh_0\Phi_+(x_1,0) - \frac{\partial}{\partial z}\Phi_+(x,-h) - k\sinh kh_0\Phi_+(x_1,0) = -A'_R(x_1)\frac{\sinh 2kh_0}{2} \\ -A'_R(x_1)kh_0,$$

i.e.

$$-\frac{\partial}{\partial z}\Phi_+(x,-h_0) = -A_R'(x_1)kh_0 - A_R'(x_1)\frac{\sinh 2kh_0}{2}. \tag{85}$$

Note that when doing this for the case of e^{-ikx_0}, we obtain

$$-\frac{\partial}{\partial z}\Phi_-(x,-h_0) = A_L'(x_1)kh_0 + A_L'(x_1)\frac{\sinh 2kh_0}{2}. \tag{86}$$

We now substitute in our expressions for Φ_+ into equation (85) to obtain

$$A_R'(x_1)\left[\frac{\sinh 2kh_0}{2} + h_0k\right] = -\frac{1}{2}e^{-i\sigma x_1}A_L(x_1)[k_wk + k^2],$$

so

$$A_R'(x_1) = -e^{-i\sigma(x_1)}\frac{A_L(x_1)(k_wk+k^2)}{\sinh 2kh_0 + 2kh_0}. \tag{87}$$

Equivalently for equation (86) we get

$$A_L'(x_1) = -e^{i\sigma(x_1)}\frac{A_R(x_1)(k_wk+k^2)}{\sinh 2kh_0 + 2kh_0}. \tag{88}$$

Now we seek solutions for the amplitudes of the form

$$A_R(x_1) = a_Re^{sx_1}, \tag{89}$$

and

$$A_L(x_1) = a_Le^{(s+i\sigma)x_1}. \tag{90}$$

Therefore we know that

$$A_R'(x_1) = sa_Re^{sx_1},$$

and

$$A_L'(x_1) = (s+i\sigma)a_Le^{(s+i\sigma)x_1}.$$

We now substitute these into (88) to get

$$sa_Re^{sx_1} = -e^{-i\sigma x_1}\frac{A_L(x_1)(k_wk+k^2)}{\sinh 2kh_0 + 2kh_0} = -e^{-i\sigma x_1}\frac{a_Le^{(s+i\sigma)x_1}(k_wk+k^2)}{\sinh 2kh_0 + 2kh_0}. \tag{91}$$

We also know that

$$a_L = -\frac{e^{i\sigma x_1}a_Re^{sx_1}(k_wk+k^2)}{(s+i\sigma)e^{(s+i\sigma)x_1}(2kh_0 + \sinh 2kh_0)}, \tag{92}$$

so we substitute equation (92) into (91) to obtain

$$sa_R e^{sx_1} = \frac{e^{i\sigma x_1} e^{-i\sigma x_1} a_R e^{sx_1}(k_w k + k^2) e^{(s+i\sigma)x_1}(k_w k + k^2)}{(s+i\sigma) e^{(s+i\sigma)x_1}(2kh_0 + \sinh 2kh_0)(\sinh 2kh_0 + 2kh_0)}.$$

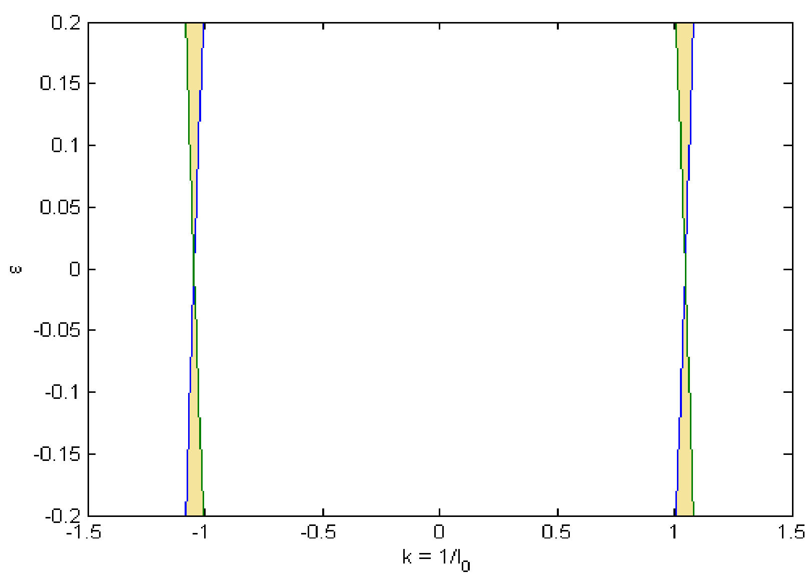
By eliminating the constants a_R and a_L, we obtain

$$(s+i\sigma)s = \frac{(k_w k + k^2)^2}{(\sinh 2kh_0 + 2kh_0)^2} = \Omega^2. \tag{93}$$

As shown in section (3.2), s can also be expressed as

$$s = \frac{1}{2}\left\{-i\sigma \pm \sqrt{-\sigma^2 + 4\Omega^2}\right\}. \tag{94}$$

To demonstrate what equation (94) is telling us, we can consider the plot below.



For this plot we have taken $h_0 = 1$ and to non dimensonalise we let $l_0 = 3h_0$, the length of one ripple. To obtain the plot for $k < 0$, we have plotted the lines $\sigma = \pm 2\Omega$. We do this by using the relationship between the wavenumber and the wall undulation, we can form a relationship between ϵ, σ and k, which is

$$\epsilon = \pm \frac{2k + kw}{\sigma}.$$

Thus we can combine Ω into this equation to obtain

$$\epsilon = \pm \frac{(2k + k_w)(\sinh 2kh_0 + 2kh_0)}{2(k_w k + k^2)}.$$

When $k > 0$, we have considered when equation (76) has taken on the plus sign and becomes

$$k = -k + k_w + \epsilon\sigma,$$

which gives us

$$(s + i\sigma)s = \Omega^2, \tag{95}$$

where

$$\Omega^2 = \frac{(k_w k - k^2)^2}{(\sinh 2kh_0 + 2kh_0)^2}.$$

Then, again, we can plot k against ϵ, where in this case

$$\epsilon = \pm\frac{(2k - k_w)(\sinh 2kh_0 + 2kh_0)}{2(k_w k - k^2)}.$$

We can analyse this plot by considering equation (94).

When $4\Omega^2 > \sigma^2$ then this implies that s is complex and of the form $(a + bi)$, where a, $b \in \mathbb{R}$. Then after considering equations (89) and (90), we can see that when the $\mathrm{Re}(s) \neq 0$, represented by the shaded yellow regions, that the waves are decaying over time. When the waves are evanescent, the ratio between the two terms ϕ_0 and $\epsilon\phi_1$ in the asymptotic expansion describing the velocity potential are unbounded. This definition exhibits Bragg resonance.

Another case to consider would be when $\sigma^2 > 4\Omega^2$. s is then imaginary and would display propagating modes. The amplitudes of these modes would be bounded and therefore never decay nor grow beyond their limit. This is a passband interaction.

The lines representing where $\sigma^2 = 4\Omega^2$ in the plot is called the transition frequency as the mode changes from one band to the other.

5 Multiple Scales for a General Function

5.1 Fourier Series

So far, we have considered the problem posed in Porter and Chamberlain (1997) with a bounded function $\zeta(x)$. Porter and Chamberlain (1997) had specified that the bounded function would be of the form

$$\zeta(x) = \sin\left(\frac{2\pi x}{l_0}\right).$$

Now we want to consider what would happen for a general function.

If we use Fourier series we can find a general, l_0-periodic function to apply to the governing equation and boundary conditions, (71). Let

$$\zeta(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi x}{l_0}\right) + b_n \sin\left(\frac{2\pi x}{l_0}\right)\right].$$

We can combine these terms to get,

$$\zeta(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{2n\pi}{l_0}x} = \sum_{n=-\infty}^{\infty} C_n e^{ink_w x}, \tag{96}$$

where the only rule posed upon the constant C_n is that it must combine with the exponential term to make the sum real. Now we can work through section 4.

5.2 Multiple Scales Applied to the Water-Wave Problem

Again, for this section, we will apply the method in Asfar and Nayfeh (1983), using the asymptotic expansion

$$\phi(x_0, x_1, z) = \phi_0(x_0, x_1, z) + \epsilon\phi_1(x_0, x_1, z) + ...$$

The governing equation and boundary conditions will be slightly different to those posed in section 4, and will now be of the form

$$\left.\begin{array}{ll} \nabla^2\phi = 0 & (-h_0 < z < 0) \\ \phi_z - \nu\phi = 0 & (z = 0) \\ \phi_z - \epsilon i m k_w C_m e^{imk_w x}\phi_x = 0 & (z = -h_0) \end{array}\right\}.$$

We can collect the terms of leading order to obtain a new set of equations, which since they are not affected by the second term for $z = -h_0$, will not change from the set of equations (71), which are

$$\left.\begin{array}{ll} \phi_{0x_0x_0} + \phi_{0zz} = 0 & (-h_0 < z < 0) \\ \phi_{0z} - \nu\phi_0 = 0 & (z = 0) \\ \phi_{0z} = 0 & (z = -h_0) \end{array}\right\}.$$

The terms of $O(\epsilon^1)$ will be different for $z = -h_0$, but otherwise remain the same, and become

$$\left.\begin{array}{ll} \phi_{1x_0x_0} + \phi_{1zz} = -2\phi_{0x_0x_1} & (-h_0 < z < 0) \\ \phi_{1z} - \nu\phi_1 = 0 & (z = 0) \\ \phi_{1z} = imk_wC_me^{imk_wx}\phi_{0x_0} - C_me^{imk_wx}\phi_{0zz} & (z = -h_0) \end{array}\right\}. \tag{97}$$

Our next task will be to use ϕ_0, equation (72), in order to find more explicit terms for our ϕ_1 and eventually find a general solution. We need to work through (97), applying ϕ_0 where applicable.

The governing equation will work out to be the same whilst the boundary condition at $z = 0$ will not change, as it has no dependence on ϕ_0. After applying our expression for ϕ_0 our boundary condition at $z = -h_0$ will become

$$\begin{aligned} \frac{\partial\phi_1}{\partial z} &= imk_wC_me^{imk_wx_0}\left(ikA_R(x_1)\cosh k(z+h_0)e^{ikx_0} - ikA_L(x_1)\cosh k(z+h_0)e^{-ikx_0}\right) \\ &- C_me^{imk_wx_0}\left(k^2A_R(x_1)\cosh k(z+h_0)e^{ikx_0} + k^2A_L(x_1)\cosh k(z+h_0)e^{-ikx_0}\right). \end{aligned}$$

After some similar manipulation as applied in section 3, we obtain

$$\frac{\partial\phi_1}{\partial z} = -e^{i(mk_w+k)x_0}\left[A_R(x_1)C_m\left(mk_wk + k^2\right)\right] - e^{i(mk_w-k)x_0}\left[A_L(x_1)C_m\left(k^2 - mk_wk\right)\right].$$

To develop this equation in terms of the slow variable x_1, we need to recall the relationship we had in section 4 and manipulate it to become

$$k = -k + mk_w + \epsilon\sigma \tag{98}$$

where $m \in \mathbb{Z}$. As we had previously done, we should develop two relationships between the wave number and wall undulation, using equation (98). We get

$$k - mk_w = -k + \epsilon\sigma$$

and

$$mk_w - k = k - \epsilon\sigma.$$

This gives us the exponential terms,

$$e^{i(k-mk_w)x_0} = e^{i(-k+\epsilon\sigma)x_0} = e^{-ikx_0+i\sigma x_1}, \tag{99}$$

and

$$e^{i(mk_w-k)x_0} = e^{i(k-\epsilon\sigma)x_0} = e^{ikx_0-i\sigma x_1}. \tag{100}$$

We can define ϕ_1 in such a way to eventually determine our amplitudes,

$$\phi_1 = i\Phi_+(x_1, z)e^{ikx_0} + i\Phi_-(x_1, z)e^{-ikx_0}.$$

We can then work through the new equations for the governing system and boundary conditions. By substituting in the expression for ϕ_0 and equating the coefficients of the exponential terms, we obtain

$$\frac{\partial^2}{\partial z^2}\Phi_+(x_1, z) - k^2\Phi_+(x_1, z) = -2kA_R'(x_1)\cosh k(z + h_0), \tag{101}$$

for e^{ikx_0}, and

$$\frac{\partial^2}{\partial z^2}\Phi_-(x_1, z) - k^2\Phi_-(x_1, z) = 2kA_L'(x_1)\cosh k(z + h_0). \tag{102}$$

for e^{-ikx_0}, as we had in section 4. For the boundary condition at $z = 0$, we again obtain the same equations as in the previous section, namely

$$\frac{\partial}{\partial z}\Phi_+(x_1, 0) - \nu\Phi_+(x_1, 0) = 0,$$

for e^{ikx_0} and

$$\frac{\partial}{\partial z}\Phi_-(x_1, 0) - \nu\Phi_-(x_1, 0) = 0,$$

for e^{-ikx_0}.

The boundary condition at the bed requires a bit more work. We will use the same method as we had in section 4. For the coefficients of e^{ikx_0} we obtain

$$i\frac{\partial}{\partial z}\Phi_+(x_1, -h_0) = -e^{-i\sigma x_1}\left[A_L(x_1)C_m\left(k^2 + mk_wk\right)\right], \tag{103}$$

and for the coefficients of e^{-ikx_0} we obtain

$$i\frac{\partial}{\partial z}\Phi_-(x_1,-h_0) = -e^{i\sigma x_1}\left[A_R(x_1)C_m\left(mk_wk+k^2\right)\right]. \tag{104}$$

Now as we did in section 4, we shall take equations (101) and (102), multiply through by $\cosh k(z+h_0)$ and integrate them from $z=-h_0$ to $z=0$, eventually giving us

$$-\frac{\partial}{\partial z}\Phi_+(x_1,-h_0) = -A_R'(x_1)kh_0 - A_R'(x_1)\frac{\sinh 2kh_0}{2}$$

in terms of the equation for e^{ikx_0} and

$$-\frac{\partial}{\partial z}\Phi_-(x_1,-h_0) = A_L'(x_1)kh_0 + A_L'(x_1)\frac{\sinh 2kh_0}{2}$$

for e^{-ikx_0}. We can now substitute equations (103) and (104) into the above equations where necessary. By doing this we would obtain,

$$e^{-i\sigma x_1}A_L(x_1)C_m\left(k^2+mk_wk\right)\frac{1}{i} = -A_R'(x_1)\left[\frac{2kh_0+\sinh 2kh_0}{2}\right],$$

which can be rearranged to give

$$A_R'(x_1) = -\frac{2e^{-i\sigma x_1}A_L(x_1)C_m\left(k^2+mk_wk\right)}{i\left(2kh_0+\sinh 2kh_0\right)},$$

and also

$$e^{i\sigma x_1}A_R(x_1)C_m\left(k^2+mk_wk\right)\frac{1}{i} = A_L'(x_1)\left[\frac{2kh_0+\sinh 2kh_0}{2}\right],$$

which gives

$$A_L'(x_1) = \frac{2e^{i\sigma x_1}A_R(x_1)C_m\left(k^2+mk_wk\right)}{i\left(2kh_0+\sinh 2kh_0\right)}.$$

Now as before we shall define functions for the amplitudes,

$$A_R(x_1) = a_Re^{sx_1}$$

and

$$A_L(x_1) = a_Le^{(s+i\sigma)x_1}.$$

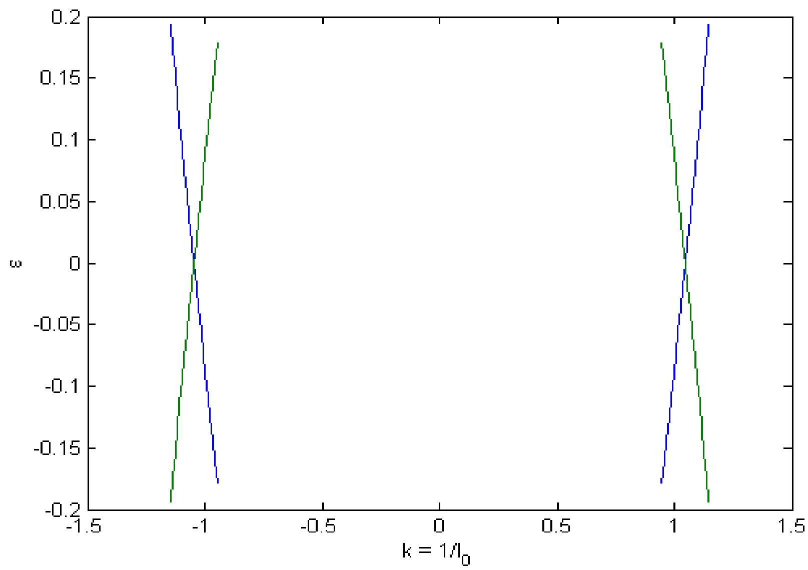
We can use these functions for the amplitudes, to substitute into our equations for $A_L'(x_1)$ and $A_R'(x_1)$, and by combining the two we will eventually obtain

$$s(s+i\sigma) = \frac{4{C_m}^2\left(k^2+mk_wk\right)^2}{\left(2kh_0+\sinh 2kh_0\right)^2} = \Omega^2.$$

This again gives us a quadratic in s which can be solved to be analysed. We end up with the equation

$$s = \frac{1}{2}\left\{-i\sigma \pm \sqrt{-\sigma^2 + 4\Omega^2}\right\}. \tag{105}$$

As before we can plot the lines $\sigma = \pm 2\Omega$ as we did in section 4, to obtain the plot,



For this plot, we had to define an l_0-periodic function and work out the Fourier series coefficient for it. The function was chosen as

$$\zeta(x) = \cos k_w x,$$

where $k_w = 2\pi/l_0$. So now we want $\cos k_w x$ to be expressed as a summation of exponential terms. Note that

$$\zeta(x) = \sum_{n=-\infty}^{\infty} C_n e^{ink_w x} = \sum_{n=-\infty}^{\infty} C_n \left(\cos nk_w x + i \sin nk_w x\right).$$

We should consider the terms separately.

$$\begin{aligned} \cos k_w x &= ...C_{-2}\left(\cos 2k_w x - i\sin 2k_w x\right) + C_{-1}\left(\cos k_w x - i \sin k_w\right) + C_0 \\ &+ C_1\left(\cos k_w x + i\sin k_w x\right) + C_2\left(\cos 2k_w x + i \sin 2k_w x\right) + ... \end{aligned}$$

So for the sum to be true, we must let $C_1 = C_{-1} = 1/2$ and the remaining coefficients equal zero.

Firstly consider when $4\Omega > \sigma^2$, this tells us that the modes are reversed and also that s is complex. As seen in sections 3 and 4, if s is complex then there is a stop band. For the four triangles made up of the lines of $\sigma = \pm\Omega$, this is where the modes are decaying and Bragg resonance is occurring, as the exponential contains a real part, which will dominate the movement of the mode.

Now we can consider when the modes are codirectional, this occurs when $4\Omega < \sigma^2$ and $\mathrm{Re}(s) = 0$. Because our exponential terms have no real parts, we know that the modes are propagating and are therefore bounded - never to decay or grow. Within this region, outside of the crosses, is a passband interaction.

When $4\Omega = \sigma^2$ a transition frequency is occurring.

This analysis would be true for any general function that can be manipulated into the form of equation (96). As this function had only the two coefficients, we would expect a function of more/ranging coefficients to have more 'crosses' and to change in gradient, depending on the coefficient.

6 Conclusion

From the use of the method of multiple scales we have been able to find a solution for the problem posed in Porter and Chamberlain (1997). In conclusion we have been able to take a governing equation and set of boundary conditions with regards to a velocity potential, and whilst showing that an asymptotic expansion has been unsuccessful, we have shown that a multiple scale expansion can be used to give accurate results, demonstrating regions of uniform propagation and regions of decay.

For further work, we could improve the accuracy of our expansion and test the approximations of multiple scales. We could do this by seeing what happens at $O(\epsilon^2)$. We would do this by considering an expansion of the form

$$\phi = \phi_0(x_0, x_1, x_2, z) + \epsilon\phi_1(x_0, x_1, x_2, z) + \epsilon^2\phi_2(x_0, x_1, x_2, z) + ...$$

where $x_0 = x$, the fast variable, $x_1 = \epsilon x$, a slow variable and $x_2 = \epsilon^2 x$ a slower variable. Then by following through the method shown in section 4, we may find some interesting results.

References

[1] D. Porter and P. G. Chamberlain (1997) *Linear Wave Scattering by Two-Dimensional Topography*

[2] O. R. Asfar and A. H. Nayfeh (1983) *The Application of the Method of Multiple Scales to Wave Propagation in Periodic Structures.* SIAM Review, **25**, 455-480.

[3] A. C. King, J. Billingham and S. R. Otto (2003). *Differential Equations: Linear, Nonlinear, Ordinary, Partial.* Cambridge University Press, Cambridge.

[4] A. Majda (2003). *Introduction to PDE's and Waves for the Atmosphere and Ocean.* American Mathematical Society, Providence, Rhode Island.

[5] P. Glaister, 2008-2009, Asymptotic Methods. *Course Notes*